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# Nontrivial solutions to boundary value problems for semilinear strongly degenerate elliptic differential equations

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**Abstract.** In this paper we study boundary value problems for semilinear equations involving strongly degenerate elliptic differential operators. Via a Pohozaev's type identity we show that if the nonlinear term grows faster than some power function then the boundary value problem has no nontrivial solution. Otherwise when the nonlinear term grows slower than the same power function, by establishing embedding theorems for weighted Sobolev spaces associated with the strongly degenerate elliptic equations, then applying the theory of critical values in Banach spaces, we prove that the problem has a nontrivial solution, or even infinite number of solutions provided that the nonlinear term is an odd function.

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## 1. Introduction

Boundary value problems for semilinear elliptic equations were studied in [1,2] (see also the references therein); and for semilinear degenerate elliptic equations in [3–18] (see also the references therein and the recent survey paper [21] and the book [22]). Operators considered in [3–18] are degenerate on smooth surfaces. In this note we consider the following boundary value problem for semilinear equation involving strongly degenerate elliptic differential operator

$$\begin{aligned} &\Delta_x u + \Delta_y u + |x|^{2\alpha}|y|^{2\beta}\Delta_z u - c(x,y,z)u + g(x,y,z,u) \\ &\quad := P_{\alpha,\beta} u - c(x,y,z)u + g(x,y,z,u) = 0 \quad \text{in} \quad \Omega \qquad (1) \\ &\qquad u = 0 \qquad \text{on} \quad \partial\Omega, \qquad (2) \end{aligned}$$

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where $x = (x_1, \ldots, x_{n_1}) \in \mathbb{R}^{n_1}, y = (y_1, \ldots, y_{n_2}) \in \mathbb{R}^{n_2}, z = (z_1, \ldots, z_{n_3}) \in \mathbb{R}^{n_3}; 0 \leqslant c(x,y,z), c(x,y,z) \in C^{0,\sigma}(\Omega), g(x,y,z,0) = 0, g(x,y,z,t) \not\equiv 0, \alpha, \beta \geq 0, \alpha + \beta > 0, 0 < \sigma \leqslant 1$ and $\Omega$ is a bounded domain with a smooth boundary in $\mathbb{R}^{n_1+n_2+n_3}$. It is obvious that the problem (1)–(2) has a solution $u \equiv 0$, called the trivial solution. We are interested in finding other nontrivial solutions to the problem (1)–(2). Note that the operator $P_{\alpha,\beta}$ under consideration is degenerate on two intersecting surfaces. Some particular cases of the present paper have been announced in [19,20]. The paper is organized as follows. In Sect. 2, if the nonlinear term grows faster than a power function and the domain $\Omega$ is $P_{\alpha,\beta}$-star-shape, we establish nonexistence theorem via an identity of Pohozaev's type. In Sect. 3 we prove some embedding theorems for weighted Sobolev spaces associated with the operator $P_{\alpha,\beta}$. Some of the results here are optimal in the sense that they can not be improved. The method used in this section can be applied to the cases considered in [8–11] and produce a better result than that of [8–11]. Note that embedding theorems for weighted Sobolev spaces associated with nonsmooth vector fields were also obtained in [23]. In Sect. 4, based on the embedding theorems in Sect. 3 and the theory of critical values of nonlinear functionals developed in [2] we obtain theorems on the existence of nontrivial solutions if the nonlinear term grows slower than a power function (the same power function as for the nonexistence results), or even infinite number of solutions provided that the nonlinear term is an odd function. In Sect. 5 we present an interesting illustrating example that exhibits the connection between the elliptic, degenerate and strongly degenerate elliptic cases.

## 2. Nonexistence results

In this section we deal with the nonexistence of nontrivial solutions to the problems (1)–(2). Let us put $G(x,y,z,t) = \int_0^t g(x,y,z,s)\, ds$ and $\nu = (\nu_x, \nu_y, \nu_z) = (\nu_{x_1}, \ldots, \nu_{x_{n_1}}, \nu_{y_1}, \ldots, \nu_{y_{n_2}}, \nu_{z_1}, \ldots, \nu_{z_{n_3}})$ be the unit outward normal on $\partial\Omega$. Set $N = n_1 + n_2 + n_3, N_{\alpha,\beta} = n_1 + n_2 + n_3(\alpha + \beta + 1)$. Let us take a fixed point $(a,b,c) = (a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2}, c_1, \ldots, c_{n_3}) \in \mathbb{R}^N$.

**Lemma 1.** *Suppose that $c(x,y,z) \equiv 0, g(x,y,z,t) = g(t)$. Let $u\,(x,y,z)$ be a solution of the problem (1)–(2), which belongs to the space $H^2(\Omega)$. Then the solution $u\,(x,y,z)$ satisfies the identity*

$$\int_\Omega \left[N_{\alpha,\beta} G\,(u) - \frac{N_{\alpha,\beta} - 2}{2} g\,(u)\,u\right] \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$+ \int_\Omega |x|^{2(\alpha-1)} |y|^{2(\beta-1)} |\nabla_z u|^2 [\alpha(x,a)|y|^2 + \beta(y,b)|x|^2]\, \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \frac{1}{2} \int_{\partial\Omega} \nu_{\alpha,\beta} \nu_{a,b,c}^{\alpha,\beta} \left(\frac{\partial u}{\partial \nu}\right)^2 \mathrm{d}S, \tag{3}$$

*where*

$$\nu_{\alpha,\beta} = |\nu_x|^2 + |\nu_y|^2 + |x|^{2\alpha}|y|^{2\beta}|\nu_z|^2,$$
$$\nu^{\alpha,\beta}_{a,b,c} = (x-a,\nu_x) + (y-b,\nu_y) + (\alpha+\beta+1)(z-c,\nu_z).$$

*Proof.* Integrating by parts gives

$$(\alpha_1+\beta_1+1)\int_\Omega G(u)\mathrm{d}x\mathrm{d}y\mathrm{d}z = -\int_\Omega \left[\frac{\alpha_1((x-a),\nabla_x u)}{n_1} + \frac{\beta_1((y-b),\nabla_y u)}{n_2} + \frac{((z-c),\nabla_z u)}{n_3}\right] g(u)\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

for any real numbers $\alpha_1, \beta_1$. From (1) it follows that

$$g(u) = -\left(\Delta_x u + \Delta_y u + |x|^{2\alpha}|y|^{2\beta}\Delta_z u\right).$$

Hence we get

$$\begin{aligned}(\alpha_1+\beta_1+1)&\int_\Omega G(u)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_\Omega \left[\frac{\alpha_1((x-a),\nabla_x u)}{n_1} + \frac{\beta_1((y-b),\nabla_y u)}{n_2} + \frac{((z-c),\nabla_z u)}{n_3}\right] \\ &\quad\times \left(\Delta_x u + \Delta_y u + |x|^{2\alpha}|y|^{2\beta}\Delta_z u\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z. \end{aligned} \tag{4}$$

A detailed computation shows that when $\alpha_1 = \frac{n_1}{n_3(\alpha+\beta+1)}, \beta_1 = \frac{n_2}{n_3(\alpha+\beta+1)}$ the right-hand side of (4) equals

$$\begin{aligned}&\frac{N_{\alpha,\beta}-2}{2n_3(\alpha+\beta+1)}\int_\Omega\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + |x|^{2\alpha}|y|^{2\beta}\left(\frac{\partial u}{\partial z}\right)^2\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &-\frac{1}{n_3(\alpha+\beta+1)}\int_\Omega |x|^{2(\alpha-1)}|y|^{2(\beta-1)}|\nabla_z u|^2[\alpha(x,a)|y|^2 + \beta(y,b)|x|^2]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &+\frac{1}{2n_3(\alpha+\beta+1)}\int_{\partial\Omega}\left(|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2\right)\nu^{\alpha,\beta}_{a,b,c}\mathrm{d}S. \end{aligned} \tag{5}$$

From the boundary condition (2) we deduce that

$$\int_\Omega g(u)u\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_\Omega \left[\left|\nabla_x u\right|^2 + \left|\nabla_y u\right|^2 + |x|^{2\alpha}|y|^{2\beta}\left|\nabla_z u\right|^2\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{6}$$

and

$$\nabla_x u = \nu_x\frac{\partial u}{\partial\nu},\quad \nabla_y u = \nu_y\frac{\partial u}{\partial\nu},\quad \nabla_z u = \nu_z\frac{\partial u}{\partial\nu} \text{ on the boundary } \partial\Omega. \tag{7}$$

Putting (4)–(7) together we finally get the desired identity (3). $\square$

From Lemma 1 we can easily deduce the following two theorems:

**Theorem 1.** *Suppose that* $c(x,y,z) \equiv 0, g(x,y,z,t) = g(t)$. *Let* $\Omega$ *be a* $P_{\alpha,\beta}$*-star-shape domain with respect to the point* $\{(0,0,0)\}$ *(i. e. the inequality* $\nu^{\alpha,\beta}_{0,0,0} > 0$ *holds almost everywhere on* $\partial\Omega$*) and* $N_{\alpha,\beta}G(t) - \frac{N_{\alpha,\beta}-2}{2}g(t)t < 0$ *when* $t \neq 0$. *Then the problem* (1)–(2) *has no nontrivial solution* $u \in H^2(\Omega)$.

**Theorem 2.** *Suppose that* $c(x,y,z) \equiv 0, g(x,y,z,t) = g(t)$. *Let* $\Omega$ *be a* $P_{\alpha,\beta}$*-star-shape domain with respect to the point* $\{(0,0,0)\}$ *and* $N_{\alpha,\beta}G(t) - \frac{N_{\alpha,\beta}-2}{2}g(t)t < 0$ *when* $t > 0$. *Then the problem* (1)–(2) *has no nontrivial positive solution* $u \in H^2(\Omega)$.

**Theorem 3.** *Suppose that* $c(x,y,z) \equiv 0$. *Let* $\Omega$ *be a* $P_{\alpha,\beta}$*-star-shape domain with respect to the point* $\{(0,0,0)\}$ *and* $g(x,y,z,t) = \lambda t + |t|^{\gamma}t$ *with* $\lambda \leq 0$, $\gamma \geq \frac{4}{N_{\alpha,\beta}-2}$. *Then the problem* (1)–(2) *has no nontrivial solution* $u \in H^2(\Omega)$.

*Proof.* Indeed, in this case $G(t) = \frac{\lambda t^2}{2} + \frac{|t|^{\gamma+2}}{\gamma+2}$. Assume that $u$ is a nontrivial solution of the boundary value problem (1)–(2). From Lemma 1, taking $(a,b,c) = (0,0,0)$, we have

$$\int_{\Omega}\left\{\lambda u^2 + \frac{[\gamma(2-N_{\alpha,\beta})+4]|u|^{\gamma+2}}{2(\gamma+2)}\right\}\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{2}\int_{\partial\Omega}\nu_{\alpha,\beta}\nu_{0,0,0}^{\alpha,\beta}\left(\frac{\partial u}{\partial\nu}\right)^2\mathrm{d}S. \tag{8}$$

If $\gamma > \frac{4}{N_{\alpha,\beta}-2}$ or $\lambda < 0$, we see that the left-hand side of (8) is negative but the right-hand side is nonnegative since $\Omega$ is $P_{\alpha,\beta}$-star-shape. This leads to a contradiction.

If $\gamma = \frac{4}{N_{\alpha,\beta}-2}$ and $\lambda = 0$, then we deduce from (8)

$$\frac{1}{2}\int_{\partial\Omega}\nu_{\alpha,\beta}\nu_{0,0,0}^{\alpha,\beta}\left(\frac{\partial u}{\partial\nu}\right)^2\mathrm{d}S = 0.$$

Again since $\Omega$ is $P_{\alpha,\beta}$-star-shape we deduce that $\left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega} \equiv 0$. Then from a uniqueness theorem it follows that $u \equiv 0$. Theorem 3 is thus proved. $\square$

**Remark 1.** Without the condition of $P_{\alpha,\beta}$-starness of the domain $\Omega$, Theorems 1–3 may not be true. Indeed, if $\Omega$ does not intersect with the hyperplanes $\{x = 0\}$ and $\{y = 0\}$ then $P_{\alpha,\beta}$ is elliptic in $\Omega$. Hence we can prove the existence of a nontrivial solution of the problem (1)–(2) for any function $g(t)$ with the growth order less than $\frac{N+2}{N-2}$ (note that $\frac{N_{\alpha,\beta}+2}{N_{\alpha,\beta}-2} < \frac{N+2}{N-2}$) by using the variational method and the classical Sobolev embedding theorem. Note that, in this case, of course, the $P_{\alpha,\beta}$-star shape condition of $\Omega$ fails. An example in Sect. 5 will illuminate this phenomenon.

## 3. Embedding theorems

The situation changes drastically if the growth rate of $g(x,y,z,t)$ in the $t$-variable is less than $\frac{N_{\alpha,\beta}+2}{N_{\alpha,\beta}-2}$ as we shall now show. Let us first introduce some auxiliary spaces.

**Definition 1.** By $S_1^p(\Omega), 1 \leq p < \infty$, we will denote the set of all functions $u \in L^p(\Omega)$ such that $\nabla_x u, \nabla_y u, |x|^{\alpha}|y|^{\beta}\nabla_z u \in L^p(\Omega)$. For the norm in $S_1^p(\Omega)$

we take

$$\|u\|^p_{S^p_1(\Omega)} = \int_\Omega \left( |u|^p + \sum_{i=1}^{n_1} \left| \frac{\partial u}{\partial x_i} \right|^p + \sum_{j=1}^{n_2} \left| \frac{\partial u}{\partial y_j} \right|^p + \sum_{l=1}^{n_3} \left| |x|^\alpha |y|^\beta \frac{\partial u}{\partial z_l} \right|^p \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

If $p = 2$ we can also define the scalar product in $S^2_1(\Omega)$ as follows

$$(u, v)_{S^2_1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla_x u, \nabla_x v)_{L^2(\Omega)} + (\nabla_y u, \nabla_y v)_{L^2(\Omega)} + \left(|x|^\alpha |y|^\beta \nabla_z u, |x|^\alpha |y|^\beta \nabla_z v\right)_{L^2(\Omega)}.$$

The space $S^p_{1,0}(\Omega)$ is defined as the closure of $C^1_0(\Omega)$ in the space $S^p_1(\Omega)$, where $C^1_0(\Omega)$ denotes the set of functions in $C^1(\Omega)$ with a compact support in $\Omega$.

**Lemma 2.** *Let* $\Pi^n = \underbrace{[-1,1] \times \cdots \times [-1,1]}_{n \text{ times}}$ *and* $f_i(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n) := f_i(t; \hat{t_i}) \in L^{n-1}(\Pi^{i-1}_{(t;\hat{t}_i)})$. *Assume that* $n \geq 2$ *then the following inequality holds*

$$\left| \int_{\Pi^n} \prod_{i=1}^n f_i(t; \hat{t_i}) dt \right|^{n-1} \leq \prod_{i=1}^n \int_{\Pi^{n-1}_{(t;\hat{t}_i)}} \left| f_i(t; \hat{t_i}) \right|^{n-1} \hat{dt_i},$$

*where* $dt = dt_1 \dots dt_n$, $\hat{dt_i} = dt_1 \dots dt_{i-1} dt_{i+1} \dots dt_n$ *and* $\Pi^{n-1}_{(t;\hat{t}_i)} = \{(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n) := (t; \hat{t_i}) \in \mathbb{R}^{n-1} : (t; \hat{t_i}) \in \underbrace{[-1,1] \times \cdots \times [-1,1]}_{(n-1) \text{ times}}\}$.

The proof is easily established by induction (see also [24], [25]).

**Proposition 1.** *Suppose that* $1 \leq p < N_{\alpha,\beta}$. *Then we have* $S^p_{1,0}(\Omega) \subset L^{\frac{pN_{\alpha,\beta}}{N_{\alpha,\beta}-p} - \varepsilon}(\Omega)$ *for every small positive* $\varepsilon$.

*Proof.* It suffices to prove that for any sufficiently small $\varepsilon$ there exists a constant $C$ such that the following inequality

$$\|u\|_{L^{\frac{pN_{\alpha,\beta}}{N_{\alpha,\beta}-p} - \varepsilon}(\Omega)} \leqslant C \|u\|_{S^1_{1,0}(\Omega)} \tag{9}$$

holds for any function $u \in C^1_0(\Omega)$. Without loss of generality we can assume that $\Omega \subset \underbrace{[-1,1] \times \cdots \times [-1,1]}_{N \text{ times}} := \Pi^N_{(x,y,z)}$. Extend the function $u$ to be equal to 0 outside of $\Omega$. We have for all $(x, y, z) \in \Omega$

$$u(x, y, z) = \int_{-1}^{x_i} \frac{\partial u}{\partial \theta} (x_1, \dots x_{i-1}, \theta, x_{i+1}, \dots, x_{n_1}, y_1, \dots y_{n_2}, z_1, \dots, z_{n_3}) \, \mathrm{d}\theta.$$

Therefore

$$|u(x, y, z)| \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\Pi^1_{(x_i)})} := A_i(x, y, z; \hat{x}_i) := A_i, \quad \forall (x, y, z) \in \Omega, i = \overline{1, n_1}.$$

Hence for all $(x, y, z) \in \Omega$

$$|u(x, y, z)|^{\delta_1} \leq A_i^{\delta_1}, \quad \delta_1 = \frac{n_1 + \alpha n_3}{n_1(n_1 + n_2 + n_3(1 + \alpha + \beta) - 1)}, i = \overline{1, n_1}.$$

Similarly, for all $(x, y, z) \in \Omega$

$$|u(x,y,z)|^{\delta_2} \leq \left\| \frac{\partial u}{\partial y_j} \right\|^{\delta_2}_{L^1(\Pi^1_{(y_j)})} := (B_j(x,y,z;\hat{y}_j))^{\delta_2} := B_j^{\delta_2},$$

$$\delta_2 = \frac{n_2 + \beta n_3}{n_2(n_1 + n_2 + n_3(1+\alpha+\beta) - 1)}, j = \overline{1, n_2},$$

$$|u(x,y,z)|^{\delta_3} \leq \left\| \frac{\partial u}{\partial z_l} \right\|^{\delta_3}_{L^1(\Pi^1_{(z_l)})} := (C_l(x,y,z;\hat{z}_l))^{\delta_3} := C_l^{\delta_3},$$

$$\delta_3 = \frac{1 - \tilde{\varepsilon}}{n_1 + n_2 + n_3(1+\alpha+\beta) - 1}, l = \overline{1, n_3},$$

where $\tilde{\varepsilon}$ is small enough (but positive). Define $\tilde{N}_\delta = n_1\delta_1 + n_2\delta_2 + n_3\delta_3$. Multiplying all the $N$ above inequalities and then integrating over the cube $\Pi^N_{(x,y,z)}$, in view of the Holder inequality with respect to the $z$-variable, we obtain

$$\begin{aligned}
\int_\Omega |u|^{\tilde{N}_\delta} \mathrm{d}x\mathrm{d}y\mathrm{d}z &\leq \int_{\Pi^N_{(x,y,z)}} |x|^{-\alpha n_3 \delta_3} |y|^{-\beta n_3 \delta_3} \prod_{i,j,l=1}^{n_1,n_2,n_3} A_i^{\delta_1} B_j^{\delta_2} C_l^{\delta_3} \mathrm{d}z\mathrm{d}x\mathrm{d}y \\
&\leq \int_{\Pi^{n_1+n_2}_{(x,y)}} |x|^{-\alpha n_3 \delta_3} |y|^{-\beta n_3 \delta_3} \left( \int_{\Pi^{n_3}_{(z)}} \prod_{l=1}^{n_3} C_l^{\frac{1}{n_3-1}} \mathrm{d}z \right)^{(n_3-1)\delta_3} \\
&\quad \times \left( \int_{\Pi^{n_3}_{(z)}} \prod_{i,j=1}^{n_1,n_2} A_i^{\frac{\delta_1}{1-(n_3-1)\delta_3}} B_j^{\frac{\delta_2}{1-(n_3-1)\delta_3}} \mathrm{d}z \right)^{1-(n_3-1)\delta_3} \mathrm{d}x\mathrm{d}y := I_1.
\end{aligned} \tag{10}$$

By applying Lemma 2 with $n = n_3$ we deduce that

$$\begin{aligned}
\left( \int_{\Pi^{n_3}_{(z)}} \prod_{l=1}^{n_3} C_l^{\frac{1}{n_3-1}} \mathrm{d}z \right)^{(n_3-1)\delta_3} &\leq \prod_{l=1}^{n_3} \left( \int_{\Pi^{n_3-1}_{(z;\hat{z}_l)}} C_l \mathrm{d}\hat{z}_l \right)^{\delta_3} \\
&= \prod_{l=1}^{n_3} \left( \int_{\Pi^{n_3}_{(z)}} |x|^\alpha |y|^\beta \left| \frac{\partial u}{\partial z_l} \right| \mathrm{d}z \right)^{\delta_3} \\
&:= \prod_{l=1}^{n_3} (\hat{C}_l(x,y))^{\delta_3} := \prod_{l=1}^{n_3} \hat{C}_l^{\delta_3}
\end{aligned}$$

and by the Holder inequality

$$\begin{aligned}
&\left( \int_{\Pi^{n_3}_{(z)}} \prod_{i,j=1}^{n_1,n_2} A_i^{\frac{\delta_1}{1-(n_3-1)\delta_3}} B_j^{\frac{\delta_2}{1-(n_3-1)\delta_3}} \mathrm{d}z \right)^{1-(n_3-1)\delta_3} \\
&\leq \prod_{i=1}^{n_1} \left( \int_{\Pi^{n_3}_{(z)}} A_i \mathrm{d}z \right)^{\delta_1} \prod_{j=1}^{n_2} \left( \int_{\Pi^{n_3}_{(z)}} B_j \mathrm{d}z \right)^{\delta_2}
\end{aligned}$$

$$= \prod_{i=1}^{n_1} \left( \int_{\Pi^{n_3}_{(z)}} \int_{-1}^{1} \left| \frac{\partial u}{\partial x_i} \right| \mathrm{d}x_i \mathrm{d}z \right)^{\delta_1} \prod_{j=1}^{n_2} \left( \int_{\Pi^{n_3}_{(z)}} \int_{-1}^{1} \left| \frac{\partial u}{\partial y_j} \right| \mathrm{d}y_j \mathrm{d}z \right)^{\delta_2}$$

$$:= \prod_{i=1}^{n_1} (\hat{A}_i(x, y; \hat{x}_i))^{\delta_1} \prod_{j=1}^{n_2} (\hat{B}_j(x, y; \hat{y}_j))^{\delta_2} := \prod_{i=1}^{n_1} \hat{A}_i^{\delta_1} \prod_{j=1}^{n_2} \hat{B}_j^{\delta_2}.$$

Hence for $p = \frac{n_1(n_1+n_2+n_3(1+\alpha+\beta)-1)}{n_3(\alpha+n_1\tilde{\varepsilon})}$, in view of the Holder inequality, we have

$$I_1 \leq \int_{\Pi^{n_2}_{(y)}} |y|^{-\beta n_3 \delta_3} \left\{ \int_{\Pi^{n_1}_{(x)}} |x|^{-\alpha n_3 \delta_3} \prod_{i,j,l=1}^{n_1,n_2,n_3} \hat{A}_i^{\delta_1} \hat{B}_j^{\delta_2} \hat{C}_l^{\delta_3} \mathrm{d}x \right\} \mathrm{d}y$$

$$\leq \left( \int_{\Pi^{n_1}_{(x)}} |x|^{-\alpha n_3 \delta_3 p} \mathrm{d}x \right)^{\frac{1}{p}} \times \int_{\Pi^{n_2}_{(y)}} |y|^{-\beta n_3 \delta_3} \left\{ \left( \int_{\Pi^{n_1}_{(x)}} \prod_{i=1}^{n_1} \hat{A}_i^{\frac{1}{n_1-1}} \mathrm{d}x \right)^{\delta_1(n_1-1)} \right.$$

$$\left. \left( \int_{\Pi^{n_1}_{(x)}} \prod_{j=1}^{n_2} \hat{B}_j \mathrm{d}x \right)^{\delta_2} \prod_{l=1}^{n_3} \left( \int_{\Pi^{n_1}_{(x)}} \hat{C}_l \mathrm{d}x \right)^{\delta_3} \right\} \mathrm{d}y$$

$$:= C \int_{\Pi^{n_2}_{(y)}} |y|^{-\beta n_3 \delta_3} \left\{ \left( \int_{\Pi^{n_1}_{(x)}} \prod_{i=1}^{n_1} \hat{A}_i^{\frac{1}{n_1-1}} \mathrm{d}x \right)^{\delta_1(n_1-1)} \right.$$

$$\left. \prod_{j=1}^{n_2} \left( \int_{\Pi^{n_1}_{(x)}} \hat{B}_j \mathrm{d}x \right)^{\delta_2} \prod_{l=1}^{n_3} \left( \int_{\Pi^{n_1}_{(x)}} \hat{C}_l \mathrm{d}x \right)^{\delta_3} \right\} \mathrm{d}y := I_2. \tag{11}$$

Again by Lemma 2 with $n = n_1$

$$\left( \int_{\Pi^{n_1}_{(x)}} \prod_{i=1}^{n_1} \hat{A}_i^{\frac{1}{n_1-1}} \mathrm{d}x \right)^{\delta_1(n_1-1)} \leq \prod_{i=1}^{n_1} \left( \int_{\Pi^{n_1-1}_{(x \setminus x_i)}} \hat{A}_i \mathrm{d}\hat{x}_i \right)^{\delta_1}$$

$$= \prod_{i=1}^{n_1} \left( \int_{\Pi^{n_1+n_3}_{(x,z)}} \left| \frac{\partial u}{\partial x_i} \right| \mathrm{d}z \mathrm{d}x \right)^{\delta_1} := \prod_{i=1}^{n_1} (\hat{\hat{A}}_i(y))^{\delta_1} = \prod_{i=1}^{n_1} \hat{\hat{A}}_i^{\delta_1}.$$

By using the Holder inequality with $q = \frac{n_2[n_1+n_2+n_3(1+\alpha+\beta)-1]}{n_3(\beta+n_2\tilde{\varepsilon})}$ we obtain

$$I_2 \leq C \left( \int_{\Pi^{n_2}_{(y)}} |y|^{-\alpha n_3 \delta_3 q} \mathrm{d}y \right)^{\frac{1}{q}} \prod_{i=1}^{n_1} \left( \int_{\Pi^{n_2}_{(y)}} \hat{\hat{A}}_i \mathrm{d}y \right)^{\delta_1}$$

$$\times \left[ \int_{\Pi^{n_2}_{(y)}} \prod_{j=1}^{n_2} \left( \int_{\Pi^{n_1}_{(x)}} \tilde{B}_j \mathrm{d}x \right)^{\frac{1}{n_2-1}} \mathrm{d}y \right]^{\delta_2(n_2-1)} \prod_{l=1}^{n_3} \left( \int_{\Pi^{n_1+n_2}_{(x,y)}} \tilde{C}_l \mathrm{d}x \mathrm{d}y \right)^{\delta_3}$$

$$\leq C\prod_{i=1}^{n_1}\left(\int_{\Pi^{n_2}_{(y)}}\hat{\hat{A}}_i\mathrm{d}y\right)^{\delta_1}\left[\int_{\Pi^{n_2}_{(y)}}\prod_{j=1}^{n_2}\left(\int_{\Pi^{n_1}_{(x)}}\tilde{B}_j\mathrm{d}x\right)^{\frac{1}{n_2-1}}\mathrm{d}y\right]^{\delta_2(n_2-1)}$$
$$\prod_{l=1}^{n_3}\left(\int_{\Pi^{n_1+n_2}_{(x,y)}}\tilde{C}_l\mathrm{d}x\mathrm{d}y\right)^{\delta_3}. \tag{12}$$

From Lemma 2 with $n = n_2$ it follows that

$$\left[\int_{\Pi^{n_2}_{(y)}}\prod_{j=1}^{n_2}\left(\int_{\Pi^{n_1}_{(x)}}\tilde{B}_j\mathrm{d}x\right)^{\frac{1}{n_2-1}}\mathrm{d}y\right]^{\delta_2(n_2-1)}\leq\prod_{j=1}^{n_2}\left(\int_{\Pi^{n_1+n_2}_{(x,y)}}\tilde{B}_j\mathrm{d}x\mathrm{d}y\right)^{\delta_2}$$
$$=\prod_{j=1}^{n_2}\left(\int_\Omega\left|\frac{\partial u}{\partial y_j}\right|\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{\delta_2}=\left\|\frac{\partial u}{\partial y_j}\right\|^{\delta_2}_{L^1(\Omega)}.$$

Combining (10), (11), (12) we obtain

$$I_2\leq C\prod_{i,j,l=1}^{n_1,n_2,n_3}\left\|\frac{\partial u}{\partial x_i}\right\|^{\delta_1}_{L^1(\Omega)}\cdot\left\|\frac{\partial u}{\partial y_j}\right\|^{\delta_2}_{L^1(\Omega)}\cdot\left\||x|^\alpha|y|^\beta\frac{\partial u}{\partial z_l}\right\|^{\delta_3}_{L^1(\Omega)}.$$

It implies that

$$\|u\|_{L^{\tilde{N}_\delta}(\Omega)}\leq C\prod_{i,j,l=1}^{n_1,n_2,n_3}\left\|\frac{\partial u}{\partial x_i}\right\|^{\frac{\delta_1}{\tilde{N}_\delta}}_{L^1(\Omega)}\cdot\left\|\frac{\partial u}{\partial y_j}\right\|^{\frac{\delta_2}{\tilde{N}_\delta}}_{L^1(\Omega)}\cdot\left\||x|^\alpha|y|^\beta\frac{\partial u}{\partial z_l}\right\|^{\frac{\delta_3}{\tilde{N}_\delta}}_{L^1(\Omega)}$$
$$\leq C\left(\sum_{i=1}^{n_1}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^1(\Omega)}+\sum_{j=1}^{n_2}\left\|\frac{\partial u}{\partial y_j}\right\|_{L^1(\Omega)}\right.$$
$$\left.+\sum_{l=1}^{n_3}\left\||x|^\alpha|y|^\beta\frac{\partial u}{\partial z_l}\right\|_{L^1(\Omega)}\right). \tag{13}$$

Next we put $|u|^\gamma$ $(\gamma\geq 1)$ into (13) and obtain

$$\||u|^\gamma\|_{L^{\tilde{N}_\delta}(\Omega)}\leq C\left(\sum_{i=1}^{n_1}\left\||u|^{\gamma-1}\frac{\partial u}{\partial x_i}\right\|_{L^1(\Omega)}+\sum_{j=1}^{n_2}\left\||u|^{\gamma-1}\frac{\partial u}{\partial y_j}\right\|_{L^1(\Omega)}\right.$$
$$\left.+\sum_{l=1}^{n_3}\left\||u|^{\gamma-1}|x|^\alpha|y|^\beta\frac{\partial u}{\partial z_l}\right\|_{L^1(\Omega)}\right)$$
$$\leq C\left(\||u|^{\gamma-1}\|_{L^{p'}(\Omega)}\right)\left(\sum_{i=1}^{n_1}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}+\sum_{j=1}^{n_2}\left\|\frac{\partial u}{\partial y_j}\right\|_{L^p(\Omega)}\right.$$
$$\left.+\sum_{l=1}^{n_3}\left\||x|^\alpha|y|^\beta\frac{\partial u}{\partial z_l}\right\|_{L^p(\Omega)}\right),$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Therefore choosing $\gamma=\frac{p}{p-(p-1)\tilde{N}_\delta}$, we get

$$\|u\|_{L^{\frac{p\tilde{N}_\delta}{p-(p-1)\tilde{N}_\delta}}(\Omega)} \leq C\left(\sum_{i=1}^{n_1}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}+\sum_{j=1}^{n_2}\left\|\frac{\partial u}{\partial y_j}\right\|_{L^p(\Omega)} + \sum_{l=1}^{n_3}\left\||x|^\alpha|y|^\beta\frac{\partial u}{\partial z_l}\right\|_{L^p(\Omega)}\right).$$

Note that $\frac{p\tilde{N}_\delta}{p-(p-1)\tilde{N}_\delta}\to\frac{pN_{\alpha,\beta}}{N_{\alpha,\beta}-p}$ when $\tilde{\varepsilon}\to 0$. The proof of the proposition is therefore completed. □

**Proposition 2.** *Assume that $1\leq p<N_{\alpha,\beta}$. Then the embedding map $S^p_{1,0}(\Omega)$ into $L^{\frac{pN_{\alpha,\beta}}{N_{\alpha,\beta}-p}-\varepsilon}(\Omega)$ is compact for every positive small $\varepsilon$.*

*Proof.* Set $k=\max\{[\alpha],[\beta]\}+1$, where $[\cdot]$ stands for the integral part of the argument. In [26] the authors have shown that $S^p_{1,0}(\Omega)$ is compactly embedded into $L^p_{\frac{1}{k+1}}(\Omega)$. Combining the above result of [26] and Propositions 1 we can easily get the claim in Proposition 2. □

**Proposition 3.** *Assume that $p>N_{\alpha,\beta}$. Then $S^p_{1,0}(\Omega)\subset C^0(\bar{\Omega})$.*

*Proof.* It suffices to prove the following estimate

$$\sup_{x\in\Omega}|u|\leq C\|u\|_{S^p_{1,0}(\Omega)} \quad \text{for every} \quad u\in C^1_0(\Omega).$$

First, we assume $Vol(\Omega)=1$. By the inequality (9) we have

$$\begin{aligned}\||u|^\gamma\|_{L^{\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-1}-\varepsilon}(\Omega)} &\leq C\,\||u|^\gamma\|_{S^1_{1,0}(\Omega)}\\ &= C\gamma\int_\Omega |u|^{\gamma-1}.\left(\sum_{i=1}^{n_1}\left|\frac{\partial u}{\partial x_i}\right|+\sum_{j=1}^{n_2}\left|\frac{\partial u}{\partial y_j}\right|+\sum_{l=1}^{n_3}|x|^\alpha|y|^\beta\left|\frac{\partial u}{\partial z_l}\right|\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\leq C\gamma\|u\|_{S^p_{1,0}(\Omega)}.\left\||u|^{\gamma-1}\right\|_{L^{\frac{p}{p-1}}(\Omega)}\end{aligned}$$

for $\gamma\geq 1$ and a small fixed positive $\varepsilon$. That is

$$\begin{aligned}\|u\|_{L^{\gamma\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-1}-\varepsilon\right)}(\Omega)} &\leq (C\gamma)^{\frac{1}{\gamma}}\|u\|^{\frac{1}{\gamma}}_{S^p_{1,0}(\Omega)}.\|u\|^{1-\frac{1}{\gamma}}_{L^{\frac{p(\gamma-1)}{p-1}}(\Omega)}\\ &\leq (C\gamma)^{\frac{1}{\gamma}}\|u\|^{\frac{1}{\gamma}}_{S^p_{1,0}(\Omega)}.\|u\|^{1-\frac{1}{\gamma}}_{L^{\frac{p\gamma}{p-1}}(\Omega)} \qquad \text{since } Vol(\Omega)=1.\end{aligned}$$

Therefore we get

$$\left\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\right\|_{L^{\gamma\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-1}-\varepsilon\right)}(\Omega)}\leq (C\gamma)^{\frac{1}{\gamma}}\left\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\right\|^{1-\frac{1}{\gamma}}_{L^{\frac{p\gamma}{p-1}}(\Omega)}.$$

Choose $\tau$ such that $\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-1} - \varepsilon > \frac{p}{p-1}$. Let us substitute for $\gamma$ the value $\xi^\rho (\rho = 1, 2, \dots)$ where $\xi = \left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-1} - \varepsilon\right)\frac{p-1}{p} > 1$ by the hypotheses of the theorem. Then we have

$$\left\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\right\|_{L^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-1}-\varepsilon\right).\xi^\rho}(\Omega)} \leq (C\xi^\rho)^{\frac{1}{\xi^\rho}} \left\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\right\|^{1-\xi^{-\rho}}_{L^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-1}-\varepsilon\right).\xi^{\rho-1}}(\Omega)}$$

for $\rho = 1, 2, \dots$. Iterating from $\rho = 1$ and using the fact that

$$\left\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\right\|_{L^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-1}-\varepsilon\right).\xi}(\Omega)} \leq 1$$

we obtain

$$\left\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\right\|_{L^{\xi^\rho}(\Omega)} \leq \left\|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\right\|_{L^{\left(\frac{N_{\alpha,\beta}}{N_{\alpha,\beta}-1}-\varepsilon\right).\xi^\rho}(\Omega)} \leq (C\xi)^{\Sigma\rho.\xi^{-\rho}} = C.$$

Consequently, as $\rho \longrightarrow \infty$, we have

$$\sup_{x\in\Omega}\left|\frac{u}{\|u\|_{S^p_{1,0}(\Omega)}}\right| \leq C,$$

and hence $\sup_{x\in\Omega}|u| \leq C\|u\|_{S^p_{1,0}(\Omega)}$. To eliminate the restriction $Vol\,(\Omega) = 1$ we consider the transformation $\Omega \longrightarrow \Omega_1$

$$x_1 = \{Vol\,(\Omega)\}^{-\frac{1}{N_{\alpha,\beta}}}\,x, y_1 = \{Vol\,(\Omega)\}^{-\frac{1}{N_{\alpha,\beta}}}\,y, z_1 = \{Vol\,(\Omega)\}^{-\frac{N_{\alpha,\beta}-n_1-n_2}{N_{\alpha,\beta}}}\,z.$$

The domain $\Omega_1$ now has the volume 1. That leads to

$$\begin{aligned}\sup_{(x,y,z)\in\Omega}|u(x,y,z)| &= \sup_{(x_1,y_1,z_1)\in\Omega_1}|u(x_1,y_1,z_1)| \\ &\leq C\,\|u(x_1,y_1,z_1)\|_{S^p_{1,0}(\Omega_1)} \leq C\|u(x,y,z)\|_{S^p_{1,0}(\Omega)}.\end{aligned}$$

□

In the next proposition we prove a precise embedding theorem. The method exploited here can be applied to produce a better result than those obtained in [8–11]. For the sake of simplicity we will prove the proposition for the case $n_1 = n_2 = n_3 = 1$. Let us define

$$I = \{(x,y,z) : a_1 < x < a_2, b_1 < y < b_2, -1 < z < 1\}.$$

In what follows we denote by $A_x, A_y, A_z$ the images of a set $A \subset \mathbb{R}^3$ under the projection map along the direction $Ox, Oy, Oz$, respectively. We first need an auxiliary lemma.

**Lemma 3.** *Let $a, b \in \mathbb{R}, 1 \leqslant p_1, p_2, p_3 < \infty; \frac{1}{p_1} + \frac{1}{p_2} = 1; p_3 > \max\{p_1, p_2\}$. The following inequality holds*

$$\begin{aligned}
&\int_I |F_1(y,z)|^{\frac{1}{p_1}} |F_2(x,z)|^{\frac{1}{p_2}} |F_3(x,y)|^{\frac{1}{p_3}} \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\leqslant \left(\int_{a_1}^{a_2} |x|^{-\frac{ap_1}{p_3-p_1}} \mathrm{d}x\right)^{\frac{p_3-p_1}{p_1p_3}} \left(\int_{b_1}^{b_2} |y|^{-\frac{bp_2}{p_3-p_2}} \mathrm{d}y\right)^{\frac{p_3-p_2}{p_2p_3}} \left[\int_{I_x} |F_1(y,z)|\mathrm{d}y\mathrm{d}z\right]^{\frac{1}{p_1}} \\
&\quad\times \left[\int_{I_y} |F_2(x,z)|\mathrm{d}x\mathrm{d}z\right]^{\frac{1}{p_2}} \left[\int_{I_z} |x|^a|y|^b|F_3(x,y)|\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{p_3}}.
\end{aligned}$$

*Proof.* If either of the factors on the right-hand side is infinite then the claim is obviously true. Hence we may suppose that all the factors on the right-hand side are finite. By Holder's inequality we have

$$\begin{aligned}
&\int_I |F_1(y,z)|^{\frac{1}{p_1}} |F_2(x,z)|^{\frac{1}{p_2}} |F_3(x,y)|^{\frac{1}{p_3}} \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_I |F_1(y,z)|^{\frac{1}{p_1}} |F_2(x,z)|^{\frac{1}{p_2}} |x|^{\frac{-a}{p_3}} |y|^{-\frac{b}{p_3}} ||x|^a|y|^b F_3(x,y)|^{\frac{1}{p_3}} \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\leqslant \int_{I_z} \left[\int_{-1}^{1} |F_1(y,z)|\mathrm{d}z\right]^{\frac{1}{p_1}} \left[\int_{-1}^{1} |F_2(x,z)|\mathrm{d}z\right]^{\frac{1}{p_2}} \\
&\quad\times |x|^{-\frac{a}{p_3}} |y|^{-\frac{b}{p_3}} ||x|^a|y|^b F_3(x,y)|^{\frac{1}{p_3}} \mathrm{d}x\mathrm{d}y \\
&\leqslant \left(\int_{a_1}^{a_2} |x|^{-\frac{ap_1}{p_3-p_1}} \mathrm{d}x\right)^{\frac{p_3-p_1}{p_1p_3}} \left[\int_{I_y} |F_2(x,z)|\mathrm{d}x\mathrm{d}z\right]^{\frac{1}{p_2}} \\
&\quad\times \int_{b_1}^{b_2} \left[\int_{-1}^{1} |F_1(y,z)|\mathrm{d}z\right]^{\frac{1}{p_1}} |y|^{-\frac{b}{p_3}} \left[\int_{a_1}^{a_2} |x|^a|y|^b|F_3(x,y)|\mathrm{d}x\right]^{\frac{1}{p_3}} \mathrm{d}y \\
&\leqslant \left(\int_{a_1}^{a_2} |x|^{-\frac{ap_1}{p_3-p_1}} \mathrm{d}x\right)^{\frac{p_3-p_1}{p_1p_3}} \left(\int_{b_1}^{b_2} |y|^{-\frac{bp_2}{p_3-p_2}} \mathrm{d}y\right)^{\frac{p_3-p_2}{p_2p_3}} \left[\int_{I_y} |F_2(x,z)|\mathrm{d}x\mathrm{d}z\right]^{\frac{1}{p_2}} \\
&\quad\times \left[\int_{I_x} |F_1(y,z)|\mathrm{d}y\mathrm{d}z\right]^{\frac{1}{p_1}} \left[\int_{I_z} |x|^a|y|^b|F_3(x,y)|\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{p_3}}
\end{aligned}$$

□

**Proposition 4.** *Assume that $1 < p < N_{\alpha,\beta}$. Then $S^p_{1,0}(\Omega) \subset L^{\frac{pN_{\alpha,\beta}}{N_{\alpha,\beta}-p}}(\Omega)$.*

*Proof.* To avoid unnecessary complications in the proof we prove this proposition only in case $n_1 = n_2 = n_3 = 1$. In this case $N_{\alpha,\beta} = \alpha + \beta + 3$. Hence as in Proposition 1 it suffices to prove the following inequality

$$\|u\|_{L^{\frac{(\alpha+\beta+3)p}{\alpha+\beta+3-p}}(\Omega)} \leqslant C\|u\|_{S^p_{1,0}(\Omega)} \quad \text{for any function} \quad u \in C^1_0(\Omega).$$

Extend the function $u$ to be equal to 0 outside $\Omega$. For $m, n = 0, 1, \ldots$ we denote

$$A^{(1)}_{m,n} = \left\{(x,y,z) \in \Pi^3_{(x,y,z)} : \frac{1}{2^{m+1}} \leqslant x \leqslant \frac{1}{2^m}, \frac{1}{2^{n+1}} \leqslant y \leqslant \frac{1}{2^n}\right\}$$
$$A^{(2)}_{m,n} = \left\{(x,y,z) \in \Pi^3_{(x,y,z)} : -\frac{1}{2^m} \leqslant x \leqslant -\frac{1}{2^{m+1}}, \frac{1}{2^{n+1}} \leqslant y \leqslant \frac{1}{2^n}\right\}$$
$$A^{(3)}_{m,n} = \left\{(x,y,z) \in \Pi^3_{(x,y,z)} : \frac{1}{2^{m+1}} \leqslant x \leqslant \frac{1}{2^m}, -\frac{1}{2^n} \leqslant y \leqslant -\frac{1}{2^{n+1}}\right\}$$
$$A^{(4)}_{m,n} = \left\{(x,y,z) \in \Pi^3_{(x,y,z)} : -\frac{1}{2^m} \leqslant x \leqslant -\frac{1}{2^{m+1}}, -\frac{1}{2^n} \leqslant y \leqslant -\frac{1}{2^{n+1}}\right\}$$

We recall a simple fact that for arbitrary finite numbers $a < b$ and any function $v \in C^1[a,b]$ the following inequality holds

$$|v(x)| \leqslant \frac{1}{b-a}\int_a^b |v(x)|\mathrm{d}x + \int_a^b |v'(x)|\mathrm{d}x$$

for every $x \in [a,b]$. Applying this inequality we have

$$\begin{aligned} |u\,(x,y,z)\,| &\leqslant 2^{m+1}\int_{\frac{1}{2^{m+1}}}^{\frac{1}{2^m}} |u(x,y,z)|\mathrm{d}x + \int_{\frac{1}{2^{m+1}}}^{\frac{1}{2^m}} \left|\frac{\partial u}{\partial x}(x,y,z)\right|\mathrm{d}x \\ &\leqslant \int_{\frac{1}{2^{m+1}}}^{\frac{1}{2^m}} \frac{2|u(x,y,z)|}{|x|}\mathrm{d}x + \int_{\frac{1}{2^{m+1}}}^{\frac{1}{2^m}} \left|\frac{\partial u}{\partial x}(x,y,z)\right|\mathrm{d}x \\ &:= I^{(1)}_{1,m,n}(y,z) + I^{(1)}_{2,m,n}(y,z) \end{aligned}$$

for any $(x,y,z) \in A^{(1)}_{m,n}$. Set

$$\frac{1}{p_1} = \frac{1+\alpha}{2+\alpha+\beta}, \frac{1}{p_2} = \frac{1+\beta}{2+\alpha+\beta}, \frac{1}{p_3} = \frac{1}{2+\alpha+\beta}, p_4 = 1 + \frac{1}{p_3}.$$

We have

$$|u\,(x,y,z)\,|^{\frac{1}{p_1}} \leqslant C\left\{[I^{(1)}_{1,m,n}(y,z)]^{\frac{1}{p_1}} + [I^{(1)}_{2,m,n}(y,z)]^{\frac{1}{p_1}}\right\}$$

for any $(x,y,z) \in A^{(1)}_{m,n}$. The constant $C$ does not depend on $m, n, u, x, y, z$ but may depend on $\alpha, \beta$. In a similar way we obtain

$$|u\,(x,y,z)\,|^{\frac{1}{p_1}} \leqslant C\left\{[I^{(i)}_{1,m,n}(y,z)]^{\frac{1}{p_1}} + [I^{(i)}_{2,m,n}(y,z)]^{\frac{1}{p_1}}\right\}$$

for any $(x,y,z) \in A^{(i)}_{m,n}, i = 2, 3, 4$. Analogously

$$|u\,(x,y,z)\,|^{\frac{1}{p_2}} \leqslant C\left\{[J^{(i)}_{1,m,n}(x,z)]^{\frac{1}{p_2}} + [J^{(i)}_{2,m,n}(x,z)]^{\frac{1}{p_2}}\right\}$$

for any $(x,y,z) \in A^{(i)}_{m,n}, i = 1, 2, 3, 4$. The matter concerning the derivative in the $z$ variable is simpler since

$$u(x,y,z) = \int_{-1}^{z} \frac{\partial u(x,y,z)}{\partial z}\mathrm{d}z$$

for any $(x, y, z) \in A^{(i)}_{m,n}, i = 1, 2, 3, 4$. Hence

$$|u\,(x, y, z)\,|^{\frac{1}{p_3}} \leqslant \left(\int_{-1}^{1}\left|\frac{\partial u}{\partial z}(x, y, z)\right| \mathrm{d}z\right)^{\frac{1}{p_3}} := [H(x, y)]^{\frac{1}{p_3}}$$

for any $(x, y, z) \in A^{(i)}_{m,n}, i = 1, 2, 3, 4$. Multiplying all the above inequalities and then integrating over the cube $A^{(i)}_{m,n}$, with the help of Lemma 3, we obtain

$$\begin{aligned}\int_{A^{(i)}_{m,n}} |u|^{p_4}\mathrm{d}x\mathrm{d}y\mathrm{d}z &\leqslant C\int_{A^{(i)}_{m,n}} \left\{[I^{(i)}_{1,m,n}(y, z)]^{\frac{1}{p_1}} + [I^{(i)}_{2,m,n}(y, z)]^{\frac{1}{p_1}}\right\}\\ &\quad\times\left\{[J^{(i)}_{1,m,n}(x, z)]^{\frac{1}{p_2}} + [J^{(i)}_{2,m,n}(x, z)]^{\frac{1}{p_2}}\right\}[H(x, y)]^{\frac{1}{p_3}}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\leqslant C(\ln 2)^{1-\frac{2}{p_3}}\sum_{k,l=1}^{2}\left[\tilde{I}^{(i)}_{k,m,n}\right]^{\frac{1}{p_1}}\left[\tilde{J}^{(i)}_{l,m,n}\right]^{\frac{1}{p_2}}\left[\tilde{H}^{(i)}_{m,n}\right]^{\frac{1}{p_3}},\end{aligned}$$

where

$$\begin{aligned}\tilde{I}^{(i)}_{k,m,n} &= \int_{(A^{(i)}_{m,n})_x} I^{(i)}_{k,m,n}(y, z)\mathrm{d}y\mathrm{d}z, \tilde{J}^{(i)}_{l,m,n} = \int_{(A^{(i)}_{m,n})_y} J^{(i)}_{l,m,n}(x, z)\mathrm{d}x\mathrm{d}z, \quad k, l = 1, 2\\ \tilde{H}^{(i)}_{m,n} &= \int_{(A^{(i)}_{m,n})_z} H(x, y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Therefore

$$\begin{aligned}\|u\|_{L^{\frac{3+\alpha+\beta}{2+\alpha+\beta}}} &= \left(\int_{\Omega} |u|^{p_4}\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{\frac{1}{p_4}} = \left(\sum_{i=1}^{4}\sum_{m,n=0}^{+\infty}\int_{A^{(i)}_{m,n}} |u|^{p_4}\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{\frac{1}{p_4}}\\ &\leqslant C(\ln 2)^{\frac{-2+p_3}{1+p_3}}\left(\sum_{i=1}^{4}\sum_{m,n=0}^{+\infty}\sum_{k,l=1}^{2}\left[\tilde{I}^{(i)}_{k,m,n}\right]^{\frac{1}{p_1}}\left[\tilde{J}^{(i)}_{l,m,n}\right]^{\frac{1}{p_2}}\left[\tilde{H}^{(i)}_{m,n}\right]^{\frac{1}{p_3}}\right)^{\frac{1}{p_4}}\\ &\leqslant C\sum_{i=1}^{4}\sum_{m,n=0}^{+\infty}\sum_{k,l=1}^{2}\left[\tilde{I}^{(i)}_{k,m,n}\right]^{\frac{1}{p_1p_4}}\left[\tilde{J}^{(i)}_{l,m,n}\right]^{\frac{1}{p_2p_4}}\left[\tilde{H}^{(i)}_{m,n}\right]^{\frac{1}{p_3p_4}}\\ &\leqslant C\sum_{i=1}^{4}\sum_{m,n=0}^{+\infty}\sum_{k,l=1}^{2}\left(\tilde{I}^{(i)}_{k,m,n} + \tilde{J}^{(i)}_{l,m,n} + \tilde{H}^{(i)}_{m,n}\right)\\ &\leqslant C\left(\left\|\frac{u(x, u, z)}{x}\right\|_{L^1(\Omega)} + \left\|\frac{u(x, u, z)}{y}\right\|_{L^1(\Omega)} + \|u(x, u, z)\|_{S^1_{1,0}(\Omega)}\right).\end{aligned} \tag{14}$$

Next put $\gamma = \frac{(2+\alpha+\beta)p}{3+\alpha+\beta-p}$ into (14) and by using the Holder and Hardy inequalities we get

$$\||u|^\gamma\|_{L^{\frac{3+\alpha+\beta}{2+\alpha+\beta}}(\Omega)} \leqslant C\left(\left\|\frac{|u|^\gamma}{x}\right\|_{L^1(\Omega)} + \left\|\frac{|u|^\gamma}{y}\right\|_{L^1(\Omega)} + \left\||u|^{\gamma-1}\frac{\partial u}{\partial x}\right\|_{L^1(\Omega)}\right.$$
$$\left. + \left\||u|^{\gamma-1}\frac{\partial u}{\partial y}\right\|_{L^1(\Omega)} + \left\||x|^\alpha|y|^\beta|u|^{\gamma-1}\frac{\partial u}{\partial z}\right\|_{L^1(\Omega)}\right) \leqslant C\,\||u|^{\gamma-1}\|_{L^{p'}(\Omega)}$$
$$\times\left(\left\|\frac{\partial u}{\partial x}\right\|_{L^p(\Omega)} + \left\|\frac{\partial u}{\partial y}\right\|_{L^p(\Omega)} + \left\||x|^\alpha|y|^\beta\frac{\partial u}{\partial z}\right\|_{L^p(\Omega)}\right), \tag{15}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. By noting that $\frac{\gamma(3+\alpha+\beta)}{2+\alpha+\beta}=(\gamma-1)p'=\frac{(\alpha+\beta+3)p}{\alpha+\beta+3-p}$ we can reduce similar terms in (15) and obtain the desired inequality. The proof of Proposition 4 is therefore complete. □

## 4. Existence results

From now on we suppose that $g(x,y,z,t)$ has only polynomial growth in $t$.

**Definition 2.** A function $u \in S^2_{1,0}(\Omega)$ is called a weak solution of the problem (1)–(2) if the identity

$$\int_\Omega \left[(\nabla_x u, \nabla_x \varphi) + (\nabla_y u, \nabla_y \varphi) + |x|^{2\alpha}|y|^{2\beta}(\nabla_z u, \nabla_z \varphi)\right] \mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$+ \int_\Omega [c(x,y,z)u\varphi - g(x,y,z,u)\varphi]\, \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

holds for every $\varphi \in C_0^\infty(\Omega)$.

We try to find weak solutions of the problem (1)–(2) as critical points of a nonlinear functional. To this end we define the functional $\Phi$ on the space $S^2_{1,0}(\Omega)$ as follows

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_\Omega \left(|\nabla_x u|^2 + |\nabla_y u|^2 + \left||x|^\alpha|y|^\beta\nabla_z u\right|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\quad + \frac{1}{2}\int_\Omega c(x,y,z)u^2\mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_\Omega G(x,y,z,u)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &:= \Phi_1(u) + \Phi_2(u) + \Phi_3(u).\end{aligned} \tag{16}$$

Without difficulty we can verify that $\Phi_1, \Phi_2$ are continuous on $S^2_{1,0}(\Omega)$ and for $u \in S^2_{1,0}(\Omega)$ we have

$$\Phi_1'(u)(v) = \int_\Omega \left((\nabla_x u, \nabla_x v) + (\nabla_y u, \nabla_y v) + |x|^{2\alpha}|y|^{2\beta}(\nabla_z u, \nabla_z v)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$
$$\Phi_2'(u)(v) = \int_\Omega c(x,y,z)uv\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

for any $v \in S^2_{1,0}(\Omega)$. By using Lemma 3 it's not difficult to obtain for $\Phi_3$ the following

**Lemma 4.** *Let* $1 < p < \infty$. *If* $g(x,y,z,t) \in C(\bar{\Omega}\times\mathbb{R})$ *satisfies the conditions*

$$|g(x,y,z,t)| \leqslant a(x,y,z) + b|t|^{p-1}, \quad \textit{where} \quad a(x,y,z) \in L^{\frac{p}{p-1}}(\Omega), \quad b \in \mathbb{R}_+,$$

*then*

$$\Phi_3 : u(x,y,z) \longrightarrow \int_\Omega \int_0^{u(x,y,z)} g(x,y,z,\xi)\mathrm{d}\xi\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

*is a (nonlinear) continuously differentiable functional on $L^p(\Omega)$. Moreover, for each fixed element $u \in L^p(\Omega)$*

$$\Phi_3'(u)(v) = \int_\Omega g(x,y,z,u(x,y,z))v(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

*for any element $v \in L^p(\Omega)$.*

**Corollary 1.** *Let $c(x,y,z) \in C(\bar{\Omega})$ and $g(x,y,z,t) \in C(\bar{\Omega} \times \mathbb{R})$ satisfy the conditions*

$$|g(x,y,z,t)| \leqslant a(x,y,z) + b|t|^p,$$

*where*

$$a(x,y,z) \in L^{\frac{p}{p-1}}(\Omega), 1 < p < \frac{N_{\alpha,\beta}+2}{N_{\alpha,\beta}-2},\ b \in \mathbb{R}_+.$$

*The function $u$ is a weak solution of the problem (1)–(2) if and only if $u$ is a critical point of the functional $\Phi$ defined by the formula (16).*

Now we impose the following hypotheses on $g(x,y,z,t)$

$(g)_1$ $g(x,y,z,t) \in C(\bar{\Omega} \times \mathbb{R})$ and $g(x,y,z,0) = 0$.

$(g)_2$ For some $1 < p < \frac{N_{\alpha,\beta}+2}{N_{\alpha,\beta}-2}$ and some positive constant $C$

$$|g(x,y,z,t)| \leqslant C\left(1 + |t|^p\right).$$

$(g)_3$ $g(x,y,z,t) = \bar{\bar{o}}(t)$ for $t \longrightarrow 0$ uniformly in $(x,y,z) \in \bar{\Omega}$.

$(g)_4$ $\lim_{t\to+\infty} \frac{g(x,y,z,t)}{t} = \infty$ or $lim_{t\to-\infty} \frac{g(x,y,z,t)}{t} = \infty$ uniformly in $(x,y,z) \in \bar{\Omega}$.

$(g)_5$ If $|t| \geq A$ for some number $A$, then

$$G(x,y,z,t) \leqslant \mu g(x,y,z,t)t$$

where $\mu \in [0, \frac{1}{2})$.

$(g)_6$ $g(x,y,z,t)$ is an odd function in $t$.

In the following lemma all the conditions $(I)_1 - (I)_5$ are taken from [2].

**Lemma 5.**
- *If $g(x,y,z,t)$ satisfies $(g)_2$ and $(g)_3$, then $\Phi$ satisfies $(I)_1$.*
- *If $g(x,y,z,t)$ satisfies $(g)_4$, then $\Phi$ satisfies $(I)_2$.*
- *If $g(x,y,z,t)$ satisfies $(g)_4$ and $(g)_6$, then $\Phi$ satisfies $(I)_5$.*
- *If $g(x,y,z,t)$ satisfies $(g)_2$ and $(g)_5$, then $\Phi$ satisfies $(I)_3$.*

Next let us introduce a set of additional notations:

$B_r = \{u \in S^2_{1,0}(\Omega) : \|u\|_{S^2_{1,0}(\Omega)} < r\}, S_r = \partial B_r, A_c = \{u \in S^2_{1,0}(\Omega) : \Phi(u) \leqslant c\},$

$\hat{A}_c = \{u \in S^2_{1,0}(\Omega) : \Phi(u) \geq c\}, K_c = \{u \in S^2_{1,0}(\Omega) : \Phi(u) = c, \Phi'(u) = 0\}.$

Assume that there is a nonzero element $e \in S^2_{1,0}(\Omega)$ such that $\Phi(e) = 0$. Let

$$\begin{aligned} \Gamma &= \{h \in C([0,1], S^2_{1,0}(\Omega)) : h(0) = 0, h(1) = e\} \\ \Gamma_* &= \{h \in C(S^2_{1,0}(\Omega), S^2_{1,0}(\Omega)) : h(0) = 0,\ h\text{-is a homeomorphism} \\ &\qquad \text{of } S^2_{1,0}(\Omega) \text{ onto } S^2_{1,0}(\Omega), \text{ and } h(B_1) \subset \hat{A}_0\} \\ \Gamma^e_* &= \{h \in \Gamma_* : h(S_1) \quad \text{separates } 0 \text{ and } \quad e\}. \end{aligned}$$

Define

$$b = \inf_{h\in\Gamma} \max_{t\in[0,1]} \Phi(h(t)), c = \sup_{h\in\Gamma^e_*} \inf_{u\in S_1} \Phi(h(u)). \tag{17}$$

**Theorem 4.** *Suppose that $g(x, y, z, t)$ satisfies the conditions $(g)_1 - (g)_5$. Then the values $b$ and $c$, defined by the formula (17) are critical values of the functional $\Phi$ in (16). The boundary value problem (1)–(2) has a nontrivial weak solution.*

*Proof.* Theorem 4 is a direct consequence of Lemma 5 and the theory of critical values of nonlinear functionals in [2]. □

If $g(x, y, z, t)$ is an odd function in $t$ then something more can be said about the number of nontrivial solutions of the problem (1)–(2). Let us introduce some further notations. Let $\Gamma^* = \{h \in C(S^1_0(\Omega), S^1_0(\Omega)) : h(0) = 0, h\text{-is}$ an odd homeomorphism of $S^1_0(\Omega)$ onto $S^1_0(\Omega)$; $\Phi(h(u)) \geq 0$ for $\|u\|_{S^1_0(\Omega)} \leqslant 1\}$ and $\Gamma_m = \{K \subset S^1_0(\Omega) : K$ is compact, symmetric with respect to the origin and $\kappa(K \cup h(S_1)) \geq m,\ \forall h \in \Gamma^*\}$. Here $\kappa(.)$ is the genus of a set and $S_1$ is the unit sphere in $S^1_0(\Omega)$. For each $m \in \mathbb{Z}_+$ let us denote by $E_m$ any $m$-dimensional subspace of $S^1_0(\Omega)$ and $E^\perp_m$ any its algebraic and topological complement, and define

$$b_m = \inf_{K\in\Gamma_m} \max_{u\in K} \Phi(u), c_m = \sup_{h\in\Gamma^*} \inf_{u\in S_1\cap E^\perp_{m-1}} \Phi(h(u)). \tag{18}$$

With the help of Lemma 5 and the results in [2] it is possible to establish the following theorem. We will omit the details of the proof of the theorem.

**Theorem 5.** *Suppose that $g(x, y, z, t)$ satisfies the conditions $(g)_1 - (g)_6$. Then for each $m \in \mathbb{Z}_+$, the values $b_m, c_m$, defined by the formula (18), are critical values of the functional $\Phi$ with $0 < c_m \leqslant b_m < \infty$. If $b_{m+1} = \cdots = b_{m+r} = b$, then $\kappa(u \in S^1_0(\Omega) : \Phi(u) = b, \Phi'(u) = 0) \geq r$. Therefore, the problem (1)–(2) has infinitely many nontrivial weak solutions.*

Next we deal with the effect of adding a linear term to the right hand side of the equation (1). More precisely, consider the boundary value problem

$$P_{\alpha,\beta}u + d(x, y, z)u + g(x, y, z, u) = 0 \quad \text{in } \Omega, \tag{19}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{20}$$

where $d$ is a nonnegative function. Recall that the eigenvalue problem

$$\begin{aligned} &P_{\alpha,\beta}v(x, y, z) + \lambda d(x, y, z)v(x, y, z) = 0 \quad \text{in } \Omega, \\ &v = 0 \quad \text{on } \partial\Omega \end{aligned}$$

possesses a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$, $\lim_{m\to\infty} \lambda_m = \infty$. Each eigenvalue $\lambda_m$ has finite simplicity and $\lambda_1$ is simple. Let $e_m$ be an eigenfunction corresponding to the eigenvalue $\lambda_m$. Put

$$\Psi(u) = \Phi_1(u) + \Phi_3(u) - \frac{1}{2}\int_\Omega d(x,y,z)u^2(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

It is easy to see that $\Phi(u) \geq \Psi(u)$. The following lemma gives us results for $\Psi$ similar to that of $\Phi$ in Lemma 5. As in Lemma 5 all the conditions $(I)_2 - (I)_6$ are taken from [2].

**Lemma 6.**
- *If $g(x,y,z,t)$ satisfies $(g)_4$, then $\Psi$ satisfies $(I)_2$.*
- *If $g(x,y,z,t)$ satisfies $(g)_2, (g)_4 - (g)_6$, then $\Psi$ satisfies $(I)_3$.*
- *If $g(x,y,z,t)$ satisfies $(g)_4, (g)_6$, then $\Psi$ satisfies $(I)_5$.*
- *If $1 < \lambda_{l+1} \neq \lambda_l$ and $g(x,y,z,t)$ satisfies $(g)_1 - (g)_3$, then $\Psi$ satisfies $(I)_6$ with $E^l = \{e_1, \ldots, e_l\}$.*

With the help of Lemma 6 it is not difficult to obtain

**Theorem 6.** *If $g(x,y,z,t)$ satisfies the conditions $(g)_1$–$(g)_6$ and $\lambda_l \leqslant 1 < \lambda_{l+1}$, then for all $m > l$ the values $b_m, c_m$, defined by the formula (18) (of course, with $\Phi$ replaced by $\Psi$), are critical values of the functional $\Psi$ with $0 < c_m \leqslant b_m$. If $b_{m+1} = \cdots = b_{m+r} = b$, then $\gamma(K_b) \geq r$. The problem (19)–(20) has infinitely many nontrivial solutions.*

## 5. Illustrating example

In this final section we consider an example of the problem (1)–(2) in which $n_1 = n_2 = n_3 = 1, \Omega = B_1(a_1, b_1, 0), g(x,y,z,t) = |t|^\gamma t$ where $(a_1, b_1) \in \mathbb{R}^2_+, 0 < \gamma$. We show that the existence of nontrivial solutions of the problem (1)–(2) heavily depends on $a_1, b_1, \gamma$. In this case

$$\nu^{\alpha,\beta}_{a,b,c} = (x-a)\nu_1 + (y-b)\nu_2 + (\alpha+\beta+1)(z-c)\nu_3.$$

First we suppose that $b_1 = 0$. On the sphere $S_1(a_1, 0, 0)$

$$\begin{aligned}\nu^{\alpha,\beta}_{0,0,0} &= \left(x - \frac{a_1(2\alpha+2\beta+1)}{2(\alpha+\beta)}\right)^2 + y^2 - \frac{a_1^2 + 4(\alpha+\beta)(\alpha+\beta+1)}{4(\alpha+\beta)^2}\\ &:= \left(x - \frac{a_1(2\alpha+2\beta+1)}{2(\alpha+\beta)}\right)^2 + y^2 - R^2.\end{aligned}$$

If $0 \leqslant a_1 < 1$ (or the ball $B_1(a_1,0,0)$ contains the origin), then $\nu^{\alpha,\beta}_{0,0,0} > 0$ on $S_1(a_1,0,0)$. Hence $B_1(a_1,0,0)$ is $P_{\alpha,\beta}$-star-shape. The problem (1)–(2) has no nontrivial solution if $\gamma > \frac{4}{\alpha+\beta+1}$. If $a_1 = 1$ (or the origin is on the boundary of $B_1(a_1,0,0)$) then $\nu^{\alpha,\beta}_{0,0,0} > 0$ on $S_1(a_1,0,0)\backslash\{0\}$. Hence $B_1(a_1,0,0)$ is still $P_{\alpha,\beta}$-star-shape. The problem (1)–(2) still has no nontrivial solution if $\gamma > \frac{4}{\alpha+\beta+1}$. If $a_1 > 1$ (the origin is outside of the ball $B_1(a_1,0,0)$), then $\nu^{\alpha,\beta}_{0,0,0}$ is positive on a part of $S_1(a_1,0,0)$ and negative on the other part of $S_1(a_1,0,0)$. Hence $B_1(a_1,0,0)$ is not $P_{\alpha,\beta}$-star-shape. The problem (1)–(2) may have a nontrivial solution. In this case the equation (1) is degenerate only in the $y$-variable since the ball does not intersect with the hyperplane $x = 0$. According to [11] if $\gamma < \frac{4}{\beta+1}$ the problem has a nontrivial solution, or else if $\gamma \geq \frac{4}{\beta+1}$ the problem has no nontrivial solution.

Now suppose that $a_1 = 2$. If $b_1 \leqslant 1$ the ball still intersects with the hyperplane $y = 0$. Hence the problem has no nontrivial solutions if $\gamma < \frac{4}{\beta+1}$. But when $b_1 > 1$ the operator is elliptic in $B_1(2, b_1, 0)$. According to [1] - [2], if $\gamma < 4$ the problem has a nontrivial solution, or else if $\gamma \geq 4$ the problem has no nontrivial solution.

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