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Infinitely many solutions for elliptic problems with variable exponent and nonlinear boundary conditions

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Abstract. We analyze a class of quasilinear elliptic problems involving a $p(\cdot)$-Laplace-type operator on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and we deal with nonlinear conditions on the boundary. Working on the variable exponent Lebesgue–Sobolev spaces, we follow the steps described by the "fountain theorem" and we establish the existence of a sequence of weak solutions.

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1. Introduction and preliminaries

We consider the problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) + b(x)|u|^{p(x)-2}u = f(x,u) & \text{for } x \in \Omega, \\ a(x,\nabla u)\cdot \nu(x) = g(x,u) & \text{for } x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a bounded domain with smooth boundary. We assume that $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, $a = a(x,\eta)$, is a Carathéodory function such that it is the continuous derivative with respect to η of a function $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\eta)$. More exactly, $a(x,\eta) = \nabla_\eta A(x,\eta)$. The mappings a and A verify the following.

(A1) The equality

$$A(x,0) = 0$$

holds for all $x \in \overline{\Omega}$.

(A2) There exists a constant $c_0 > 0$ such that

$$|a(x,\eta)| \leq c_0(1 + |\eta|^{p(x)-1}),$$

for all $x \in \overline{\Omega}$ and all $\eta \in \mathbb{R}^N$.

(A3) The inequality

$$0 \leq [a(x, \eta_1) - a(x, \eta_2)] \cdot (\eta_1 - \eta_2)$$

holds for all $x \in \overline{\Omega}$ and all $\eta_1, \eta_2 \in \mathbb{R}^N$, with equality if and only if $\eta_1 = \eta_2$.

(A4) The inequalities

$$|\eta|^{p(x)} \leq a(x, \eta) \cdot \eta \leq p(x)\ A(x, \eta)$$

hold for all $x \in \overline{\Omega}$ and all $\eta \in \mathbb{R}^N$.

(A5) The mapping A is even with respect to its second variable, that is,

$$A(x, -\eta) = A(x, \eta)$$

for all $x \in \overline{\Omega}$ and all $\eta \in \mathbb{R}^N$.

The above type of assumptions can be found in other papers too. For example, in [16] and [17] we meet the assumptions (A1)–(A4). In [16] the approach relies on the sub and supersolution method, so the author extends the sub and supersolutions concept to the Sobolev space with variable exponents. He establishes the existence of the solution between a number of sub and supersolutions and presents some of its properties. Although the author does not discuss a problem similar to (1), he treats a variational inequality that can be reduced to a variational equality and can be linked to a problem with Neumann or Robin boundary conditions. In [17] the authors establish the existence of a solution for an elliptic problem with Dirichlet boundary conditions. Their strategy relies on a variant of the mountain pass theorem of Ambrosetti and Rabinowitz combined with adequate variational methods. Our strategy is to apply a theorem related to the mountain pass theorem, namely the fountain theorem, which will be reminded in Sect. 2. This theorem requires the symmetry condition (A5) and allows us to obtain a sequence of solutions.

We note that, in [16] and [17], as well as in our paper, the study is conducted in the framework of the variable exponent Lebesgue–Sobolev spaces. However, nonstandard operators closely related to the ones described with the aid of (A1)–(A5) can appear in different situations, such is the case of classical Lebesgue–Sobolev spaces (see [14]), or the case of anisotropic Lebesgue–Sobolev spaces with variable exponent (see [3]).

There is a good reason for working under hypotheses like (A1)–(A5), here and elsewhere. This set of hypotheses is only natural when we think at the operators that can be obtained from it by making the suitable choices. Indeed, for $A(x, \eta) = \frac{1}{p(x)}|\eta|^{p(x)}$ we deduce that $a(x, \eta) = |\eta|^{p(x)-2}\eta$ and for $\eta = \nabla u$ we find the $p(\cdot)$-Laplace operator

$$\operatorname{div}(a(x, \nabla u)) = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right).$$

A second well-known example of operator arises when we choose $A(x, \eta) = \frac{1}{p(x)}[(1 + |\eta|^2)^{p(x)/2} - 1]$, thus $a(x, \eta) = (1 + |\eta|^2)^{(p(x)-2)/2}\eta$ and for $\eta = \nabla u$ we find the generalized mean curvature operator

$$\operatorname{div}(a(x,\nabla u)) = \operatorname{div}\left((1+|\nabla u|^2)^{(p(x)-2)/2}\nabla u\right).$$

We notice that in [17] it is considered a convexity-type condition in addition to (A1)–(A4). This condition permits to consider the above presented examples of operators only for the case when $p(x) \geq 2$. We, on the other hand, can manage without it, thus in our examples $p > 1$.

Another remark is that the first example enables us to connect (1) to the problem presented in [29],

$$\begin{cases} -\Delta_p u + b(x)|u|^{p-2}u = f(x,u) & \text{for } x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u) & \text{for } x \in \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. In (2) the exponent p is constant, hence they are dealing with the classical p-Laplace operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

A similar problem is considered also in [27], where p is not a constant, but a continuous function. More exactly, they consider the problem

$$\begin{cases} -\Delta_{p(x)} u + |u|^{p(x)-2}u = \lambda_1 f(x,u) & \text{for } x \in \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda_2 g(x,u) & \text{for } x \in \partial\Omega. \end{cases} \tag{3}$$

The $p(\cdot)$-Laplace operator $\Delta_{p(\cdot)}$ is an extension of the p-Laplace operator Δ_p and in the present paper we generalize it to the even more general operator described by (A1)–(A5). Thus we are entitled to say that our problem extends (2) and (3), since the constants λ_1, λ_2 are positive and the functions b, f, g fulfill conditions resembling to those used in our paper and described in what follows.

For the function $b : \Omega \to \mathbb{R}$ we assume that

(B) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.

In what concerns the rest of the functions involved in problem (1), in order to present the general hypotheses more easily, we introduce some notation. We set

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) \,:\, 1 < \min_{x\in\overline{\Omega}} h(x) < \max_{x\in\overline{\Omega}} h(x) < \infty \right\}$$

and for all $h \in C_+(\overline{\Omega})$ we denote

$$h^+ = \sup_{x\in\Omega} h(x), \qquad h^- = \inf_{x\in\Omega} h(x).$$

In addition, we denote

$$h^\star(x) = \begin{cases} Nh(x)/[N-h(x)] & \text{if } h(x) < N, \\ \infty & \text{if } h(x) \geq N, \end{cases}$$

and

$$h^\partial(x) = \begin{cases} (N-1)h(x)/[N-h(x)] & \text{if } h(x) < N, \\ \infty & \text{if } h(x) \geq N. \end{cases}$$

We consider $p \in C_+(\overline{\Omega})$ and we suppose that f, g are Carathéodory functions verifying the next conditions.

(F1) For $q \in C_+(\overline{\Omega})$ with $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, there exists $c_1 > 0$ such that

$$|f(x,t)| \leq c_1 |t|^{q(x)-1}$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$.

(F2) There exists $\alpha_1 > p^+$ such that

$$0 < \alpha_1 F(x,t) \leq t f(x,t)$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$, where

$$F(x,t) = \int_0^t f(x,s)ds.$$

(F3) The function f is odd with respect to its second variable, that is,

$$f(x,-t) = -f(x,t)$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$.

(G1) For $r \in C_+(\overline{\Omega})$ with $r(x) < p^\partial(x)$ for all $x \in \partial\Omega$, there exists $c_2 > 0$ such that

$$|g(x,t)| \leq c_2 |t|^{r(x)-1}$$

for all $x \in \partial\Omega$ and all $t \in \mathbb{R}$.

(G2) There exists $\alpha_2 > p^+$ such that

$$0 < \alpha_2 G(x,t) \leq t g(x,t)$$

for all $x \in \partial\Omega$ and all $t \in \mathbb{R}$, where

$$G(x,t) = \int_0^t g(x,s)ds.$$

(G3) The function g is odd with respect to its second variable, that is,

$$g(x,-t) = -g(x,t)$$

for all $x \in \partial\Omega$ and all $t \in \mathbb{R}$.

We point out that by our main theorem we will establish the existence of a sequence of weak solutions for (1) without adding further conditions on p, q and r. On the other hand, the corresponding result from [27] is provided under the additional assumptions that q^-, $r^- > p^+$.

As previously said, the nonlinear elliptic problem (1) will be studied in the framework of the variable exponent Lebesgue–Sobolev spaces which are briefly described in Sect. 2. The interest for such problems is based on the multiple possibilities to apply them. There are applications in elastic mechanics [30], in the mathematical modeling of non-Newtonian fluids [6,12,18–23,26] and in image restoration [5]. To be more specific, we recall that some of the first applications are concerning the electrorheological fluids, shortly, the ER fluids. These fluids are suspensions of extremely fine non-conducting particles (up to 50 μm diameter) in an electrically insulating fluid. The ER fluids have a special property: when disposed to an electromagnetic field, their apparent viscosity undergoes a significant change which is reversible. For example, a typical ER fluid can go from the consistency of a liquid to that of a gel, and back, with response times on the order of milliseconds. After he made this discovery, Winslow obtained an US patent [25] on the effect who now carries his name and wrote an article published in 1949 [26]. For more details it is enough to perform a simple search on the free encyclopedia site, Wikipedia. This is a rapid way to learn that the electrorheological fluids have applications in fast acting hydraulic valves and clutches [22], in ER brakes [21] and shock absorbers [23]. Recently, there are appearing novel uses, including use in the US army's planned future force warrior project or use in accurate abrasive polishing and tactile displays [15]. ER fluids have also been proposed to have potential applications in flexible electronics and Motorola filed a patent application for mobile device applications in 2006. Further information on properties, modelling and the applications of variable exponent spaces to these fluids can be found in [1,2,4,6,8,10,12,18–20].

These are all very interesting matters that sustain our determination to investigate problems with $p(\cdot)$-growth conditions but, as usually, there is a downside. Indeed, if we take the mathematical point of view, when dealing with variable exponents it is obvious that we are in fact dealing with more mathematical difficulties. Moreover, when we pass from the classical p-Laplace operator to the variable $p(\cdot)$-Laplace operator we get into more trouble, since the homogeneity is lost. However, in our case, we are manipulating even more general operators, as we saw above. And, as this would not be difficult enough, we also have nonlinear conditions on the boundary. Therefore, in order to overcome all this, we need some good "tools", some solid auxiliary results. We provide them in the next section.

2. Notation and auxiliary results

For the bounded domain with smooth boundary from the introductory section, $\Omega \subset \mathbb{R}^N$, $N \geq 2$, we denote by $\mathcal{M}$ one of the following two sets: Ω or $\partial\Omega$. We define the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\mathcal{M}) = \left\{ u : u \text{ is a measurable real–valued function such that} \int_{\mathcal{M}} |u|^{p(x)} < \infty \right\}$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\mathcal{M})} = \inf \left\{ \mu > 0 \ : \quad \int_{\mathcal{M}} \left| \frac{u(x)}{\mu} \right|^{p(x)} \leq 1 \right\}.$$

The space $\left(L^{p(\cdot)}(\mathcal{M}), \|\cdot\|_{L^{p(\cdot)}(\mathcal{M})}\right)$ has many qualities. It is a separable and reflexive Banach space ([13, Theorem 2.5, Corollary 2.7]) and the inclusion between spaces generalizes naturally: if $0 < |\mathcal{M}| < \infty$ and $p_1, p_2 \in C_+(\overline{\mathcal{M}})$ are such that $p_1 \leq p_2$ in $\mathcal{M}$, then the embedding $L^{p_2(\cdot)}(\mathcal{M}) \hookrightarrow L^{p_1(\cdot)}(\mathcal{M})$ is continuous ([13, Theorem 2.8]). Furthermore, for all $u \in L^{p(\cdot)}(\mathcal{M})$ and $v \in L^{p'(\cdot)}(\mathcal{M})$

we have the following Hölder-type inequality (see [13, Theorem 2.1])

$$\left| \int_{\mathcal{M}} uv\, dx \right| \leq 2\, \|u\|_{L^{p(\cdot)}(\mathcal{M})} \|v\|_{L^{p'(\cdot)}(\mathcal{M})} \tag{4}$$

where we denoted by $L^{p'(\cdot)}(\mathcal{M})$ the conjugate space of $L^{p(\cdot)}(\mathcal{M})$, obtained by conjugating the exponent pointwise, that is, $1/p(x) + 1/p'(x) = 1$ ([13, Corollary 2.7]).

We denote by $\rho_{\mathcal{M},\, p(\cdot)} : L^{p(\cdot)}(\mathcal{M}) \to \mathbb{R}$ the mapping defined by

$$\rho_{\mathcal{M},\, p(\cdot)}(u) = \int_{\mathcal{M}} |u|^{p(x)}\ ,$$

which is called the $p(\cdot)$-modular of the $L^{p(\cdot)}(\mathcal{M})$ space. This is very helpful when working on the generalized Lebesgue spaces. We present some of its properties ([11, Theorem 1.3, Theorem 1.4]). If $u \in L^{p(\cdot)}(\mathcal{M})$ and $p < \infty$ then,

$$\|u\|_{L^{p(\cdot)}(\mathcal{M})} < 1\ (=1;\ > 1) \quad \Leftrightarrow \quad \rho_{\mathcal{M},\, p(\cdot)}(u) < 1\ (=1;\ > 1)$$

$$\|u\|_{L^{p(\cdot)}(\mathcal{M})} > 1 \quad \Rightarrow \quad \|u\|^{p^-}_{L^{p(\cdot)}(\mathcal{M})} \leq \rho_{\mathcal{M},\, p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\mathcal{M})} \tag{5}$$

$$\|u\|_{L^{p(\cdot)}(\mathcal{M})} < 1 \quad \Rightarrow \quad \|u\|^{p^+}_{L^{p(\cdot)}(\mathcal{M})} \leq \rho_{\mathcal{M},\, p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\mathcal{M})} \tag{6}$$

$$\|u\|_{L^{p(\cdot)}(\mathcal{M})} \to 0\ (\to \infty) \quad \Leftrightarrow \quad \rho_{\mathcal{M},\, p(\cdot)}(u) \to 0\ (\to \infty). \tag{7}$$

If, in addition, $(u_n)_n \subset L^{p(\cdot)}(\mathcal{M})$, then

$$\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\mathcal{M})} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho_{\mathcal{M},\, p(\cdot)}(u_n - u) = 0$$

$$\Leftrightarrow \quad (u_n)_n \text{ converges to } u \text{ in measure and } \lim_{n\to\infty} \rho_{\mathcal{M},\, p(\cdot)}(u_n) = \rho_{\mathcal{M},\, p(\cdot)}(u). \tag{8}$$

For Ω previously introduced we define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$,

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) :\ |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

endowed with the norm

$$\|u\| = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

where by $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ we understand $\|\, |\nabla u| \|_{L^{p(\cdot)}(\Omega)}$. The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space ([13, Theorem 1.3]). Moreover, we have the following embedding result and trace theorem.

Theorem 1. ([8, Proposition 2.4]) *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain with smooth boundary. If $p,\, q \in C(\overline{\Omega})$ satisfy the condition*

$$1 \leq q(x) < p^\star(x), \qquad \forall\, x \in \overline{\Omega},$$

then there is a compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.

Theorem 2. ([7, Corollary 2.4]) *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain with smooth boundary. Suppose that $p \in C_+(\overline{\Omega})$ and $r \in C(\overline{\Omega})$ satisfy the condition*

$$1 \leq r(x) < p^\partial(x), \qquad \forall\, x \in \partial\Omega.$$

Then there is a compact boundary trace embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$.

We refer to [7] for further details concerning the extension of the classical trace to Lebesgue–Sobolev spaces with variable exponent. For simplicity, when we refer to the trace of u we will write u instead of $u|_{\partial\Omega}$ or γu.

The third useful theorem concerns the application a.

Theorem 3. ([16, Theorem 4.1]) *The mapping a is an operator of type (S_+), that is, if $u_n \rightharpoonup u$ (weakly) in $W^{1,p(\cdot)}(\Omega)$ and*

$$\limsup_{n\to\infty} \int_\Omega a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u)\, dx \leq 0,$$

then $u_n \to u$ (strongly) in $W^{1,p(\cdot)}(\Omega)$.

The next theorem that needs to be recalled here is the fountain theorem. But first we have to present the general context. It is known ([28, Sect. 17]) that for a separable and reflexive Banach space there exist $\{e_n\}_{n=1}^\infty \subset X$ and $\{f_n\}_{n=1}^\infty \subset X^\star$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}$$

$$X = \overline{\operatorname{span}}\{e_n : n = 1, 2, \ldots,\} \quad \text{and} \quad X^\star = \overline{\operatorname{span}}\{f_n : n = 1, 2, \ldots,\}.$$

(Note that, since [28] is written in Chinese, the above assertions are based on the affirmations made by Fan and Han in [9], or by Yao in [27].) For $k = 1, 2, \ldots$ we denote

$$X_k = \operatorname{span}\{e_k\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j \quad \text{and} \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{9}$$

For a separable reflexive Banach space X and for X_k, Y_k, Z_k previously defined we give the statement of the fountain theorem (see for example [24]) that we are going to use in Sect. 3.

Theorem 4. *Assume that $I \in C^1(X, \mathbb{R})$ and that for each $k = 1, 2, \ldots,$ there exist $\rho_k > \gamma_k > 0$ such that*

(H1) $\inf_{u \in Z_k,\, \|u\|_X = \gamma_k} I(u) \to \infty$ *as* $k \to \infty$.
(H2) $\max_{u \in Y_k,\, \|u\|_X = \rho_k} I(u) \leq 0$.
(H3) *I satisfies the $(PS)_c$ condition for every $c > 0$ (that is, any sequence $(u_n)_n \subset X$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in $X^\star$ as $n \to \infty$ contains a subsequence converging to a critical point of I).*

Then I has a sequence of critical values tending to $+\infty$.

In the present paper we choose $X = W^{1,p(\cdot)}(\Omega)$ which, as said above, is a separable reflexive Banach space. To apply Theorem 4 we will operate with energetic functionals and rely on the critical point theory. This is why we provide properties of some of the functionals that will be involved in our future calculus. We start with $\Lambda : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\Lambda(u) = \int_\Omega A(x, \nabla u)\, dx.$$

Proposition 1. ([17, Lemma 1]) (i) *The functional* Λ *is well-defined on* $W^{1,p(\cdot)}(\Omega)$.

(ii) *The functional* Λ *is of class* $C^1(W^{1,p(\cdot)}(\Omega), \mathbb{R})$ *and*

$$\langle \Lambda^{'}(u), v\rangle = \int_\Omega a(x, \nabla u) \cdot \nabla v\, dx,$$

for all u, $v \in W^{1,p(\cdot)}(\Omega)$.

Remark 1. The study from [17] is conducted for $\Lambda : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$, where $W_0^{1,p(\cdot)}(\Omega)$ represents the Sobolev space with zero boundary values defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$. Since the calculus is almost identical, it is omitted for brevity.

In addition, we denote by $J : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ the functional defined by

$$J(u) = \int_\Omega \left[|\nabla u|^{p(x)} + |u|^{p(x)} \right] dx.$$

Proposition 2. ([9, Proposition 2.3]) *If* $u \in W^{1,p(\cdot)}(\Omega)$ *then*

$$\|u\| > 1 \Rightarrow \|u\|^{p^-} \le J(u) \le \|u\|^{p^+}$$

$$\|u\| < 1 \Rightarrow \|u\|^{p^+} \le J(u) \le \|u\|^{p^-}.$$

Other two functionals that have interesting properties are ψ, φ: $W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, dx, \qquad \varphi(u) = \int_{\partial\Omega} \frac{1}{r(x)} |u|^{r(x)}\, dS. \tag{10}$$

Proposition 3. *The functionals* $\psi, \varphi : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ *are weakly-strongly continuous, that is,* $u_n \rightharpoonup u$ *(weakly) implies* $\psi(u_n) \to \psi(u)$*, respectively,* $\varphi(u_n) \to \varphi(u)$*, as* $n \to \infty$.

Proof. Let $(u_n)_n \subset W^{1,p(\cdot)}(\Omega)$ be such that $u_n \rightharpoonup u$ (weakly) in $W^{1,p(\cdot)}(\Omega)$. By Theorem 1 and Theorem 2, we have

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \text{ compactly}$$

and

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega) \text{ compactly.}$$

Therefore, passing eventually to a subsequence, we get the strong convergence of $(u_n)_n$ to u in $L^{q(\cdot)}(\Omega)$ and $L^{r(\cdot)}(\partial\Omega)$. This only means that

$$\|u_n - u\|_{L^{q(\cdot)}(\Omega)} \to 0, \quad \text{respectively,} \quad \|u_n - u\|_{L^{r(\cdot)}(\partial\Omega)} \to 0$$

as $n \to \infty$. Using (8) we have arrived to

$$\lim_{n\to\infty} \rho_{\Omega,\,q(\cdot)}(u_n) = \rho_{\Omega,\,q(\cdot)}(u), \quad \text{respectively,} \quad \lim_{n\to\infty} \rho_{\partial\Omega,\,r(\cdot)}(u_n) = \rho_{\partial\Omega,\,r(\cdot)}(u).$$

Since

$$|\psi(u_n) - \psi(u)| \le \frac{1}{q^-} \left|\rho_{\Omega,\,q(\cdot)}(u_n) - \rho_{\Omega,\,q(\cdot)}(u)\right| \quad \text{and}$$
$$|\varphi(u_n) - \varphi(u)| \le \frac{1}{r^-} \left|\rho_{\partial\Omega,\,r(\cdot)}(u_n) - \rho_{\partial\Omega,\,r(\cdot)}(u)\right|,$$

our proof is complete. □

We remind now the following.

Proposition 4. ([9, Lemma 3.3]) *For a separable reflexive Banach space X and for X_k, Y_k, Z_k defined by (9) assume that $\Phi : X \to \mathbb{R}$ is weakly-strongly continuous, $\Phi(0) = 0$ and $\gamma > 0$ is a given positive number. Then*

$$\sup_{u\in Z_k,\, \|u\|_X \le\gamma} |\Phi(u)| \to 0$$

as $k \to \infty$.

For $X = W^{1,p(\cdot)}(\Omega)$ with X_k, Y_k, Z_k defined by (9) and ψ, φ defined by (10), we set

$$\beta_k = \sup_{u\in Z_k,\, \|u\|\le 1} |\psi(u)|, \qquad \theta_k = \sup_{u\in Z_k,\, \|u\|\le 1} |\varphi(u)|. \tag{11}$$

By Proposition 3 and Proposition 4 we deduce our final auxiliary result.

Proposition 5. *If $k \to \infty$, then*

$$\beta_k \to 0 \quad \text{and} \quad \theta_k \to 0.$$

Now we have the necessary tools to obtain the desired multiplicity result that will be stated and proved in the third section.

3. Main result

Everywhere below we work under the hypotheses described in the previous sections. Since we are preoccupied with the existence of multiple weak solutions of problem (1), we begin by giving the definition of such solutions.

Definition 1. *A function $u \in W^{1,p(\cdot)}(\Omega)$ which verifies*

$$\int_\Omega a(x, \nabla u) \cdot \nabla v\, dx + \int_\Omega b(x)|u|^{p(x)-2}uv\, dx - \int_\Omega f(x,u)v\, dx - \int_{\partial\Omega} g(x,u)v\, dS = 0$$

for all $v \in W^{1,p(\cdot)}(\Omega)$ is called a weak solution of (1).

We associate to problem (1) the energetic functional $I : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \int_\Omega A(x, \nabla u)\, dx + \int_\Omega \frac{b(x)}{p(x)} |u|^{p(x)}\, dx - \int_\Omega F(x,u)\, dx - \int_{\partial\Omega} G(x,u)\, dS.$$

Due to Proposition 1, by a standard calculus it can be shown that I is well defined and $I \in C^1(W^{1,p(\cdot)}(\Omega); \mathbb{R})$ with

$$\begin{aligned}\langle I'(u), v\rangle = &\int_\Omega a(x, \nabla u) \cdot \nabla v\, dx + \int_\Omega b(x)|u|^{p(x)-2}uv\, dx - \int_\Omega f(x,u)v\, dx \\ &- \int_{\partial\Omega} g(x,u)v\, dS\end{aligned}$$

for all u, $v \in W^{1,p(\cdot)}(\Omega)$. Hence any critical point $u \in W^{1,p(\cdot)}(\Omega)$ of I is a weak solution to problem (1).

Our main result is given by the following theorem.

Theorem 5. *Problem (1) admits a sequence $(u_n)_n$ of weak solutions such that $I(u_n) \to \infty$ when $n \to \infty$.*

The idea of the proof of Theorem 5 is to show that (H1)–(H3) are fulfilled. To this end we will prove three corresponding lemmas.

Lemma 1. *For every $k \in \mathbb{N}$, there exists $\gamma_k > 0$ such that $\inf_{u\in Z_k,\, \|u\|=\gamma_k} I(u) \to \infty$, when $k \to \infty$.*

Proof. By (F1), (F3) and (G1), (G3) we deduce that there exist c_1, $c_2 > 0$ such that

$$F(x,u) \le \frac{c_1}{q(x)} |u|^{q(x)} \quad \forall x \in \Omega \tag{12}$$

and

$$G(x,u) \le \frac{c_2}{r(x)} |u|^{r(x)} \quad \forall x \in \partial\Omega, \tag{13}$$

where q, $r \in C_+(\overline{\Omega})$ with $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ and $r(x) < p^\partial(x)$ for all $x \in \partial\Omega$.

On the other hand, by (A4),

$$\Lambda(u) \ge \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)}\, dx. \tag{14}$$

Clearly, by (B) and (12)–(14),

$$\begin{aligned}I(u) \ge &\frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)}\, dx + \frac{b_0}{p^+} \int_\Omega |u|^{p(x)}\, dx - c_1 \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, dx \\ &- c_2 \int_{\partial\Omega} \frac{1}{r(x)} |u|^{r(x)}\, dS.\end{aligned} \tag{15}$$

Next, by (15), Proposition 2, (10) and (11),

$$I(u) \ge \frac{1}{p^+} \min\{1; b_0\} \|u\|^{p^-} - c_1 \|u\|^{q^+} \beta_k - c_2 \|u\|^{r^+} \theta_k, \tag{16}$$

for $u \in Z_k$ with $\|u\| > 1$. Depending on the order relation between p^- and q^+, respectively, p^- and r^+, we distinguish several cases to treat separately.

Case 1: $p^- < q^+$ and $p^- < r^+$. We consider the following two equations:

$$\frac{1}{4p^+}\min\{1; b_0\}t^{p^-} - c_1 t^{q^+}\beta_k = 0, \tag{17}$$

$$\frac{1}{4p^+}\min\{1; b_0\}t^{p^-} - c_2 t^{r^+}\theta_k = 0. \tag{18}$$

Let $\xi_k \neq 0$ be the solution of (17) and $\zeta_k \neq 0$ be the solution of (18). Then

$$\xi_k = \left(\frac{1}{4p^+ c_1 \beta_k}\min\{1; b_0\}\right)^{\frac{1}{q^+ - p^-}}$$

and

$$\zeta_k = \left(\frac{1}{4p^+ c_2 \theta_k}\min\{1; b_0\}\right)^{\frac{1}{r^+ - p^-}}.$$

Using Proposition 5 we obtain that

$$\lim_{k\to\infty} \xi_k = \infty \quad \text{and} \quad \lim_{k\to\infty} \zeta_k = \infty.$$

We choose $\gamma_k = \min\{\xi_k, \zeta_k\}$, for $u \in Z_k$ with $\|u\| = \gamma_k$ and by (16)–(18) we have

$$I(u) \geq \frac{1}{2p^+}\min\{1; b_0\}\gamma_k^{p^-} - \widetilde{c} \to \infty \quad \text{as } k \to \infty,$$

where $\widetilde{c}$ is a positive constant. With that, the proof of the first case is complete.

Case 2: $p^- > q^+$ and $p^- < r^+$. By Proposition 5 we deduce that, for sufficiently large k, $\beta_k < \frac{1}{2p^+ c_1}\min\{1; b_0\}$, thus (16) yields

$$I(u) \geq \frac{1}{2p^+}\min\{1; b_0\}\|u\|^{p^-} - c_2\theta_k\|u\|^{r^+},$$

for $u \in Z_k$ with $\|u\| > 1$. We choose $\gamma_k = \zeta_k$, that is, the nonzero solution of (18), and we discover that, for $u \in Z_k$ with $\|u\| = \gamma_k$,

$$I(u) \geq \frac{1}{4p^+}\min\{1; b_0\}\gamma_k^{p^-} \to \infty \quad \text{as } k \to \infty, \tag{19}$$

hence the proof of our second case is complete.

Case 3: $p^- < q^+$ and $p^- > r^+$. Proposition 5 implies that, for sufficiently large k, $\theta_k < \frac{1}{2p^+ c_2}\min\{1; b_0\}$, so we notice that, using (16),

$$I(u) \geq \frac{1}{2p^+}\min\{1; b_0\}\|u\|^{p^-} - c_1\beta_k\|u\|^{q^+},$$

for $u \in Z_k$ with $\|u\| > 1$. We choose γ_k to be the nonzero solution of (17), that is, ξ_k. For $u \in Z_k$ with $\|u\| = \gamma_k$ we have arrived at relation (19) and the proof of our third case is complete.

Case 4: $p^- > q^+$ and $p^- > r^+$. Again, by Proposition 5, for sufficiently large k, $\beta_k < \frac{1}{4p^+c_1}\min\{1; b_0\}$ and $\theta_k < \frac{1}{4p^+c_2}\min\{1; b_0\}$. Therefore by (16),

$$I(u) \geq \frac{1}{2p^+}\min\{1; b_0\}\|u\|^{p^-}.$$

By choosing $(\gamma_k)_k$ such that $\gamma_k \to \infty$ as $k \to \infty$, for $u \in Z_k$ with $\|u\| = \gamma_k$, we have that $I(u) \to \infty$ and the proof of our last case is complete. □

Remark 2. The proof of the fourth case can be done without relying on Proposition 5. Indeed, using (12)–(14), (B) and Proposition 2 for $u \in W^{1,p(\cdot)}(\Omega)$ with $\|u\| > 1$ we get that

$$I(u) \geq \frac{1}{p^+}\min\{1; b_0\}\|u\|^{p^-} - \frac{c_1}{q^-}\int_\Omega |u|^{q(x)}\,dx - \frac{c_2}{r^-}\int_{\partial\Omega}|u|^{r(x)}\,dS. \quad (20)$$

Relations (5) and (6) imply that

$$\int_\Omega |u|^{q(x)}\,dx \leq \|u\|^{q^+}_{L^{q(\cdot)}(\Omega)} + \|u\|^{q^-}_{L^{q(\cdot)}(\Omega)}.$$

Taking into account Theorem 1 we can establish the existence of a constant $c_3 > 0$ such that

$$\int_\Omega |u|^{q(x)}\,dx \leq c_3\|u\|^{q^+} \quad \text{for } \|u\| > 1. \quad (21)$$

In the same manner, using Theorem 2 instead of Theorem 1, we can find a constant $c_4 > 0$ such that

$$\int_{\partial\Omega} |u|^{r(x)}\,dS \leq c_4\|u\|^{r^+} \quad \text{for } \|u\| > 1. \quad (22)$$

We introduce (21), (22) into (20) and, for $u \in Z_k$ with $\|u\| = \gamma_k > 1$, we have

$$I(u) \geq \frac{1}{p^+}\min\{1; b_0\}\gamma_k^{p^-} - \frac{c_1c_3}{q^-}\gamma_k^{q^+} - \frac{c_2c_4}{r^-}\gamma_k^{r^+}.$$

We choose $(\gamma_k)_k$ such that $\gamma_k \to \infty$ as $k \to \infty$ and we infer that $I(u) \to \infty$ as $k \to \infty$ because $q^+ < p^-$ and $r^+ < p^-$.

Let us pass now to the proof of our second lemma, corresponding to hypothesis (H2) from the fountain theorem.

Lemma 2. *For every $k \in \mathbb{N}$ there exists $\rho_k > \gamma_k$ (γ_k given by Lemma 1) such that the following inequality*

$$\max_{u \in Y_k,\, \|u\| = \rho_k} I(u) \leq 0$$

takes place.

Proof. The assumptions (A1) and (A2) yield that, for all $\eta \in \mathbb{R}^N$ and all $x \in \overline{\Omega}$,

$$A(x, \eta) = \int_0^1 a(x, t\eta)\cdot\eta\,dt \leq c_0\left(|\eta| + \frac{1}{p(x)}|\eta|^{p(x)}\right).$$

Therefore, by (B),

$$I(u)\le c_0\|\nabla u\|_{L^1(\Omega)}+\frac{\max\{c_0;\|b\|_{L^\infty(\Omega)}\}}{p^-}J(u)-\int_\Omega F(x,u)\,dx-\int_{\partial\Omega}G(x,u)\,dS. \tag{23}$$

Since $1 < p(x)$ we have the continuous embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$ hence there exists $c_5 > 0$ such that

$$\|\nabla u\|_{L^1(\Omega)} \le c_5\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \le c_5\|u\|. \tag{24}$$

From (F2) and (G2) it follows that there exist M_1, $M_2 > 0$ such that

$$M_1|u|^{\alpha_1} \le F(x,u) \quad \forall x \in \Omega \tag{25}$$

and

$$M_2|u|^{\alpha_2} \le G(x,u) \quad \forall x \in \partial\Omega. \tag{26}$$

Due to Proposition 2 and relations (23)–(26), for $u \in Y_k$ with $\|u\| > 1$ we obtain

$$I(u) \le c_0c_5\|u\| + \frac{\max\{c_0;\|b\|_{L^\infty(\Omega)}\}}{p^-}\|u\|^{p^+} - M_1\|u\|^{\alpha_1}_{L^{\alpha_1}(\Omega)} - M_2\|u\|^{\alpha_2}_{L^{\alpha_2}(\partial\Omega)}.$$

But $1 < p^+ < \alpha_i$, $i \in \{1,2\}$, and Y_k is a finite dimensional space, thus all the norms are equivalent and we can conclude that, for $u \in Y_k$ with $\|u\| = \rho_k > \gamma_k$ big enough, $I(u) \le 0$. □

Finally, we give the third lemma, corresponding to hypothesis (H3).

Lemma 3. *The functional I satisfies the $(PS)_c$ condition for every $c > 0$.*

Proof. Let $c > 0$ and $(u_n)_n \subset W^{1,p(\cdot)}(\Omega)$ be such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \text{ in } \left(W^{1,p(\cdot)}(\Omega)\right)^\star \quad \text{as } n \to \infty. \tag{27}$$

To show that $(u_n)_n$ is strongly convergent in $W^{1,p(\cdot)}(\Omega)$ we start by showing that $(u_n)_n$ is bounded. Arguing by contradiction, we assume that, passing eventually to a subsequence still denoted by $(u_n)_n$,

$$\|u_n\| \to \infty \quad \text{as } n \to \infty.$$

We denote $\alpha = \min\{\alpha_1,\alpha_2\}$ and we choose $\tau \in (p^+,\alpha)$. Then, for sufficiently large n, by (27) and (A4) we deduce that

$$\begin{aligned} c+1+\|u_n\| &\ge I(u_n) - \frac{1}{\tau}\langle I'(u_n),u_n\rangle \\ &\ge \left(\frac{1}{p^+}-\frac{1}{\tau}\right)\int_\Omega \left(|\nabla u_n|^{p(x)} + b(x)|u_n|^{p(x)}\right)\,dx \\ &\quad - \int_\Omega \left(F(x,u_n) - \frac{1}{\tau}f(x,u_n)u_n\right)\,dx \\ &\quad - \int_{\partial\Omega}\left(G(x,u_n) - \frac{1}{\tau}g(x,u_n)u_n\right)\,dS. \end{aligned} \tag{28}$$

Taking into consideration (F2) and (G2), the inequality (28) reduces to

$$c+1+\|u_n\| \geq \left(\frac{1}{p^+}-\frac{1}{\tau}\right)\int_\Omega \left(|\nabla u_n|^{p(x)}+b(x)|u_n|^{p(x)}\right)dx. \tag{29}$$

Furthermore, due to (B) and Proposition 2, (29) becomes

$$c+1+\|u_n\| \geq \left(\frac{1}{p^+}-\frac{1}{\tau}\right)\min\{1,b_0\}\|u_n\|^{p^-},$$

for sufficiently large n. We let n go to ∞ and we divide by $\|u_n\|^{p^-}$ in the above relation. This way, we have obtained a contradiction, therefore $(u_n)_n$ is bounded in $W^{1,p(\cdot)}(\Omega)$. However, $W^{1,p(\cdot)}(\Omega)$ is a reflexive Banach space, thus, up to a subsequence, $(u_n)_n$ is weakly convergent to some u_0 in $W^{1,p(\cdot)}(\Omega)$. Using (27) we infer that

$$\lim_{n\to\infty}\langle I'(u_n),u_n-u_0\rangle=0,$$

more precisely,

$$\lim_{n\to\infty}\left[\int_\Omega a(x,\nabla u_n)\cdot(\nabla u_n-\nabla u_0)\,dx+\int_\Omega b(x)|u_n|^{p(x)-2}u_n(u_n-u_0)\,dx \right.$$
$$\left. -\int_\Omega f(x,u_n)(u_n-u_0)\,dx-\int_{\partial\Omega} g(x,u_n)(u_n-u_0)\,dS\right]=0. \tag{30}$$

At the same time, by (B) and the Hölder-type inequality (4),

$$\left|\int_\Omega b(x)|u_n|^{p(x)-2}u_n(u_n-u_0)\,dx\right|$$
$$\leq 2\|b\|_{L^\infty(\Omega)}\left\||u_n|^{p(x)-1}\right\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)}\|u_n-u_0\|_{L^{p(\cdot)}(\Omega)}. \tag{31}$$

Since $p(x)<p^\star(x)$, for all $x\in\overline{\Omega}$, then, by Theorem 1, the space $W^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{p(\cdot)}(\Omega)$. Hence

$$u_n\to u_0 \quad \text{in } L^{p(\cdot)}(\Omega). \tag{32}$$

By (31), (7) and (32) we deduce that

$$\lim_{n\to\infty}\int_\Omega b(x)|u_n|^{p(x)-2}u_n(u_n-u_0)\,dx=0. \tag{33}$$

In a similar manner, using (F1), (4), Theorem 1 and (7), we get

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u_0)\,dx=0, \tag{34}$$

while using (G1), (4), Theorem 2 and (7), we get

$$\lim_{n\to\infty}\int_{\partial\Omega} g(x,u_n)(u_n-u_0)\,dS=0. \tag{35}$$

Combining (30) with (33)–(35) we arrive at

$$\lim_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot(\nabla u_n-\nabla u_0)\,dx=0.$$

This means that, since a is a (S_+)-type operator, by Theorem 3, $(u_n)_n$ is strongly convergent to u_0 in $W^{1,p(\cdot)}(\Omega)$. From (27) it follows that u_0 is a critical point for I and the proof is complete. □

Proof of Theorem 5. The proof follows immediately from Lemma 1, Lemma 2, Lemma 3 and Theorem 4. □

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