Nonlinear Differ. Equ. Appl. 19 (2012), 459–483

1021-9722/12/040459-25
published online October 12, 2011
DOI 10.1007/s00030-011-0138-x

Nonlinear Differential Equations and Applications NoDEA

Global existence and asymptotic behavior of solutions to the homogeneous Neumann problem for ILW equation on a half-line

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Abstract. We consider the homogeneous Neumann initial-boundary value problem for intermediate long-wave (ILW) equation on a half-line. We study traditionally important problems of the theory of nonlinear partial differential equations, such as global in time existence of solutions to the initial-boundary value problem and the asymptotic behavior of solutions for large time.

Keywords. Dispersive nonlinear evolution equation, Large time asymptotics, Initial-boundary value problem.

Mathematics Subject Classification (1991). Primary 35Q35

1. Introduction

We consider the homogeneous Neumann initial-boundary value problem on a half-line for the intermediate long-wave equation (ILW)

$$\begin{cases} u_t + uu_x + \mathbb{K}u = 0, \quad x > 0, t > 0, \\ u(x,0) = u_0(x), \quad x > 0, \\ u_x(0,t) = 0, \quad t > 0, \end{cases} \tag{1.1}$$

where

$$\mathbb{K}u = -\frac{1}{2\sigma} PV \int_0^{+\infty} \left(\coth \frac{\pi(x-s)}{2\sigma} - sign(x-s) \right) u_{ss}(s,t) ds,$$

for $\sigma > 0$. Here and below PV means the Cauchy principal value of the singular integral.

The ILW equation describes long internal gravity waves in a stratified fluid with a finite depth σ [23]. In the shallow water limit $\sigma \to 0$ ILW equation reduces to the Korteveg–de Vriez (KdV) equation

$$u_t + u_x u + u_{xxx} = 0 \tag{1.2}$$

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and in the deep water limit, $\sigma \to \infty$, reduces to the Benjamin-Ono (BO) equation

$$u_t + u_x u + \frac{1}{\pi} PV \int_{-\infty}^{+\infty} \frac{u_{ss}(s,t)}{x-s} ds = 0. \tag{1.3}$$

The remarkable fact is that all three equations are integrable in the sense that they admit multi-soliton solutions, satisfy the infinite number of conservation laws and are solvable by the inverse scattering transform (see, for example, [1]). The Cauchy problem for Benjamin-Ono equation was studied by many authors. The existence of solutions in the usual Sobolev spaces $\mathbf{H}^{s,0}$ was proved in [2,13,21,22,28,30] and the smoothing properties of solutions were studied in [14,15]. In paper [19] it was obtained asymptotics of small solutions for BO equations. The well-posedness on the line and the periodic domain for KDV equation was studied extensively (see, for example, [4,25,26] and references cited therein). The asymptotic behavior of small solutions was obtained in the case of the generalized KdV equations in [5,20,29].

As far as we know the initial boundary-value problem for Eq. (1.1) was not studied up to now. The boundary value problems are more natural for applications, however their mathematical investigations are more complicated. For example, it is necessary to answer the question of the well-posedness of the problem, in particular, how many boundary values should be given in the problem for its solvability and the uniqueness of the solution. And after that it is also interesting to study the influence of the boundary data on the qualitative properties of the solution.

For integrable nonlinear evolution PDEs such as the BO and the KdV formulated on the line, it is possible to obtain the long time asymptotics of the solution using the so-called Inverse Scattering transform machinery. In this direction, the rigorous asymptotic results can be obtained using the so-called Deift–Zhou approach [6], which is a nonlinear analogue of the classical steepest descent method. A new method for analyzing initial-boundary value problems, based on ideas of the inverse scattering method, was introduced in [9] (see also [10]). Using the Fokas method it is possible, just like the case of problems on the line, to obtain the long time asymptotics of the solution (see [8]), using again the Deift–Zhou approach [11]. For a particular class of boundary conditions, called linearisable, the above method can also be used to establish existence; however, in general it requires certain a priori PDE estimates. The general theory of the initial-boundary value (IBV) problem for KdV equation was developed in the book [17]. Dirichlet problem for the Benjamin-Ono equation on the half- line was considered in paper [18].

In the present paper we consider the Neumann problem (1.1) for ILW equation on half-line. We study traditionally important problems of a theory of nonlinear partial differential equations, such as well-posedness and global in time existence of solutions to the initial-boundary value problem. This type of existence results play a crucial role in the rigorous investigation of integrable equations. Our main goal is to obtain the large time asymptotics of solutions.

This problem is very difficult. Note that even for the case of the Cauchy problem for the famous KdV equation the study of the large time asymptotics of solutions has not been reached completely. From the heuristic point of view the quadratic nonlinearity of the shallow water type uu_x is sub critical for large time: the nonlinear term decays more slowly than the linear part of the equation. In the case of the Neumann boundary-value problem (1.1) we expect that the nonlinear term of ILW equation behaves as critical in the contrary to the corresponding Cauchy problem since the solutions of (1.1) have more rapid time-decay. Our approach is based on the estimates of the integral equation in the weighted Sobolev spaces and weighted $\mathbf{L}^2$ space is used to get smooth solutions [16]. The main difficulty for nonlocal equation (1.1) on a half-line is that the equation is dispersive and its symbol $K(p) = -p\,|p|\,(\coth 2\sigma\,|p| - \frac{1}{2\sigma|p|})$ is non homogeneous. As far as we know even the case of the Cauchy problem for Eq. (1.1) was not studied well. Another difficulty is that the symbol is non analytic, therefore we can not apply the Laplace theory directly. The integral formula is obtained by using the Hilbert transform with respect to the space variable. The Hilbert transform requires the boundary data $u(0,t), u_x(0,t)$ and so $u(0,t)$ should be determined by the given data. To achieve this we need to solve the nonlinear singular integro-differential equation with Hilbert kernel.

To state precisely the results of the present paper we give some notations. We denote $\langle t\rangle = 1+t$, $\{t\} = \frac{t}{\langle t\rangle}$. Direct Laplace transformation $\mathcal{L}_{x\to\xi}$ is

$$\widehat{u}(\xi) \equiv \mathcal{L}_{x\to\xi} u = \int_0^{+\infty} e^{-\xi x} u(x)\,dx$$

and the inverse Laplace transformation $\mathcal{L}^{-1}_{\xi\to x}$ is defined by

$$u(x) \equiv \mathcal{L}^{-1}_{\xi\to x}\widehat{u} = (2\pi i)^{-1}\int_{-i\infty}^{i\infty} e^{\xi x} u(\xi)\,d\xi.$$

Weighted Lebesgue space $\mathbf{L}^{q,a}(\mathbf{R}^+) = \{\varphi \in \mathcal{S}'; \|\varphi\|_{\mathbf{L}^{q,a}} < \infty\}$, where

$$\|\varphi\|_{\mathbf{L}^{q,a}} = \left(\int_0^{+\infty} \langle x\rangle^{aq} |\varphi(x)|^q\,dx\right)^{\frac{1}{q}}$$

for $a > 0$, $1 \le q < \infty$ and

$$\|\varphi\|_{\mathbf{L}^\infty} = \text{ess.}\sup_{x\in\mathbf{R}^+} |\varphi(x)|.$$

Sobolev space

$$\mathbf{H}^1_p(R^+) = \{\varphi \in \mathcal{S}'; \|\langle\partial_x\rangle\,\varphi\|_{\mathbf{L}^p} < \infty\}.$$

and weighted Sobolev space $\mathbf{H}^{1,\mu}(\mathbf{R}^+)$

$$\mathbf{H}^{1,\mu}_p(R^+) = \{\varphi \in \mathcal{S}'; \left\|(x^\mu + x^{1+\mu}\partial_x)\varphi\right\|_{\mathbf{L}^p} < \infty\}.$$

Now we state the main results.

Theorem 1. *Suppose that the initial data* $u_0 \in \mathbf{L}^{1,\frac{21}{4}}(\mathbf{R}^+) \cap \mathbf{H}_2^{1,\frac{7}{2}}(\mathbf{R}^+)$ *and the norm*

$$\|u_0\|_{\mathbf{L}^{1,\frac{21}{4}}} + \|u_0\|_{\mathbf{H}_2^{1,\frac{7}{2}}} \leq \varepsilon,$$

where $\varepsilon > 0$ *is small enough and*

$$\int_0^{+\infty} x u_0(x) dx = \theta < 0.$$

Then there exists a unique solution

$$u \in \mathbf{C}([0, +\infty), \mathbf{H}_2^{1,\frac{7}{2}}(\mathbf{R}^+))$$

of the initial-boundary value problem (1.1) *satisfying the boundary condition such that* $u_x(0,t) = 0$ *for* $t > 0$. *Moreover the solution has the following asymptotics*

$$u(x,t) = \theta (1 + \eta \log t)^{-1} (\sigma t)^{-\frac{2}{3}} Ai'\left(\frac{x}{\sqrt[3]{\sigma t}}\right) + O\left(\varepsilon^2 (\sigma t)^{-\frac{2}{3}} (1 + \eta \log t)^{-\frac{6}{5}}\right)$$

for $t \to +\infty$ *uniformly with respect to* $x > 0$,*where*

$$\eta = -\theta \int_0^{+\infty} Ai'^2(x) dx > 0$$

and $Ai(q)$ *is Airy function*

$$Ai(q) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-z^3 + zq} dz.$$

We note here that solutions of our problem (1.1) decay faster than those of the linear problem and $Ai^{(2)}(0) = 0$, then we see that the main term satisfies the Neumann boundary condition for $t > 0$.

2. Some notations

2.1. Green operator

Setting

$$K(p) = -p|p|(\coth 2\sigma |p| - \frac{1}{2\sigma |p|}), \; K_1(p) = -\frac{\sigma p^3}{\sigma p + 1}, \tag{2.1}$$
$$K_1(k(\xi)) = -\xi, \mathrm{Re}k(\xi) > 0 \text{ for } \xi \in \mathcal{C}_1,$$

we define the following sectionally analytic functions

$$I_1(z,\xi,y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{-qy}}{q - z} \frac{1}{Y^+(q,\xi)} dq, \tag{2.2}$$

$$I_2(p,\xi) = \frac{\xi}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q - z} \left(\frac{1}{Y^+(q,\xi)} - \frac{1}{Y^-(q,\xi)}\right) \frac{1}{q} dq + \frac{K_1(p)}{p} \frac{1}{Y^-(z,\xi)}. \tag{2.3}$$

Here and below

$$Y^{\pm} = e^{\Gamma^{\pm}} w^{\pm}, \tag{2.4}$$

$\Gamma^+(p, \xi)$ and $\Gamma^-(p, \xi)$ are a left and right limiting values of sectionally analytic function $\Gamma(z, \xi)$, given by formula

$$\Gamma(z, \xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q - z} \ln \left\{ \left(\frac{(K(q) + \xi)}{K_1(q) + \xi} \right) \frac{w^-(q)}{w^+(q)} \right\} dq, \tag{2.5}$$

where

$$w^-(z) = \left(\frac{z}{z + k(\xi)} \right)^{\frac{1}{2}}, \quad w^+(z) = \left(\frac{z}{z - k(\xi)} \right)^{\frac{1}{2}}.$$

We make a cut in the plane z from point $k(\xi)$ to point $-\infty$ through 0. Owing to the manner of performing the cut the functions $w^-(z)$, $K_1(z)$ are analytic for Re $z > 0$ and the function $w^+(z)$ is analytic for Re $z < 0$.

Denote by

$$\mathcal{G}(t)\phi = \int_0^{+\infty} G(x, y, t)\phi(y)dy, \tag{2.6}$$

where the function $G(x, y, t)$ is given by formula

$$G(x, y, t) = -\frac{1}{2\pi i} \frac{1}{2\pi i} \int_{\mathcal{C}_1} d\xi e^{\xi t} \tag{2.7}$$
$$\times \int_{-i\infty}^{i\infty} e^{px} \frac{Y^+(p, \xi)}{K(p) + \xi} \left(I_1^-(p, \xi, y) - \frac{I_2^-(p, \xi)}{I_2^-(k(\xi), \xi)} I_1^-(k(\xi), \xi, y) \right) dp.$$

for $x > 0, \ y > 0, \ t > 0$. The contour $\mathcal{C}_1$ is defined as

$$\mathcal{C}_1 = \left\{ p \in \left(\infty e^{-i(\frac{\pi}{2} + \varepsilon)}, 0 \right) \bigcup \left(0, \infty e^{i(\frac{\pi}{2} + \varepsilon)} \right) \right\}, \tag{2.8}$$

where $\varepsilon > 0$ can be chosen such that all functions under integration are analytic and Re $k(\xi) > 0$ for $\xi \in \mathcal{C}_1$. All the integrals are understood in the sense of the principal values.

2.2. Strategy of the proof of Theorem 1

For the convenience of the reader we now state of our strategy of the proof. In Sect. 3 we construct the integral formula of the solution of the linear problem corresponding to (1.1)

$$u(x, t) = \mathcal{G}(t)u_0 = \int_0^{+\infty} G(x, y, t)u_0(y)dy, \tag{2.9}$$

where $G(x, y, t)$ is Green function, given by (2.7). Lemma 1 is devoted to the study of the asymptotic behavior of solutions to the linear problem by using the integral formula (2.9). We will show that for $t > 1, 0 < \delta < 1$

$$\mathcal{G}(t)u_0 = \theta(\sigma t)^{-\frac{2}{3}} Ai' \left(\frac{x}{\sqrt[3]{\sigma t}} \right) + O((\sigma t)^{-\frac{2+\delta}{3}}),$$

where $\theta = \int_0^{+\infty} x u_0 dx$. Therefore the nonlinear term in the equation $u_t + uu_x + \mathbb{K}u = 0$ has the same decay rate as linear terms. Section 4 attempts to prove a global result and to establish asymptotic formulas of solutions. Here as in book [16] we change variable $u = e^{-\phi(t)}v$. Then for the new function v we get the following problem

$$\begin{cases} v_t - \phi_t v + e^{-\phi(t)} v v_x + \mathbb{K}v = 0, & t > 0, x > 0, \\ v(x,0) = e^{\phi(0)} u_0(x), & x > 0, \\ v_x(0,t) = 0, & t > 0. \end{cases}$$

In order to obtain an additional time decay rate we assume that the real-valued function $\phi(t)$ satisfies the following condition

$$\int_0^{+\infty} x \left(-\phi_t v + e^{-\phi(t)} v v_x\right) dx = 0, \phi(0) = 0,$$

which implies that

$$\begin{aligned} e^{\phi(t)} &= g(t) = 1 + \theta^{-1} \int_0^t d\tau \int_0^{+\infty} x v v_x dx \\ &= 1 - \theta^{-1} \int_0^t \|v(\tau)\|^2 d\tau. \end{aligned}$$

We look for the solution v in the neighborhood of the first approximation

$$\mathcal{G}(t) u_0 \approx \theta (\sigma t)^{-\frac{2}{3}} Ai' \left(\frac{x}{\sqrt[3]{\sigma t}}\right) = \theta G_0.$$

We put $r = v - \mathcal{G}(t) u_0$ then we get the following integral formula

$$\begin{cases} r = \int_0^t g(\tau)^{-1} G(x, t-\tau)(G_0 \theta^2 \|G_0\|^2 + G_0 G_{0x}) d\tau + R_1 \\ g = 1 - \theta \int_0^t \|G\|^2 d\tau + R_2, \end{cases}$$

where R_1, R_2 are considered as remainder terms in our function space defined later. By Lemma 2 in Sect. 3 we obtain

$$g(t) = \frac{C}{1 + \eta \log(1+t)} \text{ and } \|r\|_{L^\infty} \le C t^{-\frac{2}{3}} g(t)^{-\frac{1}{5}},$$

where under condition $\theta < 0$

$$\eta = -\theta \int_0^{+\infty} Ai'^2(z) dz > 0.$$

Hence from the representation $u = \frac{v}{g} = \frac{r + \Phi}{g}$ we get the result of Theorem 1.

3. Preliminaries

We consider the following linear initial-boundary value problem on half-line

$$\begin{cases} u_t + \mathbb{K}u = 0, \ t > 0, \ x > 0, \\ u(x,0) = u_0(x), \ x > 0, \\ u_x(0,t) = 0, \ t > 0, \end{cases} \tag{3.1}$$

where for $\sigma > 0$

$$\mathbb{K}u = \frac{1}{2\sigma} PV \int_0^{+\infty} \left(\coth \frac{\pi(x-s)}{2\sigma} - sign(x-s)\right) u_{ss}(s,t) ds.$$

Proposition 1. *Let the initial data $u_0 \in \mathbf{L}^1$. Then there exists a unique solution $u(x,t)$ of the initial-boundary value problem (3.1), which has integral representation*

$$u(x,t) = \mathcal{G}(t)u_0, \tag{3.2}$$

where operator $\mathcal{G}(t)$ is given by (2.6).

Proof. To derive an integral representation for the solutions of the problem (3.1) we suppose that there exists a solution $u(x,t)$ of problem (3.1), which is continued by zero outside of $x > 0$:

$$u(x,t) = 0 \quad \text{for all} \quad x < 0.$$

Let $\phi(p)$ be a function of the complex variable p, which obeys the Hölder condition for all finite p and tends to 0 as $p \to \pm i\infty$. We define the operator

$$\mathbb{P}\phi(z) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\phi(q)dq.$$

We have for the Laplace transform

$$\mathcal{L}\left\{\mathbb{K}\mathrm{u}\right\} = \mathbb{P}\left\{K(p)\left(\mathcal{L}\left\{u\right\} - \frac{u(0,t)}{p} - \frac{u_x(0,t)}{p^2}\right)\right\},$$

where $K(p) = -p\left|p\right|(\coth 2\sigma\left|p\right| - \frac{1}{2\sigma|p|})$.

Since $\mathcal{L}\left\{u\right\}$ is analytic for all $\mathrm{Re}\ q > 0$ we have

$$\widehat{u}(p,t) = \mathcal{L}\left\{u\right\} = \mathbb{P}\widehat{u}(p,t). \tag{3.3}$$

Therefore applying the Laplace transform with respect to x to problem (3.1) we obtain for $t > 0$

$$\mathbb{P}^{-}\left\{\widehat{u}_t + K(p)\left(\widehat{u}(p,t) - \frac{u(0,t)}{p} - \frac{1}{p^2}u_x(0,t)\right)\right\} = 0. \tag{3.4}$$

We rewrite the Eq. (3.4) in the form

$$\widehat{u}_t + K(p)\left(\widehat{u}(p,t) - \frac{u(0,t)}{p} - \frac{1}{p^2}u_x(0,t)\right) = \Phi(p,t), \tag{3.5}$$

with some function $\Phi(p,t) = O(\langle p\rangle^{-1})$ such that

$$\mathbb{P}^{-}\left\{\Phi(p,t)\right\} = 0. \tag{3.6}$$

Applying the Laplace transformation with respect to time variable to problem (3.5) we find for $\mathrm{Re}\ p > 0$

$$\widehat{\widehat{u}}(p,\xi) = \tfrac{1}{K(p)+\xi}\left(\widehat{u}_0(p) + K(p)\tfrac{\widehat{u}(0,\xi)}{p} + \tfrac{K(p)}{p^2}\widehat{u}_x(0,\xi) + \widehat{\Phi}(p,\xi)\right). \tag{3.7}$$

Here the functions $\widehat{\widehat{u}}(p,\xi), \widehat{\Phi}(p,\xi), \widehat{u}(0,\xi)$ and $\widehat{u}_x(0,\xi)$ are the Laplace transforms for $\widehat{u}(p,t), \Phi(p,t), u(0,t)$ and $u_x(0,t)$ with respect to time, respectively. We will find the function $\widehat{\Phi}(p,\xi)$ using the analytic properties of function $\widehat{\widehat{u}}$ in the right-half complex planes $\mathrm{Re}\ p > 0$ and $\mathrm{Re}\ \xi > 0$. We have for $\mathrm{Re}\ p = 0$

$$\widehat{\widehat{u}}(p,\xi) = -\frac{1}{\pi i}VP\int_{-i\infty}^{i\infty} \frac{1}{q-p}\widehat{\widehat{u}}(q,\xi)dq. \tag{3.8}$$

Taking into account the assumed condition (3.6) and making use of Sokhotzki–Plemelj formula we perform the condition (3.8) in the form of non-homogeneous Riemann problem

$$\Omega^{+}(p,\xi)=\frac{K(p)+\xi}{\xi}\Omega^{-}(p,\xi)-K(p)\Lambda^{+}(p,\xi), \tag{3.9}$$

where the sectionally analytic functions functions $\Omega(z,\xi)$ and $\Lambda(z,\xi)$ given by formulas

$$\Omega(z,\xi)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{K(q)}{K(q)+\xi}\widehat{\Phi}(q,\xi)dq, \tag{3.10}$$

$$\Lambda(z,\xi)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{K(q)+\xi}\left(\widehat{u}_0(q)+\frac{K(q)}{q}\widehat{u}(0,\xi)+\frac{K(q)}{q^2}\widehat{u}_x(0,\xi)\right)dq. \tag{3.11}$$

It is required to find two functions for some fixed point ξ, Re $\xi>0$: $\Omega^{+}(z,\xi)$, analytic in Re $z<0$ and $\Omega^{-}(z,\xi)$, analytic in Re $z>0$, which satisfy on the contour Re $p=0$ the relation (3.9).

Note that bearing in mind formula (3.10) we can find unknown function $\widehat{\Phi}(p,\xi)$ which involved in the formula (3.7) by the relation

$$\widehat{\Phi}(p,\xi)=\frac{K(p)+\xi}{K(p)}\left(\Omega^{+}(p,\xi)-\Omega^{-}(p,\xi)\right). \tag{3.12}$$

Setting

$$K_1(p)=-\frac{\sigma p^3}{\sigma p+1}$$

we introduce the function

$$\widetilde{W}(p,\xi)=\left(\frac{K(p)+\xi}{K_1(p)+\xi}\right)\frac{w^{-}(p,\xi)}{w^{+}(p,\xi)},$$

where $w^{\pm}$ were defined by (2.5). We observe that the function $\widetilde{W}(p,\xi)$ given on the contour Re $p=0$, satisfies the Hölder condition and under the assumption Re $K_1(p)>0$ does not vanish for any Re $\xi>0$. Also we have

$$\text{Ind.}\widetilde{W}(p,\xi)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\ln\widetilde{W}(p,\xi)=0.$$

Therefore the function $\widetilde{W}(p,\xi)$ can be uniquely represented as the ratio of the functions $X^{+}(p)$ and $X^{-}(p)$, constituting the boundary values of functions, $X^{+}(z)$ and $X^{-}(z)$, analytic in the left and right complex semi-plane and having in these domains no zero

$$\widetilde{W}(p,\xi)=\frac{X^{+}(p,\xi)}{X^{-}(p,\xi)}. \tag{3.13}$$

These functions are determined by formula

$$X^{\pm}(p,\xi)=e^{\Gamma^{\pm}(p,\xi)},\quad \Gamma(z,\xi)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\widetilde{W}(q,\xi)dq.$$

Now we return to the nonhomogeneous Riemann problem (3.9). Multiplying and dividing the expression $\frac{K(p)+\xi}{\xi}$ by $\frac{1}{K_1(p)+\xi}\frac{w^-(p)}{w^+(p)}$ and making use of the formula (3.13) we get

$$\frac{K(p)+\xi}{\xi} = \frac{Y^+(p,\xi)}{Y^-(p,\xi)}\left(\frac{K_1(p)+\xi}{\xi}\right), \tag{3.14}$$

where

$$Y^{\pm}(p,\xi) = X^{\pm}(p,\xi)w^{\pm}(p).$$

Replacing in Eq. (3.9) the coefficient of the Riemann problem by (3.14) we reduce the nonhomogeneous Riemann problem (3.9) in the form

$$\frac{\Omega_1^+(p,\xi)}{Y^+(p,\xi)} + U^+(p,\xi) = \frac{\Omega_1^-(p,\xi)}{Y^-(p,\xi)} + U^-(p,\xi). \tag{3.15}$$

where

$$U(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}g_1(q,\xi)dq. \tag{3.16}$$

$$g_1(p,\xi) = \left(\frac{1}{Y^+(p,\xi)} - \frac{1}{Y^-(p,\xi)}\right)\left(\widehat{u}_0(p) - \frac{\xi}{q}\widehat{u}(0,\xi) - \frac{\xi}{q^2}\widehat{u}_x(0,\xi)\right), \tag{3.17}$$

$$\Omega_1^+(p,\xi) = \Omega^+(p,\xi) - \xi\Lambda^+(p,\xi), \tag{3.18}$$

$$\Omega_1^-(p,\xi) = (K_1(p)+\xi)\left(\xi^{-1}\Omega^-(p,\xi) - \Lambda_2^-(p,\xi)\right) - \Lambda_3^-(p,\xi). \tag{3.19}$$

$$\Lambda_2(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{K_1(q)-K(q)}{(K(q)+\xi)(K_1(q)+\xi)} \left(\widehat{u}_0(q) - \frac{\xi}{q}\widehat{u}(0,\xi) - \frac{\xi\widehat{u}_x(0,\xi)}{q^2}\right)dq. \tag{3.20}$$

$$\Lambda_3(z,\xi) = -k'(\xi)\left(\frac{K_1(z)+\xi}{z-k(\xi)}\right) \left(\widehat{u}_0(k(\xi)) - \frac{\xi}{k(\xi)}\widehat{u}(0,\xi) - \frac{\xi}{k(\xi)^2}\widehat{u}_x(0,\xi)\right). \tag{3.21}$$

Here $k(\xi)$ is one of roots of equation $K_1(z) = -\xi$ such that $\mathrm{Re}\, k(\xi) > 0$ for all $\mathrm{Re}\,\xi > 0$.

The relation (3.15) indicates that the function $\frac{\Omega_1^+}{Y^+} + U^+$, analytic in $\mathrm{Re}\, z < 0$, and the function $\frac{\Omega_1^-}{Y^-} + U^-$, analytic in $\mathrm{Re} z > 0$, constitute the analytic continuation of each other through the contour $\mathrm{Re}\, z = 0$. Consequently, they are branches of unique analytic function in the entire plane. According to Liouville theorem this function is some arbitrary constant A. Thus, bearing in mind the representations (3.18) and (3.19) we get

$$\Omega^+(p,\xi) = Y^+\left(A - U^+\right) + \xi\Lambda^+, \tag{3.22}$$

$$\Omega^-(p,\xi) = \frac{\xi}{K_1(p)+\xi}Y^-\left(A-U^-\right) + \xi\left(\Lambda_3^+ + \Lambda_2^-\right).$$

Since there exists only one root $k(\xi)$ of equation $K_1(z) = -\xi$ such that $\mathrm{Re}\ k(\xi) > 0$ for all $\mathrm{Re}\ \xi > 0$, in the expression for the function $\Omega^-(z,\xi)$ the factor $\frac{\xi}{K_1(z)+\xi}$ has a pole in the point $z = k(\xi)$. Also the function $\xi\Lambda_1^+$ has a pole in the point $z = k(\xi)$. Thus the problem (3.9) is soluble only when the functions $U^-(z,\xi)$ and $\xi\Lambda_1^+$ satisfies additional conditions

$$\mathrm{Res}_{p=k(\xi)}\left\{\frac{1}{K_1(p)+\xi}Y^-\left(A-U^-\right)+\Lambda_3^+(p,\xi)\right\}=0, \tag{3.23}$$

where $\Lambda_3^+(p,\xi)$ was defined by (3.21). Multiplying the last relation by $\frac{1}{Y^-(k(\xi),\xi)}$ and taking limit $\xi\to\infty$ we get that $A = 0$. This implies that for solubility of the non-homogeneous problem (3.9) it is necessary and sufficient that the following condition is satisfied

$$Y^-(k(\xi),\xi)U^-(k(\xi),\xi)-\widehat{u}_0(k(\xi))+\frac{\xi}{k(\xi)}\widehat{u}(0,\xi)+\frac{\xi}{k^2(\xi)}\widehat{u}_x(0,\xi)=0. \tag{3.24}$$

Thus we need to put in the problem (3.1) only one boundary data. The rest boundary data can be find from condition (3.24). In the case of Neumann problem we put $u_x(0,t) = 0$. Then solving last equation, after some calculation we obtain for the Laplace transform of $u(0,t)$

$$\widehat{u}(0,\xi)=-\Psi^{-1}(\xi)\frac{1}{2\pi i}Y^-(k(\xi),\xi)\int_{-i\infty}^{i\infty}\frac{\widehat{u}_0(q)}{q-k(\xi)}\frac{1}{Y^+(q,\xi)}dq, \tag{3.25}$$

where

$$\Psi(\xi)=\frac{\xi}{k(\xi)}-\frac{\xi}{2\pi i}Y^-(k(\xi),\xi)\int_{-i\infty}^{i\infty}\frac{1}{q-k(\xi)}\frac{1}{q}\left(\frac{1}{Y^+(q,\xi)}-\frac{1}{Y^-(q,\xi)}\right)dq.$$

Now we return to problem (3.9). From (3.22) under the condition (3.25) the difference limiting values of solution of (3.9) are given by formula

$$\Omega^+(p,\xi)-\Omega^-(p,\xi)=-\frac{K(p)}{K(p)+\xi}Y^+(p,\xi)U^+(p,\xi) \tag{3.26}$$

Replacing the difference $\Omega^+(p,\xi)-\Omega^-(p,\xi)$ in the relation (3.12) by formula (3.26) we get

$$\widehat{\Phi}(p,\xi)=-Y^+(p,\xi)U^+(p,\xi)$$

It is easily to observe that $\widehat{\Phi}(p,\xi)$ is boundary value of the function analytic in the left complex semi-plane and therefore satisfies our basic assumption (3.6). Having determined the function $\widehat{\Phi}(p,\xi)$ from (3.7) we determine required function $\widehat{\widehat{u}}$

$$\widehat{\widehat{u}}=\frac{1}{K_1(p)+\xi}\left(\widehat{u}_0(p)+\frac{K_1(p)}{p}\widehat{u}(0,\xi)-Y^-U^-\right). \tag{3.27}$$

Under condition (3.25) the function $\widehat{\widehat{u}}$ is the limiting value of an analytic function in $\mathrm{Re}\ z > 0$. Note the fundamental importance of the proven fact, that the solution $\widehat{\widehat{u}}$ constitutes an analytic function in $\mathrm{Re}\ z > 0$ and, as a consequence, its inverse Laplace transform vanish for all $x < 0$.

Taking inverse Laplace transform of (3.27) with respect to time and inverse Fourier transform with respect to space variables we obtain

$$u(x,t) = \mathcal{G}(t)u_0 = \int_0^{\infty} G(x,y,t)u_0(y)dy,$$

where the function $G(x,y,t)$ was defined by formula (2.7). Proposition 1. □

Now we collect some preliminary estimates of the Green operator $\mathcal{G}(t)$, given by (2.7).

Lemma 1. *The estimates are valid for $m \geq 0, t > 0$*

$$\left\| (\cdot)^{\frac{m}{2}} \partial_x^{(n)} \mathcal{G}(t) f \right\|_{\mathbf{L}^2} \tag{3.28}$$

$$\leq C \begin{cases} t^{-\frac{4+2n}{6}} \|f\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + t^{-\frac{3+2n-m}{6}} \|f\|_{\mathbf{H}_1^{0,1}} + t^{-\frac{3+3n}{6}} \|f\|_{\mathbf{H}_1^{0,\frac{m+2+n}{2}}}, \\ t^{-\frac{n+1}{3}} \|f\|_{\mathbf{H}_1^{0,\frac{m+1}{2}}} + t^{-\frac{1+2n-m}{6}} \|f\|_{\mathbf{L}^1} \\ +t^{-\frac{3n+2}{6}+\frac{1}{12}} \|f\|_{\mathbf{H}_1^{0,\frac{m+2+n}{2}-\frac{3}{4}}} + t^{-\frac{3+2n-m}{6}} \|f\|_{\mathbf{H}_1^{0,1}} \end{cases}$$

$$+ \langle t \rangle^{-\gamma} (t^{-\frac{1}{2}\left(\frac{1}{2}-\frac{m}{2}+n\right)} \|f\|_{\mathbf{L}^1} + t^{-\frac{1}{2}\left(\frac{1}{2}+n\right)} \|f\|_{\mathbf{H}_1^{0.\frac{m}{2}}}) \tag{3.29}$$

if $n = 1, 2$, $\gamma > 0$ and

$$\left\| x^{\frac{m}{2}} \mathcal{G}(t) f \right\|_{\mathbf{L}^2} \tag{3.30}$$

$$\leq C \begin{cases} t^{-\frac{2}{3}} \|f\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + t^{-\frac{3-m}{6}} \|f\|_{\mathbf{H}_1^{0,1}} + t^{-\frac{3}{4}} \|f\|_{\mathbf{H}_1^{0,\frac{m+2}{2}+\frac{3}{4}}}, \\ t^{-\frac{1}{3}} \|f\|_{\mathbf{H}_1^{0,\frac{m+1}{2}}} + t^{-\frac{1-m}{6}} \|f\|_{\mathbf{L}^1} \\ +t^{-\frac{1}{2}} \|f\|_{\mathbf{H}_1^{0,\frac{m+2}{2}}} + t^{-\frac{3-m}{6}} \|f\|_{\mathbf{H}_1^{0,1}} \end{cases}$$

$$+ \langle t \rangle^{-\gamma} (t^{-\frac{1}{2}\left(\frac{1}{2}-\frac{m}{2}\right)} \|f\|_{\mathbf{L}^1} + t^{-\frac{1}{4}} \|f\|_{\mathbf{H}_1^{0.\frac{m}{2}}}) \tag{3.31}$$

provided that the right-hand sides are finite. Moreover we have

$$\mathcal{G}(t) f = \theta(\sigma t)^{-\frac{2}{3}} Ai' \left(\frac{x}{\sqrt[3]{\sigma t}} \right) + R(x,t), \tag{3.32}$$

where for $t > 1, 0 < \delta < 1, \gamma > 0$

$$\left\| \partial_x^{(n)} R(t) \right\|_{\mathbf{H}_2^{0,\frac{m}{2}}} \leq Ct^{-\frac{3-m+2n+2\delta}{6}} (\|\phi\|_{\mathbf{H}_1^{0,\frac{21}{4}}} + \|\phi\|_{\mathbf{H}_2^{1,\frac{7}{2}}})$$
$$+ \langle t \rangle^{-\gamma} (t^{-\frac{1}{2}\left(\frac{1}{2}-\frac{m}{2}\right)} \|f\|_{\mathbf{L}^1} + t^{-\frac{1}{4}} \|f\|_{\mathbf{H}_1^{0.\frac{m}{2}}}),$$

$$\theta = \int_0^{+\infty} x f(x) dx.$$

Proof. Let the contours $\mathcal{C}_i$ are defined as

$$\mathcal{C}_1 = \left\{ p \in \left(\infty e^{-i(\frac{\pi}{2}+\varepsilon)}, 0 \right) \bigcup \left(0, \infty e^{i(\frac{\pi}{2}+\varepsilon)} \right) \right\}, \tag{3.33}$$

$$\mathcal{C}_2 = \left\{ q \in \left(\infty e^{-i(\frac{\pi}{2}+2\varepsilon)}, 0 \right) \bigcup \left(0, \infty e^{i(\frac{\pi}{2}+2\varepsilon)} \right) \right\}, \tag{3.34}$$

$$\mathcal{C}_3 = \left\{ q \in \left(\infty e^{-i(\frac{\pi+\varepsilon}{2})}, 0 \right) \bigcup \left(0, \infty e^{i(\frac{\pi+\varepsilon}{2})} \right) \right\}, \tag{3.35}$$

where $\varepsilon > 0$ can be chosen such that all functions under integration are analytic and Re $k(\xi) > 0$ for $\xi \in \mathcal{C}_1$. Via Sokhotskii–Plemeli formula and Cauchy Theorem we have

$$G(x,y,t) = \sum_{j=1}^{3} J_j(x,y,t), \tag{3.36}$$

where

$$J_1(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t}e^{-py}dp, \tag{3.37}$$

$$J_2(x,y,t) = -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\mathcal{C}_1} d\xi e^{\xi t}\int_{\mathcal{C}_2} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}I_1^+(p,\xi,y)dp, \tag{3.38}$$

$$\begin{aligned} &J_3(x,y,t) \\ &\quad = \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\mathcal{C}_1} d\xi e^{\xi t}\frac{I_1^-(k(\xi),\xi,y)}{I_2^-(k(\xi),\xi)}\int_{-i\infty}^{i\infty} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}I_2^-(p,\xi)dp. \end{aligned} \tag{3.39}$$

The functions I_1, I_2 and $Y^{\pm}(z,\xi)$ were defined by formulas (2.2)–(2.3) and (2.4).

For subsequent considerations it is required to investigate the behavior of the function $\Gamma(z,\xi)$ given by (2.5). Set

$$g(p,\xi) = \ln\left\{\left(\frac{K(p)+\xi}{K_1(p)+\xi}\right)\frac{w^-(p)}{w^+(p)}\right\} \neq 0, \quad \text{Re } p = 0, \quad \text{Re } \xi < 0.$$

Observe that the function $g(p,\xi)$ obeys the Hölder condition for all finite p and tends to a definite limit $\phi(\infty,\xi)$ as $p \to \pm i\infty$,

$$\begin{aligned} g(\infty,\xi) &= \lim_{p\to\pm i\infty} \ln\left\{\left(\frac{K(p)+\xi}{K_1(p)+\xi}\right)\frac{w^-(p)}{w^+(p)}\right\} \\ &= 0. \end{aligned}$$

Also it can be easily obtain that for large p and some fixed ξ the following inequality holds

$$|g(p,\xi) - g(\infty,\xi)| \le C\left(\frac{|\xi|^{\mu}}{|p|^{2\mu}}\right), \quad \mu > 0. \tag{3.40}$$

By properties of Cauchy type integral the above estimates imply the following results for all $\xi \in \mathcal{C}_1$

$$\left|Y^{\pm}(z,\xi)\right| \le C, Y^{\pm}(\infty,\xi) = 1,$$

and

$$Y^{\pm}(p,\xi) - 1 = O\left(\frac{|\xi|^{\mu}}{|p|^{2\mu}}\right), \mu \in [0,1], \tag{3.41}$$

In the neighborhood of $p = 0$ we have

$$\begin{aligned} K(p)+\xi &= \sigma p^3 + \xi + O(p^4), \\ K_1(p)+\xi &= -\sigma p^3 + \xi + O(p^4). \end{aligned}$$

Also for small ξ by construction $k(\xi) = k_1(\xi) + O(\xi)$, where

$$k_1(\xi) = \left(\frac{1}{\sigma}\xi\right)^{\frac{1}{3}}.$$

Note that $k_1(\xi)$ is root of the equation $-\sigma q^3 + \xi = 0$ such that $\operatorname{Re} k_1(\xi) > 0$. In order to separate the principal part of the expansion of

$$\Gamma^+(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \frac{K(q)+\xi}{K_1(q)+\xi}\left(\frac{q-k(\xi)}{q+k(\xi)}\right)^{\frac{1}{2}} dq$$

near points $p = 0$, $\xi = 0$ we introduce the new function $\widetilde{\Gamma}^+(z,\xi)$ by the following relation

$$\widetilde{\Gamma}^+(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \frac{\sigma q^3+\xi}{-\sigma q^3+\xi}\left(\frac{(q-k_1(\xi))}{(q+k_1(\xi))}\right)^{\frac{1}{2}} dq.$$

It can be proved by a direct calculation for small $|z| < 1, |\xi| < 1$

$$\widetilde{\Gamma}^+(z,\xi) - \Gamma^+(z,\xi) = O(z\xi^{\frac{1}{3}}).$$

Integrating by parts and using Cauchy Theorem we get

$$\begin{aligned}\widetilde{\Gamma}^+(z,\xi) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \frac{\sigma q^3+\xi}{-\sigma q^3+\xi}\left(\frac{(q-k_1(\xi))}{(q+k_1(\xi))}\right)^{\frac{1}{2}} dq \\ &= -\frac{1}{2}\ln z + \ln(\phi_1(\xi)-z) + \ln(\phi_2(\xi)-z) - \frac{1}{2}\ln(k_1(\xi)-z),\end{aligned} \tag{3.42}$$

where $\phi_1(\xi)$ and $\phi_2(\xi)$ are roots of the equation $\sigma q^3 + \xi = 0$ such that $\operatorname{Re}\,\phi_1(\xi) > 0, \operatorname{Re}\,\phi_2(\xi) > 0$ in $\operatorname{Re}\xi > 0$:

$$\phi_1(\xi) = \left(\xi\sigma^{-1}\exp(i\pi)\right)^{\frac{1}{3}}, \phi_2(\xi) = \left(\xi\sigma^{-1}\exp(-i\pi)\right)^{\frac{1}{3}}. \tag{3.43}$$

On the basis last relations for points of the contours close the points $p = 0$ and $\xi = 0$ integrand of $J_2(x,y,t)$ has the following representation

$$e^{\xi t}e^{px}\frac{1}{(p+k_1(\xi))(p-k_1(\xi))}\widetilde{I}^+(p,\xi,y), \tag{3.44}$$

where

$$\begin{aligned}\widetilde{I}^+(p,\xi,y) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{-qy}(q-k_1)(q+k_1)}{(q-z)(\sigma q^3+\xi)} dq \\ &= \sum_{j=1}^{2} e^{-\phi_j(\xi)y}\frac{\phi_j^2-k_1^2}{\phi_j-p}\phi_j'(\xi)\end{aligned}$$

On integrating this function by extending the limits to $-\infty$ and $+\infty$ we obtain the contributions to $J_2(x,y,t)$ from the neighborhoods of $\xi = 0$ and $p = 0$

$$\begin{aligned}&-\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\frac{1}{(p+k_1(\xi))(p-k_1(\xi))}\widetilde{I}^+(p,\xi,y)dp \\ &= -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1} d\xi e^{\xi t}e^{-k_1(\xi)x}\frac{1}{2k_1(\xi)}\sum_{j=1}^{2} e^{-\phi_j(\xi)y}(\phi_j-k_j)\phi_j'(\xi)d\xi.\end{aligned} \tag{3.45}$$

Via (3.42) we have that

$$\begin{aligned}&\frac{\xi}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{q}\left(\frac{1}{Y^+}-\frac{1}{Y^-}\right)dq\\&=\frac{\xi}{Y^-z}+\frac{\xi}{z}res_{p=0}\frac{1}{Y^+(p,\xi)}\\&=\frac{\xi}{Y^-z}+\frac{1}{z}k_1^2(\xi).\end{aligned}$$

On the basis last relations for points of the contours close the points $p=0$ and $\xi=0$ integrand of $J_3(x,y,t)$ has the following representation

$$e^{\xi t}e^{px}\frac{1}{(p+k_1(\xi))(p-k_1(\xi))}\frac{k_1(\xi)}{p}\widetilde{I}(k_1(\xi),\xi,y),$$

where

$$\widetilde{I}(k_1(\xi),\xi,y)=\sum_{j=1}^{2}e^{-\phi_j(\xi)y}(\phi_j+k_1)\phi_j'(\xi).$$

On integrating this function by extending the limits to $-\infty$ and $+\infty$ we obtain the contributions to $J_3(x,y,t)$ from the neighborhoods of $\xi=0$ and $p=0$

$$\begin{aligned}&-\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1}d\xi e^{\xi t}\int_{-i\infty}^{i\infty}e^{px}\frac{1}{(p+k_1(\xi))(p-k_1(\xi))}\widetilde{I}^+(p,\xi,y)dp \qquad (3.46)\\&=-\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1}d\xi e^{\xi t}e^{-k_1(\xi)x}\frac{1}{2k_1(\xi)}\sum_{j=1}^{2}e^{-\phi_j(\xi)y}(\phi_j+k_1)\phi_j'(\xi)d\xi.\end{aligned}$$

Substituting (3.45) and (3.46) into (3.36) we rewrite the Green function in the form

$$G(x,y,t)=F(x,y,t)+R(x,y,t), \tag{3.47}$$

where the function $F(x,y,t)$ given by

$$\begin{aligned}F(x,y,t)&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{-\sigma p^3t+p(x-y)}dp \qquad (3.48)\\&\quad-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{\xi t-k_1(\xi)x}k_1^{-1}(\xi)\sum_{j=1}^{2}e^{-\phi_j(\xi)y}\phi_j(\xi)\phi_j'(\xi)d\xi\end{aligned}$$

and $R(x,y,t)$ is the contribution to $G(x,y,t)$ over the rest of the range of integration:

$$R(x,y,t))=\sum M_j(x,y,t) \tag{3.49}$$

$$M_1(x,y,t)=\frac{1}{2\pi i}\int_{-i\infty,|p|>1}^{i\infty}e^{px-K(p)t}e^{-py}dp, \tag{3.50}$$

$$M_2(x,y,t) = -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\mathcal{C}_1,|\xi|<1} d\xi e^{\xi t}\int_{\mathcal{C}_2,|p|>1} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}I_1^+(p,\xi,y)dp \tag{3.51}$$
$$-\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\mathcal{C}_1,|\xi|>1} d\xi e^{\xi t}\int_{\mathcal{C}_2,|p|>1} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}I_1^+(p,\xi,y)dp,$$

$$M_3(x,y,t) = \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\mathcal{C}_1,|\xi|<1} d\xi e^{\xi t}\frac{I_1^-(k(\xi),\xi,y)}{I_2^-(k(\xi),\xi)}$$
$$\int_{-i\infty,|p|>1}^{i\infty} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}I_2^-(p,\xi)dp \tag{3.52}$$
$$+\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\mathcal{C}_1,|\xi|>1} d\xi e^{\xi t}\frac{I_1^-(k(\xi),\xi,y)}{I_2^-(k(\xi),\xi)}$$
$$\int_{-i\infty,|p|>1}^{i\infty} e^{px}\frac{Y^+(p,\xi)}{K(p)+\xi}I_2^-(p,\xi)dp.$$

Denote by

$$\Phi(x,t)f(\tau) = \int_0^{+\infty} F(x,y,t)f(y,\tau)dy \tag{3.53}$$

and

$$\mathcal{R}(t)f = \theta(x)\int_0^{+\infty} R(x,y,t)f(y)dy. \tag{3.54}$$

In the book [17] it was proved that operator $\Phi(x,t)f(\tau)$ is Green operator of the Neumann problem for linear KdV equation $u_t + \sigma u_{xxx} = 0$ and satisfies all estimates of Lemma 1.

To estimate operator $\mathcal{R}\phi$ we use standard method of "partition of unity" and Watson Lemma [7]. Firstly we estimate the function $M_1(x,y,t)$. Sea $x-y>0$. We have

$$e^{-K(p)t+p(x-y)}pdp = \frac{1}{(1-K'(p)t+(x-y))}de^{-K(p)t+p(x-y)}p$$

Therefore integrating by part we get

$$M_1(x,y,t) = -\frac{1}{2\pi i}\int_{-i\infty,|p|>1}^{i\infty} e^{-K(p)t+p(x-y)}\Theta(p,t,x,y)dp,$$

where

$$\Theta(p,t,x,y) = \frac{K''(p)t}{(1-K'(p)t+(x-y))^2} - \frac{1}{p}\frac{1}{(1-K'(p)t+(x-y))}.$$

In the case of $x-y<0$ we change contour of the integration to obtain

$$|M_1(x,y,t)| \le C\int_{\mathcal{L},|p|>1} e^{-C|p|^2t - C|p||x-y|}dp,$$

where

$$\mathcal{L} = \left\{z\in\mathbb{C}, z=\rho e^{\pm i\phi}, \phi = \frac{\pi}{2}-\varepsilon, \varepsilon>0.\right\}$$

Here we are used that for $p \in L, \mathrm{Re}K(p) > 0, \mathrm{Re}p > 0$ and

$$K(p) = O(\langle p \rangle^2).$$

Since $1 - K'(p)t + (x-y) \neq 0$ for $\mathrm{Re}p = 0, x - y > 0$ and

$$K'(p) = O(\langle p \rangle), K''(p) = O(1), \partial_p^l K(p) = O(\langle p \rangle^{-l+2}), l > 2$$

we get

$$\|\partial_x^n M_1(\cdot, y, t)\|_{L^{s,\mu}} \leq C t^{-\frac{1}{2}(n+1-\frac{1}{s}-\mu)} \langle t \rangle^{-\gamma},$$

for $s > 1$, where $\gamma > 0$. Using obtained estimate we get

$$\begin{aligned} &\left\| \int_0^{+\infty} M_1(x-y,t) f(y) dy \right\|_{L^{2,\frac{m}{2}}} \\ &\leq C \left(\|M_1(\cdot,t)\|_{L^{2,\frac{m}{2}}} \|f\|_{L^1} + \|M_1(\cdot,t)\|_{L^2} \|f\|_{L^{1,\frac{m}{2}}} \right) \\ &\leq C t^{-\frac{1+2n-2m}{4}} \langle t \rangle^{-\gamma} \|f\|_{L^1} + t^{-\frac{1+2n}{4}} \langle t \rangle^{-\gamma} \|f\|_{L^{1,\frac{m}{2}}}. \end{aligned} \tag{3.55}$$

Now we estimate $M_2(x, y, t)$. In order to separate the principal part of the expansion of $\mathcal{M}_2(x, y, t)$ near point $\xi = 0$ we use the following obvious relations for $|p| > 1$

$$\begin{aligned} K(p) &= -p|p| + O(p^{-1}), \\ K_1(p) &= -p^2 + O(p^{-1}), \\ k(\xi) &= O(\xi^{\frac{1}{3}}). \end{aligned}$$

Therefore

$$\begin{aligned} &\left\| \partial_x^n \int_0^{+\infty} M_2(\cdot, y, t) f(y) dy \right\|_{L^{2,\mu}} \\ &\leq \frac{1}{2\pi i} \frac{1}{2\pi i} e^{-\delta t} \int_{C_1} e^{\xi t} d\xi \int_{C_2, |p|>1}^{i\infty} \frac{|p|^n}{|p|^{\frac{1+2\mu}{2}}} \frac{1}{(|p|^2 + \xi)} dp \\ &\quad \times \int_{C_3, |q|>1} \frac{1}{|q-p| \, |q|^{\frac{1}{2}}} dq \\ &\leq C e^{-\delta t} t^{-\frac{1}{2}\left(1 - \frac{1+\mu 2}{2} + n\right)} \|f\|_{L^1}, \end{aligned} \tag{3.56}$$

for some small $\delta > 0, n = 0, 1, 2$. In the same way can be estimated operator M_3. Thus from estimates (3.56) and (3.55) it follows that

$$\|\partial_x^n \mathcal{R} f\|_{L^{2,\frac{m}{2}}} \leq \langle t \rangle^{-\gamma} (t^{-\frac{1}{2}\left(\frac{1}{2} - \frac{m}{2} + n\right)} \|f\|_{L^1} + t^{-\frac{1}{2}\left(\frac{1}{2}+n\right)} \|f\|_{L^{1,\frac{m}{2}}}) \tag{3.57}$$

Lemma is proved. □

Lemma 2. *Let the function $f(x,t)$ such that $\int_0^{+\infty} x f(x,t) dx = 0$ and satisfy the estimate*

$$\left\| (\cdot)^{\frac{l}{2}} f(\cdot, t) \right\|_{\mathbf{L}^1} \leq C \varepsilon^{\beta} (1+t)^{-\frac{8-l}{6}} g(t)^{\alpha}, 0 \leq l \leq 5,$$

where $\alpha = \frac{4}{5}, 0 < \beta$. *We also assume that the function* $g(t)$ *satisfies the inequalities for* $\eta > 0$

$$\frac{1}{2}\left(1+\eta\log\left(1+t\right)\right) < g(t) < 2\left(1+\eta\log\left(1+t\right)\right).$$

Then the following estimates are valid for $n = 0, 1, t > 0$

$$\left\|(\cdot)^{\frac{m}{2}}\int_0^t g^{-1}\left(\tau\right)\partial_x^{(n)}\mathcal{G}\left(t-\tau\right)f\left(\tau\right)d\tau\right\|_{\mathbf{L}^2} \leq C\varepsilon^{\beta}g^{-1+\alpha}\left(t\right)t^{-\frac{3-m+2n}{6}},$$

where $0 \leq m \leq 2$.

Proof. Via formula (3.47) we obtain the following representation

$$\mathcal{G}(x,t-\tau)f(\tau) = \int_0^{+\infty}\Phi(x,y,t-\tau)f(y,\tau)dy + \int_0^{+\infty}\mathcal{R}(x,y,t-\tau)f(y,\tau)dy. \tag{3.58}$$

From book [17] we have

$$\left\|(\cdot)^{\frac{m}{2}}\int_0^t g^{-1}\left(\tau\right)\partial_x^{(n)}\Phi\left(t-\tau\right)f\left(\tau\right)d\tau\right\|_{\mathbf{L}^2} \leq C\varepsilon^{\beta}g^{-1+\alpha}\left(t\right)t^{-\frac{3-m+2n}{6}}.$$

Now we estimate second term of (3.58). From (3.57) we have for $t > 0$, $m \geq 0$

$$\left\|\partial_x^n\mathcal{R}\phi\right\|_{L^{2,\frac{m}{2}}} \leq \langle t\rangle^{-\gamma}\left(t^{-\frac{1}{2}\left(\frac{1}{2}-\frac{m}{2}+n\right)}\left\|\phi\right\|_{L^1} + t^{-\frac{1}{2}\left(\frac{1}{2}+n\right)}\left\|\phi\right\|_{L^{1,\frac{m}{2}}}\right).$$

Also from the condition of the lemma for the function $g(t)$ we have for $t > 1:$ $t^{-\alpha} < \frac{1}{g(t)}$, $\alpha > 0$ and $\sup_{\tau\in\left[\sqrt{t},t\right]}\frac{1}{g(\tau)} < \frac{C}{g(t)}$. Hence we get for $t > 4$, $n = 0, 1$

$$\begin{aligned}
&\left\|\int_0^t(\cdot)^{\frac{m}{2}}g^{-1}(\tau)\partial_x^n\mathcal{R}(t-\tau)f(\tau)d\tau\right\|_{\mathbf{L}^2} \\
&\leq C\int_0^{\sqrt{t}}t^{-\gamma}\left\|f(\tau)\right\|_{\mathbf{L}^{1,\frac{m}{2}}}d\tau + Cg^{-1}\left(t\right)\int_{\sqrt{t}}^{\frac{t}{2}}t^{-\gamma}\left\|f(\tau)\right\|_{\mathbf{L}^{1,\frac{m}{2}}}d\tau \\
&\quad + C\sup_{\frac{t}{2}<\tau<t}g^{-1}\left(\tau\right)\left\|f(\tau)\right\|_{\mathbf{L}^{1,\frac{m}{2}}}\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}\left(\frac{1}{2}+n\right)}d\tau \\
&\quad + C\sup_{\frac{t}{2}<\tau<t}g^{-1}\left(\tau\right)\left\|f(\tau)\right\|_{\mathbf{L}^1}\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}\left(\frac{1}{2}-\frac{m}{2}+n\right)}d\tau.
\end{aligned}$$

By the previous assumptions we have

$$\left\|x^{\frac{m}{2}}f(t)\right\|_{\mathbf{L}^1} \leq C\varepsilon^{\beta}\left(1+t\right)^{-\frac{5}{6}}t^{\frac{m-3}{6}}g\left(t\right)^{\alpha},\ 0 \leq m \leq 5;$$

hence for $n = 0, 1$

$$\left\|\int_0^t(\cdot)^{\frac{m}{2}}g^{-1}(\tau)\partial_x^n\mathcal{R}(t-\tau)f(\tau)d\tau\right\|_{\mathbf{L}^2} \leq C\varepsilon^{\beta}g^{-1+\alpha}\left(t\right)t^{-\frac{3-m+2n}{6}}.$$

Lemma 2 is proved. □

By the contraction mapping principle we can prove the following theorem.

Theorem 2. *Let $u_0(x) \in \mathbf{H}_2^{1,\frac{7}{2}}(\mathbf{R}^+)$. Then there exist a positive time T and a unique solution of the problem (1.1) such that*

$$u(t) \in C([0,T]; \mathbf{H}_2^{1,\frac{7}{2}}(\mathbf{R}^+)).$$

4. Proof of Theorem 1

We suppose that for a sufficiently small $\varepsilon > 0$

$$\|u_0\|_{\mathbf{H}_1^{0,\frac{21}{4}}} + \|u_0\|_{\mathbf{H}_2^{1,\frac{7}{2}}} \le \varepsilon$$

and

$$\int_0^{+\infty} x u_0(x) dx = \theta < 0.$$

We denote

$$\eta = -\theta \int_0^{+\infty} Ai'^2(z) dz > 0.$$

We let $u(x,t) = e^{-\phi(t)} v(x,t)$. Then we get from (1.1)

$$\begin{cases} v_t - \phi_t v + e^{-\phi(t)} v v_x + \mathbb{K} v = 0, \quad t > 0, x > 0, \\ v(x,0) = e^{\phi(0)} u_0(x), \ x > 0, \\ v_x(0,t) = 0, \ t > 0. \end{cases} \tag{4.1}$$

Now we assume that the real-valued function $\phi(t)$ satisfies the following condition

$$\int_0^{+\infty} x \left(-\phi_t v + e^{-\phi(t)} v v_x \right) dx = 0. \tag{4.2}$$

Since $v_x(0,t) = 0$ we have

$$\int_0^{+\infty} x \mathbb{K} v(x,t) dx = 0.$$

Indeed, by Plancherel Theorem we have

$$\begin{aligned} \int_0^{+\infty} x \mathbb{K} v(x,t) dx &= \int_{-i\infty}^{i\infty} \frac{1}{p^2} \lim_{z \to p, \mathrm{Re} z > 0} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} K(q) (\widehat{v}(q,t) - \frac{v(0,t)}{q}) dq \\ &= \pi i \lim_{z \to 0, \mathrm{Re} z > 0} \int_{-i\infty}^{i\infty} \frac{1}{(q-z)^2} K(q) (\widehat{v}(q,t) - \frac{v(0,t)}{q}) dq \\ &= \pi i \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \sigma q (\widehat{v}(q,t) - \frac{v(0,t)}{q}) dq = \pi \sigma i v_x(0,t) = 0. \end{aligned}$$

Then via (4.1) we have for all $t > 0$

$$\frac{d}{dt} \int_0^{+\infty} x v(x,t) dx = - \int_0^{+\infty} x \mathbb{K} v(x,t) dx = 0.$$

Therefore choosing $\phi(0) = 0$ we get by (4.2)

$$\phi_t e^{\phi(t)} = \theta^{-1} \int_0^{+\infty} x v v_x dx.$$

Integrating with respect to time we get

$$e^{\phi(t)} = g(t) = 1 + \theta^{-1} \int_0^t d\tau \int_0^{+\infty} x v v_x dx$$
$$= 1 - \frac{1}{2}\theta^{-1} \int_0^t \|v(\tau)\|_{\mathbf{L}^2}^2 d\tau.$$

Thus we get the following problem

$$\begin{cases} v_t + g(t)^{-1}(\frac{v}{2\theta}\|v(t)\|_{\mathbf{L}^2}^2 + v v_x) + \mathbb{K}v = 0, \quad t > 0, x > 0, \\ g(t) = 1 - \frac{\lambda}{2\theta}\int_0^t \|v(\tau)\|_{\mathbf{L}^2}^2 d\tau, \\ v(x,0) = u_0(x), \ x > 0, \\ v_x(0,t) = 0, \ g(0) = 1, t > 0. \end{cases} \tag{4.3}$$

We consider the integral equation associated with (4.3) which is written as

$$v(x,t) = \mathcal{G}(x,t)u_0(x)$$
$$+ \lambda \int_0^t g(\tau)^{-1}\mathcal{G}(x,t-\tau)(\frac{v}{2\theta}\|v(\tau)\|_{\mathbf{L}^2}^2 + v v_x) d\tau.$$

Changing the variables

$$v(x,t) = \mathcal{G}(x,t)u_0(x) + r(x,t) \tag{4.4}$$

we get the system of integral equations $(r,g) = (\mathbb{M}_1(r,g), \mathbb{M}_2(r,g))$ for the first approximation of perturbation theory

$$\begin{cases} \mathbb{M}_1(r,g) = \lambda \int_0^t g(\tau)^{-1}\mathcal{G}(x,t-\tau)(\frac{v}{2\theta}\|v(\tau)\|_{\mathbf{L}^2}^2 + v v_x) d\tau. \\ \mathbb{M}_2(r,g) = 1 - \lambda \frac{1}{2\theta}\int_0^t \|v(\tau)\|_{\mathbf{L}^2}^2 d\tau. \end{cases} \tag{4.5}$$

We prove that $(\mathbb{M}_1(r,g), \mathbb{M}_2(r,g))$ is a contradiction mapping in the set

$$\mathbf{X}_\varepsilon = \Big\{ r \in C([0,+\infty), \mathbf{X}); g \in C(0,+\infty), |||r|||_{\mathbf{X}_1} \le \varepsilon, |||r|||_{\mathbf{X}_2} \le \varepsilon^{\frac{3}{4}},$$
$$r_x(0,t) = 0, \tfrac{1}{2}(1 + \eta \log t) < g(t) < 2(1 + \eta \log t), t > 0 \Big\}$$

with the norm

$$|||r|||_{\mathbf{X}} = |||r|||_{\mathbf{X}_1} + |||r|||_{\mathbf{X}_2}$$

$$|||r|||_{\mathbf{X}_1} = \sum_{n=0}^{1}\sum_{m=0}^{2} \sup_{t>0} g(t)^{\frac{1}{5}} (1+t)^{\frac{3+2n-m}{6}} \left\| x^{\frac{m}{2}} \partial_x^{(n)} r(t) \right\|_{\mathbf{L}^2},$$
$$|||r|||_{\mathbf{X}_2} = \sum_{n=0}^{1}\sum_{m=3}^{7} \sup_{t>0} g(t)^{-1} (1+t)^{\frac{3+2n-m}{6}} \left\| x^{\frac{m}{2}} \partial_x^{(n)} (\mathcal{G}(t)u_0 + r(t)) \right\|_{\mathbf{L}^2}.$$

We assume that

$$|||r|||_{\mathbf{X}_1} \le \varepsilon, |||r|||_{\mathbf{X}_2} \le \varepsilon^{\frac{3}{4}}. \tag{4.6}$$

First we prove that the mapping transforms the set X_ε into itself if $\varepsilon > 0$ is small. From local existence Theorem2 we have

$$|||\mathbb{M}_1(r,g)|||_{\mathbf{X}_1} \le \varepsilon, |||\mathbb{M}_1(r,g)|||_{\mathbf{X}_2} \le \varepsilon^{\frac{3}{4}}$$

for $t < 1$. Now we consider case of $t > 1$. From Lemma 1 we have for $t > 1$ and $0 \leq m \leq 7$

$$\left\| (\cdot)^{\frac{m}{2}} \partial_x^{(n)} \mathcal{G}(t) u_0(\cdot) \right\|_{\mathbf{L}^2} \leq C \varepsilon t^{-\frac{3-m+2n}{6}}$$

since $u_0 \in \mathbf{H}_1^{0,\frac{21}{4}}$. Thus via (4.4) and (4.6) we have

$$\begin{aligned} &\sum_{n=0}^{1} \sum_{m=0}^{2} \sup_{t>0} (1+t)^{\frac{3+2n-m}{6}} \left\| (\cdot)^{\frac{m}{2}} \partial_x^{(n)} v(t) \right\|_{\mathbf{L}^2} \\ &\quad \leq \sum_{n=0}^{1} \sum_{m=0}^{2} \sup_{t>0} (1+t)^{\frac{3+2n-m}{6}} \left\| (\cdot)^{\frac{m}{2}} \partial_x^{(n)} \mathcal{G}(t) u_0(\cdot) \right\|_{\mathbf{L}^2} \\ &\qquad + \sum_{n=0}^{1} \sum_{m=0}^{2} \sup_{t>0} (1+t)^{\frac{3+2n-m}{6}} \left\| (\cdot)^{\frac{m}{2}} \partial_x^{(n)} r(\cdot,t) \right\|_{\mathbf{L}^2} \leq C\varepsilon \end{aligned}$$

and

$$\sum_{n=0}^{1} \sum_{m=3}^{7} \sup_{t>0} g(t)^{-1} (1+t)^{\frac{3+2n-m}{6}} \left\| (\cdot)^{\frac{m}{2}} \partial_x^{(n)} v(\cdot,t) \right\|_{\mathbf{L}^2} \leq C\varepsilon^{\frac{3}{4}}. \tag{4.7}$$

We have by the Schwarz inequality with $\rho = \|xf\|_{\mathbf{L}^2} / \|f\|_{\mathbf{L}^2}$

$$\begin{aligned} \|f\|_{\mathbf{L}^1} &\leq \left(\int (\rho^2 + x^2) |f|^2 \, dx \right)^{\frac{1}{2}} \left(\int (\rho^2 + x^2)^{-1} dx \right)^{\frac{1}{2}} \\ &\leq \rho^{-\frac{1}{2}} \left(\int (1+x^2)^{-1} dx \right)^{\frac{1}{2}} \left(\int (\rho^2 + x^2) |f|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \sqrt{2} \left(\int (1+x^2)^{-1} dx \right)^{\frac{1}{2}} \|f\|_{\mathbf{L}^2}^{\frac{1}{2}} \|xf\|_{\mathbf{L}^2}^{\frac{1}{2}}. \end{aligned}$$

Therefore we get for $t > 0$

$$\begin{aligned} &\left\| (\cdot)^{\frac{m+3}{2}} \left(-\frac{v(\cdot,t)}{\theta} \int_0^{+\infty} y v v_y dy + v v_x(\cdot,t) \right) \right\|_{\mathbf{L}^1} \\ &\quad \leq C\theta^{-1} \left\| \langle \cdot \rangle^{\frac{m+5}{2}} v(\cdot,t) \right\|_{\mathbf{L}^2}^{\frac{1}{2}+\frac{1}{2}\frac{m+1}{m+3}} \| \langle \cdot \rangle v(\cdot,t) \|_{\mathbf{L}^2}^{\frac{1}{2}\frac{2}{m+3}} \|v(t)\|_{\mathbf{L}^2}^2 \\ &\qquad + C \left\| (\cdot)^{\frac{m+1}{2}} v(\cdot,t) \right\|_{\mathbf{L}^2} \|(\cdot) v_x(\cdot,t)\|_{\mathbf{L}^2} \\ &\quad \leq C\varepsilon^{\frac{9}{5}} (1+t)^{-\frac{5-m}{6}} (\log(1+t))^{\frac{4}{5}} \end{aligned}$$

for $m = 0, 1, 2$. Also by the construction of v we see that

$$\int_0^{+\infty} x \left(-\frac{v}{\theta} \int_0^{+\infty} y v v_y dy + v v_x \right) dx = 0.$$

Thus using results of Lemma 2 we have for $t>1$, $\alpha=\frac{4}{5}$, $m=0,1,2$

$$\begin{aligned}
&\sum_{n=0}^{1}\sum_{m=0}^{2}\sup_{t>1} g(t)^{1-\alpha}t^{\frac{3-m+2n}{6}}\left\|(\cdot)^{\frac{m}{2}}\partial_x^{(n)}\mathbb{M}_1(r,g)\right\|_{\mathbf{L}^2}\\
&\le C\sum_{n=0}^{1}\sum_{m=0}^{2}\sup_{t>1} g(t)^{1-\alpha}t^{\frac{3-m+2n}{6}}\\
&\quad\times\left\|(\cdot)^{\frac{m}{2}}\int_0^t g(\tau)^{-1}\partial_x^{(n)}G(\cdot,t-\tau)\left(-\frac{v}{\theta}\int_0^{+\infty} yvv_y dy+vv_x\right)d\tau\right\|_{\mathbf{L}^2}\\
&\le C\varepsilon^{\frac{9}{5}}\le\varepsilon. \qquad (4.8)
\end{aligned}$$

For higher order m, we turn to the original equation

$$u(x,t)=\mathcal{G}(x,t)u_0(x)+\int_0^t\mathcal{G}(x,t-\tau)uu_x d\tau. \qquad (4.9)$$

By Lemma 1 and the local existence theorem we get for $3\le m\le 7$

$$\begin{aligned}
&\left\|(\cdot)^{\frac{m}{2}}u(\cdot,t)\right\|_{\mathbf{L}^2}\\
&\le C(1+t)^{-\frac{3-m}{6}}\|u_0\|_{\mathbf{H}_1^{0,\frac{m+2}{2}+\frac{3}{4}}}\\
&\quad+C\varepsilon^{\frac{3}{2}}\int_0^t\left((t-\tau)^{-\frac{2}{3}}(1+\tau)^{-\frac{5-m}{6}}+(t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{5-m}{6}+\frac{1}{12}}\right)d\tau\\
&\quad+C\varepsilon^2\int_0^t(t-\tau)^{-\frac{3-m}{6}}(1+\tau)^{-1}(1+\eta\log(1+\tau))^{-2}d\tau\\
&\le C(1+t)^{-\frac{3-m}{6}}\|u_0\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}}+C\varepsilon(1+t)^{-\frac{3-m}{6}}\le C\varepsilon(1+t)^{-\frac{3-m}{6}}.
\end{aligned}$$

In the same way

$$\begin{aligned}
&\left\|x^{\frac{m}{2}}\partial_x u(t)\right\|_{\mathbf{L}^2}\\
&\le C(1+t)^{-\frac{5-m}{6}}\|u_0\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}}\\
&\quad+C\int_0^{\frac{t}{2}}\left((t-\tau)^{-1}\|uu_x\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}}+(t-\tau)^{-\frac{5-m}{6}}\|uu_x\|_{\mathbf{H}_1^{0,1}}\right)d\tau\\
&\quad+C\int_{\frac{t}{2}}^t\left((t-\tau)^{-\frac{2}{3}}\|uu_x\|_{\mathbf{H}_1^{0,\frac{m+1}{2}}}+(t-\tau)^{-\frac{3-m}{6}}\|uu_x\|_{\mathbf{L}^1}\right.\\
&\quad\left.+(t-\tau)^{-\frac{5}{6}+\frac{1}{12}}\|uu_x\|_{\mathbf{H}_1^{0,\frac{m+3}{2}-\frac{3}{4}}}+(t-\tau)^{-\frac{5-m}{6}}\|uu_x\|_{\mathbf{H}_1^{0,1}}\right)d\tau.
\end{aligned}$$

Therefore by a direct calculation

$$\begin{aligned}
&\left\| x^{\frac{m}{2}}\partial_x u\left(t\right)\right\|_{\mathbf{L}^2} \\
&\le C\left(1+t\right)^{-\frac{5-m}{6}}\left\|u_0\right\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + C\varepsilon^2\int_0^{\frac{t}{2}}\left(t-\tau\right)^{-1}\left(1+\tau\right)^{-\frac{5-m}{6}}d\tau \\
&\quad + C\varepsilon^2\int_0^{\frac{t}{2}}\left(t-\tau\right)^{-\frac{5-m}{6}}\left(1+\tau\right)^{-1}\left(1+\eta\log\left(1+\tau\right)\right)^{-2}d\tau \\
&\quad + C\varepsilon^{\frac{3}{2}}\int_{\frac{t}{2}}^{t}\Big(\left(t-\tau\right)^{-\frac{2}{3}}\left(1+\tau\right)^{-\frac{3-m}{6}} + \left(t-\tau\right)^{-\frac{3-m}{6}}\left(1+\tau\right)^{-\frac{4}{3}} \\
&\quad + \left(t-\tau\right)^{-\frac{5}{6}+\frac{1}{12}}\left(1+\tau\right)^{-1-\frac{1}{12}+\frac{m}{6}}\Big)d\tau \\
&\quad + \varepsilon^2\int_{\frac{t}{2}}^{t}\left(t-\tau\right)^{-\frac{5-m}{6}}\left(1+\tau\right)^{-1}\left(1+\eta\log\left(1+\tau\right)\right)^{-2}d\tau \\
&\le C\left(1+t\right)^{-\frac{5-m}{6}}\left\|u_0\right\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + C\varepsilon\left(1+t\right)^{-\frac{5-m}{6}} \le C\varepsilon\left(1+t\right)^{-\frac{5-m}{6}}.
\end{aligned}$$

Thus we get

$$\sum_{n=0}^{1}\sum_{m=3}^{7}\sup_{t>0}\left(1+t\right)^{\frac{3+2n-m}{6}}\left\|\left(\cdot\right)^{\frac{m}{2}}\partial_x^{(n)}u\left(\cdot,t\right)\right\|_{\mathbf{L}^2} \le C\varepsilon$$

which implies

$$\sum_{n=0}^{1}\sum_{m=3}^{7}\sup_{t>0} g(t)^{-1}\left(1+t\right)^{\frac{3+2n-m}{6}}\left\|\left(\cdot\right)^{\frac{m}{2}}\partial_x^{(n)}\mathbb{M}_2(r,g)\right\|_{\mathbf{L}^2} \le C\varepsilon \le \varepsilon^{\frac{3}{4}}.$$

Also from Lemma 2 we have

$$\sum_{n=0}^{1}\sum_{m=3}^{7}\sup_{t>0} g(t)^{-1}\left(1+t\right)^{\frac{3+2n-m}{6}}\left\|\left(\cdot\right)^{\frac{m}{2}}\partial_x^{(n)}\mathcal{R}(t)u_0(\cdot)\right\|_{\mathbf{L}^2} \le C\varepsilon$$

Then using last obtained estimate and (4.8) we can prove (see [17]) for $t > 0$

$$\frac{1}{2}(1+\eta\log(1+t)) < g(t) < 2(1+\eta\log(1+t)).$$

Thus $(\mathbb{M}_1(r,g), \mathbb{M}_2(r,g))$ transform the set X_ε into itself. Similarly, we can prove that the transformation $(\mathbb{M}_1(r,g), \mathbb{M}_2(r,g))$ is the contraction mapping. Hence there exists a unique solution (r,g) of the system of integral equations (4.5) in the set X and for $t > 1$ and

$$\begin{aligned}
\left\|r(t)\right\|_{\mathbf{L}^\infty} &= \left\|v\left(t\right)-\mathcal{G}\left(t\right)u_0\right\|_{\mathbf{L}^\infty} \\
&\le C\left\|r\left(t\right)\right\|_{\mathbf{L}^2}^{\frac{1}{2}}\left\|r_x\left(t\right)\right\|_{\mathbf{L}^2}^{\frac{1}{2}} \le Ct^{-\frac{2}{3}}g^{-1+\frac{4}{5}}(t).
\end{aligned}$$

From Lemma 1 we have

$$\mathcal{G}(t)u_0 = \theta(\sigma t)^{-\frac{2}{3}}Ai'\left(\frac{x}{\sqrt[3]{\sigma t}}\right) + R(x,t),$$

where for $t > 1, 0 < \delta < 1$

$$\left\|\partial_x^{(n)}R(t)\right\|_{\mathbf{H}_2^{0,\frac{m}{2}}} \le C\varepsilon t^{-\frac{3-m+2n+2\delta}{6}}$$

and therefore

$$\|R\|_{\mathbf{L}^\infty} \leq C\|R\|_{\mathbf{L}^2}^{\frac{1}{2}} \|R_x\|_{\mathbf{L}^2}^{\frac{1}{2}} \leq Ct^{-\frac{2+\delta}{3}}.$$

Thus we obtain the following asymptotics of solutions for $t \to +\infty$ uniformly with respect to $x > 0$ such that

$$\begin{aligned} u(x,t) &= g^{-1}(t)v(x,t) = g^{-1}(t)\left(r(x,t) + \mathcal{G}(t)u_0\right) \\ &= \theta\left(\eta\log t\right)^{-1}(\sigma t)^{-\frac{2}{3}} Ai'\left(\frac{x}{\sqrt[3]{\sigma t}}\right) + g^{-1}(t)R(x,t) + g^{-1}(t)r(x,t) \\ &= \theta\left(\eta\log t\right)^{-1}(\sigma t)^{-\frac{2}{3}} Ai'\left(\frac{x}{\sqrt[3]{\sigma t}}\right) + O\left(t^{-\frac{2}{3}}(\eta\log t)^{-\frac{6}{5}}\right). \end{aligned}$$

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Received: 12 January 2011.
Accepted: 20 September 2011.