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Well-posedness of the Cauchy problem for nonlinear parabolic equations with variable density in the hyperbolic space

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Abstract. We investigate the well-posedness of the Cauchy problem for a class of nonlinear parabolic equations with variable density in the hyperbolic space. We state sufficient conditions for uniqueness or non-uniqueness of bounded solutions, depending on the behavior of the density at *infinity*. Nonuniqueness relies on the prescription at infinity of suitable conditions of Dirichlet type, and possibly *inhomogeneous*.

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1. Introduction

We study uniqueness and nonuniqueness of bounded solutions to nonlinear Cauchy problems with variable density of the following form:

$$\begin{cases} a\,\partial_t u = \Delta_g\big[\phi(u)\big] & \text{in } \mathbb{H}^n \times (0,T] =: H_T \\ u = u_0 & \text{in } \mathbb{H}^n \times \{0\}\,; \end{cases} \tag{1.1}$$

here $\mathbb{H}^n$ is the n-dimensional hyperbolic space ($n \geq 2$), Δ_g is the Laplace–Beltrami operator in $\mathbb{H}^n$, a, referred to as a density, is a positive function of the space variables, u_0 is bounded. A typical choice for the function ϕ is $\phi(u) = |u|^{m-1}u$ with $m \geq 1$. More precisely, we always assume the following:

$$(H_0) \qquad \begin{cases} \text{(i)} & a \in C(\mathbb{H}^n)\,,\ a > 0 \quad \text{in } \mathbb{H}^n\,; \\ \text{(ii)} & \phi \in C^1(\mathbb{R}),\ \phi(0) = 0,\ \phi'(s) > 0 \text{ for any } s \in \mathbb{R}\setminus\{0\}\,, \\ & \phi' \text{ decreasing in } (-\delta, 0) \text{ and increasing in } (0,\delta), \\ & \text{if } \phi'(0) = 0\ (\delta > 0)\,; \\ \text{(iii)} & u_0 \in C(\mathbb{H}^n),\ u_0 \text{ is bounded.} \end{cases}$$

The Cauchy problem (1.1) can be regarded as the counterpart in $\mathbb{H}^n$ of the Cauchy problem

$$\begin{cases} a\,\partial_t u = \Delta\big[\phi(u)\big] & \text{in } \mathbb{R}^n \times (0,T] \\ u = u_0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases} \tag{1.2}$$

which has been the object of detailed investigations (*e.g.*, see [2,4,6,13,18,21]). It is worth mentioning that problem (1.2) arises in situations of physical interest (see [15]) in a wide variety of fields that involve diffusion processes. When such diffusion processes are considered on the hyperbolic space $\mathbb{H}^n$, then it is natural to introduce problem (1.1) to describe them.

In the literature, the case of linear diffusion has been particularly addressed. In fact, problem (1.1) with $a \equiv 1$ and $\phi(u) = u$ has been largely studied (*e.g.*, see [5,7–10] and references therein) by means of analytical and probabilistic methods.

Notice that also qualitative properties of solutions to semilinear elliptic equations on $\mathbb{H}^n$ have been investigated (see [3,16]). Of course, parabolic and elliptic equations on the hyperbolic space have attracted so much attention, since $\mathbb{H}^n$ is the model of complete Riemannian manifolds with constant negative sectional curvature (see [1,8]).

Recall that, as is well-known, problem (1.2) is well-posed in the class of bounded solutions when a is bounded and $n \leq 2$ (see [11]), or when $n \geq 3$ and a is constant (see [2,4,12]). Moreover, for $n \geq 3$, uniqueness of bounded solutions to problem (1.2), not satisfying any extra constraints at infinity, has been showed if $a(x) \to 0$ slowly, or $a(x)$ is bounded from below by a positive constant or it goes to infinity as $|x| \to \infty$ (see [13,19]).

On the contrary, for $n \geq 3$, existence of bounded solutions to problem (1.2), satisfying at infinity some additional conditions, has been proved, if $a(x) \to 0$ sufficiently fast as $|x| \to \infty$ (see [6,13,14,19,21]). Clearly, these existence results imply nonuniqueness of bounded solutions to the Cauchy problem (1.2).

Such uniqueness and nonuniqueness results have also a deep probabilistic interpretation in the case $\phi(u) = u$. In fact, uniqueness for problem (1.2) holds true, when the Markov process associated to the operator $\frac{1}{a}\Delta$ starting from any point of $\mathbb{R}^n$ is recurrent (see [8]), thus, loosely speaking, it cannot attain the *infinity*, therefore we cannot prescribe conditions at infinity. Instead, when this Markov process is transient, and so it can reach the *infinity*, we can prescribe some conditions at infinity in a proper sense, whence nonuniqueness follows.

The aim of this paper is to prove the following:

- (i) uniqueness of bounded solutions to problem (1.1) not satisfying at infinity any additional constraints, when $a(x)$ is bounded from below by a positive constant or it goes to infinity as x tends to *infinity* and $n \geq 2$;
- (ii) existence of bounded solutions to problem (1.1) satisfying at infinity some extra conditions, when $a(x)$ goes to zero as x tends to *infinity* and $n \geq 2$.

Such results extend to the nonlinear problem (1.1) those already established, in [17], for problem (1.1) with $\phi(u) = u$ and $n \geq 2$, by means of different methods which cannot be applied to the nonlinear problem (1.1) with general ϕ. Moreover, for $a(x) \equiv 1$, $\phi(u) = u$ and $n \geq 2$, uniqueness of bounded solutions to the Cauchy problem (1.1) has already been obtained in [8]. Finally, let us mention that support properties of solutions to problem (1.1) has been studied in [20], provided that *supp* u_0 is compact.

The paper is organized as follows. In Sect. 2 we state the uniqueness and nonuniqueness results, while in Sect. 3 we introduce some preliminary notions of hyperbolic geometry which will be used in the sequel. Finally, in Sects. 4 and 5 we shall prove uniqueness and nonuniqueness results, respectively.

2. Results

To state our results, we need to introduce some notations. For any $r > 0$ let $B_r := \{x \in \mathbb{R}^n \,|\, |x| < r\}$; here and hereafter $|\cdot|$ denotes the standard euclidian norm.

In the sequel we shall consider the *Poincaré* ball model of the hyperbolic space $\mathbb{H}^n$. To be specific, we set $\mathbb{H}^n \equiv B_1$ endowed with the Riemannian metric

$$g_{ij} := \frac{4}{(1-|x|^2)^2}\delta_{ij} \quad (x \in B_1;\ i,j = 1,\ldots,n); \tag{2.3}$$

moreover, let

$$g^{ij} := (g_{ij})^{-1}, \quad g := det(g_{ij}) \quad (i,j = 1,\ldots,n).$$

As usual (see [1]), we set

$$\partial B_1 \equiv \partial_\infty \mathbb{H}^n \equiv \{\infty\}. \tag{2.4}$$

2.1. Uniqueness results

At first, let us state a standard existence result of solutions to problem (1.1).

Theorem 2.1. *Let assumption* (H_0) *be satisfied. Then there exists a bounded solution of problem* (1.1).

The following uniqueness result will be proved.

Theorem 2.2. *Let assumption* (H_0) *be satisfied. Moreover, suppose that*

$$(H_1) \qquad \begin{cases} \text{there exist } C_1 > 0 \text{ and } \alpha \geq 0 \text{ such that} \\ a(x) \geq C_1(1-|x|)^{-\alpha} \text{ for any } x \in \mathbb{H}^n. \end{cases}$$

Then there exists at most one bounded solution of problem (1.1).

Remark 2.3. Hypothesis (H_1) can be expressed also as follows:

$$\begin{cases} \text{there exist } C_1 > 0 \text{ and } \alpha \geq 0 \text{ such that} \\ a(x) \geq C_1 e^{\alpha\rho(x)} \ \text{ for any } \ x \in \mathbb{H}^n\,; \end{cases}$$

here $\rho(x)$ denotes the *geodesic distance* in $\mathbb{H}^n$ of $x \in \mathbb{H}^n$ from 0 (see (3.1) below).

Remark 2.4. In Theorem 2.2 hypothesis (H_1) can be replaced by the following weaker condition

$$(H_1)^* \qquad \begin{cases} \text{there exist } \hat{\varepsilon} \in (0,1) \text{ and } \underline{a} \in C([0,\hat{\varepsilon}];[0,+\infty)) \text{ such that} \\ \text{(i)} \quad a(x) \geq \underline{a}(1-|x|) > 0 \text{ for any } x \in B_{1-\hat{\varepsilon}}, \text{ and} \\ \text{(ii)} \quad \int_0^{\hat{\varepsilon}} \frac{\underline{a}(\eta)}{\eta}\, d\eta = \infty. \end{cases}$$

2.2. Nonuniqueness results

We shall prove the following result which regards existence of solutions to problem (1.1) satisfying additional conditions at *infinity* (see (2.4)).

Theorem 2.5. *Let assumption* (H_0) *be satisfied and* $A \in Lip([0,T])$ *with* $A(0)=0$. *Moreover, suppose that*

$$(H_2) \qquad \begin{cases} \text{there exist } C_2 > 0 \text{ and } \alpha < 0 \text{ such that} \\ a(x) \leq C_2(1-|x|)^{-\alpha} \ \text{ for any } x \in \mathbb{H}^n. \end{cases}$$

Then there exists a bounded solution u *of problem* (1.1) *such that*

$$\lim_{|x|\to 1} |U(x,t) - A(t)| = 0 \tag{2.5}$$

uniformly with respect to $t \in [0,T]$; *here*

$$U(x,t) := \int_0^t \phi(u(x,\tau))\, d\tau \quad ((x,t) \in H_T). \tag{2.6}$$

Remark 2.6. Hypothesis (H_2) can be expressed also as follows:

$$\begin{cases} \text{there exist } C_2 > 0 \text{ and } \alpha < 0 \text{ such that} \\ a(x) \leq C_2 e^{\alpha\rho(x)} \ \text{ for any } x \in \mathbb{H}^n\,; \end{cases}$$

here $\rho(x)$ denotes the *geodesic distance* in $\mathbb{H}^n$ of $x \in \mathbb{H}^n$ from 0 (see (3.1) below).

Consider the following condition:

$$(H_2^*) \qquad \text{there exists } x_0 \in \mathbb{H}^n \text{ such that } \int_{\mathbb{H}^n} G_g(x_0,y)a(y)\, d\mu(y) < \infty,$$

where G_g is the Green function for Δ_g in $\mathbb{H}^n$ (see [8]) and $d\mu$ is the Riemannian volume element (see (3.6) below).

Remark 2.7. If assumption $(H_2)^*$ is satisfied, then we have $\int_{\mathbb{H}^n} G_g(x,y)a(y)\, d\mu(y) < \infty$ for any $x \in \mathbb{H}^n$ (see [7]). Furthermore, since

$$G_g(0,x) = \int_{\rho(x)}^{\infty} \frac{d\xi}{(\sinh\xi)^{n-1}} \quad (x \in \mathbb{H}^n \setminus \{0\}),$$

it is direct to see that hypothesis (H_2) implies $(H_2)^*$.

If in Theorem 2.5 assumption (H_2) is replaced by the weaker condition $(H_2)^*$, then we obtain next

Theorem 2.8. *Let assumptions* $(H_0), (H_2)^*$ *be satisfied and* $A \in Lip([0,T])$ *with* $A(0) = 0$*. Then there exists a sequence* $\{x_m\} \subseteq \mathbb{H}^n$, $|x_m| \to 1$ *as* $m \to \infty$ *and a bounded solution* u *of problem* (1.1) *satisfying*

$$\lim_{m\to+\infty} |U(x_m,t) \; - \; A(t)| \; = \; 0 \tag{2.7}$$

uniformly with respect to $t \in [0,T]$, *with* U *defined in* (2.6).

Remark 2.9. Clearly, both Theorems 2.5 and 2.8 imply nonuniqueness of bounded solutions to problem (1.1).

3. Mathematical background

The geodesic distance between any $x \in \mathbb{H}^n$ and 0 is given by

$$\rho(x) := \int_0^{|x|} \frac{2}{1-s^2}\, ds = \log\left(\frac{1+|x|}{1-|x|}\right) \quad (x \in \mathbb{H}^n \equiv B_1)\,; \tag{3.1}$$

therefore

$$|x| \; = \; \tanh\left(\frac{\rho(x)}{2}\right), \quad \frac{2}{1-|x|^2} \; = \; 2\left[\cosh\left(\frac{\rho(x)}{2}\right)\right]^2 \quad (x \in \mathbb{H}^n\,). \tag{3.2}$$

For any $r > 0$ define

$$\mathfrak{B}_r := \{x \in \mathbb{H}^n \,|\, \rho(x) \; < \; r\}\,, \;\; \mathfrak{S}_r := \{x \in \mathbb{H}^n \,|\, \rho(x) \; = \; r\,\}. \tag{3.3}$$

Hence for any $r \in (0,1)$

$$B_r = \mathfrak{B}_{\log\left(\frac{1+r}{1-r}\right)}. \tag{3.4}$$

Let (ρ,θ) be polar geodesic coordinates in $\mathbb{H}^n$. Then we have

$$ds^2 := \sum_{i,j=1}^{n} g_{ij}\; dx_i dx_j = d\rho^2 \; + (\sinh\rho)^2\, d\theta;$$

moreover,

$$\Delta_g = \frac{\partial^2}{\partial\rho^2} + (n-1)\coth\rho\frac{\partial}{\partial\rho} + \frac{1}{(\sinh\rho)^2}\,\Delta_\theta, \tag{3.5}$$

where Δ_θ is the Laplace–Beltrami operator on the $(n-1)-$dimensional sphere of $\mathbb{R}^n$.

In $\mathbb{H}^n$ the Riemannian volume element is defined by

$$d\mu := \sqrt{g}\, dx_1 dx_2 \ldots dx_n \; = \; \frac{2^n}{(1-|x|^2)^n} dx_1 \ldots dx_n, \tag{3.6}$$

$dx_1 dx_2 \ldots dx_n \equiv dx$ being the Lebesgue measure in $\mathbb{R}^n$. Furthermore, let $d\mu'$ be the Riemannian area element on submanifold of $\mathbb{H}^n$ of codimension 1. We have for any $r > 0$

$$\mu'(\mathfrak{S}_r) = \omega_n (\sinh r)^{n-1}, \tag{3.7}$$

where ω_n is the area of the unit sphere of $\mathbb{R}^n$.

For every $x \in \mathbb{H}^n$, let $T_x\mathbb{H}^n$ denote the tangent space to $\mathbb{H}^n$ at x. As usual, let $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ be the standard basis of $T_x\mathbb{H}^n$. The inner Riemannian product $\langle \cdot, \cdot \rangle_g$ for every $x \in \mathbb{H}^n$ is given by:

$$\langle \beta, \xi \rangle_g \equiv \langle \beta, \xi \rangle_{g,x} = \sum_{i,j=1}^{n} g_{ij}\, \beta_i\, \xi_j \tag{3.8}$$

for any $\beta, \xi \in T_x\mathbb{H}^n$, with $\beta = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i}$ and $\xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$ for some $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

Recall that the gradient $\nabla_g u \equiv \big((\nabla_g u)_1, \dots, (\nabla_g u)_n\big)$ is given by

$$(\nabla_g u)_i := \sum_{j=1}^{n} g^{ij} \frac{\partial u}{\partial x_j} \quad (i = 1, \dots, n)\,; \tag{3.9}$$

furthermore,

$$\|\nu\|_g := \sqrt{\langle \nu, \nu \rangle_g} \quad (\nu \in T_x\mathbb{H}^n, x \in \mathbb{H}^n).$$

Let us make the following definition.

Definition 3.1. A *solution* of problem (1.1) is a function $u \in C\,(\mathbb{H}^n \times (0,T])$ such that

$$\int_0^\tau \int_{\Omega_1} \{a\, u\, \partial_t \psi + \phi(u)\Delta_g \psi\}\, d\mu dt = \int_{\Omega_1} a\, \big[u(x,\tau)\psi(x,\tau) - u_0(x)\psi(x,0)\big]\, d\mu + \int_0^\tau \int_{\partial\Omega_1} \phi(u) \langle \nabla_g \psi,\, \nu\rangle_g d\mu'\, dt \tag{3.10}$$

for any precompact set $\Omega_1 \subseteq \mathbb{H}^n$ with smooth boundary $\partial\Omega_1$, $\overline{\Omega}_1 \subseteq \mathbb{H}^n$, $\tau \in (0,T]$, $\psi \in C^{2,1}(\overline{\Omega_1} \times [0,\tau]), \psi \geq 0, \psi = 0$ in $\partial\Omega_1 \times [0,\tau]$; ν denotes the outer normal to Ω_1 satisfying $\|\nu\|_g = 1$.

Supersolutions (subsolutions) of (1.1) are defined replacing "=" by "≤" ("≥", respectively) in (3.10).

4. Proof of Theorems 2.1 and 2.2

Proof of Theorem 2.1. The conclusion follows by the same argument as in the proof of Theorem 2.5 in [18]. □

In what follows we regard $B_{1-\varepsilon}$ $(\varepsilon \in (0,1))$ as a subset of $\mathbb{H}^n$, in view of the identification $\mathbb{H}^n \equiv B_1$. For any $\varepsilon \in (0,1)$ the operator Δ_g is uniformly elliptic in $B_{1-\varepsilon}$, hence there exists a unique classical solution ψ_ε of the elliptic problem:

$$\begin{cases} \Delta_g U = -F & \text{in } B_{1-\varepsilon} \\ U = 0 & \text{on } \partial B_{1-\varepsilon}\,, \end{cases} \tag{4.1}$$

where $F \in C_0^\infty(B_1)$. In the proof of Theorem 2.2 we will use next lemma.

Lemma 4.1. *Let assumption* (H_0) *be satisfied,* $\varepsilon_0 \in (0,1)$*. For any* $\varepsilon \in (0,\varepsilon_0)$ *let* ψ_ε *be the solution to problem* (4.1) *with* $F \in C^\infty(B_1)$, $F \geq 0$, *supp* $F \subseteq B_{1-\varepsilon_0}$*. Then the following statements hold true:*

(i) *for any* $\varepsilon_1, \varepsilon_2 \in (0,\varepsilon_0)$, $\varepsilon_1 \geq \varepsilon_2$ *we have*

$$0 < \psi_{\varepsilon_1} \leq \psi_{\varepsilon_2} \quad \text{in } B_{1-\varepsilon_1}; \tag{4.2}$$

(ii) *there exists* $C > 0$ *such that for any* $\varepsilon \in (0,\varepsilon_0/2)$ *we have*

$$-C\left(\frac{1-|x|^2}{2|x|}\right)^{n-1} \leq \langle \nabla_g \psi_\varepsilon \,,\, \nu_\varepsilon \rangle_g < 0 \quad \text{for any } x \in \partial B_{1-\varepsilon} \tag{4.3}$$

ν_ε *denoting the outer normal to* $B_{1-\varepsilon}$ *at* $\partial B_{1-\varepsilon}$*, with* $\|\nu_\varepsilon\|_g = 1$*;*

(iii) *there exist* $\bar{C} > 0$ *and* $\bar{\varepsilon} \in (0,\varepsilon_0)$ *such that*

$$\psi_0(x) \geq \bar{C}(1-|x|^2)^{n-1} \quad \text{for any } x \in B_1 \setminus B_{1-\bar{\varepsilon}}\,, \tag{4.4}$$

where

$$\psi_0 := \lim_{\varepsilon\to 0} \psi_\varepsilon \quad \text{in } B_1. \tag{4.5}$$

Proof. (*i*) By the strong maximum principle $\psi_\varepsilon > 0$ in $B_{1-\varepsilon}$ for any $\varepsilon \in (0,\varepsilon_0)$; hence the function $\psi_{\varepsilon_1} - \psi_{\varepsilon_2}$ $(0 < \varepsilon_2 \leq \varepsilon_1 < \varepsilon_0)$ is a subsolution of problem

$$\begin{cases} \Delta_g U = 0 & \text{in } B_{1-\varepsilon_1} \\ U = 0 & \text{on } \partial B_{\varepsilon_1}. \end{cases} \tag{4.6}$$

Then again by the maximum principle we get (4.2).

(*ii*) Clearly, by the strong maximum principle there holds $\langle \nabla \psi_\varepsilon \,,\, \nu_\varepsilon \rangle_{\mathbb{R}^n} < 0$ on ∂B_ε for any $\varepsilon \in (0,\varepsilon_0/2)$. Hence

$$\langle \nabla_g \psi, \nu_\varepsilon \rangle_g < 0. \tag{4.7}$$

It is well-known that (see [8])

$$\psi_0(x) = \int_{\mathbb{H}^n} G_g(x,y)\, F(y)\, d\mu(y) \quad (x \in \mathbb{H}^n \equiv B_1)\,,$$

where ψ_0 is defined in (4.5). From (*i*), for any $\varepsilon \in (0,\varepsilon_0)$, we have

$$\psi_\varepsilon \leq \psi_0 \quad \text{in } B_{1-\varepsilon}. \tag{4.8}$$

For any $\varepsilon \in (0,\varepsilon_0)$ let us introduce the problem

$$\begin{cases} \Delta_g U = 0 & \text{in } B_{1-\varepsilon} \setminus B_{1-\varepsilon_0} \\ U = 0 & \text{on } \partial B_{1-\varepsilon} \\ U = M & \text{on } \partial B_{1-\varepsilon_0}, \end{cases} \tag{4.9}$$

where $M := \max_{\partial B_{1-\varepsilon_0}} \psi_0$.

Observe that (see [8], Sect. 4.2) the function

$$\check{Z}(x) := \int_{\rho(x)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}}\, d\xi \quad (x \in \mathbb{H}^n)$$

solves

$$\Delta_g \check{Z} = 0 \quad \text{in } \mathbb{H}^n \setminus \{0\}. \tag{4.10}$$

Define

$$Z(x) \equiv Z(\rho(x)) := C_\varepsilon \left[\int_{\rho(x)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}}\, d\xi - \int_{\tilde{\rho}(1-\varepsilon)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}}\, d\xi \right]$$
$$(x \in B_{1-\varepsilon} \setminus B_{1-\varepsilon_0}),$$

where

$$C_\varepsilon := \frac{M}{\int_{\tilde{\rho}(1-\varepsilon_0)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}}\, d\xi - \int_{\tilde{\rho}(1-\varepsilon)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}}\, d\xi},$$

$$\tilde{\rho}(s) := \log\left(\frac{1+s}{1-s}\right) \quad (s \in [0,1)).$$

From (4.10) immediately follows that Z is a supersolution to problem (4.9) for any $\varepsilon \in (0, \varepsilon_0/2)$. On the other hand, ψ_ε ($\varepsilon \in (0, \varepsilon_0)$) is a subsolution to the same problem. By comparison principles we get for any $\varepsilon \in (0, \varepsilon_0/2)$

$$Z \geq \psi_\varepsilon \quad \text{in} \,; B_{1-\varepsilon} \setminus B_{1-\varepsilon_0}.$$

Since for any $\varepsilon \in (0, \varepsilon_0/2)$

$$Z = \psi_\varepsilon = 0 \quad \text{on } \partial B_{1-\varepsilon},$$

we can infer that

$$\langle \nabla_g \psi_\varepsilon, \nu_\varepsilon \rangle_g \geq \langle \nabla_g Z, \nu_\varepsilon \rangle_g = \frac{\partial Z(\rho)}{\partial \rho} = -\frac{C_\varepsilon}{[\sinh(\rho(x))]^{n-1}}$$
$$\geq -C_{\varepsilon_0/2} \left(\frac{1-|x|^2}{2|x|}\right)^{n-1} \quad (x \in \partial B_{1-\varepsilon}),$$

This combined with (4.7) gives (4.3).

(iii) Let $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$. Clearly, the function ψ_{ε_1} is a supersolution to problem

$$\begin{cases} \Delta_g U = -F & \text{in } B_{1-\varepsilon_1} \setminus B_{1-\varepsilon_2} \\ U = 0 & \text{on } \partial B_{1-\varepsilon_1} \\ U = m & \text{on } \partial B_{1-\varepsilon_2}, \end{cases} \tag{4.11}$$

where $m := \min_{\partial B_{1-\varepsilon_2}} \psi_{\varepsilon_1}$. From *(i)* it follows that for any $0 < \varepsilon < \varepsilon_1$

$$\psi_\varepsilon \geq \psi_{\varepsilon_1} \quad \text{in} \quad B_{1-\varepsilon_1} \setminus B_{1-\varepsilon_2}.$$

Then, for any $\varepsilon \in (0, \varepsilon_2)$, ψ_ε is a supersolution to problem

$$\begin{cases} \Delta_g U = -F & \text{in } B_{1-\varepsilon} \setminus B_{1-\varepsilon_2} \\ U = 0 & \text{on } \partial B_{1-\varepsilon} \\ U = m & \text{on } \partial B_{1-\varepsilon_2}. \end{cases} \tag{4.12}$$

Observe that

$$\tilde{\psi}_\varepsilon := m \frac{\int_{\rho(x)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}}\, d\xi \, - \, \int_{\tilde{\rho}(1-\varepsilon)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}}\, d\xi}{\int_{\tilde{\rho}(1-\varepsilon_2)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}}\, d\xi \, - \, \int_{\tilde{\rho}(1-\varepsilon)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}}\, d\xi} \quad (x \in B_{1-\varepsilon} \setminus B_{1-\varepsilon_2}),$$

is a subsolution to problem (4.12). By comparison principles we have

$$\tilde{\psi}_\varepsilon \le \psi_\varepsilon \quad \text{in } \ B_{1-\varepsilon} \setminus B_{1-\varepsilon_2}.$$

Sending $\varepsilon \to 0$ we obtain

$$\frac{m}{\int_{\tilde{\rho}(1-\varepsilon_2)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}} d\xi} \int_{\rho(x)}^{+\infty} \frac{1}{(\sinh \xi)^{n-1}} d\xi \le \psi_0 \quad \text{in } \ B_1 \setminus B_{1-\varepsilon_2}.$$

Then easy computations yield (4.4) with $\bar{\varepsilon} = \varepsilon_2$.

□

Let us prove the following

Lemma 4.2. *Let assumptions* $(H_0), (H_1)^*$ *be satisfied. Let* v_1, v_2 *be any two bounded solutions of problem* (1.1) *such that* $v_1 \ge v_2$ *in* H_T. *Then*

$$\liminf_{\varepsilon \to 0} \varepsilon^{n-1} \int_0^T \int_{\partial B_{1-\varepsilon}} \{\phi(v_1) - \phi(v_2)\}\, d\mu'\, dt = 0. \tag{4.13}$$

Proof. From problem (1.1) we have

$$\begin{cases} a\,(\partial_t v_1 - \partial_t v_2) = \Delta\big[\phi(v_1) - \phi(v_2)\big] \text{ in } H_T \\ v_1 - v_2 = 0 \quad\quad\quad\quad\quad\quad\quad\quad \text{in } \mathbb{H}^n \times \{0\}. \end{cases} \tag{4.14}$$

Then equality (3.10) with $\Omega_1 = B_{1-\varepsilon}$ $(\varepsilon \in (0, \varepsilon_0/2))$, $\tau \in (0, T]$ yields

$$\begin{aligned} \int_{B_{1-\varepsilon}} a(x)\big[v_1(x,\tau) - v_2(x,\tau)\big]\psi_\varepsilon(x)\, d\mu + \int_0^\tau \int_{B_{1-\varepsilon}} \big[\phi(v_1) - \phi(v_2)\big] F(x)\, d\mu\, dt \\ = -\int_0^\tau \int_{\partial B_{1-\varepsilon}} \{\phi(v_1) - \phi(v_2)\}\langle \nabla_g \psi_\varepsilon, \nu_\varepsilon\rangle_g\, d\mu'\, dt \end{aligned} \tag{4.15}$$

where ψ_ε denotes the solution of problem (4.1). Set

$$\varphi(\varepsilon) := \int_0^T \int_{\partial B_{1-\varepsilon}} \{\phi(v_1) - \phi(v_2)\}\, d\mu'\, dt \quad (\varepsilon \in (0, \varepsilon_0/2)).$$

Suppose, by absurd, that

$$\liminf_{\varepsilon \to 0} \varepsilon^{n-1}\varphi(\varepsilon) \ge \gamma$$

for some $\gamma > 0$; then there exists $\breve{\varepsilon} \in (0, \varepsilon_0/2)$ such that

$$\varepsilon^{n-1}\varphi(\varepsilon) \geq \frac{\gamma}{2} \quad \text{for any } \varepsilon \in (0, \breve{\varepsilon}). \tag{4.16}$$

Let $M := \max\{\sup_{H_T}|v_1|, \sup_{H_T}|v_2|\}$. Since $F \geq 0$, $v_1 \geq v_2$ and $(H_0) - (ii)$ holds true, from (4.3) and (4.15) we have

$$\begin{aligned}
&\int_0^T \int_{B_{1-\varepsilon}} a(x)\left[v_1(x,\tau) - v_2(x,\tau)\right]\psi_\varepsilon(x)\, d\mu d\tau \\
&\leq \int_0^T \int_0^\tau \int_{\partial B_{1-\varepsilon}} \{\phi(v_1) - \phi(v_2)\}\, |\langle \nabla_g \psi_\varepsilon, \nu_\varepsilon\rangle_g|\, d\mu' dt d\tau \\
&\leq 2 \max_{-M\leq r\leq M}|\phi(r)| \int_0^T \int_0^\tau \int_{\partial B_{1-\varepsilon}} |\langle \nabla_g \psi_\varepsilon, \nu_\varepsilon\rangle_g|\, d\mu' dt d\tau \\
&\leq 2 \max_{-M\leq r\leq M}|\phi(r)| T^2 \mu'(\partial B_{1-\varepsilon}) C \left[\frac{(2-\varepsilon)\varepsilon}{2(1-\varepsilon)}\right]^{n-1} \\
&= 2 \max_{-M\leq r\leq M}|\phi(r)| T^2 C \mu'(\mathfrak{S}_{\log\left(\frac{2-\varepsilon}{\varepsilon}\right)}) \left[\frac{(2-\varepsilon)\varepsilon}{2(1-\varepsilon)}\right]^{n-1} \\
&= 2 \max_{-M\leq r\leq M}|\phi(r)| T^2 C \omega_n \left[\frac{2(1-\varepsilon)}{(2-\varepsilon)\varepsilon}\right]^{n-1} \left[\frac{(2-\varepsilon)\varepsilon}{2(1-\varepsilon)}\right]^{n-1} \\
&= 2 \max_{-M\leq r\leq M}|\phi(r)| T^2 C \omega_n
\end{aligned} \tag{4.17}$$

for any $\varepsilon \in (0, \varepsilon_0/2)$; here use of (3.7) has been made.

Letting $\varepsilon \to 0$ in (4.17), by (4.2), (4.5) and the monotone convergence theorem we have

$$\int_0^T \int_\Omega a(x)\left[v_1(x,\tau) - v_2(x,\tau)\right]\psi_0(x)\, dx d\tau \leq 2 \max_{-M\leq r\leq M}|\phi(r)| T^2 C \omega_n. \tag{4.18}$$

On the other hand, we have

$$\begin{aligned}
&\int_0^T \int_{B_1} a(x)\left[v_1(x,\tau) - v_2(x,\tau)\right]\psi_0(x)\, d\mu d\tau \\
&\geq \frac{1}{L} \int_0^T \int_{B_1\setminus B_{1-\varepsilon}} a(x)\left[\phi(v_1) - \phi(v_2)\right]\, \psi_0(x)\, d\mu\, d\tau;
\end{aligned} \tag{4.19}$$

here $\tilde{\varepsilon} := \min\{\hat{\varepsilon}, \bar{\varepsilon}, \breve{\varepsilon}\}$, $L := \max_{s\in[-M,M]} \phi'(s)$.

Clearly, we have:

$$\int_{B_{1-\varepsilon}} d\mu = \int_0^{1-\varepsilon} \int_{\partial B_r} \frac{2}{1-r^2}\, dr\, d\mu', \tag{4.20}$$

where $d\mu'$ is the Riemannian area element on ∂B_r $(0 < \varepsilon < 1, 0 < r < 1-\varepsilon)$.

By $(H_1)^*$, (4.16) and (4.20) we have

$$\begin{aligned}
&\frac{1}{L}\int_0^T\int_{B_1\setminus B_{1-\tilde\varepsilon}} a(x)\big[\phi(v_1)-\phi(v_2)\big]\psi_0(x)\,d\mu d\tau\\
&\geq \frac{\bar C}{L}\int_0^T\int_{B_1\setminus B_{1-\tilde\varepsilon}} \underline{a}(1-|x|)\big[\phi(v_1)-\phi(v_2)\big](1-|x|^2)^{n-1}\,d\mu d\tau\\
&= \frac{2\bar C}{L}\int_0^{\tilde\varepsilon}\int_{\partial B_{1-\varepsilon}} \frac{\underline{a}(\varepsilon)}{\varepsilon(2-\varepsilon)}(2-\varepsilon)^{n-1}\varepsilon^{n-1}\int_0^T[\phi(v_1)-\phi(v_2)]\,d\mu' d\varepsilon d\tau\\
&= \frac{2\bar C}{L}\int_0^{\tilde\varepsilon}\frac{\underline{a}(\varepsilon)}{\varepsilon}(2-\varepsilon)^{n-2}\varepsilon^{n-1}\varphi(\varepsilon)\,d\varepsilon \geq \frac{(2-\tilde\varepsilon)^{n-2}\bar C\gamma}{L}\int_0^{\tilde\varepsilon}\frac{\underline{a}(\varepsilon)}{\varepsilon}\,d\varepsilon=\infty.
\end{aligned}$$

The previous inequalities and (4.19) yield

$$\int_0^T\int_{B_1} a(x)\big[v_1(x,\tau)-v_2(x,\tau)\big]\psi_0(x)\,d\mu\,d\tau \;=\;\infty,$$

in contrast with (4.18). Hence (4.13) follows. The proof is complete. □

For any $\varepsilon\in(0,\varepsilon_0/2)$ consider the auxiliary problem

$$\begin{cases} a\,u_t=\Delta_g\big[\phi(u)\big] & \text{in } B_{1-\varepsilon}\times(0,T]=:H_{\varepsilon,T}\\ u\,=\zeta & \text{in } \partial B_{1-\varepsilon}\times(0,T)\\ u\,=u_0 & \text{in } B_{1-\varepsilon}\times\{0\},\end{cases}\tag{4.21}$$

where $\zeta\in L^\infty(\partial B_{1-\varepsilon}\times(0,T))$.

Definition 4.3. A *supersolution* of problem (4.21) is a function $u\in C\big(B_{1-\varepsilon}\times(0,T]\big)$ such that

$$\begin{aligned}\int_0^\tau\int_{\Omega_1}\{a\,u\,\partial_t\psi+\phi(u)\Delta_g\psi\}\,d\mu dt &\leq \int_{\Omega_1} a\,\left[u(x,\tau)\psi(x,\tau)-u_0(x)\psi(x,0)\right]d\mu\\ &\quad+\int_0^\tau\int_{\partial\Omega_1\cap\partial B_{1-\varepsilon}}\phi(\zeta)\langle\nabla_g\psi,\,\nu\rangle_g d\mu'\,dt\end{aligned}$$

for any open set $\Omega_1\subseteq B_{1-\varepsilon}$ with smooth boundary $\partial\Omega_1$, $\tau\in(0,T]$, $\psi\in C^{2,1}(\overline{\Omega_1}\times[0,\tau])$, $\psi\geq0$, $\psi=0$ in $\partial\Omega_1\times[0,\tau]$; here ν denotes the outer normal to Ω_1 with $\|\nu\|_g=1$. *Solutions* and *subsolutions* are defined accordingly.

Since Δ_g is a uniformly elliptic operator in $B_{1-\varepsilon}$ for any $\varepsilon\in(0,1)$, existence, uniqueness and comparison results for problem (4.21) can be proved by standard methods.

Now we can prove Theorem 2.2.

Proof of Theorem 2.2. We give the proof replacing (H_1) by the weaker condition $(H_1)^*$, introduced in Remark 2.4.

Let u_1, u_2 be any two bounded solutions of problem (1.1); set

$$M:=\max\{\sup_{H_T}|u_1|,\,\sup_{H_T}|u_2|\}.$$

For any $\varepsilon \in (0, \varepsilon_0/2)$ let u_ε be the unique solution of problem (4.21) with $\zeta \equiv -M$. By comparison results we have:

$$-M \leq u_\varepsilon \leq u_1 \quad \text{and} \quad -M \leq u_\varepsilon \leq u_2 \quad \text{in} \quad H_{\varepsilon,T}. \tag{4.22}$$

By usual compactness arguments there exists a subsequence $\{u_{\varepsilon_m}\} \subseteq \{u_\varepsilon\}$ which converges uniformly in any compact subset of $\mathbb{H}^n \times (0,T]$. Set

$$\underline{u} := \lim_{m\to\infty} u_{\varepsilon_m} \quad \text{in } \mathbb{H}^n \times (0,T].$$

The function $\underline{u}$ is a solution of problem (1.1); moreover, from (4.22) we obtain

$$-M \leq \underline{u} \leq u_1 \quad \text{and} \quad -M \leq \underline{u} \leq u_2 \quad \text{in} \quad H_T. \tag{4.23}$$

Set $w = u_1$ or $w = u_2$ for simplicity. The conclusion will follow, if we show that

$$\int_0^T \int_{\mathbb{H}^n} [\phi(w) - \phi(\underline{u})]\, F\, d\mu\, dt \,=\, 0 \tag{4.24}$$

for any $F \in C_0^\infty(\mathbb{H}^n)$.

In fact, in view of assumption $(H_0) - (ii)$ and the arbitrariness of F, (4.24) implies

$$u_1 \,=\, \underline{u} \,=\, u_2 \quad \text{in} \quad H_T, \tag{4.25}$$

whence the conclusion.

Let us prove equality (4.24). Without loss of generality, we suppose $supp\, F \subseteq (B_{1-\varepsilon_0})$, $F \geq 0$, $F \not\equiv 0$.

In view of the inequality $w \geq \underline{u}$ (see (4.23)), arguing as in the proof of Lemma 4.2, we obtain

$$\begin{aligned} &\int_{B_{1-\varepsilon}} a(x)\, [w(x,\tau) - \underline{u}(x,\tau)]\, \psi_\varepsilon(x)\, d\mu + \int_0^\tau \int_{B_{1-\varepsilon}} [\phi(w) - \phi(\underline{u})]\, F(x)\, d\mu\, dt \\ &\quad = - \int_0^\tau \int_{\partial B_{1-\varepsilon}} \{\phi(w) - \phi(\underline{u})\}\, \langle \nabla_g \psi_\varepsilon, \nu_\varepsilon\rangle_g\, d\mu'\, dt \end{aligned} \tag{4.26}$$

where ψ_ε denotes the solution of (4.1), $\varepsilon \in (0, \varepsilon_0/2)$, $\tau \in (0,T]$. Since $F \geq 0$, $\psi_\varepsilon \geq 0$, $w \geq \underline{u}$ and $(H_0) - (ii)$ holds true, equality (4.26) with $\tau = T$ gives

$$\begin{aligned} &\int_0^T \int_{B_1} [\phi(w) - \phi(\underline{u})]\, F(x)\, d\mu\, dt \\ &\qquad \leq \liminf_{\varepsilon\to 0} \left| \int_0^T \int_{\partial B_{1-\varepsilon}} \{\phi(w) - \phi(\underline{u})\}\, \langle \nabla_g \psi_\varepsilon, \nu_\varepsilon\rangle_g\, d\mu'\, dt \right|. \end{aligned} \tag{4.27}$$

Hence, if we can prove that

$$\liminf_{\varepsilon\to 0} \left| \int_0^T \int_{\partial B_{1-\varepsilon}} \{\phi(w) - \phi(\underline{u})\}\, \langle \nabla_g \psi_\varepsilon, \nu_\varepsilon\rangle_g\, d\mu'\, dt \right| \,=\, 0\,, \tag{4.28}$$

the conclusion follows.

Equality (4.28) is just a consequence of (4.13) with $v_1 = w$ and $v_2 = u$ together with (4.3). The proof is complete. $\square$

5. Proof of Theorem 2.5

We shall consider equation

$$\Delta_g V = -a \quad \text{in} \quad \mathbb{H}^n. \tag{5.1}$$

Let us give the following

Definition 5.1. A *supersolution* to equation (5.1) is a function $U \in C(B_1)$ such that

$$\int_{\Omega_1} U\,\Delta_g\psi\,d\mu \le \int_{\partial\Omega_1} U\,\langle\nabla_g\psi,\nu\rangle_g\,d\mu' - \int_{\Omega_1} a\,\psi\,d\mu \tag{5.2}$$

for any open set $\Omega_1 \subseteq B_1$ with smooth boundary $\partial\Omega_1$, $\overline{\Omega}_1 \subseteq B_1$, $\psi \in C^2(\overline{\Omega_1})$, $\psi \ge 0$, $\psi = 0$ in $\partial\Omega_1$; here ν denotes the outer normal to Ω_1, $\|\nu\|_g = 1$. *Subsolutions* and *solutions* of equation (5.1) are defined accordingly.

The following lemma will be used.

Lemma 5.2. *Let assumption* (H_2) *be satisfied. Then there exists a positive supersolution* V *to equation* (5.1) *such that*

$$\lim_{|x|\to 1} V(x) = 0. \tag{5.3}$$

Proof of Lemma 5.2. Let

$$\beta \in \left(0, \min\left\{\frac{n-1}{2}, -\alpha\right\}\right], \quad C := \frac{2C_2}{\beta}. \tag{5.4}$$

Define

$$V(x) := C(1-|x|^2)^\beta \quad (x \in \mathbb{H}^n).$$

For any $x \in \mathbb{H}^n$ we have:

$$\frac{\partial V(x)}{\partial x_i} = -2C\beta(1-|x|^2)^{\beta-1}x_i\,,$$

$$\frac{\partial^2 V(x)}{\partial x_i \partial x_j} = 2C\beta(1-|x|^2)^{\beta-2}\left[2(\beta-1)x_i x_j - (1-|x|^2)\delta_{ij}\right],$$

where $i, j = 1, \dots, n$. From (5.4) and the fact that $n \ge 2$ we obtain:

$$\begin{aligned}\Delta_g V(x) &= \frac{C}{2}\beta(1-|x|^2)^\beta\left[2(\beta-1)|x|^2 - n(1-|x|^2) - 2(n-2)|x|^2\right]\\ &\le \frac{C}{2}\beta(1-|x|^2)^\beta\left[(2\beta+2-n)|x|^2 - n\right]\\ &\le \frac{C}{2}\beta(1-|x|^2)^\beta\left[(2\beta+1-n)|x|^2 - n + 1\right]\\ &\le -C_2(1-|x|^2)^\beta.\end{aligned}$$

This inequality, combined with hypothesis (H_2) and (5.4), implies that

$$\Delta_g V(x) \le -a(x) \quad \text{for any} \quad x \in \mathbb{H}^n.$$

Moreover, clearly, condition (5.3) is satisfied. Hence the proof is complete. □

Observe that for any $A \in Lip([0,T])$ the derivative A' exists almost everywhere in $[0,T]$ and belongs to $L^\infty((0,T))$.

For any $\varepsilon \in (0, \varepsilon_0/2)$ we will make use of the following auxiliary problems

$$\begin{cases} a\,\partial_t u = \Delta_g\,[\phi(u)] & \text{in } H_{\varepsilon,T} \\ u = \phi^{-1}(A') & \text{in } \partial B_{1-\varepsilon} \times (0,T) \\ u = u_0 & \text{in } B_{1-\varepsilon} \times \{0\} \end{cases} \tag{5.5}$$

and

$$\begin{cases} \Delta_g U = f & \text{in } B_{1-\varepsilon} \\ U = \gamma & \text{in } \partial B_{1-\varepsilon}, \end{cases} \tag{5.6}$$

where $f \in C(B_{1-\varepsilon})$ and $\gamma \in C(\partial B_{1-\varepsilon})$.

Definition 5.3. A *supersolution* of problem (5.6) is a function $U \in C(B_{1-\varepsilon})$ such that

$$\int_{\Omega_1} U\,\Delta_g\psi\,d\mu \le \int_{\Omega_1} f\,\psi\,d\mu + \int_{\partial\Omega_1\cap\partial B_{1-\varepsilon}} \gamma\,\langle\nabla_g\psi,\,\nu\rangle_g\,d\mu'$$

for any open set $\Omega_1 \subseteq B_{1-\varepsilon}$ with smooth boundary $\partial\Omega_1$, $\psi \in C^2(\overline{\Omega_1}), \psi \ge 0, \psi = 0$ in $\partial\Omega_1$; here ν denotes the outer normal to Ω_1 with $\|\nu\|_g = 1$. *Solutions* and *subsolutions* are defined accordingly.

Proof of Theorem 2.5. For any $\varepsilon \in (0, \varepsilon_0/2)$ let u_ε be the unique solution to problem (5.5); then by comparison results we have

$$|u_\varepsilon| \le \sup_{\mathbb{H}^n} |u_0| + \max_{-\|A'\|_\infty \le r \le \|A'\|_\infty} |\phi^{-1}(r)| =: M \quad \text{in } H_{\varepsilon,T}. \tag{5.7}$$

By usual compactness arguments there exists a subsequence $\{u_{\varepsilon_m}\} \subseteq \{u_\varepsilon\}$, which converges uniformly in any compact subset of $\mathbb{H}^n \times (0,T]$ to a solution u of problem (1.1).

Define U as in (2.6) and

$$U_\varepsilon(x,t) := \int_0^t \phi(u_\varepsilon(x,\tau))d\tau \quad ((x,t) \in H_{\varepsilon,T})\,. \tag{5.8}$$

Observe that $U_{\varepsilon_m} \to U$ in $\mathbb{H}^n \times (0,T]$ as $m \to \infty$.

It is straightforward to show that for any $t \in (0,T]$ the function $U_\varepsilon(\cdot,t)$ satisfies the problem (5.6) with $f = -\rho[u_0 - u(\cdot,t)]$, $\gamma = A(t)$. In fact, by Definition 4.3 we obtain

$$\begin{aligned} \int_{\Omega_1} U_\varepsilon(x,t)\Delta_g\psi(x)\,dx = & \int_{\Omega_1} a(x)\,[u_\varepsilon(x,t) - u_0(x)]\,\psi(x)\,d\mu \\ & + \int_{\partial\Omega_1\cap\partial B_{1-\varepsilon}} A(t)\,\langle\nabla_g\psi(x), \nu(x)\rangle_g\,d\mu' \end{aligned} \tag{5.9}$$

for any Ω_1 and $\psi = \psi(x)$ as in Definition 5.3 and $t \in (0,T]$.

We shall prove that

$$-2MV(x)+A(t)\leq U(x,t)\leq 2MV(x)+A(t)\quad ((x,t)\in H_T)\,, \tag{5.10}$$

where V is the supersolution introduced in Lemma 5.2.

From (5.3) and (5.10) the equality (2.5), thus the conclusion follows.

It remains to prove (5.10). It is easily seen that for any fixed $t\in(0,T]$ the function $2MV-U_\varepsilon+A$ is a supersolution, while the function $-2MV-U_\varepsilon+A$ is a subsolution of problem (5.6) with $f=\gamma\equiv 0$ for any $\varepsilon\in(0,\varepsilon_0/2)$. In fact, fix any Ω_1, ψ as in Definition 5.3 and $t\in(0,T]$. Then by (5.9) and (5.2) we have

$$\begin{aligned}\int_{\Omega_1}(2MV-U_\varepsilon+A)\,\Delta_g\psi\,dx\,&\leq\int_{\partial\Omega_1\setminus\partial B_{1-\varepsilon}}(2MV-U_\varepsilon+A)\langle\nabla_g\psi,\nu\rangle_g\,d\mu'\\&+\int_{\partial\Omega_1\cap\partial B_{1-\varepsilon}}2MV\langle\nabla_g\psi,\nu\rangle_g\,d\mu'-\int_{\Omega_1}(2M+u_\varepsilon-u_0)a\,\psi\,dx\,=\end{aligned}\tag{5.11}$$

for any $\varepsilon\in(0,\varepsilon_0)$.

Notice that

$$2MV\,\geq\,0\quad\text{on}\quad\partial B_{1-\varepsilon}\,,\,2M\geq u_0-u_\varepsilon\quad\text{in}\quad\Omega_1 \tag{5.12}$$

for any $\varepsilon\in(0,\varepsilon_0)$. Moreover, it is easily checked that

$$\langle\nabla_g\psi,\nu\rangle_g\leq 0\quad\text{on}\quad\partial\Omega_1. \tag{5.13}$$

From (5.11) to (5.13) we obtain

$$\int_{\Omega_1}(2MV-U_\varepsilon+A)\,\Delta_g\psi\,d\mu\leq\int_{\partial\Omega_1\setminus\partial B_{1-\varepsilon}}(2MV-U_\varepsilon+A)\langle\nabla_g\psi,\nu\rangle_g\,d\mu' \tag{5.14}$$

for any $\varepsilon\in(0,\varepsilon_0)$. This shows that the function $2MV-U_\varepsilon+A$ is a supersolution to problem (5.6) with $f=\gamma\equiv 0$ for any $\varepsilon\in(0,\varepsilon_0)$ (see Definition 5.3). It is similarly seen that $-2MV-U_\varepsilon+A$ is a subsolution of the same problem for any $\varepsilon\in(0,\varepsilon_0)$.

By comparison principles we obtain

$$-2MV(x)+A(t)\leq U_\varepsilon(x,t)\leq 2MV(x)+A(t)\qquad ((x,t)\in[H_{\varepsilon,T})$$

for any $\varepsilon\in(0,\varepsilon_0)$. This implies (5.10), thus the proof is complete. □

To prove Theorem 2.8 we need a preliminary result.

Lemma 5.4. *Let there exist a nonnegative supersolution to equation* (5.1) *such that*

$$\inf_{\mathbb{H}^n}V\,=\,0. \tag{5.15}$$

Then there exists a sequence $\{x_m\}\subseteq\mathbb{H}^n$, $|x_m|\to 1$ *as* $m\to\infty$ *such that*

$$\lim_{m\to\infty}V(x_m)\,=\,0.$$

Lemma 5.4 can be proved arguing as in the proof Lemma 2.6 in [18].

Proof of Theorem 2.8. The function

$$V(x):=\int_{\mathbb{H}^n}G_g(x,y)\,a(y)\quad d\mu(y)-\inf_{x\in\mathbb{H}^n}\left(\int_{\mathbb{H}^n}G_g(x,y)\,a(y)\quad d\mu(y)\right)\quad (x\in\mathbb{H}^n)$$

is a nonnegative solution to equation (5.1) satisfying condition (5.15) (see [8]). Then by Lemma 5.4 there exists a sequence $\{x_m\} \subseteq \mathbb{H}^n$, $|x_m| \to 1$ as $m \to \infty$ such that $V(x_m) \to 0$ as $m \to \infty$. The conclusion follows repeating the proof of Theorem 2.5. □

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