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Nonlinear Differential Equations and Applications NoDEA

On a class of non-uniformly elliptic equations

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Abstract. In this paper we study the existence and uniqueness of both weak solutions and entropy solutions for the Dirichlet boundary value problem of a class of non-uniformly elliptic equations. A comparison result is also discussed. Some well-known elliptic equations are the special cases of this equation.

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1. Introduction

Suppose that Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$. The following exponential energy functional

$$E(u) = \int_{\Omega} \exp(|\nabla u|^2)\, dx \tag{1.1}$$

originates from the exponential harmonic mappings. It has been studied in [10,16,17], especially for the regularity theory. Naito [17] proved the existence, uniqueness and C^{α} regularity of the minimizer. Duc and Eells [10], Lieberman [16] proved the C^{∞} and $C^{1,\alpha}$ regularity of the minimizer, respectively. Besides, Siepe [21] studied the Lipschitz regularity of the minimizers of the more general functional

$$\int_{\Omega} \Phi(\nabla u)\, dx,$$

under the main assumption that implies the Δ_2 condition on Φ: there exists a positive number $K > 2$ such that

$$\Phi(2\xi) \leq K\Phi(\xi).$$

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This condition implies that $\Phi(\xi)$ would be controlled by a polynomial of $|\xi|$. Therefore, Δ_2 condition is not satisfied by function $\Phi(\xi) = \exp(|\xi|^2)$.

Motivated by the above results, in this paper we are concerned with the following non-uniformly elliptic Dirichlet boundary problem

$$\begin{cases} -\mathrm{div}\,(D_\xi \Phi(\nabla u)) = f - \mathrm{div} g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Phi : \mathbb{R}^N \mapsto \mathbb{R}_+$ is a C^1 nonnegative, strictly convex function, $D_\xi \Phi : \mathbb{R}^N \to \mathbb{R}$ represents the gradient of $\Phi(\xi)$ with respect to ξ and ∇u represents the gradient with respect to the spatial variables x. Without loss of generality we may assume that $\Phi(0) = 0$.

Our main assumptions are that $\Phi(\xi)$ satisfies the super-linear condition (or 1-coercive condition, see [15], Chapter E)

$$\lim_{|\xi| \to \infty} \frac{\Phi(\xi)}{|\xi|} = \infty, \tag{1.3}$$

and the symmetric condition: there exists a positive number $C > 0$ such that

$$\Phi(-\xi) \le C\Phi(\xi), \quad \xi \in \mathbb{R}^N. \tag{1.4}$$

For the right-hand side of Eq. (1.2) we assume that

$$f \in L^1(\Omega) \quad \text{and} \quad g \in (L^\infty(\Omega))^N. \tag{1.5}$$

There are numerous examples of $\Phi(\xi)$ satisfying structure assumptions (1.3) and (1.4) as well as $\Phi(\xi) = \exp(|\xi|^2) - 1$. The well-known are listed as follows.

Example 1.

$$\Phi(\xi) = \frac{1}{p}|\xi|^p, \quad p > 1.$$

In this case, Eq. (1.2) is the p-Laplacian equation.

Example 2.

$$\Phi(\xi) = \frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2} + \cdots + \frac{1}{p_N}|\xi_N|^{p_N}, \qquad p_i > 1, \quad i = 1, 2, \ldots, N,$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$. In this case, Eq. (1.2) is the anisotropic p-Laplacian equation.

Example 3.

$$\Phi(\xi) = |\xi| \log(1 + |\xi|).$$

(See [8, Chapter 4; 12].)

Example 4.

$$\Phi(\xi) = |\xi| L_k(|\xi|),$$

where $L_i(s) = \log(1 + L_{i-1}(s))\,(i = 1, 2, \ldots, k)$ and $L_0(s) = \log(1 + s)$ for $s \ge 0$. The corresponding elliptic problems are introduced in Prandtl-Eyring fluids and plastic materials with logarithmic hardening law. (See [14].)

The main purpose of this paper is to establish the existence and uniqueness of solutions for problem (1.2) under the integrability condition (1.5). In this case, it is reasonable to work with entropy solutions or renormalized solutions, which need less regularity than the usual weak solutions. The notion of entropy solutions was first proposed by Bénilan et al. [3] for the nonlinear elliptic problems. It was then adapted to the study of some nonlinear elliptic and parabolic problems. We refer to [2,4,5,18] for details. To do this, inspired by the ideas in [6], we first employ a unifying method to prove the existence and uniqueness of weak solutions for problem (1.2) under the integrability condition that $f \in L^N(\Omega)$. It is worth pointing out that we do not assume polynomial or exponential growth for function Φ as in [1,7,16]. Generally speaking, finding solutions for such problems or deriving the Euler–Lagrange equations for minimizers of variational problems is not a trivial fact when function $\Phi(\xi)$ does not satisfy the Δ_2-condition. Based on the above result we further study the existence and uniqueness of entropy solutions for problem (1.2) by using the approximation techniques.

Now we state our main results. The first two theorems are about the existence and uniqueness of weak solutions and entropy solutions. The third one is about the comparison principle.

Theorem 1.1. *Let the structure assumptions* (1.3) *and* (1.4) *be satisfied. If* $f \in L^N(\Omega)$ *and* $g \in (L^\infty(\Omega))^N$, *then there exists a unique weak solution for problem* (1.2).

Theorem 1.2. *Let the structure assumptions* (1.3) *and* (1.4) *be satisfied. If* $f \in L^1(\Omega)$ *and* $g \in (L^\infty(\Omega))^N$, *then there exists a unique entropy solution for problem* (1.2).

Theorem 1.3. *Suppose that* u *is an entropy solution of problem* (1.2) *with* $g = 0$. *If* $f \geq 0$, *then we have* $u \geq 0$.

The rest of this paper is organized as follows. In Sect. 2, we state some basic results that will be used later. We will prove the main results in Sect. 3. In the following C will represent a generic constant that may change from line to line even if in the same inequality.

2. Preliminaries

Let $\Phi(\xi)$ be a nonnegative convex function. We define the polar function of $\Phi(\xi)$ as

$$\Psi(\eta) = \sup_{\xi \in \mathbb{R}^N} \{\eta \cdot \xi - \Phi(\xi)\}, \tag{2.1}$$

which is also known as the Legendre transform of $\Phi(\xi)$. It is obvious that $\Psi(\eta)$ is a convex function. In the following we will list several lemmas.

Definition 2.1 [15, Definition 4.1.3]. Let $C \subset \mathbb{R}^N$ be convex. The mapping $F : C \to \mathbb{R}^N$ is said to be monotone (rest. strictly monotone)] on C when, for all x and x' in C,

$$\langle F(x) - F(x'), x - x' \rangle \geq 0,$$
$$[resp.\ \langle F(x) - F(x'), x - x' > 0 \rangle \quad \text{whenever } x \neq x'].$$

Lemma 2.2 [15, Theorem 4.1.4]. *Let f be a function differentiable on an open set $\Omega \subset \mathbb{R}^N$ and let C be a convex subset of Ω. Then, f is convex (resp. strictly convex) on C if and only if its gradient ∇f is monotone (resp. strictly monotone) on C.*

Lemma 2.3. *Suppose that $\Phi(\xi)$ is a convex C^1 function with $\Phi(0) = 0$. Then we have, for all $\xi, \zeta \in \mathbb{R}^N$,*

$$\Phi(\xi) \leq \xi \cdot D\Phi(\xi), \tag{2.2}$$
$$(D\Phi(\xi) - D\Phi(\zeta)) \cdot (\xi - \zeta) \geq 0. \tag{2.3}$$

Lemma 2.4 [6] *Suppose that $\Phi(\xi)$ is a nonnegative convex C^1 function and $\Psi(\eta)$ is its polar function. Then we have, for $\xi, \eta, \zeta \in \mathbb{R}^N$,*

$$\xi \cdot \eta \leq \Phi(\xi) + \Psi(\eta), \tag{2.4}$$
$$\Psi(D\Phi(\zeta)) + \Phi(\zeta) = D\Phi(\zeta) \cdot \zeta. \tag{2.5}$$

Lemma 2.5 [13, Chapter 3]. *Suppose that $\Phi(\xi)$ is a nonnegative convex function with $\Phi(0) = 0$, which satisfies (1.3). Then its polar function $\Psi(\eta)$ in (2.1) is a well-defined, nonnegative function in $\mathbb{R}^N$, which also satisfies (1.3).*

Lemma 2.6 [19, Chapter 4]. *Let $D \subset \mathbb{R}^N$ be measurable with finite Lebesgue measure and $f_k \in L^1(D)$ and $g_k \in L^1(D)\,(k = 1, 2, \ldots)$, and*

$$|f_k(x)| \leq g_k(x), \qquad a.e.\ x \in D,\ k = 1, 2, \ldots.$$

If

$$\lim_{k\to\infty} f_k(x) = f(x), \qquad \lim_{k\to\infty} g_k(x) = g(x), \quad a.e.\ x \in D,$$

and

$$\lim_{k\to\infty} \int_D g_k(x)\,dx = \int_D g(x)\,dx < \infty,$$

then we have

$$\lim\nolimits_{k\to\infty} \int_D f_k(x)\,dx = \int_D f(x)\,dx.$$

Lemma 2.7 [11, Proposition 9.1c]. *Let $D \subset \mathbb{R}^N$ be measurable with finite Lebesgue measure, and let $\{f_n\}$ be a sequence of functions in $L^p(D)\,(p \geq 1)$ such that*

$$f_n \rightharpoonup f \quad weakly\ in\ L^p(D),$$
$$f_n \to g \quad a.e.\ in\ D.$$

Then $f = g$ a.e. in D.

Lemma 2.8 [9, Chapter 3; 20]. *Suppose that $\Phi(\xi)$ is a nonnegative convex function satisfying (1.3). Let $D \subset \mathbb{R}^N$ be a measurable with finite Lebesgue measure $|D|$ and let $\{f_k\} \subset L^1(D;\mathbb{R}^N)$ be a sequence satisfying that*

$$\int_D \Phi(f_k)\,dx \le C, \tag{2.6}$$

where C is a positive constant. Then there exist a subsequence $\{f_{k_j}\} \subset \{f_k\}$ and a function $f \in L^1(D;\mathbb{R}^N)$ such that

$$f_{k_j} \rightharpoonup f \quad \text{weakly in } L^1(D;\mathbb{R}^N) \text{ as } j \to \infty \tag{2.7}$$

and

$$\int_D \Phi(f)\,dx \le \liminf_{j\to\infty} \int_D \Phi(f_{k_j})\,dx \le C. \tag{2.8}$$

3. The proofs of main results

3.1. Weak solutions

In this section we will give a reasonable definition for weak solutions and prove the existence and uniqueness of weak solutions for problem (1.2).

Definition 3.1. A function $u \in W_0^{1,1}(\Omega)$ with $D_\xi\Phi(\nabla u)\cdot\nabla u \in L^1(\Omega)$ is called a weak solution of problem (1.2) if for every $\varphi \in C_0^1(\Omega)$, we have

$$\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla\varphi\,dx = \int_\Omega f\varphi\,dx + \int_\Omega g\cdot\nabla\varphi\,dx. \tag{3.1}$$

Proof of Theorem 1.1. We consider the variational problem

$$\min\{J(v) | v \in V\},$$

where $V = \{v \in W_0^{1,1}(\Omega) | \Phi(\nabla v) \in L^1(\Omega)\}$, and functional J is

$$J(v) = \int_\Omega \Phi(\nabla v)\,dx - \int_\Omega fv\,dx - \int_\Omega g\cdot\nabla v\,dx.$$

We will establish that $J(v)$ has a minimizer $u(x)$ in V and then prove that the minimizer satisfies the Euler–Lagrange equation of functional J weakly.

Due to (1.3), for every $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$|\xi| \le \delta\Phi(\xi) + C_\delta. \tag{3.2}$$

By Hölder's and Sobolev's inequalities and (3.2), we have

$$\begin{aligned}\left|\int_\Omega fv\,dx\right| &\le \|f\|_{L^N(\Omega)}\|v\|_{L^{1^*}(\Omega)}\\ &\le C\|f\|_{L^N(\Omega)}\|\nabla v\|_{L^1(\Omega)}\\ &\le C\delta\|\Phi(\nabla v)\|_{L^1(\Omega)} + C_\delta\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\left|\int_\Omega g\cdot\nabla v\,dx\right| &\le \|g\|_{L^\infty(\Omega)}\|\nabla v\|_{L^1(\Omega)}\\ &\le C\delta\|\Phi(\nabla v)\|_{L^1(\Omega)} + C_\delta.\end{aligned} \tag{3.4}$$

Choosing δ sufficiently small, we conclude from (3.3) and (3.4) that

$$J(v) \geq \frac{1}{2}\int_\Omega \Phi(\nabla v)\,dx - C(\Phi, \|f\|_{L^N(\Omega)}, \|g\|_{L^\infty(\Omega)}) \geq -C.$$

Thus we get

$$-C \leq \inf_{v\in V} J(v) \leq J(0) = 0.$$

Then we can find a minimizing sequence $\{v_n\}_{n=1}^\infty \subset V$ such that

$$\lim_{n\to\infty} J(v_n) = \inf_{v\in V} J(v).$$

It follows that, for $n = 1, 2, \ldots,$

$$\int_\Omega \Phi(\nabla v_n)\,dx \leq C.$$

From Lemma 2.8 we may choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ and a function $u \in W_0^{1,1}(\Omega)$ such that

$$\nabla v_{n_i} \rightharpoonup \nabla u \quad \text{weakly in } L^1(\Omega).$$

Since

$$\int_\Omega \Phi(\nabla u)\,dx \leq \liminf_{i\to\infty} \int_\Omega \Phi(\nabla v_{n_i})\,dx,$$

we know that $u \in V$, and $J(v)$ is weakly lower semi-continuous on V, which ensures that

$$J(u) \leq \liminf_{i\to\infty} J(v_{n_i}) = \inf_{v\in V} J(v).$$

This implies that $u \in V$ is a minimizer of the functional $J(v)$ in V, i.e.,

$$J(u) = \inf_{v\in V} J(v).$$

Furthermore, since $u \in V$ is a minimizer, we have $\lambda u \in V$, $\lambda \in (0,1)$, and

$$J(u) \leq J(\lambda u),$$

which implies

$$\int_\Omega \Phi(\nabla u)\,dx - \int_\Omega fu\,dx - \int_\Omega g\cdot\nabla u\,dx \leq \int_\Omega \Phi(\lambda\nabla u)\,dx - \lambda\int_\Omega fu\,dx - \lambda\int_\Omega g\cdot\nabla u.$$

Recalling (2.3), we know

$$\Phi(\nabla u) - \Phi(\lambda\nabla u) \geq (1-\lambda)D_\xi\Phi(\lambda\nabla u)\cdot\nabla u.$$

Then

$$(1-\lambda)\int_\Omega D_\xi\Phi(\lambda\nabla u)\cdot\nabla u\,dx \leq (1-\lambda)\int_\Omega fu\,dx + (1-\lambda)\int_\Omega g\cdot\nabla u\,dx.$$

Dividing the above inequality by $1-\lambda$, and passing to limits as $\lambda \to 1$, we get

$$\liminf_{\lambda\to 1}\int_\Omega D_\xi\Phi(\lambda\nabla u)\cdot\nabla u\,dx \leq \int_\Omega fu\,dx + \int_\Omega g\cdot\nabla u\,dx.$$

Since $D_\xi\Phi(\lambda\nabla u)\cdot\nabla u \geq 0$, by Fatou's Lemma we conclude that

$$\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla u\,dx \leq \int_\Omega fu\,dx + \int_\Omega g\cdot\nabla u\,dx.$$

Using (2.2) and the same argument as (3.4), we have

$$\frac{1}{2}\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla u\,dx \leq C.$$

It follows from (2.5) that $D_\xi\Phi(\nabla u)\cdot\nabla u \in L^1(\Omega)$ and $\Psi(D_\xi\Phi(\nabla u)) \in L^1(\Omega)$.

For some fixed $\varphi(x) \in C_0^1(\Omega)$, we know that $J(u) \leq J(\lambda u+(1-\lambda)\varphi), \forall\lambda \in (0,1)$. Denote $\xi_\lambda = \lambda\nabla u + (1-\lambda)\nabla\varphi$. In light of (2.3), we find

$$\Phi(\nabla u) - \Phi(\xi_\lambda) \geq (1-\lambda)D_\xi\Phi(\xi_\lambda)\cdot(\nabla u - \nabla\varphi),$$

and deduce as above to have

$$\begin{aligned}&\int_\Omega D_\xi\Phi(\xi_\lambda)\cdot(\nabla u-\nabla\varphi)\,dx\\ &\quad\leq \int_\Omega fu\,dx - \int_\Omega f\varphi\,dx + \int_\Omega g\cdot\nabla u\,dx - \int_\Omega g\cdot\nabla\varphi\,dx. \qquad (3.5)\end{aligned}$$

Consider

$$g(\lambda) = \Phi(\xi_\lambda) = \Phi(\lambda\nabla u + (1-\lambda)\nabla\phi).$$

It is obvious that g is a convex function in $\mathbb{R}$. Then by the monotonicity of a convex function's derivative, we know

$$g'(0) \leq g'(\lambda) \leq g'(1), \quad \lambda \in (0,1),$$

which yields that

$$D_\xi\Phi(\nabla\phi)\cdot(\nabla u-\nabla\varphi) \leq D_\xi\Phi(\xi_\lambda)\cdot(\nabla u-\nabla\varphi) \leq D_\xi\Phi(\nabla u)\cdot(\nabla u-\nabla\phi). \qquad (3.6)$$

Recalling (2.4), (1.4) and (2.5), we have

$$\begin{aligned}|D_\xi\Phi(\nabla u)\cdot\nabla\varphi| &\leq \Psi(D_\xi\Phi(\nabla u)) + \Phi(\nabla\varphi) + \Phi(-\nabla\varphi)\\ &\leq \Psi(D_\xi\Phi(\nabla u)) + (C+1)\Phi(\nabla\varphi). \qquad (3.7)\end{aligned}$$

As $\Psi(D_\xi\Phi(\nabla u)) \in L^1(\Omega)$ and $\varphi \in C_0^1(\Omega)$, it is easy to know $D_\xi\Phi(\nabla\varphi)\cdot(\nabla u - \nabla\varphi) \in L^1(\Omega)$ and $D_\xi\Phi(\nabla u)\cdot(\nabla u-\nabla\varphi) \in L^1(\Omega)$. By Lebesgue dominated convergence theorem, we have

$$\int_\Omega \lim_{\lambda\to 1} D_\xi\Phi(\xi_\lambda)\cdot(\nabla u-\nabla\varphi)\,dx = \lim_{\lambda\to 1}\int_\Omega D_\xi\Phi(\xi_\lambda)\cdot(\nabla u-\nabla\varphi)\,dx.$$

Moreover, recalling (3.5) we have

$$\int_\Omega D_\xi\Phi(\nabla u)\cdot(\nabla u-\nabla\varphi)\,dx \leq \int_\Omega fu\,dx - \int_\Omega f\varphi\,dx + \int_\Omega g\cdot\nabla u\,dx - \int_\Omega g\cdot\nabla\varphi\,dx.$$

Denote

$$A_0 = \int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla u\,dx - \int_\Omega fu\,dx - \int_\Omega g\cdot\nabla u\,dx.$$

Then we conclude that, for every $\varphi(x) \in C_0^1(\Omega)$,

$$\int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla \varphi \, dx - \int_\Omega f \varphi \, dx - \int_\Omega g \cdot \nabla \varphi \, dx \geq A_0.$$

By a scaling argument, it follows that

$$\int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla \varphi \, dx - \int_\Omega f \varphi \, dx - \int_\Omega g \cdot \nabla \varphi \, dx = 0.$$

It means that $u(x)$ is a weak solution of problem (1.2).

Suppose that there exists another weak solution $\tilde{u}$ for problem (1.2). Then, for every $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega D_\xi \Phi(\nabla \tilde{u}) \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx + \int_\Omega g \cdot \nabla \varphi \, dx,$$

which follows that

$$\int_\Omega [D_\xi \Phi(\nabla u) - D_\xi \Phi(\nabla \tilde{u})] \cdot \nabla \varphi \, dx = 0. \tag{3.8}$$

Recalling (2.4) and (2.5) we observe that

$$\begin{aligned} |D_\xi \Phi(\nabla \tilde{u}) \cdot \nabla u| &\leq \Phi(\nabla u) + \Phi(-\nabla u) + \Psi(D_\xi \Phi(\nabla \tilde{u})) \\ &\leq (C+1)\Phi(\nabla u) + D_\xi \Phi(\nabla \tilde{u}) \cdot \nabla \tilde{u} \in L^1(\Omega). \end{aligned}$$

Making use of the approximation argument, we conclude that $w = u - \tilde{u}$ can be a test function in (3.8). Therefore, we deduce that

$$\int_\Omega [D_\xi \Phi(\nabla u) - D_\xi \Phi(\nabla \tilde{u})] \cdot (\nabla u - \nabla \tilde{u}) \, dx = 0.$$

By the strict convexity of Φ, we have $\nabla u = \nabla \tilde{u}$ a.e. in Ω. Recalling the fact that $u, \tilde{u} \in W_0^{1,1}(\Omega)$, we know that $u = \tilde{u}$ a.e. in Ω. This finishes the proof. □

3.2. Entropy solutions

In this section we are ready to prove the existence and uniqueness of entropy solutions for problem (1.2). We would like to point out that our approach is much influenced by [3].

Let T_k denote the truncation function at height $k \geq 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \geq k, \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leq -k. \end{cases}$$

Next, we define the very weak gradient of a measurable function u with $T_k(u) \in W_0^{1,1}(\Omega)$. As a matter of the fact, working as in Lemma 2.1 of [3] we can prove the following result:

Proposition 3.2. *For every measurable function u on Ω such that $T_k(u)$ belongs to $W_0^{1,1}(\Omega)$ for every $k > 0$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$, such that*

$$\nabla T_k(u) = v \chi_{\{|u|<k\}}, \quad \textit{almost everywhere in } \Omega \textit{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the weak gradient of u.

From the above Proposition, we denote $v = \nabla u$, which is called the very weak gradient of u. The notion of the very weak gradient allows us to give the following definition of entropy solutions for problem (1.2).

Definition 3.3. A measurable function u with $T_k(u) \in W_0^{1,1}(\Omega)$ is an entropy solution to problem (1.2) if the following conditions are satisfied:

(i) $\int_\Omega D_\xi \Phi(\nabla T_k(u)) \cdot \nabla T_k(u)\, dx < \infty$;

(ii) For every $k > 0$ and every function $\phi \in C_0^1(\Omega)$,

$$\int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla T_k(u - \phi)\, dx \le \int_\Omega f T_k(u - \phi)\, dx + \int_\Omega g \cdot \nabla T_k(u - \phi)\, dx \tag{3.9}$$

holds.

Proof of Theorem 1.2. (1) Existence of entropy solutions.

We first introduce the approximate problems. Let $\{f_n\}, \{g_n\} \subset C_0^\infty(\Omega)$ be two sequences of functions strongly convergent to f in $L^1(\Omega)$ and to g in $(L^\infty(\Omega))^N$ such that

$$\|f_n\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}, \quad \|g_n\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)}. \tag{3.10}$$

Let us consider the approximate problems

$$\begin{cases} -\mathrm{div}\,(D_\xi \Phi(\nabla u_n)) = f_n - \ \mathrm{div} g_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

By virtue of Theorem 1.1 we can find $u_n \in W_0^{1,1}(\Omega)$ with

$$\int_\Omega D_\xi \Phi(\nabla u_n) \cdot \nabla u_n\, dx < \infty, \tag{3.12}$$

that is a weak solution of problem (3.11) in the sense of Definition 3.1. Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a measurable function u, which is an entropy solution of problem (1.2). We will divide the proof into several steps.

Using an approximation argument, we can choose $T_k(u_n)$ as a test function in (3.11) to have

$$\begin{aligned} &\int_\Omega D_\xi \Phi(\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, dx \\ &\quad = \int_\Omega f_n T_k(u_n)\, dx + \int_\Omega g_n \cdot \nabla T_k(u_n)\, dx. \end{aligned} \tag{3.13}$$

It follows from (3.10) that

$$\begin{aligned} &\int_\Omega D_\xi \Phi(\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, dx \\ &\quad \le k\|f_n\|_{L^1(\Omega)} + \|g_n\|_{L^\infty(\Omega)} \int_\Omega |\nabla T_k(u_n)|\, dx \\ &\quad \le k\|f\|_{L^1(\Omega)} + \|g\|_{L^\infty(\Omega)} \int_\Omega |\nabla T_k(u_n)|\, dx. \end{aligned} \tag{3.14}$$

Recalling (2.2) and using the similar estimate as in (3.4), we have

$$\int_\Omega \Phi(\nabla T_k(u_n))\,dx \le \int_\Omega D_\xi\Phi(T_k(\nabla u_n))\cdot\nabla T_k(u_n)\,dx \le C(k+1), \quad (3.15)$$

which implies from (3.2) that

$$\int_\Omega |\nabla T_k(u_n)|\,dx \le C\delta k + C_\delta, \quad (3.16)$$

that is $T_k(u_n)$ is bounded in $W_0^{1,1}(\Omega)$.

Step 1. We shall prove the convergence in measure of $\{u_n\}$ and we shall find a subsequence which is almost everywhere convergent in Ω.

Recalling Sobolev embedding theorem, we have the following continuous embedding

$$W_0^{1,1}(\Omega) \hookrightarrow L^{1^*}(\Omega),$$

where $1^* = \frac{N}{N-1}$. It follows from (3.16) that for every $k>1$,

$$\|T_k(u_n)\|_{L^{1^*}(\Omega)} \le C\|\nabla T_k(u_n)\|_{L^1(\Omega)} \le C\delta k + C_\delta.$$

Noting that $\{|u_n|\ge k\} = \{|T_k(u_n)|\ge k\}$, we have

$$\text{meas}\{|u_n|>k\} \le \left(\frac{\|T_k(u_n)\|_{L^{1^*}(\Omega)}}{k}\right)^{1^*} \le \left(C\delta + \frac{C_\delta}{k}\right)^{1^*}. \quad (3.17)$$

For every fixed $\epsilon>0$, and every $k>0$, we know that

$$\{|u_n-u_m|>\epsilon\} \subset \{|u_n|>k\}\cup\{|u_m|>k\}\cup\{|T_k(u_n)-T_k(u_m)|>\epsilon\}.$$

Recalling the convergence of $\{T_k(u_n)\}$ in $L^q(\Omega)$ with $q\in[1,1^*)$ and (3.17), we conclude that

$$\limsup_{n,m\to\infty} \text{meas}\{|u_n-u_m|>\epsilon\} \le \left(C\delta + \frac{C_\delta}{k}\right)^{1^*}.$$

First sending $k\to\infty$ and then $\delta\to 0$, we conclude that

$$\limsup_{n,m\to\infty} \text{meas}\{|u_n-u_m|>\epsilon\} = 0,$$

which implies the convergence in measure of $\{u_n\}$, and then we find an a.e. convergent subsequence (still denoted by $\{u_n\}$) in Ω such that

$$u_n \to u \quad \text{a.e. in } \Omega. \quad (3.18)$$

Recalling (3.15) and Lemma 2.8, we may draw a subsequence (we also denote it by the original sequence for simplicity) such that

$$\nabla T_k(u_n) \rightharpoonup \eta_k, \quad \text{weakly in } L^1(\Omega) \quad (3.19)$$

and

$$\int_\Omega \Phi(\eta_k)\,dx \le Ck.$$

In view of (3.18), we conclude that $\eta_k = \nabla T_k(u)$ a.e. in Ω.

Step 2. We shall prove that the sequence $\{\nabla u_n\}$ converges almost everywhere in Ω to ∇u (up to a subsequence).

We first claim that $\{\nabla u_n\}$ is a Cauchy sequence in measure. Let $\delta > 0$, and denote

$$E_1 := \{x \in \Omega : |\nabla u_n| > h\} \cup \{|\nabla u_m| > h\},$$
$$E_2 := \{x \in \Omega : |u_n - u_m| > 1\}$$

and

$$E_3 := \{x \in \Omega : |\nabla u_n| \le h, |\nabla u_m| \le h, |u_n - u_m| \le 1, |\nabla u_n - \nabla u_m| > \delta\},$$

where h will be chosen later. It is obvious that

$$\{x \in \Omega : |\nabla u_n - \nabla u_m| > \delta\} \subset E_1 \cup E_2 \cup E_3.$$

For $k \ge 0$, we can write

$$\{x \in \Omega : |\nabla u_n| \ge h\} \subset \{x \in \Omega : |u_n| \ge k\} \cup \{x \in \Omega : |\nabla T_k(u_n)| \ge h\}.$$

Thus, applying (3.16) and (3.17), there exist constants $C > 0$ such that

$$\text{meas}\{x \in \Omega : |\nabla u_n| \ge h\} \le \left(C\delta + \frac{C_\delta}{k}\right)^{1^*} + \frac{C\delta k + C_\delta}{h}.$$

Without loss of generality, we can assume that $0 < \delta < 1$, then we have

$$\text{meas}\{x \in \Omega : |\nabla u_n| \ge h\} \le C\delta^{1^*} + \frac{C_\delta}{k^{1^*}} + \frac{Ck}{h} + \frac{C_\delta}{h}.$$

By choosing $k = C_\delta h^{\frac{1}{1+1^*}}$, we deduce that

$$\text{meas}\{x \in \Omega : |\nabla u_n| \ge h\} \le C\delta + C_\delta(h^{-\frac{1^*}{1+1^*}} + h^{-1}).$$

Let $\varepsilon > 0$. We may choose $\delta = \varepsilon/6$ and let $h = h(\varepsilon)$ large enough such that

$$\text{meas}(E_1) \le \varepsilon/3, \text{ for all } n, m \ge 0. \tag{3.20}$$

On the other hand, by Step 1, we know that $\{u_n\}$ is a Cauchy sequence in measure. Then there exists $N_1(\varepsilon) \in \mathbb{N}$ such that

$$\text{meas}(E_2) \le \varepsilon/3, \quad \text{for all } n, m \ge N_1(\varepsilon). \tag{3.21}$$

Moreover, since Φ is C^1 and strictly convex, then from Lemma 2.2 and Definition 2.1, there exists a real valued function $m(h, \delta) > 0$ such that

$$(D\Phi(\xi) - D\Phi(\zeta)) \cdot (\xi - \zeta) \ge m(h, \delta) > 0, \tag{3.22}$$

for all $\xi, \zeta \in \mathbb{R}^N$ with $|\xi|, |\zeta| \le h, |\xi - \zeta| \ge \delta$. By taking $T_1(u_n - u_m)$ as a test function in the approximation equation (3.11) and integrating on E_3, we obtain

$$\begin{aligned} &m(h,\delta)\text{meas}(E_3) \\ &\quad \le \int_{E_3} [D_\xi\Phi(\nabla u_n) - D_\xi\Phi(\nabla u_m)] \cdot (\nabla u_n - \nabla u_m)\, dx \\ &\quad = \int_{E_3} [f_n - f_m] T_1(u_n - u_m)\, dx + \int_{E_3} [g_n - g_m] \cdot (\nabla u_n - \nabla u_m)\, dx \\ &\quad \le \|f_n - f_m\|_{L^1(\Omega)} + 2h\text{meas}(\Omega)\|g_n - g_m\|_{L^\infty(\Omega)} := \alpha_{n,m}, \end{aligned}$$

which implies that

$$\text{meas}(E_3) \le \frac{\alpha_{n,m}}{m(h,\delta)} \le \varepsilon/3,$$

for all $n, m \ge N_2(\varepsilon, \delta)$. It follows from (3.20) and (3.21) that

$$\text{meas}\{x \in \Omega : |\nabla u_n - \nabla u_m| > \delta\} \le \varepsilon, \quad \text{for all } n, m \ge \max\{N_1, N_2\},$$

that is $\{\nabla u_n\}$ is a Cauchy sequence in measure. Then we may choose a subsequence (denote it by the original sequence) such that

$$\nabla u_n \to v \quad \text{a.e. in } \Omega.$$

Thus, from Proposition 3.2 and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $L^1(\Omega)$, we deduce from Lemma 2.7 that v coincides with the very weak gradient of u. Therefore, we have

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{3.23}$$

Step 3. We shall prove that u is an entropy solution.

Now we choose $v_n = T_k(u_n - \phi)$ as a test function in (3.11) for $k > 0$ and $\phi \in C_0^1(\Omega)$. We note that, if $L = k + \|\phi\|_{L^\infty(\Omega)}$, then

$$\begin{aligned}\int_\Omega & D_\xi \Phi(\nabla u_n) \cdot \nabla T_k(u_n - \phi)\, dx \\ &= \int_\Omega D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_k(T_L(u_n) - \phi)\, dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega & D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_k(T_L(u_n) - \phi)\, dx \\ &= \int_\Omega f_n T_k(u_n - \phi)\, dx + \int_\Omega g_n \cdot \nabla T_k(u_n - \phi)\, dx. \end{aligned} \tag{3.24}$$

The first term on the left hand side of (3.24) can be written as

$$\begin{aligned}\int_\Omega & D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_k(T_L(u_n) - \phi)\, dx \\ &= \int_{\{|T_L(u_n) - \phi| \le k\}} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_L(u_n)\, dx \\ &\quad - \int_{\{|T_L(u_n) - \phi| \le k\}} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla \phi\, dx.\end{aligned}$$

From (2.2), we have

$$D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_L(u_n) \ge 0,$$

it follows from Fatou's lemma and (3.23) that

$$\begin{aligned}\int_{\{|T_L(u) - \phi| \le k\}} & D_\xi \Phi(\nabla T_L(u)) \cdot \nabla T_L(u)\, dx \\ &\le \liminf_{n \to \infty} \int_{\{|T_L(u_n) - \phi| \le k\}} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_L(u_n)\, dx.\end{aligned}$$

In view of (3.14) and (2.5), we know that

$$\int_\Omega \Psi(D_\xi\Phi(\nabla T_L(u_n)))\,dx \le C. \tag{3.25}$$

Applying Lemmas 2.5, 2.8 and (3.23), we conclude that (up to a subsequence)

$$D_\xi\Phi(\nabla T_L(u_n)) \rightharpoonup D_\xi\Phi(\nabla T_L(u)) \quad \text{weakly in } L^1(\Omega). \tag{3.26}$$

Denote

$$\xi_n = D_\xi\Phi(\nabla T_L(u_n)), \quad E_n = \{x\in\Omega : |T_L(u_n)-\phi|\le k\}$$

and

$$E = \{x\in\Omega : |T_L(u)-\phi|\le k\}$$

for simplicity. We can write

$$\int_{E_n} \xi_n\cdot\nabla\phi\,dx = \int_E \xi_n\cdot\nabla\phi\,dx + \int_{E_n\setminus E}\xi_n\cdot\nabla\phi\,dx := I_1+I_2.$$

From (3.26), we have

$$\lim_{n\to\infty} I_1 = \int_{\{|T_L(u)-\phi|\le k\}} D_\xi\Phi(\nabla T_L(u))\cdot\nabla\phi\,dx.$$

Recalling Lemma 2.5, we know that Ψ also satisfies the super-linear condition (1.3). Then for every $\varepsilon>0$, there exists a constant $M>0$ such that

$$|s|\le\varepsilon\Psi(s), \quad \text{for all } |s|>M.$$

It follows that from (3.25) that

$$\begin{aligned}
|I_2| &\le C(\|\nabla\phi\|_{L^\infty(\Omega)})\int_\Omega |\xi_n|\chi_{E_n\setminus E}\,dx\\
&= C\left(\int_{\{|\xi_n|\le M\}}|\xi_n|\chi_{E_n\setminus E}\,dx + \int_{\{|\xi_n|\ge M\}}|\xi_n|\chi_{E_n\setminus E}\,dx\right)\\
&\le C\left(M\text{meas}(E_n\setminus E) + \varepsilon\int_\Omega \Psi(\xi_n)\,dx\right)\\
&\le CM\text{meas}(E_n\setminus E) + C\varepsilon.
\end{aligned}$$

Moreover, by the arbitrariness of ε, we get

$$\lim_{n\to\infty}|I_2| = 0.$$

Thus we obtain

$$\int_{\{|T_L(u_n)-\phi|\le k\}} D_\xi\Phi(\nabla T_L(u_n))\cdot\nabla\phi\,dx \to \int_{\{|T_L(u)-\phi|\le k\}} D_\xi\Phi(\nabla T_L(u))\cdot\nabla\phi\,dx.$$

Using the strong convergence of f_n and g_n, (3.18), (3.19) and the Lebesgue dominated convergence theorem, we can pass to the limits as $n\to\infty$ in the other terms of (3.24) to conclude

$$\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla T_k(u-\phi)\,dx \le \int_\Omega fT_k(u-\phi)\,dx + \int_\Omega g\cdot\nabla T_k(u-\phi)\,dx,$$

for all $k > 0$ and $\phi \in C_0^1(\Omega)$. Therefore, we finish the proof of the existence of entropy solutions.

(2) Uniqueness of entropy solutions.

Suppose that v is another entropy solution for problem (1.2). Our aim is to prove that $u = v$ a.e. in Ω. We write the entropy inequality (3.9) corresponding to solution u with test function $T_h(v)$ and v with test function $T_h(u)$. Add up both results, we have

$$\begin{aligned} &\int_{\{|u-T_h(v)|\le k\}} [D_\xi(\Phi\nabla u) - g]\cdot \nabla T_k(u - T_h(v))\,dx \\ &\quad + \int_{\{|v-T_h(u)|\le k\}} [D_\xi(\Phi\nabla v) - g]\cdot \nabla T_k(v - T_h(u))\,dx \\ &\le \int_\Omega f\left[T_k(u-T_h(v)) + T_k(v - T_h(u))\right]\,dx. \end{aligned} \tag{3.27}$$

First we consider the right-hand side of inequality (3.27). Noting that

$$T_k(u-T_h(v)) + T_k(v-T_h(u)) = 0 \quad \text{in} \quad \{|u|\le h, |v|\le h\},$$

then we have

$$\begin{aligned} &\left|\int_\Omega f\left[T_k(u-T_h(v)) + T_k(v-T_h(u))\right]\,dx\right| \\ &\quad \le 2k\left(\int_{\{|u|>h\}} |f|\,dx + \int_{\{|v|>h\}} |f|\,dx\right). \end{aligned}$$

Since both meas $\{|u| > h\}$ and meas $\{|v| > h\}$ tend to 0 as h goes to infinity from (3.17), the right-hand side of inequality (3.27) tends to 0 as h goes to infinity.

For the left-hand side of (3.27), let us define

$$\begin{aligned} A_0 &= \{x\in\Omega : |u-v|\le k, |u|\le h, |v|\le h\}, \\ A_1 &= \{x\in\Omega : |u-T_h(v)|\le k, |v|>h\}, \\ A_1' &= \{x\in\Omega : |v-T_h(u)|\le k, |u|>h\}, \\ A_2 &= \{x\in\Omega : |u-T_h(v)|\le k, |v|\le h, |u|>h\}, \\ A_2' &= \{x\in\Omega : |v-T_h(u)|\le k, |u|\le h, |v|>h\}, \end{aligned}$$

then we have $\{x\in\Omega : |u-T_h(v)|\le k\} = A_0\cup A_1\cup A_2$ and $\{x\in\Omega : |v-T_h(u)|\le k\} = A_0\cup A_1'\cup A_2'$. On the set A_0 the left-hand side of (3.27) is equal to

$$\int_{A_0} [D_\xi\Phi(\nabla u) - D_\xi\Phi(\nabla v)]\cdot\nabla(u-v)\,dx.$$

On the set A_1, for the first term of the left-hand side in (3.27), from (2.2) we have

$$\int_{A_1} D_\xi\Phi(\nabla u)\cdot\nabla u\,dx \ge -\int_{A_1} g\cdot\nabla u\,dx.$$

Since $g \in L^\infty(\Omega)$, we have

$$\left| \int_{A_1} g \cdot \nabla u \, dx \right| \le \|g\|_{L^\infty(A_1)} \|\nabla u\|_{L^1(A_1)}.$$

Since $|A_1|$ tends to zero as $h \to \infty$, we know that $\|g\|_{L^\infty(A_1)}$ tends to zero as $h \to \infty$. If we prove that $\|\nabla u\|_{L^1(A_1)}$ is bounded, then the term with A_1 will converge to zero. We decompose A_1 as

$$A_1 = \{v > h, |u-h| \le k\} \cup \{v \le -h, |u+h| \le k\} = A_1^+ \cup A_1^-.$$

On A_1^+ (and the same for A_1^-) we have $-k \le u - h \le k$, and then $h - k \le u \le h + k$. Hence $A_1^+ \subseteq B_{h-k,2k}$, where $B_{h,k} = \{h \le |u| \le h + k\}$. Choosing $\phi = T_h(u)$ in (3.9), we get

$$\int_{B_{h,k}} D_\xi \Phi(\nabla u) \cdot \nabla u \, dx \le k \|f\|_{L^1(\Omega)} + \|g\|_{L^\infty(\Omega)} \int_{B_{h,k}} |\nabla u| \, dx.$$

It follows from (2.2) and the similar estimate as in (3.4) that

$$\int_{B_{h,k}} \Phi(\nabla u) \, dx \le C(k+1).$$

Moreover, we have

$$\int_{B_{h,k}} |\nabla u| \, dx \le C(k+1).$$

Thus we deduce that

$$\|\nabla u\|_{L^1(A_1)} \le \int_{B_{h-k,2k}} |\nabla u| \, dx \le C(k+1).$$

Therefore, we have

$$\limsup_{h\to\infty} \int_{A_1} [D_\xi \Phi(\nabla u) - g] \cdot \nabla u \, dx \ge 0.$$

On the set A_2, we have

$$\int_{A_2} [D_\xi \Phi(\nabla u) - g] \cdot \nabla (u - v) \, dx \ge - \int_{A_2} D_\xi \Phi(\nabla u) \cdot \nabla v \, dx + \int_{A_2} g \cdot \nabla v \, dx - \int_{A_2} g \cdot \nabla u \, dx. \quad (3.28)$$

Reasoning as before, the second and third terms of the right-hand side in (3.28) tend to zero as h tends to infinity. Furthermore, by (2.4), (1.4) and (2.5) we know that

$$\begin{aligned}&\int_{A_2} |D_\xi \Phi(\nabla u)\cdot \nabla v|\,dx\\&\quad\le \int_{A_2} \Psi(D_\xi\Phi(\nabla u))\,dx + \int_{A_2}\Phi(\nabla v)\,dx + \int_{A_2}\Phi(-\nabla v)\,dx\\&\quad\le \int_{\{h<|u|<h+k\}} D_\xi\Phi(\nabla u)\cdot\nabla u\,dx + (C+1)\int_{\{h-k<|v|<h\}} D_\xi\Phi(\nabla v)\cdot\nabla v\,dx.\end{aligned}$$

For given $k>0$, define the function $T_{h,1}(s) = T_1(s - T_h(s))$ by

$$T_{h,1}(s) = \begin{cases} s - h\text{sign}(s) & \text{if } h<|s|<h+1,\\ \text{sign}(s) & \text{if } |s|\ge h+1,\\ 0 & \text{if } |s|\le h.\end{cases}$$

Choosing $k=1$ and $\phi = T_h(u)$ in the inequality (3.9), we obtain

$$\begin{aligned}&\int_{\{h\le|u|\le h+1\}} D_\xi\Phi(\nabla u)\cdot\nabla u\,dx\\&\quad\le \int_\Omega f T_{h,1}(u)\,dx + \int_{\{h\le|u|\le h+1\}} g\cdot\nabla u\,dx\\&\quad\le \int_{\{|u|>h\}} |f|\,dx + \|g\|_{L^\infty(\Omega)}\int_{\{h\le|u|\le h+1\}}|\nabla u|\,dx.\end{aligned}$$

It follows from (3.17) and the above argument that

$$\lim_{h\to\infty}\int_{\{h<|u|<h+1\}} D_\xi\Phi(\nabla u)\cdot\nabla u\,dx = 0. \tag{3.29}$$

Therefore, we conclude from (3.29) that

$$\lim_{h\to\infty}\int_{A_2} D_\xi\Phi(\nabla u)\cdot\nabla v\,dx = 0.$$

Due to the above reason, the same estimates can be done on A_1' and A_2'.

Summing up the results obtained for A_0, A_1, A_1', A_2 and A_2', we deduce that

$$\lim_{h\to\infty}\int_{A_0}[D_\xi\Phi(\nabla u) - D_\xi\Phi(\nabla v)]\cdot\nabla(u-v)\,dx = 0,$$

that is, for every $k>0$,

$$\int_{\{|u-v|\le k\}}[D_\xi\Phi(\nabla u) - D_\xi\Phi(\nabla v)]\cdot\nabla(u-v)\,dx = 0.$$

Thus from the strict convexity of function $\Phi(\xi)$, we have $\nabla u = \nabla v$ a.e. in Ω. Furthermore, Sobolev embedding theorem implies that

$$\|T_k(u-v)\|_{L^{1^*}(\Omega)} \le C\|\nabla T_k(u-v)\|_{L^1(\Omega)} = 0, \quad \text{for all } k>0,$$

and hence $u=v$ a.e. in Ω. This completes the proof. □

Next, we begin to prove a comparison result.

Proof of Theorem 1.3. Choosing $T_l(u^+)$ as a test function in (3.9) of Definition 3.3, we obtain

$$\int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla T_k(u - T_l(u^+))\, dx \le \int_\Omega f T_k(u - T_l(u^+))\, dx. \tag{3.30}$$

For the right hand side of (3.30), we can write

$$\begin{aligned}\int_\Omega f T_k(u - T_l(u^+))\, dx = &\int_{\{u \ge l\}} f T_k(u - T_l(u^+))\, dx \\ &+ \int_{\{0<u<l\}} f T_k(u - T_l(u^+))\, dx \\ &+ \int_{\{u \le 0\}} f T_k(u - T_l(u^+))\, dx.\end{aligned}$$

If $0 < u < l$, then $u - T_l(u^+) = 0$ and

$$\int_{\{0<u<l\}} f T_k(u - T_l(u^+))\, dx = 0.$$

If $u \le 0$, then $u - T_l(u^+) = u$ and

$$\int_{\{u \le 0\}} f T_k(u - T_l(u^+))\, dx \le 0$$

due to the fact that $f \ge 0$. If $u \ge l$, then $u^+ = u$. Therefore, we have

$$\int_\Omega f T_k(u - T_l(u^+))\, dx \le \int_{\{u \ge l\}} f T_k(u - T_l(u))\, dx. \tag{3.31}$$

On the other hand, we can split the left hand side of (3.30) into three terms

$$\begin{aligned}&\int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla T_k(u - T_l(u^+))\, dx \\ &= \int_{\{u \le 0\}} D_\xi \Phi(\nabla u) \cdot \nabla T_k(u)\, dx + \int_{\{l<u<l+k\}} D_\xi \Phi(\nabla u) \cdot \nabla (u - l)\, dx \\ &\quad + \int_{\{u \ge l+k\}} D_\xi \Phi(\nabla u) \cdot \nabla l\, dx.\end{aligned}$$

Recalling $D_\xi \Phi(\nabla u) \cdot \nabla u \ge 0$ and (2.4), we deduce that

$$\begin{aligned}\int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla T_k(u - T_l(u^+))\, dx &\ge \int_\Omega D_\xi \Phi(\nabla T_k(u^-)) \cdot \nabla T_k(u^-)\, dx \\ &\ge \int_\Omega \Phi(\nabla T_k(u^-))\, dx.\end{aligned}$$

It follows from (3.30) and (3.31) that

$$\int_\Omega \Phi(\nabla T_k(u^-))\, dx \le \int_{\{u \ge l\}} f T_k(u - T_l(u))\, dx.$$

Using the Lebesgue dominated theorem and letting $l \to \infty$ in the above inequality, we conclude that

$$\int_\Omega \Phi(\nabla T_k(u^-))\,dx \le 0,$$

which implies from the nonnegativity of $\Phi(\xi)$ that $\nabla T_k(u^-) = 0$ and $T_k(u^-) = C$. Then from $T_k(u^-) \in W_0^{1,1}(\Omega)$ we have $u^- = 0$ a.e. in Ω, i.e., $u \ge 0$ a.e. in Ω. This finishes the proof. □

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