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Infinitely many solutions for a fourth-order elastic beam equation

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Abstract. Existence results of infinitely many solutions for a fourth-order nonlinear boundary value problem are established. No symmetric condition on the nonlinear term is assumed. The main tool is an infinitely many critical points theorem.

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1. Introduction

The aim of the present paper is to investigate the existence of infinitely many solutions for the following fourth-order problem

$$\begin{cases} u^{iv} + Au'' + Bu = \lambda f(t,u) & \text{in } [0,1] \\ u(0) = u(1) = 0 \\ u''(0) = u''(1) = 0, \end{cases} \tag{D_λ}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a function, A and B are real constants and λ is a positive parameter. Owing to the importance of fourth-order two-point boundary value problems in describing a large class of elastic deflection, there is a wide literature that deals with multiplicity results for such a problem (see, for instance, [4,5,8], and the references therein). Infinitely many solutions for problem (D_λ) are obtained in [1] where $A = B = 0$ and f is monotone increasing in u. In [6], using the Morse theory, the authors obtain multiple solutions under the following assumptions on f:

(H_1) there exist $\nu > 2$ and $M > 0$ such that:

$$0 < \nu \int_0^u f(t,s)\,ds \leq uf(t,u) \quad \text{for all } |u| \geq M,\ t \in [0,1]$$

(H_2) $f(t,u)$ is odd in u for all $t \in [0,1]$.

In [7] these conditions, that implies a superquadratic behaviour of $\int_0^u f(t,s)\,ds$ at infinity, are combined with the assumption that $A < 0 < B$ in order to obtain infinitely many solutions.

In the present paper, we establish some multiplicity results for problem (D_λ) under rather different assumptions on the function f and parameters A and B. It is worth noticing that in our results neither symmetric nor monotonic condition on the nonlinear term is assumed. We require that f has a suitable oscillating behaviour either at infinity or at zero. In the first case, we obtain an unbounded sequence of solutions (Theorem 3.1); in the second case, we obtain a sequence of non-zero solutions strongly converging at zero (Theorem 3.2). We emphasize that our theorems and results in [1,6,7] and [8] are mutually independent as Example 3.1 shows (see Remark 3.5).

As example, we state here a special case of our results.

Theorem 1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative and continuous function. Assume that*

$$\liminf_{\xi\to+\infty} \frac{\int_0^\xi g(x)\,dx}{\xi^2} = 0$$

and

$$\limsup_{\xi\to+\infty} \frac{\int_0^\xi g(x)\,dx}{\xi^2} = +\infty.$$

Then, the problem

$$\begin{cases} u^{iv} = g(u) & \text{in } [0,1] \\ u(0) = u(1) = 0 \\ u''(0) = u''(1) = 0, \end{cases}$$

admits a sequence of pairwise distinct positive classical solutions.

The note is arranged as follows. In Sect. 2, we recall some basic definitions and our abstract framework, while Sect. 3 is devoted to infinitely many solutions for the problem (D_λ).

2. Preliminaries and basic notations

Here and in the sequel, let A and B be two real constants such that

$$\max\left\{\frac{A}{\pi^2}, -\frac{B}{\pi^4}, \frac{A}{\pi^2} - \frac{B}{\pi^4}\right\} < 1 \tag{2.1}$$

(For instance, if $A \le 0$ and $B \ge 0$, then (2.1) is verified). Moreover, put

$$\sigma := \max\left\{\frac{A}{\pi^2}, -\frac{B}{\pi^4}, \frac{A}{\pi^2} - \frac{B}{\pi^4}, 0\right\}$$

and

$$\theta := \sqrt{1-\sigma}.$$

Let $X := W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ the Sobolev space endowed with the norm

$$\|u\| = \left(\int_0^1 (|u''|^2 - A|u'|^2 + B|u|^2)\, dx \right)^{1/2}.$$

that is equivalent to the usual one (see [3]).
The following Lemma is useful for proving our main result.

Lemma 1. (Proposition 2.1 of [3]). *Let $u \in X$. Then*

$$\|u\|_\infty \leq \frac{1}{2\pi\theta}\|u\|.$$

For the function f we assume the following conditions:

(a) $t \to f(t,x)$ is measurable for every $x \in \mathbb{R}$;
(b) $x \to f(t,x)$ is continuous for almost every $t \in [0,1]$;
(c) for every $\rho > 0$ there exists a function $l_\rho \in L^2([0,1])$ such that

$$\sup_{|x|\leq\rho} |f(t,x)| \leq l_\rho(t)$$

for almost every $t \in [0,1]$.

It is easy to see that if $f(t,x)x < 0$ for every $t \in [0,1]$ and $x \neq 0$, the problem (D_λ) has only the trivial solution.
Let us recall that a weak solution of (D_λ) is a function $u \in X$ such that

$$\int_0^1 [u''(t)v''(t) - Au'(t)v'(t) + Bu(t)v(t)]\, dt - \lambda \int_0^1 f(t,u(t))v(t)\, dt = 0 \quad \forall v \in X. \tag{2.2}$$

Moreover, a function $u : [0,1] \to \mathbb{R}$ is said a generalized solution to problem (D_λ) if $u \in C^3([0,1]), u''' \in AC([0,1]), u(0) = u(1) = 0, u''(0) = u''(1) = 0$, and $u^{iv} + Au'' + Bu = \lambda f(t,u)$ for almost every $t \in [0,1]$. If f is continuous in $[0,1] \times \mathbb{R}$, therefore each generalized solution u is a classical solution.

The assumptions on f implies that a weak solutions to problem (D_λ) is a generalized one (see [3, Proposition 2.2]).

In the next section we shall prove our results applying the following smooth version of Theorem 2.1 of [2], which is a more precise version of Ricceri's Variational Principle [9, Theorem 2.5].

Theorem 2.1. *Let X be a reflexive real Banach space; let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *for every* $r > \inf_X \Phi$ *and every* $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, *the restriction of the functional* $I_\lambda = \Phi - \lambda\Psi$ *to* $\Phi^{-1}(]-\infty, r[)$ *admits a global minimum, which is a critical point (local minimum) of* I_λ *in* X.

(b) *If* $\gamma < +\infty$ *then, for each* $\lambda \in \left]0, \frac{1}{\gamma}\right[$ *the following alternative holds: either*

(b_1) I_λ *has a global minimum*

or

(b_2) *there exists a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that* $\lim_{n\to\infty} \Phi(u_n) = +\infty$.

(c) *If* $\delta < +\infty$ *then, for each* $\lambda \in \left]0, \frac{1}{\delta}\right[$ *the following alternative holds: either*

(c_1) *there exists a global minimum of* Φ *which is a local minimum of* I_λ

or

(c_2) *there exists a sequence of pairwise distinct critical points (local minima) of* I_λ, *which weakly converges to a global minimum of* Φ.

3. Main results

In this section we present our main results. Put

$$F(t, \xi) = \int_0^\xi f(t, x)\, dx$$

for each $(t, \xi) \in [0, 1] \times \mathbb{R}$, and

$$k = 2\theta^2\pi^2 \left(\frac{2048}{27} - \frac{32}{9}A + \frac{13}{40}B \right)^{-1}. \quad (1)$$

It is easy to check that $0 < k < \frac{1}{2}$.

Our first result is as follows.

Theorem 3.1. *Assume that*

(i) $F(t, \xi) \geq 0$ *for all* $(t, \xi) \in \left([0, \frac{3}{8}] \cup [\frac{5}{8}, 1]\right) \times \mathbb{R}$;

(ii) $\liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|x|\leq\xi} F(t, x)\, dt}{\xi^2} < k \limsup_{\xi\to+\infty} \frac{\int_{3/8}^{5/8} F(t, \xi)\, dt}{\xi^2}$;

Then, for every

$$\lambda \in \Lambda := \left] \frac{2\theta^2\pi^2}{k} \frac{1}{\limsup_{\xi\to+\infty} \frac{\int_{3/8}^{5/8} F(t,\xi)\, dt}{\xi^2}}, \frac{2\theta^2\pi^2}{\liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|x|\leq\xi} F(t,x)\, dt}{\xi^2}} \right[,$$

the problem (D_λ) *has an unbounded sequence of generalized solutions in* X.

Proof. Let us define in X two functionals Φ and Ψ by setting, for each $u \in X$,

$$\Phi(u) = \frac{1}{2}||u||^2 \quad \text{and} \quad \Psi(u) = \int_0^1 F(t, u(t))\, dt.$$

Clearly, Φ is continuous and convex, hence it is weakly sequentially lower semicontinuous. Moreover, Φ is continuously Gâteaux differentiable and its Gâteaux derivative admits a continuous inverse. The functional Ψ is well-defined, continuously Gâteaux differentiable and with compact derivative, hence it is sequentially weakly continuous. In particular, one has

$$\Phi'(u)(v) = \int_0^1 [u''(t)v''(t) - Au'(t)v'(t) + Bu(t)v(t)]\, dt$$

and

$$\Psi'(u)(v) = \lambda \int_0^1 f(t, u(t))v(t)\, dt$$

for all $v \in X$. Clearly, the weak solutions to problem (D_λ) are exactly the critical points of the functional $I_\lambda := \Phi - \lambda\Psi$ and they are also generalized solutions. So, our end is to apply Theorem 2.1 to Φ and Ψ.

Now, we wish to prove that

$$\gamma < +\infty.$$

Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to +\infty$ and

$$\lim_{n\to\infty} \frac{\int_0^1 \max_{|x|\le\xi_n} F(t,x)\, dt}{\xi_n^2} = \liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|x|\le\xi} F(t,x)\, dt}{\xi^2}$$

Put $r_n = 2\pi^2\theta^2\xi_n^2$ for all $n \in \mathbb{N}$. By Lemma 1, for all $u \in X$ one has

$$\max_{t\in[0,1]} |v(t)| \le \xi_n$$

for all $v \in X$ such that $\|v\|^2 < 2r_n$. Hence one has

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v) - \Psi(u)}{r_n - \Phi(u)} \\ &\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v)}{r_n} \\ &\le \frac{\int_0^1 \max_{|x|\le\xi_n} F(t,x)\, dt}{2\pi^2\theta^2\xi_n^2}.\end{aligned}$$

So,

$$\gamma \le \frac{1}{2\pi^2\theta^2} \liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|x|\le\xi} F(t,x)\, dt}{\xi^2} < +\infty. \tag{2}$$

From assumption (ii) and (2) one has

$$\Lambda \subseteq \left]0, \frac{1}{\gamma}\right[.$$

Fix $\lambda \in \Lambda$, the previous inequality assures that the conclusion (b) of Theorem 2.1 can be used and either I_λ has a global minimum or there exists a sequence $\{u_n\}$ of solutions of problem (D_λ) such that $\lim_{n\to\infty} \|u_n\| = +\infty$.

The other step is to verify that the functional I_λ has no global minimum.

By the choice of λ, we can choose a constant h such that, for each $n \in \mathbb{N}$,

$$\sup_{\eta \geq n} \frac{\int_{3/8}^{5/8} F(t,\eta)\, dt}{\eta^2} > h > \frac{2\pi^2\theta^2}{\lambda k} \tag{3}$$

and so there exists $\eta_n \geq n$ such that

$$\frac{\int_{3/8}^{5/8} F(t,\eta_n)\, dt}{\eta_n^2} > h.$$

Thus, if we consider a function $v_n \in X$ defined by setting

$$v_n(x) = \begin{cases} -\dfrac{64\eta_n}{9}\left(x^2 - \dfrac{3}{4}x\right) & x \in \left[0, \dfrac{3}{8}\right] \\ \eta_n & x \in \left]\dfrac{3}{8}, \dfrac{5}{8}\right] \\ -\dfrac{64\eta_n}{9}\left(x^2 - \dfrac{5}{4}x + \dfrac{1}{4}\right) & x \in \left]\dfrac{5}{8}, 1\right] \end{cases}$$

one has

$$\Phi(v_n) = \frac{2\pi^2\theta^2}{k}\eta_n^2$$

and, bearing in mind (i),

$$\begin{aligned} I_\lambda(v_n) = \Phi(v_n) - \lambda\Psi(v_n) &= \frac{2\pi^2\theta^2}{k}\eta_n^2 - \lambda\int_0^1 F(t, v_n(t))\, dt \\ &\leq \frac{2\pi^2\theta^2}{k}\eta_n^2 - \lambda\int_{3/8}^{5/8} F(t,\eta_n)\, dt \\ &< \eta_n^2\left(\frac{2\pi^2\theta^2}{k} - \lambda h\right). \end{aligned} \tag{4}$$

Putting together (3) and (4) we get that the functional I_λ is unbounded from below and then it has no global minimum.

Therefore, Theorem 2.1 assures that there is a sequence $\{u_n\} \subseteq X$ of critical points of I_λ such that $\lim_{n\to+\infty} \|u_n\| = +\infty$ and, taking into account the considerations made in Section 2, the theorem is completely proved. □

Remark 3.1. Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is nonnegative in $[0,1] \times \mathbb{R}$. Clearly, condition (i) holds, and condition (ii) becomes:

(ii')

$$\liminf_{\xi\to+\infty}\frac{\int_0^1 F(t,\xi)\,dt}{\xi^2} < k\limsup_{\xi\to+\infty}\frac{\int_{3/8}^{5/8} F(t,\xi)\,dt}{\xi^2}.$$

In this case, the interval Λ is $\left]\frac{2\theta^2\pi^2}{k}\frac{1}{\limsup_{\xi\to+\infty}\frac{\int_{3/8}^{5/8} F(t,\xi)\,dt}{\xi^2}}, \frac{2\theta^2\pi^2}{\liminf_{\xi\to+\infty}\frac{\int_0^1 F(t,\xi)\,dt}{\xi^2}}\right[$.

Further, if $\liminf_{\xi\to+\infty}\frac{\int_0^1 F(t,\xi)\,dt}{\xi^2} = 0$ and $\limsup_{\xi\to+\infty}\frac{\int_{3/8}^{5/8} F(t,\xi)\,dt}{\xi^2} = +\infty$, then (ii') holds and problem (D_λ) has an unbounded sequence of generalized solutions in X.

Remark 3.2. Theorem 1.1 in Introduction is an immediatly consequence of Theorem 3.1 taking into account Remark 3.1.

Remark 3.3. We explicity observe that assumption (ii) in Theorem 3.1 could be replaced by the more general

(iii) there exist two sequence $\{a_n\}$ and $\{b_n\}$ with $|a_n| < \sqrt{k}\, b_n$ for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} b_n = +\infty$ such that

$$\lim_{n\to+\infty}\frac{\int_0^1 \max_{|x|\le b_n} F(t,x)\,dt - \int_{3/8}^{5/8} F(t,a_n)\,dt}{b_n^2 - \frac{a_n^2}{k}} < k\limsup_{\xi\to+\infty}\frac{\int_{3/8}^{5/8} F(t,\xi)\,dt}{\xi^2}.$$

Clearly, from (iii) we obtain (ii), by choosing $a_n = 0$ for all $n \in \mathbb{N}$. Moreover, if we assume (iii) instead of (ii) and put $r_n = 2\pi^2\theta^2 b_n^2$ for all $n \in \mathbb{N}$, following the same reasoning made inside in Theorem 3.1, we obtain

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)}\frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\Psi(v) - \Psi(u)}{r_n - \Phi(u)} \\
&\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\Psi(v) - \int_0^1 F(t,v_n(t))\,dt}{r_n - \frac{\|v_n\|^2}{2}} \\
&\le \frac{1}{2\pi^2\theta^2}\frac{\int_0^1 \max_{|x|\le b_n} F(t,x)\,dt - \int_{3/8}^{5/8} F(t,a_n)\,dt}{b_n^2 - \frac{a_n^2}{k}}
\end{aligned}$$

where

$$v_n(x) = \begin{cases} -\frac{64a_n}{9}\left(x^2 - \frac{3}{4}x\right) & x \in \left[0, \frac{3}{8}\right] \\ a_n & x \in \left]\frac{3}{8}, \frac{5}{8}\right] \\ -\frac{64a_n}{9}\left(x^2 - \frac{5}{4}x + \frac{1}{4}\right) & x \in \left]\frac{5}{8}, 1\right]. \end{cases}$$

The conclusion is the same of Theorem 3.1 with Λ replaced by

$$\Lambda' := \left] \frac{2\theta^2\pi^2}{k} \frac{1}{\limsup\limits_{\xi\to+\infty} \frac{\int_{3/8}^{5/8} F(t,\xi)\,dt}{\xi^2}}, \frac{2\theta^2\pi^2}{\lim\limits_{n\to+\infty} \frac{\int_0^1 \max_{|x|\le b_n} F(t,x)\,dt - \int_{3/8}^{5/8} F(t,a_n)\,dt}{b_n^2 - \frac{a_n^2}{k}}} \right[$$

Now, we point out some consequences of Theorem 3.1.

Corollary 3.1. *Assume that*

(i$_1$) $F(t,\xi) \ge 0$ *for all* $(t,\xi) \in \left([0,\frac{3}{8}] \cup [\frac{5}{8},1]\right) \times \mathbb{R}$*;*

(i$_2$) $\liminf\limits_{\xi\to+\infty} \frac{\int_0^1 \max_{|x|\le\xi} F(t,x)\,dt}{\xi^2} < 2\theta^2\pi^2$

(i$_3$) $\limsup\limits_{\xi\to+\infty} \frac{\int_{3/8}^{5/8} F(t,\xi)\,dt}{\xi^2} > \frac{2\theta^2\pi^2}{k}$.

Then, the problem

$$\begin{cases} u^{iv} + Au'' + Bu = f(t,u) \quad in\ [0,1] \\ u(0) = u(1) = 0 \\ u''(0) = u''(1) = 0, \end{cases}$$

has an unbounded sequence of generalized solutions in X*.*

Corollary 3.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a nonnegative continuous function such that*

(h$_1$)

$$\liminf_{\xi\to+\infty} \frac{\int_0^\xi f(x)\,dx}{\xi^2} < \frac{k}{4} \limsup_{\xi\to+\infty} \frac{\int_0^\xi f(x)\,dx}{\xi^2}.$$

Then, for every

$$\lambda \in \Lambda := \left] \frac{8\theta^2\pi^2}{k} \frac{1}{\limsup\limits_{\xi\to+\infty} \frac{\int_0^\xi f(x)\,dx}{\xi^2}}, \frac{2\theta^2\pi^2}{\liminf\limits_{\xi\to+\infty} \frac{\int_0^\xi f(x)\,dx}{\xi^2}} \right[,$$

the problem

$$\begin{cases} u^{iv} + Au'' + Bu = \lambda f(u) \quad \text{in}\ [0,1] \\ u(0) = u(1) = 0 \\ u''(0) = u''(1) = 0, \end{cases} \tag{D_λ}$$

has an unbounded sequence of classical solutions in X*.*

Arguing as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 2.1 instead of (b), one establishes the following result.

Theorem 3.2. *Assume that*

(j) $F(t,\xi) \geq 0$ *for all* $(t,\xi) \in \left([0,\frac{3}{8}] \cup [\frac{5}{8},1]\right) \times \mathbb{R}$;

(jj) $\liminf_{\xi\to 0^+} \frac{\int_0^1 \max_{|x|\leq\xi} F(t,x)\,dt}{\xi^2} < k \limsup_{\xi\to 0^+} \frac{\int_{3/8}^{5/8} F(t,\xi)\,dt}{\xi^2}$;

Then, for every

$$\lambda \in \Lambda := \left] \frac{2\theta^2\pi^2}{k} \frac{1}{\limsup_{\xi\to 0^+} \frac{\int_{3/8}^{5/8} F(t,\xi)\,dt}{\xi^2}}, \frac{2\theta^2\pi^2}{\liminf_{\xi\to 0^+} \frac{\int_0^1 \max_{|x|\leq\xi} F(t,x)\,dt}{\xi^2}} \right[,$$

The problem (D_λ) *has a sequence of non-zero generalized solutions, which strongly converges to 0 in* X.

Proof. We take X, Φ and Ψ as in the proof of Theorem 3.1. In a similar way we prove that $\delta < +\infty$. Now, fixed h such that

$$\limsup_{\eta\to 0^+} \frac{\int_{3/8}^{5/8} F(t,\eta)\,dt}{\eta^2} > h > \frac{2\pi^2\theta^2}{\lambda k}.$$

For each $n \in \mathbb{N}$, there exists $\eta_n \leq \frac{1}{n}$ such that

$$\frac{\int_{3/8}^{5/8} F(t,\eta_n)\,dt}{\eta_n^2} > h.$$

If we take w_n as in the proof of Theorem 3.1, of course the sequence $\{w_n\}$ strongly converges to 0 in X and $I_\lambda(w_n) < 0$ for each $n \in \mathbb{N}$. Since $I_\lambda(0) = 0$, that means that 0 is not a local minimum of I_λ. The part (c) of Theorem 2.1 ensures that there exists a sequence $\{u_n\}$ in X of critical points of I_λ such that $\lim_{n\to+\infty} \|u_n\| = 0$ and the proof is complete. □

Remark 3.4. Clearly, for Theorem 3.2 we obtain similar consequences to those of Theorem 3.1. For instance, also in this case, assumption (jj) in Theorem 3.2 could be replaced by the more general

(jjj) there exist two sequence $\{a_n\}$ and $\{b_n\}$ with $|a_n| < \sqrt{k}\, b_n$ for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} b_n = 0$ such that

$$\lim_{n\to+\infty} \frac{\int_0^1 \max_{|x|\leq b_n} F(t,x)\,dt - \int_{3/8}^{5/8} F(t,a_n)\,dt}{b_n^2 - \frac{a_n^2}{k}} < k \limsup_{\xi\to 0^+} \frac{\int_{3/8}^{5/8} F(t,\xi)\,dt}{\xi^2}.$$

Example 3.1. Let α be a positive constant and let $f : \mathbb{R} \to \mathbb{R}$ be the nonnegative and continuous function defined as follows

$$f(\xi) = \begin{cases} 0 \quad \text{if } \xi \in \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} [n!n,\ (n+1)!] \\ 4\alpha[(n+1)^2 - 1] \min\{\xi - n!n,\ (n+1)! - \xi\} \quad \text{if } \xi \in [n!n,\ (n+1)!]. \end{cases}$$

Put

$$a_n = n! \qquad b_n = n!n,$$

for each $n \in \mathbb{N}$.

One has

$$F(a_n) = \sum_{i=1}^{n-1} \int_{b_i}^{a_{i+1}} f(\xi)\, d\xi = \sum_{i=1}^{n-1} 4\alpha \left[(i+1)^2 - 1\right] \left[(a_{i+1} - b_i)\left(\frac{a_{i+1} + b_i}{2} - (i!)i\right) \frac{1}{2}\right]$$
$$= \sum_{i=1}^{n-1} \alpha \left[(i+1)^2 - 1\right] (i!)^2 = \alpha[(n!)^2 - 1]$$

for every $n \in \mathbb{N}$. Therefore, one has

$$\lim_{n \to +\infty} \frac{F(a_n)}{a_n^2} = \alpha$$

and, by simple calculations, we obtain

$$\limsup_{n \to +\infty} \frac{F(\xi)}{\xi^2} = \alpha.$$

On the other hand, since

$$F(b_n) = \sum_{i=1}^{n-1} \int_{b_i}^{a_{i+1}} f(\xi)\, d\xi = \alpha[(n!)^2 - 1],$$

one has

$$\lim_{n \to +\infty} \frac{F(b_n)}{b_n^2} = 0,$$

for which

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0.$$

Therefore,

$$0 = \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} < k\alpha = k \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}.$$

Hence, owing to Corollary 3.2, for each $\lambda > \dfrac{8\theta^2\pi^2}{\alpha k}$, the problem

$$\begin{cases} u^{iv} + Au'' + Bu = \lambda f(u) \quad \text{in } [0,1] \\ u(0) = u(1) = 0 \\ u''(0) = u''(1) = 0, \end{cases} \tag{P_λ}$$

admits a sequence of pairwise distinct classical solutions.

Remark 3.5. We point out that [1, Theorem 4.1] cannot be applied to the problem (P_λ) in Example 3.1 since the function f is not increasing and [6, Theorem 1.3], as well as [7, Theorem 3.2] and [8, Theorem 3.4], cannot be applied to the same problem since f is not odd.

Now, we give an application of Theorem 3.2.

Example 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows

$$f(\xi) = \begin{cases} \xi\,(2a - 2\sin(\ln(|\xi|)) - \cos(\ln(|\xi|)) & \text{if } \xi \in \mathbb{R}\backslash\{0\} \\ 0 & \text{if } \xi = 0, \end{cases}$$

where a is a constant such that

$$1 < a < \frac{4+k}{4-k}. \tag{5}$$

Then, for every $\lambda \in \Lambda := \left] \frac{8\theta^2\pi^2}{k(a+1)}, \frac{2\theta^2\pi^2}{a-1} \right[$, the problem

$$\begin{cases} u^{iv} + Au'' + Bu = \lambda f(u) & \text{in } [0,1] \\ u(0) = u(1) = 0 \\ u''(0) = u''(1) = 0 \end{cases}$$

admits a sequence of pairwise distinct classical solutions strongly converging at 0 in X.

Indeed,

$$F(\xi) = \begin{cases} \xi^2(a - \sin(\ln(|\xi|)) & \text{if } \xi \in \mathbb{R}\backslash\{0\} \\ 0 & \text{if } \xi = 0, \end{cases}$$

$$\liminf_{\xi\to 0^+} \frac{\max_{|x|\le\xi} F(x)}{\xi^2} = a - 1 \text{ and } \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = a + 1.$$

Hence, from (5) one has

$$\liminf_{\xi\to 0^+} \frac{\max_{|x|\le\xi} F(x)}{\xi^2} < \frac{k}{4} \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2}$$

and, owing to Theorem 3.2, our claim is proved.

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