Nonlinear Differ. Equ. Appl. 19 (2012), 417–436
© 2011 Springer Basel AG
1021-9722/12/040417-20
published online October 12, 2011
DOI 10.1007/s00030-011-0136-z

Nonlinear Differential Equations and Applications NoDEA

Multiple variational solutions to nonlinear Steklov problems

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Abstract. We consider the problem of finding a harmonic function u in a bounded domain $\Omega \subset \mathbb{R}^N, N \geq 2$, satisfying a nonlinear boundary condition of the form $\partial_\nu u(x) = \lambda\,(\eta(x)u(x) + \mu(x)h(u(x)), x \in \partial\Omega$ where μ and η are bounded functions and h is a $\mathcal{C}^1$ odd function with subcritical growth at infinity and such that $\lim_{s\to\infty} h'(s) = +\infty$. By using variants of the mountain pass lemma based on index theory, we discuss existence and multiplicity of non trivial solutions to the problem for every value of λ.

1. Statement of the problem and main result

In this paper we discuss the following problem: find nontrivial solutions u to the system

$$\begin{aligned} &\Delta u(x) = 0 \quad \text{in } \Omega \\ &\partial_\nu u(x) = \lambda\,[\eta(x)u(x) + \mu(x)h(u(x))] \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N, N \geq 2, \lambda \in \mathbb{R}$ and η, $\mu \in L^\infty(\partial\Omega)$; h is an odd $\mathcal{C}^1$ function on $\mathbb{R}$:

$$h(-u) = -h(u), \tag{1.2}$$

with *superlinear, subcritical growth at infinity.*

Furthermore, we require

$$h'(0) = 0, \tag{1.3}$$

$$\lim_{s\to\infty} h'(s) = +\infty, \tag{1.4}$$

By assuming other (more technical) conditions, we will obtain *existence and multiplicity* of solutions to (1.1) for every $\lambda \neq 0$. The complete list of the hypotheses and a precise statement are given in Sect. 5.

Motivations for studying (1.1) come both from applications and from more theoretical contexts.

For $N \geq 3$, a boundary condition of the form

$$\partial_\nu u(x) = \frac{N-2}{2} \left[\eta(x)u(x) + \mu(x)|u(x)|^{\beta-1}u(x)\right] \quad \text{with } \beta \in (1, \tfrac{N}{N-2}), \tag{1.5}$$

has been considered in [1], where, however, the interest is focused on positive solutions. Indeed, positive harmonic functions satisfying (1.5) arise in connection with the problem of finding conformal metrics with prescribed mean curvature on the boundary of a Riemannian manifold.

Positive solutions to problem (1.1) (and to the analogous problem for a uniformly elliptic operator) were found in [2] with additional hypotheses on the weights η, μ and by assuming that h has a precise power-like growth at infinity.

In dimension $N = 2$, harmonic functions satisfying a boundary condition of the form

$$\partial_\nu u(x) = \lambda\mu(x)\left(e^{\alpha u} - e^{-(1-\alpha)u}\right), \tag{1.6}$$

are of interest in the study of mathematical models of corrosion (see [3], where μ is either identically 1 or the characteristic function of a subset of $\partial\Omega$). In [4] the general condition (1.6), with μ changing sign along $\partial\Omega$, is considered. The case $\alpha = 1/2$ in (1.6) takes on a special interest, both in applications and for theoretical reasons; in [5] the authors prove the existence of infinitely many solutions for any positive λ (assuming $\mu(x) \equiv 1$) by applying variational methods, relying on index theory, which are suitable for even functionals (see [6, chapter 5]).

The boundary condition (1.1) with a symmetric h, considered in this paper, includes the case $\alpha = 1/2$ of (1.6) (choose $\eta = \mu$ and $h(u) = 2\sinh(u/2) - u$) as well as (1.5).

We show that problem (1.1) characterizes the critical points of an *even* functional in $H^1(\Omega)$, which satisfies suitable estimates near to the origin and along certain directions at infinity. Then, by exploiting known critical point theorems for symmetric functionals, based on the topological notions of *index* and *pseudo-index*, [7,8], we obtain existence and multiplicity results.

We stress that these results are obtained under weaker assumptions on the function h compared to the non symmetric case (with $\eta = \mu$) considered in [4]; in particular, we do not require positivity of the derivative $h'(s)$ for any s, but only (1.3), (1.4).

As in [4], the approach to the non linear problem relies on the solution of a related linear Steklov eigenvalue problem with indefinite weight on the boundary. In the next section, we summarize (without proofs) the crucial results about this problem. In Sect. 3 we provide the main estimates necessary to develop the variational procedure. In Sect. 4 we prove that our functional satisfies the Palais–Smale condition. Section 5 contains the main theorem, which states existence of infinitely many solutions to problem (1.1) for every $\lambda \neq 0$.

Finally, we prove additional results on existence of solutions in the non symmetric case and on existence of positive solutions, which also follow from the estimates of Sects. 3 and 4. These results complete those obtained previously in [4].

2. The linear eigenvalue problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and consider the following linear Steklov eigenvalue problem in $H^1(\Omega)$:

$$\begin{aligned} \Delta u(x) = 0 \quad &\text{in } \Omega \\ \gamma(\partial_\nu u)(x) = \lambda\, \sigma(x)\gamma(u)(x) \quad &\text{on } \partial\Omega \end{aligned} \tag{2.1}$$

where $\lambda \in \mathbb{R}, \sigma \in L^\infty(\partial\Omega)$ and γ denotes the trace operator on $\partial\Omega$.

We recall that, for a Lipschitz domain Ω, the trace on $\partial\Omega$ of the normal derivative of a $H^1(\Omega)$ function satisfying $\Delta u \in L^2(\Omega)$ (in the weak sense) is well defined as an element of the Sobolev space $H^{-1/2}(\partial\Omega)$. For a general overview of Sobolev spaces and traces of functions see [9].

It is easily seen that the solutions to (2.1) belong to the subspace $H^1_\sigma \subset H^1(\Omega)$ defined as follows:

$$H^1_\sigma \equiv \left\{ u \in H^1(\Omega), \quad \int_{\partial\Omega} \sigma\, \gamma(u) = 0 \right\}. \tag{2.2}$$

Assuming that

$$\int_{\partial\Omega} \sigma \neq 0, \tag{2.3}$$

it turns out [4] that the Dirichlet norm $\int_\Omega |\nabla u|^2$ is equivalent to the H^1 norm in H^1_σ and that (2.1) is equivalent to the following variational problem:
Find $u \in H^1_\sigma, u \neq 0$, such that

$$\int_\Omega \nabla u\, \nabla v = \lambda \int_{\partial\Omega} \sigma\gamma(u)\gamma(v) \tag{2.4}$$

holds for every $v \in H^1_\sigma$.

Moreover, the expression

$$\|u\|_1^2 = \int_\Omega |\nabla u|^2 + \left(\int_{\partial\Omega} \sigma\gamma(u) \right)^2, \tag{2.5}$$

defines an equivalent norm in $H^1(\Omega)$. We consider the scalar product in $H^1(\Omega)$ associated to this equivalent norm; then, we have the following result [4]:

Proposition 2.1. *Assume (2.3). Then, problem (2.1) has infinitely many eigenvalues λ_n, each of finite multiplicity and such that $|\lambda_n| \to +\infty$. Moreover, the following orthogonal decomposition holds:*

$$H^1 = H^1_0 \oplus c \oplus V_\sigma \oplus V_0, \tag{2.6}$$

where c are constants eigenfunctions corresponding to the null eigenvalue $\lambda_0 = 0$, the subspace V_σ is spanned by the eigenfunctions u_n satisfying the variational equations

$$\int_\Omega \nabla u_n \nabla v = \lambda_n \int_{\partial\Omega} \sigma\gamma(u_n)\gamma(v), \quad \lambda_n \neq 0, \tag{2.7}$$

for every $v \in H^1$ and V_0 is spanned by (harmonic) functions w such that

$$\int_{\partial\Omega} \sigma\gamma(w)\gamma(v) = 0 \tag{2.8}$$

for every $v \in H^1$.

Notice that a non trivial w satisfying (2.8) can only exist if $\sigma\gamma(w) = c\sigma = 0$, which implies that the function σ vanishes on a subset of positive Hausdorff measure of $\partial\Omega$; otherwise, V_0 is empty.

In the sequel, we will list all the eigenvalues to problem (2.1) as follows

$$...\lambda_{-2} \leq \lambda_{-1} \leq 0 \leq \lambda_1 \leq \lambda_2...$$

The eigenvalue $\lambda_0 = 0$ corresponds to the constant solutions of the homogeneous Neumann problem.

By (2.7), we can take all the u_n orthogonal and normalized with respect to the scalar product associated to the Dirichlet norm $\int_\Omega |\nabla u|^2$ and even to the equivalent norm (2.5) by defining $u_0 = \left(\int_{\partial\Omega} \sigma\right)^{-1}$; then, we have

$$\int_\Omega \nabla u_n \, \nabla u_m = \int_{\partial\Omega} \sigma \, \gamma(u_n)\gamma(u_m) = 0, \tag{2.9}$$

for $n \neq m$.

Note that from the relations

$$\int_\Omega |\nabla u_n|^2 = \lambda_n \int_{\partial\Omega} \sigma \, \gamma(u_n)^2, \tag{2.10}$$

we get the inequalities

$$\int_{\partial\Omega} \sigma \, \gamma(u_n)^2 > 0, \quad \text{for } n > 0; \qquad \int_{\partial\Omega} \sigma \, \gamma(u_n)^2 < 0, \quad \text{for } n < 0. \tag{2.11}$$

Remark 2.2. It is worth stressing that, if σ has definite sign, $\sigma \geq 0$ say, the problem is coercive for $\lambda < 0$ and we have infinitely many positive eigenvalues (this is the case of the classical Steklov problem); an analogous assertion holds if $\sigma \leq 0$. But, as soon as σ is positive on some subset of $\partial\Omega$ and negative on some other subset, both subsets beeing of positive measure, there are infinitely many positive *and* negative eigenvalues (see [4, remark 2.6]).

Remark 2.3. When $\int_{\partial\Omega} \sigma = 0$ (and σ is non trivial) it has been proved in [10] theorem 1.1 that there are still unbounded sequences of positive and negative eigenvalues to problem (2.1). However, the decomposition (2.6) does not hold in that case (see [4, remark 2.7]).

Remark 2.4. *On the regularity of eigenfunctions.*

Global regularity of the eigenfunctions of (2.1) depends on the indefinite weight σ and on the regularity of the boundary $\partial\Omega$. For the subsequent discussion of the nonlinear problem, it is important to guarantee that the eigenfunctions u_m are bounded. Recall that any solution of (2.4) belongs to $H^1(\Omega)$ and is harmonic in Ω; hence the trace of its normal derivative, being proportional to $\sigma\gamma(u)$, belongs to $L^2(\partial\Omega)$, so that we have $u_m \in H^{3/2}(\Omega)$ even in a Lipschitz domain [11]. Then, in case of dimension $N = 2$ we get $u_m \in \mathcal{C}(\overline{\Omega})$ (by Sobolev imbedding) without any additional assumption. For $N \geq 3$, more regularity of σ and of the boundary $\partial\Omega$ will be required in order to achieve $u_m \in H^s(\Omega)$ with $s > N/2$, which implies the continuity of u_m up to the boundary. For the sake of brevity, we will assume when it is needed in the following that σ and $\partial\Omega$ are smooth enough to satisfy such conditions without entering into further details (see e.g. [9]).

3. Main estimates

We now discuss the solvability of the non linear problem (1.1); since both η and μ are indefinite, we may assume $\lambda > 0$.

Let us define $H(u) = \int_0^u h(t)dt$ and consider the *even functional*

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega} \eta u^2 - \lambda \int_{\partial\Omega} \mu H(u) \tag{3.1}$$

where $u \in H^1$. From now on, the trace operator γ will be dropped to make the notation shorter.

We now make the following assumption on the nonquadratic term of the functional:

Let $0 < \epsilon < \frac{2}{N-2}$ *(*$\epsilon > 0$ *for* $N = 2$*) and* $q \geq \frac{2(N-1)}{2-\epsilon(N-2)}$ *(any* $q > 1$ *for* $N = 2$*). We suppose that*

$$|H(u)| \leq |u|^{2+\epsilon}\tilde{H}(u), \tag{3.2}$$

where $\tilde{H} : H^1(\Omega) \to L^q(\partial\Omega)$ *is bounded.*

For $N \geq 3$, it is readily verified that the above condition holds if there is $C > 0$ such that $|h(u)| \leq C|u|^{1+\epsilon}$; in the case $N = 2$, it can be shown [5, lemma 2.1] that the function $H(u) = \cosh u - 1 - u^2/2$ also satisfies the assumption.

Define further

$$S_R = \{v \in H^1 : \ \|v\|_1 = R\}; \tag{3.3}$$

where $\| \|_1$ is the equivalent norm defined in (2.5) with $\sigma = \eta$, that is:

$$\|u\|_1^2 = \int_\Omega |\nabla u|^2 + \left(\int_{\partial\Omega} \eta\, u\right)^2. \tag{3.4}$$

Then, we have

Lemma 3.1. *Let (3.2) hold; then, for every $\lambda > 0$, there exists $R > 0$ and a closed subspace $V^+ \subseteq H^1(\Omega)$ with codim $V^+ < \infty$ such that*

$$E_\lambda(v) \geq c_0 > 0,$$

for every $v \in S_R \cap V^+$.

Proof. By (3.2) we have

$$E_\lambda(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega} \eta u^2 - \lambda\|\mu\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}|u|^{2+\epsilon}\tilde{H}(u), \tag{3.5}$$

and the integral in the last term can be bounded as follows

$$\left|\int_{\partial\Omega} u^{2+\epsilon}\tilde{H}(u)\right| \leq \|\tilde{H}(u)\|_{L^q(\partial\Omega)}\left(\int_{\partial\Omega}|u|^{(2+\epsilon)p}\right)^{1/p} \leq C\,\|u\|_1^{2+\epsilon} = C\,R^{2+\epsilon}, \tag{3.6}$$

where the last estimate follows by $(2+\epsilon)p = (2+\epsilon)\frac{q}{q-1} \leq \frac{2(N-1)}{N-2}$.

Let us now consider the quadratic part of the functional. Assume first $\int_{\partial\Omega}\eta \neq 0$ and denote by $\lambda_n^0, u_n^0, n \in \mathbb{Z}$, the eigenvalues and the eigenfunctions of the linear problem (2.1) with $\sigma = \eta$; by Remark 2.2, if the sequence of positive eigenvalues $\lambda_n^0, n > 0$ is not missing, then it is unbounded. In that case, there exists a non negative integer k such that

$$\lambda_k^0 \leq \lambda < \lambda_{k+1}^0 \tag{3.7}$$

Then, by recalling the decomposition (2.6) (with $\sigma = \eta$) we set

$$V^+ = H_0^1 \oplus c \oplus V_0 \oplus \mathrm{span}_{n<0}\{u_n^0\} \oplus \mathrm{span}_{n\geq k+1}\{u_n^0\}, \tag{3.8}$$

if $\int_{\partial\Omega}\eta < 0$, and

$$V^+ = H_0^1 \oplus V_0 \oplus \mathrm{span}_{n<0}\{u_n^0\} \oplus \mathrm{span}_{n\geq k+1}\{u_n^0\}, \tag{3.9}$$

if $\int_{\partial\Omega}\eta > 0$.

In case the sequence of positive eigenvalues is missing (that is, if $\eta \leq 0$ a.e. in $\partial\Omega$ and $\int_{\partial\Omega}\eta < 0$) we simply take $V^+ = H^1(\Omega)$.

Let us first consider the subspace (3.8) and decompose an element $u \in S_R \cap V^+$ as $u = c \oplus \tilde{u}$.

By the right hand side of (3.8) and by recalling the second of inequalities (2.11) and the definition of the subspace V_0, we get

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega}\eta u^2 &= \frac{1}{2}\int_\Omega |\nabla \tilde{u}|^2 - \frac{\lambda}{2}\int_{\partial\Omega}\eta\tilde{u}^2 - \frac{\lambda}{2}c^2\int_{\partial\Omega}\eta\\ &\geq \frac{1}{2}\left(1-\frac{\lambda}{\lambda^0_{k+1}}\right)\int_\Omega |\nabla\tilde{u}|^2 - \frac{\lambda}{2}c^2\int_{\partial\Omega}\eta\\ &\geq \frac{1}{2}\min\left[(1-\lambda/\lambda^0_{k+1}),\, \lambda\left(-\int_{\partial\Omega}\eta\right)^{-1}\right]R^2 \qquad (3.10)\end{aligned}$$

for every $u \in S_R \cap V^+$.

In case $\int_{\partial\Omega}\eta > 0$, the above estimate holds with $c = 0$ for every $u = \tilde{u}$ with norm R in the subspace (3.9), where the equivalent norm is simply the Dirichlet norm.

Finally, if $\int_{\partial\Omega}\eta = 0$, we just observe that

$$E_\lambda(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega}(\eta+\delta)u^2 - \lambda\int_{\partial\Omega}\mu H(u)$$

for every constant $\delta > 0$, so that we can define V^+ as in (3.9) with u^0_n the eigenfunctions of problem (2.1) with $\sigma = \eta + \delta$.

Then, the lemma follows by taking R small enough. $\square$

Remark 3.2. By the proof of the previous lemma, if $\int_{\partial\Omega}\eta \neq 0$ and λ satisfies (3.7) for some $k \geq 0$, we can choose an 'optimal' subspace V^+ such that: codim $V^+ = k$ if $\int_{\partial\Omega}\eta < 0$, codim $V^+ = k+1$ if $\int_{\partial\Omega}\eta > 0$. Furthermore, if $\int_{\partial\Omega}\eta < 0$ and $0 < \lambda < \lambda^0_1$, it follows by (3.8), (3.10) that we can take $V^+ = H^1$ (hence, codim $V^+ = 0$); clearly, the same holds for any positive λ in case of non trivial $\eta \leq 0$.

On the other hand, when $\int_{\partial\Omega}\eta = 0$, the precise evaluation of the codimension of the optimal V^+ is more delicate, as it involves a comparison between the eigenvalues of the linear problem, respectively with $\sigma = \eta$ and $\sigma = \eta + \delta$, for $\delta \to 0$.

We stress that the case $\eta = 0$ is included in the previous lemma; however, a sharper result can be achieved by more restrictive assumptions on the non linear term:

Proposition 3.3. *Let us consider the functional* (3.1) *with* $\eta = 0$ *and let* ϵ, q, *be defined above equation* (3.2); *suppose that*

(i)

$$\lim_{u\to 0}\frac{H(u)}{|u|^{2+\epsilon}} = \kappa \neq 0; \qquad (3.11)$$

(ii)

$$\kappa\int_{\partial\Omega}\mu < 0;$$

(iii)

$$|h(u)| \leq |u|^{1+\epsilon}\tilde{h}(u), \qquad (3.12)$$

where $\tilde{h} : H^1(\Omega) \to L^q(\partial\Omega)$ *is bounded.*

Then, for every $\lambda > 0$*, there exists* $R > 0$ *such that*

$$E_\lambda(v) \geq c_0 > 0,$$

for every $v \in S_R$*.*

Proof. Let us write $u = c \oplus \tilde{u}$, where we are now using the orthogonal decomposition (2.6) with $\sigma = \mu$. Then, we have

$$\begin{aligned} E_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} \mu H(u) \\ &= \frac{1}{2}\int_\Omega |\nabla \tilde{u}|^2 - \lambda H(c)\int_{\partial\Omega} \mu - \lambda \int_{\partial\Omega} \mu[H(c+\tilde{u}) - H(c)] \\ &= \frac{1}{2}\int_\Omega |\nabla \tilde{u}|^2 - \lambda H(c)\int_{\partial\Omega} \mu - \lambda \int_{\partial\Omega} \mu \int_0^1 h(c+t\tilde{u})\tilde{u}\, dt \\ &\geq \frac{1}{2}\int_\Omega |\nabla \tilde{u}|^2 + \frac{\lambda\kappa}{2}\left(-\int_{\partial\Omega} \mu\right)|c|^{2+\epsilon} - \lambda\|\mu\|_\infty \int_{\partial\Omega}\int_0^1 \tilde{h}(c+t\tilde{u})|c \\ &\quad + t\tilde{u}|^{1+\epsilon}\,|\tilde{u}|\, dt \end{aligned}$$

for *small enough* c by (3.11), (3.12). Note that the second term of the above expression is positive, by the assumption $\kappa \int_{\partial\Omega} \mu < 0$. Now, by using in the last term the inequality

$$|c+t\tilde{u}|^{1+\epsilon} \leq 2^{1+\epsilon}\left(|c|^{1+\epsilon} + |\tilde{u}|^{1+\epsilon}\right)$$

and by the properties of $\tilde{h}$ stated in (3.12), we obtain the following bound

$$E_\lambda(u) \geq \frac{1}{2}\|\nabla\tilde{u}\|_{L^2}^2 + d_1|c|^{2+\epsilon} - d_2|c|^{1+\epsilon}\|\nabla\tilde{u}\|_{L^2} - d_3\|\nabla\tilde{u}\|_{L^2}^{2+\epsilon}$$

for some positive constants d_1, d_2, d_3. Finally, by Young inequality

$$|c|^{1+\epsilon}\|\nabla\tilde{u}\|_{L^2} \leq \frac{1+\epsilon}{2+\epsilon} r^{\frac{2+\epsilon}{1+\epsilon}}\,|c|^{2+\epsilon} + \frac{1}{2+\epsilon} r^{-(2+\epsilon)}\,\|\nabla\tilde{u}\|_{L^2}^{2+\epsilon}, \quad \forall r > 0,$$

we get

$$\begin{aligned} E_\lambda(u) \geq \frac{1}{2}\|\nabla\tilde{u}\|_{L^2}^2 &- \left(d_3 + \frac{d_2}{2+\epsilon} r^{-(2+\epsilon)}\right)\|\nabla\tilde{u}\|_{L^2}^{2+\epsilon} \\ &+ \left(d_1 - d_2\frac{1+\epsilon}{2+\epsilon} r^{\frac{2+\epsilon}{1+\epsilon}}\right)|c|^{2+\epsilon} \end{aligned}$$

Then, by choosing r and $\|\nabla\tilde{u}\|_{L^2}$ sufficiently small, we get $E_\lambda(u) \geq c_0 > 0$ and the lemma follows. □

Remark 3.4. We remark that in related problems with indefinite superlinear terms [2,12], it is commonly assumed that the principal eigenvalue of a uniformly elliptic operator in Ω ($-\Delta$ in our case) under a homogeneous boundary condition on $\partial\Omega$ ($\partial_\nu u - \lambda\eta u = 0$) is strictly positive. We do not need such condition in Lemma 3.1 (see, however, the discussion at the end of Sect. 5); furthermore, the assumptions of Proposition 3.3 should be compared with those of theorem 1.3 in [2].

We are now going to construct closed, *finite dimensional* subspaces, $V^- \subset H^1(\Omega)$ such that:

- $\dim V^- > \operatorname{codim} V^+$;
- $E_\lambda(u) \le c_\infty < \infty$ for every $u \in V^-$.

To this aim, let us denote by $\lambda_n^\infty, u_n^\infty, n \in \mathbb{Z}$, the eigenvalues and the eigenfunctions of the linear problem (2.1) with $\sigma = \mu$ and assume that

μ has a non trivial positive part,

so that the sequence of positive eigenvalues $\lambda_n^\infty, n > 0$, is unbounded.

Moreover, let $u_{n_i}^\infty, 1 \le i \le l$ be *any* finite sequence of l eigenfunctions, with $l > \operatorname{codim} V^+$, corresponding to *non negative* eigenvalues

$$0 \le \lambda_{n_1}^\infty \le \lambda_{n_2}^\infty \le \ldots \le \lambda_{n_l}^\infty. \tag{3.13}$$

Then we define:

$$V^- = \operatorname{span}_{1\le i\le l} \{u_{n_i}^\infty\}, \tag{3.14}$$

The next lemma provides the key estimates at infinity on the functional (3.1).

Lemma 3.5. *Let condition (1.4) holds and let V^- be defined by (3.14), with $\lambda_{n_1}^\infty > 0$ if $\int_{\partial\Omega} \mu \le 0$.*

Then, for every $\lambda > 0$ we have $E_\lambda(u) < 0$ for any $u \in V^-$ with large enough norm. As a consequence, there exists $c_\infty < \infty$ such that

$$E_\lambda(u) \le c_\infty \qquad \forall u \in V^-.$$

Proof. Let us first consider the case of strictly positive eigenvalues. For notational simplicity we now set $u_{n_i}^\infty = u_i, \lambda_{n_i}^\infty = \lambda_i > 0$ $(1 \le i \le l)$.

Thus, we can write any $u \in V^-$ in the form

$$u = \sum_{i=1}^{l} t_i u_i$$

and consider the function

$$\begin{aligned} f(t_1,\ldots,t_l) \equiv E_\lambda(u) &= \sum_{i=1}^{l} \frac{t_i^2}{2}\int_\Omega |\nabla u_i|^2 - \frac{\lambda}{2}\int_{\partial\Omega} \eta u^2 - \lambda \int_{\partial\Omega} \mu H(u) \\ &= \frac{1}{2}\sum_{i=1}^{l} t_i^2 - \frac{\lambda}{2}\int_{\partial\Omega} \eta u^2 - \lambda\int_{\partial\Omega} \mu H(u), \end{aligned} \tag{3.15}$$

where we used orthogonality and normalization of u_i with respect to the inner product defined by the Dirichlet norm (recall that $\|\nabla u_i\|_{L^2} = \|u_i\|_1 = 1, \forall i$).

Let $1 \le j \le l$ and consider the variational equation

$$\int_\Omega \nabla u_j \, \nabla h(u) = \lambda_j \int_{\partial\Omega} \mu \, u_j \, h(u) \tag{3.16}$$

where $h(u) = h(\sum_{i=1}^{l} t_i u_i)$; we stress that, by the regularity results at the end of the previous section, u is a bounded continuous function on Ω, so that

$h(u) \in H^1(\Omega)$ and (3.16) holds by Proposition 2.1. Multiplying by t_j and summing up from $j = 1$ to l, we find

$$\sum_{j=1}^{l} t_j \int_\Omega \nabla u_j \, \nabla u \, h'(u) = \sum_{j=1}^{l} \lambda_j t_j \int_{\partial\Omega} \mu \, u_j \, h(u),$$

that is

$$\int_\Omega |\nabla u|^2 h'(u) = \sum_{j=1}^{l} \lambda_j t_j \int_{\partial\Omega} \mu \, u_j \, h(u). \tag{3.17}$$

Let us now calculate

$$\begin{aligned}
&\left(\sum_{j=1}^{l} \lambda_j t_j \partial_{t_j}\right) f(t_1, \ldots, t_l) \\
&\quad = \sum_{j=1}^{l} \lambda_j t_j^2 - \lambda \sum_{j=1}^{l} \lambda_j t_j \int_{\partial\Omega} \eta \, u_j \, u - \lambda \sum_{j=1}^{l} \lambda_j t_j \int_{\partial\Omega} \mu \, u_j \, h(u) \\
&\quad = \sum_{j=1}^{l} \lambda_j t_j^2 - \lambda \sum_{i,j=1}^{l} \lambda_j t_i t_j \int_{\partial\Omega} \eta \, u_i \, u_j - \lambda \sum_{j=1}^{l} \lambda_j t_j \int_{\partial\Omega} \mu \, u_j \, h(u)
\end{aligned}$$

(by elementary estimates on the quadratic terms and by (3.17))

$$\begin{aligned}
&\le \lambda_l (1 + C_l \|\eta\|_\infty) \sum_{j=1}^{l} t_j^2 - \lambda \int_\Omega |\nabla u|^2 h'(u) \\
&= \int_\Omega |\nabla u|^2 \left[\lambda_l (1 + C_l \|\eta\|_\infty) \, - \lambda \, h'(u)\right],
\end{aligned} \tag{3.18}$$

where C_l only depends on the eigenfunctions $u_1, \ldots, u_l$.

Since $h'(s) \to +\infty$ for $|s| \to +\infty$, the expression in the square brackets is ≤ -1 for $|u| \ge L$, where L depends on h', η, C_l, λ_l and λ. Let us define $\rho = \sqrt{t_1^2 + \cdots + t_l^2} \ge 0$ and let $\mathbf{a} = (\alpha_1, \ldots, \alpha_l)$ be a point on the unit sphere $\mathbb{S}^l \subset \mathbb{R}^{l+1}$. Then we can write

$$u = \rho(\alpha_1 u_1 + \ldots + \alpha_l u_l) \equiv \rho \, \omega_{\mathbf{a}}; \tag{3.19}$$

hence, the integrand in (3.18) may be non negative (actually, greater than -1) only on the sublevel sets

$$\Omega_{\rho,\mathbf{a}} = \{x \in \Omega, \, |\omega_{\mathbf{a}}(x)| \le L/\rho\}, \tag{3.20}$$

where it is bounded by $[\lambda_l(1 + C_l\|\eta\|_\infty) - \lambda \min h'] \, |\nabla u|^2$.

As $\omega_{\mathbf{a}}$ is harmonic in Ω, its nodal set has vanishing measure in $\mathbb{R}^N$ (in a neighborhood of a regular point, it is a $(N-1)-$dimensional manifold, while the critical zero set $\omega_{\mathbf{a}}^{-1}(0) \cap \nabla\omega_{\mathbf{a}}^{-1}(0)$ has vanishing $(N-1)-$dimensional Hausdorff measure [13,14]). Then, for every $\epsilon > 0$, one can find $\delta_\epsilon > 0$ such that $|\Omega_{\rho,\mathbf{a}}| \le \epsilon$, for $\rho \ge L/\delta_\epsilon$ and *for every* $\mathbf{a}$. The proof depends on the compactness of the unit sphere $\mathbb{S}^l$ and is reported in [4], lemma 3.3.

By our previous choice of L, we have $\lambda_l(1 + C_l\|\eta\|_\infty) - \lambda \, h'(u) < -1$ in $\Omega \backslash \Omega_{\rho,\mathbf{a}}$.

Then, for $\rho > L/\delta_\epsilon$, the right hand side of (3.18) is bounded as follows:

$$\begin{aligned}
&\int_\Omega |\nabla u|^2 \left[\lambda_l(1+C_l\|\eta\|_\infty)\ -\lambda\, h'(u)\right] \\
&\quad \le [\lambda_l(1+C_l\|\eta\|_\infty) - \lambda \min h'] \int_{\Omega_{\rho,\mathbf{a}}} |\nabla u|^2 - \int_{\Omega\setminus\Omega_{\rho,\mathbf{a}}} |\nabla u|^2 \\
&\quad = [\lambda_l(1+C_l\|\eta\|_\infty) - \lambda \min h' + 1] \int_{\Omega_{\rho,\mathbf{a}}} |\nabla u|^2 - \int_\Omega |\nabla u|^2 \\
&\quad \le \left[C \int_{\Omega_{\rho,\mathbf{a}}} (|\nabla u_1|^2 + \ldots |\nabla u_l|^2) - 1 \right] \rho^2,
\end{aligned} \tag{3.21}$$

for some positive constant C, where $|\Omega_{\rho,\mathbf{a}}| \le \epsilon$.

If ϵ has been chosen sufficiently small, the constant in the square brackets is strictly negative for $\rho > L/\delta_\epsilon$ and for every $\mathbf{a}$, that is for $\|u\|_1 = \sqrt{t_1^2 + \cdots t_l^2}$ large enough.

Note that the quantity estimated in (3.18) is (proportional to) the derivative of the function f in the direction of the vector $(\lambda_1 t_1, \ldots, \lambda_l t_l)$; hence, for $\sqrt{t_1^2 + \ldots t_l^2} > L/\delta_\epsilon$, the function is *strictly decreasing* along the curves $t_1 = c_1 e^{\lambda_1 s}, \ldots, t_l = c_l e^{\lambda_l s}$, $s \in \mathbb{R}$, (orthogonal to the hypersurfaces $\lambda_1 t_1^2 + \cdots + \lambda_l t_l^2 = cost.$).

We conclude that $f(u) < 0$ for $u \in V^-$ with $\|u\|_1$ large enough; since f is continuous and V^- has finite dimension, we have

$$\sup_{u\in V^-} f(u) = c_\infty < \infty.$$

We now prove that for $\int_{\partial\Omega} \mu > 0$ we may allow $\lambda_{n_1}^\infty = 0$ in (3.13). We first note that,

$$E_\lambda(c) = -\lambda c^2 \int_{\partial\Omega} \eta - \lambda H(c) \int_{\partial\Omega} \mu.$$

Hence, by the above assumption on μ and by the superlinearity of H', we have

$$\lim_{|c|\to\infty} \frac{E_\lambda(c)}{c^2} = -\lim_{|c|\to\infty} \lambda \frac{H(c)}{c^2} \int_{\partial\Omega} \mu = -\infty. \tag{3.22}$$

We now estimate the functional E_λ on vectors (of the l-dimensional subspace V^-) of the form $c+u$, where $c \in \mathbb{R}$ and $u = t_2 u_2 + \cdots + t_l u_l$, with $u_i = u_{n_i}^\infty$ eigenfunctions corresponding to strictly positive eigenvalues $\lambda_i = \lambda_{n_i}^\infty, i = 2, \ldots, l$; we now set $\rho = \sqrt{t_2^2 + \cdots + t_l^2}$. Then we can write

$$E_\lambda(c+u) = \frac{1}{2}\rho^2 - \frac{\lambda}{2}\int_{\partial\Omega} \eta (c+u)^2 - \lambda \int_{\partial\Omega} \mu H(c+u)$$

(by elementary estimates of the second term)

$$\le \frac{1}{2}(1+\lambda\|\eta\|_\infty C_l)\,\rho^2 + \lambda\|\eta\|_\infty \sqrt{C_l|\partial\Omega|}\, c\,\rho - \frac{\lambda}{2}c^2 \int_{\partial\Omega} \eta - \lambda \int_{\partial\Omega} \mu H(c+u).$$

Let us now define $u(\tau) = t_2\tau^{\lambda_2}u_1 + \cdots + t_l\tau^{\lambda_l}u_l$, with $0 \le \tau \le 1$. Then,

$$\int_{\partial\Omega} \mu H(c+u) = H(c)\int_{\partial\Omega}\mu + \int_0^1 d\tau \int_{\partial\Omega} \mu \frac{du}{d\tau}\, h(c+u(\tau)).$$

Now, from the variational equation

$$\int_\Omega \nabla u_j \nabla h(c+u(\tau)) = \lambda_j \int_{\partial\Omega} \mu u_j h(c+u(\tau)), \quad (j = 2, \dots, l)$$

we get as before

$$\begin{aligned}\int_{\partial\Omega} \mu \frac{du}{d\tau}\, h(c+u(\tau)) &= \sum_{i=2}^{l} t_i \tau^{\lambda_i - 1}\lambda_i \int_{\partial\Omega} \mu u_i h(c+u(\tau)) \\ &= \frac{1}{\tau}\int_\Omega |\nabla u(\tau)|^2 h'(c+u(\tau)) \ge \tau^{2\lambda_l - 1} \min h' \int_\Omega |\nabla u|^2,\end{aligned}$$

as $0 \le \tau \le 1$ and $\lambda_l \ge \cdots \ge \lambda_2 > 0$.

Thus,

$$\int_{\partial\Omega} \mu H(c+u) \ge H(c)\int_{\partial\Omega}\mu + \frac{1}{2\lambda_l}\min h' \int_\Omega |\nabla u|^2. \tag{3.23}$$

Putting this in the previous estimate, we have the bound

$$E_\lambda(c+u) \le \frac{1}{2}\left(1 + \lambda\|\eta\|_\infty C_l - \min h' \frac{\lambda}{\lambda_l}\right)\rho^2 + \lambda\|\eta\|_\infty \sqrt{C_l|\partial\Omega|}\, c\,\rho + E_\lambda(c) \tag{3.24}$$

By this estimate and by (3.22), we conclude that there exists a positive constant M (depending on $\lambda, \lambda_l, \mu, \eta$ and h) such that $E_\lambda(c+u) \le 0$ for

$$0 \le \rho \le M H(c)^{1/2}. \tag{3.25}$$

Note that, if the above inequality is not satisfied, the ratio $H(c)^{1/2}/\rho$ is bounded; hence, outside the region (3.25), $c/\rho \to 0$ for $\rho \to +\infty$.

Let us now calculate the derivative of the function

$$f(c, t_2, \dots t_l) = E_\lambda(c + t_2u_2 + \cdots + t_lu_l)$$

along the curves $(c, t_2e^{\lambda_2 s}, \dots, t_le^{\lambda_l s})$ at $s = 0$; as in equation (3.18) we get:

$$\left(\sum_{j=2}^{l} \lambda_j t_j \partial_{t_j}\right) f(c, t_2, \dots, t_l) \le \int_\Omega |\nabla u|^2 \left[\lambda_l(1 + C_l\|\eta\|_\infty) - \lambda\, h'(u)\right],$$

where $u = c + t_2u_2 + \cdots + t_lu_l$. By recalling (3.19), we now write $u = c + \rho\,\omega_{\mathbf{a}}$ and observe that, by the above discussion, $u/\rho \approx \omega_{\mathbf{a}}$ for large ρ outside the region (3.25). Then, by the same arguments that lead to (3.21), we find that f is definitively decreasing to $-\infty$ along the trajectories above. Summing up, we conclude that $E_\lambda(c + \rho\,\omega_{\mathbf{a}}) \le 0$ for $\sqrt{c^2 + \rho^2}$ large enough; hence, the lemma follows. □

4. The Palais–Smale condition

The results of the previous section suggest that it could be possible to prove existence and even multiplicity of solutions to problem (1.1) for every λ, by applying minimax methods for even functionals, like the Symmetric Mountain Pass Lemma (see [6, theorem 6.3]) or the pseudo-index theory of [8]; what we are left to show is that the functional

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega} \eta u^2 - \lambda \int_{\partial\Omega} \mu H(u)$$

satisfies the Palais–Smale condition (as before, we may assume $\lambda > 0$). The main difficulty in this task is to prove the boundedness of a Palais–Smale sequence, since the usual inequality assumptions relating E_λ and E'_λ could not be helpful in presence of an indefinite weight; in addition, we would like to include the case of exponential growth of the nonlinear term for $N = 2$. We will obtain the desired result with two different kinds of hypotheses which are illustrated below; similar assumptions have been previously introduced in [4], where they are also compared to other existing hypotheses implying the Palais–Smale condition for functionals with indefinite nonlinarities.

Let us introduce the following decomposition of the boundary $\partial\Omega$:

$$\partial\Omega = \Gamma_+ \cup \Gamma_- \cup \Gamma_0, \tag{4.1}$$

where $\mu > 0$ on $\Gamma_+, \mu < 0$ on Γ_- and $\mu = 0$ on Γ_0; we stress that each of the sets Γ_+, Γ_- and Γ_0 may have vanishing measure, but $|\Gamma_+ \cup \Gamma_-| > 0$. We further define $\mu_\pm = \max\{\pm\mu, 0\}$. Finally, we recall that $H' = h$ satisfies $h'(s) \to +\infty$ for $|s| \to +\infty$.

We now state our first set of assumptions:

Condition PS1.

1. $$\overline{\Gamma}_+ \cap \overline{\Gamma}_- = \emptyset$$
   and $\eta \leq 0$ on Γ_0;
2. There exist constants $q > 2$ and $R_0 > 0$ such that
   $$qH(u) \leq uh(u), \tag{4.2}$$
   for $|u| \geq R_0$.
3. There exist positive constants C such that
   $$|h'(u)| \leq Ce^{\alpha|u|} \tag{4.3}$$
   for some $\alpha \in \mathbb{R}$ if $N = 2$;
   $$|h(u)| \leq C(1 + |u|^\beta)$$
   (with $\beta < \frac{N}{N-2}$) if $N \geq 3$.

Remark 4.1. Condition 2 is satisfied in particular by functions of the form $H(u) = \frac{1}{q}|u|^q + G(u)$ where $2 < q \leq \frac{2(N-1)}{N-2}$ if $N \geq 3$, any $q > 2$ if $N = 2$ and G is such that: $|G'(u)| \leq C_1|u|^{\gamma_1}, \gamma_1 > 1$ for $|u| << 1$, and $|G'(u)| \leq C_2|u|^{\gamma_2}, \gamma_2 \leq 1$ for $|u| >> 1$. Moreover, in case $N = 2$, it is readily verified that $H(u) = \cosh u - 1 - u^2/2$ also satisfies (4.2) with $q = 4$ (and any R_0).

Finally, condition 3 specifies the previously mentioned assumption that h has subcritical growth at infinity; note that for $N = 2$ a bound similar to (4.3) obviously holds for $|h|$; however, the estimate for h' is needed in the proof of Proposition 4.2 below.

We can now state:

Proposition 4.2. *Let $z_m \in H^1(\Omega)$ be a sequence such that $E_\lambda(z_m) \to c$ and $E'_\lambda(z_m) \to 0$ in $H^1(\Omega)'$. Assume that (PS1) holds. Then, the sequence $\|z_m\|$ is bounded and the functional (3.1) satisfies the Palais–Smale condition.*

Proof of Proposition 4.2. The main arguments of the proof are similar to those in proposition 4.2 of [4], where the quadratic boundary term with the weight η was missing in the functional E_λ; hence, we will discuss in detail only the non trivial changes in the proof due to this additional term.

Here we denote by $\| \|_1$ anyone of the equivalent norms (2.5) (the particular choice of the function σ is not relevant in the sequel). Assume by contradiction (considering a subsequence if necessary) that $\|z_m\|_1 \to +\infty$ and define $v_m = t_m^{-1} z_m$, where $t_m = \|z_m\|_1$.

Since v_m is bounded, there is a subsequence (still denoted by v_m) such that v_m converges weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and $v_m|_{\partial\Omega}$ converges strongly in $L^2(\partial\Omega)$.

Substituting $z_m = t_m v_m$ in the condition $E'_\lambda(z_m)v = o(1)\|v\|_1, v \in H^1(\Omega)$, we get

$$\int_\Omega \nabla v_m \nabla v - \lambda \int_{\partial\Omega} \eta\, v_m\, v - \lambda \int_{\partial\Omega} \mu \frac{h(t_m v_m)}{t_m} v = o(1)\|v\|_1/t_m \qquad (4.4)$$

By exploiting assumption 1 in the above relation (with a suitable choice of test functions v) it can be shown that $v_m|_{\Gamma_\pm} \to 0$ a.e. (see the proof of proposition 4.2 of [4]).

Thus, we have $v_m \rightharpoonup w$ in $H^1(\Omega)$, with w *harmonic function* in Ω satisfying $w|_{\Gamma_\pm} = 0$. Moreover, by choosing v in (4.4) with supp $v|_{\partial\Omega} \subset \Gamma_0$ and taking again the limit for $m \to \infty$ we also find (in a weak sense): $(\partial_\nu w - \eta w)|_{\Gamma_0} = 0$. Again by assumption 1, we conclude that $w = 0$, so that $v_m \to 0$ in $L^2(\Omega)$. Now, by this conclusion and by assumption 2, one can prove: $\|\nabla v_m\|_{L^2(\Omega)} \to 0$, and therefore $v_m \to 0$ in $H^1(\Omega)$, thus contradicting $\|v_m\|_1 = 1$.

Thus, the sequence $\|z_m\|_1$ is bounded and in particular we have $z_m = c_m \oplus \tilde{z}_m$ with $|c_m|$ bounded sequence and $\tilde{z}_m$ bounded in $H^1_\sigma(\Omega)$ (see (2.2) where, as previously remarked, we can choose any σ with non vanishing mean); by the Lax-Milgram theorem, the linear map $L : H^1_\sigma(\Omega) \to H^1_\sigma(\Omega)'$ defined by $L(u)v = \int_\Omega \nabla u \nabla v$ is boundedly invertible.

Finally, by assumption 3, the operator defined through the bilinear form

$$\int_{\partial\Omega} [\eta u + \mu h(u)]v$$

maps bounded sets in $H^1(\Omega)$ to relatively compact sets in $H^1(\Omega)'$ (see [15], appendix B for $N \geq 3$; for $N = 2$, the result follows by an obvious extension of the arguments in [5], Lemmas 2.1 and 2.2). By standard results [6], Chapter

II, Proposition 2.2, it follows that $\tilde{z}_m$ is relatively compact in $H^1_\sigma(\Omega)$; then, the same holds for z_m in $H^1(\Omega)$. □

We now describe a second set of assumptions, which imply the Palais–Smale condition only for small enough λ, but without assuming that the weight μ vanishes on a nontrivial subset of $\partial\Omega$. In this case, the hypotheses are more readily stated by letting λ of indefinite sign.

Condition PS2. Assume $N \geq 3$ and

1.
$$(\beta+1)H(u) - uh(u) = \frac{\beta-1}{2}Au^2 + g(u), \tag{4.5}$$
with $1 < \beta < N/(N-2)$, $A \in \mathbb{R}$ and $g(u)/u^2 = o(1)$ for large $|u|$, $\int_{|u|\geq R} \frac{|g(u)|}{|u|^3} \leq \infty$ for every $R > 0$.
2.
$$\lambda \int_{\partial\Omega} (\eta + A\mu) < 0 \quad \text{and} \quad \lambda_{-1} < \lambda < \lambda_1; \tag{4.6}$$
here λ_1 and λ_{-1} denote the lowest positive eigenvalue and the highest negative eigenvalue of the linear problem (2.1), with $\sigma = \eta + A\mu$.

Remark 4.3. Assumption 1 of PS2 is satisfied, e.g. by the functions

$$H(u) = \frac{1}{\beta+1}|u|^{\beta+1} + G(u),$$

where G' is asymptotically linear, with $\lim_{u\to+\infty} G'(u)/u = A$ and $\frac{(\beta+1)G(u)-uG'(u)}{u^3} - \frac{\beta-1}{2u}A$ is integrable at infinity.

We now have:

Proposition 4.4. *Let $z_m \in H^1(\Omega)$ be a sequence such that $E_\lambda(z_m) \to c$ and $E'_\lambda(z_m) \to 0$ in $H^1(\Omega)'$. Assume that (PS2) holds and let $\| \|_1$ denote the norm (2.5) with $\sigma = \eta + A\mu$.*

Then, the sequence $\|z_m\|_1$ is bounded and the functional (3.1) satisfies the Palais–Smale condition.

Proof. We first show that, assuming h superlinear, (4.5) implies that h is also subcritical. By defining

$$K(u) = H(u) - \frac{A}{2}u^2,$$

one readily finds that K solves the linear equation

$$K'(u) = \frac{\beta+1}{u}K(u) - \frac{g(u)}{u}.$$

Since h is superlinear, we have $K(u) \geq u^2$ for $|u| \geq R$ large enough; then, dividing the above equation by K, integrating for $u \geq R$ and taking the exponential of both members, we get

$$K_-(R)u^{\beta+1} \leq K(u) \leq K_+(R)u^{\beta+1}, \tag{4.7}$$

for $u \geq R$, where

$$K_{\pm}(R) = \frac{K(R)}{R^{\beta+1}} e^{\pm \int_R^{+\infty} \frac{|g(u)|}{u^3}}$$

An analogous estimate holds for $u \leq -R$.

Furthermore, by (4.5) we also have

$$\begin{aligned} C + o(1)\|z_m\|_1 &= (\beta+1)E_\lambda(z_m) - E'_\lambda(z_m)z_m, \\ &= \frac{\beta-1}{2}\left[\int_\Omega |\nabla z_m|^2 - \lambda \int_{\partial\Omega} \eta z_m^2\right] \\ &\quad -\lambda \int_{\partial\Omega} \mu[(\beta+1)H(z_m) - z_m h(z_m)] \\ &\geq \frac{\beta-1}{2}\left[\int_\Omega |\nabla z_m|^2 - \lambda \int_{\partial\Omega} (\eta + A\mu) z_m^2\right] - C(\epsilon) - \epsilon\|z_m\|_1^2, \end{aligned}$$

for any positive ϵ and suitably chosen $C(\epsilon) > 0$. Then, by writing $z_m = \tilde{z}_m + c_m$, with $\tilde{z}_m \in H^1_{\eta+A\mu}$ and by rescaling ϵ, we conclude that there exists a constant $K(\epsilon)$ such that

$$K(\epsilon) + o(1)\|z_m\|_1 \geq \int_\Omega |\nabla \tilde{z}_m|^2 - \lambda \int_{\partial\Omega} (\eta + A\mu)\tilde{z}_m^2 - \lambda c_m^2 \int_{\partial\Omega} (\eta + A\mu) - \epsilon\|z_m\|_1^2$$

Thus, by recalling (2.5) and Theorem 2.1, we get

$$K(\epsilon) + o(1)\|z_m\|_1 \geq \min\left\{1 - \frac{\lambda}{\lambda_{\pm 1}} - \epsilon; \lambda\left(-\int_{\partial\Omega} (\eta + A\mu)\right)^{-1} - \epsilon\right\} \|z_m\|_1^2$$

where we take λ_1 for $\lambda > 0$ and λ_{-1} for $\lambda < 0$. From the previous estimates the boundedness of a Palais–Smale sequence follows.

Finally, by the previously established subcritical growth of h, the operator defined through the bilinear form $\int_{\partial\Omega}[\eta u + \mu h(u)]v$ maps bounded sets in $H^1(\Omega)$ to relatively compact sets in $H^1(\Omega)'$, so that Palais–Smale condition again follows by [6], Chapter II, Proposition 2.2. □

Remark 4.5. In the case $\eta = 0$ and $A = 0$, e.g. for the problem (1.1) with the boundary condition $\partial_\nu u = \mu|u|^{\beta-1}u$, one can show that the Palais–Smale condition holds *for every* $\lambda \neq 0$, provided that $\int_{\partial\Omega} \mu \neq 0$ and $|h'(u)| \leq C(1 + |u|^{\beta-1})$.

The proof is exactly as in the case $A = 0$ of proposition 4.3 of [4].

5. Main theorem and final comments

We recall here the general assumptions about problem (1.1):

- $\Omega \subset \mathbb{R}^N, N \geq 2$, is a bounded open set with sufficiently regular boundary $\partial\Omega$;
- h is a $\mathcal{C}^1$ function on $\mathbb{R}$ satisfying (1.2), (1.3), (1.4);
- η and μ belong to $L^\infty(\partial\Omega)$;

- for $N \geq 3$ we suppose that $\partial\Omega$ and μ are regular enough to guarantee that the eigenfunctions u_k^∞ (of the linear problem (2.1) with $\sigma = \mu$) belong to $H^s(\Omega)$ with $s > N/2$, so that $u_k^\infty \in \mathcal{C}(\overline{\Omega})$.

We can now state our main result;

Theorem 5.1. *Let the general assumptions listed above hold; in addition, assume that* (3.2) *is satisfied,* μ *has non trivial positive part if* $\lambda > 0$ (*non trivial negative part if* $\lambda < 0$) *and that either* $PS1$ *or* $PS2$ *holds.*

Then, there exist infinitely many solutions of problem (1.1).

Proof. Recall that we may assume $\lambda > 0$ since η and μ are indefinite. By Lemmas 3.1 and 3.5, for any positive integer m there exist two closed subspaces V^+, V^- of $H^1(\Omega)$ with $\dim V^- -$ codim$V^+ = m$, and positive constants R, c_0, c_∞ (the last one depending on m) such that:

$$a)\quad E_\lambda(u) \geq c_0 \quad \forall u \in V^+, \quad \|u\|_1 = R; \qquad b)\quad E_\lambda(u) \leq c_\infty \quad \forall u \in V^-.$$

Then, by theorem 2.4 of [8], the functional E_λ possesses at least m distinct pairs of critical points, corresponding to critical levels $c_k, k = 1, 2, \dots, m$, such that

$$c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_m \leq c_\infty.$$

Since this conclusion holds for arbitrary m, we get infinitely many critical points; hence, problem (1.1) has infinitely many solutions in $H^1(\Omega)$. By the regularity results of [16], if Ω is smooth and $\eta, \mu \in \mathcal{C}^\infty(\partial\Omega)$, we have $u \in \mathcal{C}^\infty(\overline{\Omega})$. □

Remark 5.2. We recall that in the degenerate case $c_k = \ldots = c_{k+r} = c$ (with $k \geq 1$ and $k + r \leq m$) it was shown in [8] that $i_2(K_c) \geq r + 1 \geq 2$, where K_c is the set of critical points at level c and i_2 is the Krasnoselski genus; since a finite set (not containing the origin) has genus 1, it follows that E_λ has infinitely many critical points at level c.

Remark 5.3. As discussed in the introduction, the special two-dimensional case where $\eta = \mu$ and $h(u) = \sinh u - u$ in (1.1) represents a problem in corrosion modelling which was previously considered in [5] for $\mu = 1$. Then, Theorem 5.1 shows that such problem has infinitely many solutions (for every positive λ) provided that μ has a non trivial positive part and that the supports of the positive and negative parts of μ are disjoint (recall assumption PS1). That generalizes the results of [5].

Here is another application to a special problem:

Corollary 5.4. *Let* Ω, μ *be as in Theorem* 5.1 *and suppose further that* $\lambda \int_\Omega \mu < 0$.

Then, the problem

$$\begin{aligned} &\Delta u(x) = 0 \quad \text{in } \Omega \\ &\partial_\nu u(x) = \lambda\, \mu(x) |u(x)|^{\beta-1} u(x), \quad \text{on } \partial\Omega \end{aligned} \tag{5.1}$$

with $0 < \beta < \frac{N}{N-2}$ ($\beta > 0$ *if* $N = 2$), *admits infinitely many nontrivial solutions.*

Proof. One readily verifies that for every $\lambda \neq 0$, the functional

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{\beta+1}\int_{\partial\Omega} \mu |u|^{\beta+1},$$

satisfies the assumptions of Proposition 3.3, so that condition a) of Theorem 5.1 holds with $V^+ = H^1(\Omega)$. Furthermore, the Palais–Smale condition is also satisfied by Remark 4.5. Then, the result follows as in the previous theorem. □

As one can observe in the previous proof, it may happen that the bound a) of Theorem 5.1 holds on the entire space $H^1(\Omega)$. In this case, the estimates obtained in Lemma 3.5 suggest that we could apply the standard Mountain Pass Lemma, which does not require symmetry assumptions.

We recall below all the conditions, found in Sect. 3 each of which guarantees that $V^+ = H^1(\Omega)$; assuming $\lambda > 0$, they are:

1. $\eta \leq 0$, not identically vanishing;
2. $\int_{\partial\Omega} \eta < 0$ and $0 < \lambda < \lambda_1^0$, where λ_1^0 is the first positive eigenvalue of the linear problem with weight η;
3. $\eta = 0$ and Proposition 3.3 holds.

Clearly, if $\lambda < 0$, we have to reverse the inequalities in 1 and 2 and to replace λ_1^0 with the first negative eigenvalue λ_{-1}^0.

Then, we can prove some existence and positivity results which complete those obtained in [4] for the non symmetric case and extend (with slightly different assumptions) some results in [2].

Proposition 5.5. *Let Ω, η and μ satisfy the same assumptions as in Theorem 5.1; assume further that either condition 1 or condition 2 above holds if $\lambda > 0$ (or the corresponding ones if $\lambda < 0$). Moreover, let $h \in \mathcal{C}^1(\mathbb{R})$ satisfy*

$$h(0) = h'(0) = 0; \qquad \lim_{|u|\to\infty} h'(u) = +\infty.$$

Then, if either $PS1$ or $PS2$ holds, problem (1.1) admits a non trivial solution.

Proof. By our assumptions, there exist $R > 0, c_0 > 0$ such that $E_\lambda(u) \geq c_0$ for every $u \in H^1(\Omega)$ with $\|u\|_1 = R$. Moreover, let u_n^∞ be an eigenfunction of the linear problem with weight μ, corresponding to a positive eigenvalue λ_n (if $\int_{\partial\Omega} \mu > 0$ we can also take u_n^∞ constant and $\lambda_n = 0$). Then, by the proof of Lemma 3.5 it follows that $E_\lambda(tu_n) \to -\infty$ as $|t| \to \infty$. By the Mountain Pass Lemma [6, theorem 6.1] we obtain the existence of a nontrivial solution u to (1.1). □

Remark 5.6. If η is proportional to μ, the above result completes the set of intervals of the parameter λ for which there is a solution to the non symmetric problem studied in [4].

As it was remarked in [12], Sect. 3 the mountain pass procedure yields a critical point $u \geq 0$, not identically vanishing, whenever the functional satisfies $E_\lambda(u) = E_\lambda(|u|)$ and $E_\lambda(u_1) < 0$ for some non negative $u_1 \in H^1(\Omega)$. The first condition holds for a symmetric term $H(u)$ in $E_\lambda(u)$ (note that, if one is only interested in positive solutions, one can always assume that $u \mapsto h(u)$ is a $\mathcal{C}^1$ *odd* function defined in $\mathbb{R}$, as $h(0) = h'(0) = 0$). By the proof of the previous proposition, for $\lambda > 0$ the second condition can be satisfied by taking as u_1 any large enough positive constant if $\int_{\partial\Omega} \mu > 0$ or the eigenfunction (suitably rescaled) associated to the first positive eigenvalue of the linear problem with weight μ (see [10, theorem 1.2]) if $\int_{\partial\Omega} \mu < 0$. Then, by the strong maximum principle, we conclude:

Corollary 5.7. *With the same assumptions of Proposition 5.5, there exists a positive solution $u^+ > 0$ in Ω to problem (1.1).*

We recall that a positive solution to the same problem was found in [2], theorem 1.2, by assuming the positivity of the principal eigenvalue of $-\Delta$ under the boundary condition $\partial_\nu u = \lambda\eta u$ on $\partial\Omega$; by considering the variational characterization of such eigenvalue, it can be shown that this assumption is precisely equivalent to assumption 2 above.

The same arguments for the existence of a positive solution apply to the following result, concerning the case $\eta = 0$ in the non symmetric case:

Proposition 5.8. *Let Ω, μ and h satisfy the same assumptions as in Proposition 5.5; assume further that condition 3 above holds. Then, if either $PS1$ or $PS2$ holds, problem (1.1) (with $\eta = 0$) has a non trivial (positive) solution.*

Proof. The proof follows as in Proposition 5.5 and in the subsequent corollary. □

References

[1] Escobar, J.: Conformal metrics with prescribed mean curvature on the boundary. Calc. Var. **4**, 559–592 (1996)

[2] Zhu, M.: On elliptic problems with indefinite superlinear boundary conditions. J. Differ. Equ. **193**(1), 180–195 (2003)

[3] Vogelius, M., Xu, J.-M.: A nonlinear elliptic boundary value problem related to corrosion modeling. Q. Appl. Math. **56**, 479–505 (1998)

[4] Pagani, C.D., Pierotti, D.: Variational methods for nonlinear Steklov eigenvalues problems with an indefinite weight function. Calc. Var. **39**(1), 35–58 (2010)

[5] Kavian, O., Vogelius, M.: On the existence and 'blow-up' of solutions to a two-dimensional nonlinear boundary-value problem arising in corrosion modeling. Proc. Royal Soc. Edinb. Sect. A **133**, 119–149 (2003). Corrigendum to the same, Proc. Royal Soc. Edinb. Sect. A **133**, 729–730 (2003)

[6] Struwe, M.: Variational Methods and Applications to Nonlinear Partial Differential Equations and Hamiltonian systems. Springer, Berlin (1990)

[7] Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)

[8] Bartolo, P., Benci, V., Fortunato, D.: Abstract critical point theorems and applications to some nonlinear problems with “strong” resonance at infinity. Nonlinear Anal. Theory Methods Appl. **7**(9), 981–1012 (1983)

[9] Grisvard, P.: Elliptic problems in nonsmooth domains. Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA (1985)

[10] Torné, O.: Steklov problem with an indefinite weight for the $p-$Laplacian. Electron. J. Differ. Equ. **87**, 1–8 (2005)

[11] Jerison, D.S., Kenig, C.E.: The Neumann problem on Lipschitz domains. Bull. AMS **4**(2), 203–207 (1981)

[12] Berestycki, H., Capuzzo Dolcetta, I., Nirenberg, L.: Variational methods for indefinite superlinear homogeneous elliptic problems. NoDEA **2**, 553–572 (1995)

[13] Caffarelli, L., Friedman, A.: Partial regularity of the zero set of linear elliptic equations. J. Differ. Equ. **60**, 420–439 (1985)

[14] Hardt, R., Hoffmann-Ostenhof, M., Hoffmann-Ostenhof, T., Nadirashvili, N.: Critical sets of solutions to elliptic equations. J. Differ. Geom. **51**, 359–373 (1999)

[15] Rabinowitz, P.H.: Minimax methods in critical point theory with applications to differential equations. CBMS Regional Conference Series in Mathematics, vol. 65, AMS, Providence (1986)

[16] Cherrier, P.: Problèmes de Neumann nonlinéaires sur les variétés riemanniennes. J. Funct. Anal. **57**, 154–207 (1984)

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Received: 10 January 2011.
Accepted: 14 September 2011.