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Nonlinear Differential Equations and Applications NoDEA

Existence of solutions for a perturbation sublinear elliptic equation in $\mathbb{R}^N$

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Abstract. In this paper we consider the following problem

$$\begin{cases} -\Delta u = u - |u|^{-2\theta} u + f \\ u \in H^1(\mathbb{R}^N) \cap L^{2(1-\theta)}(\mathbb{R}^N) \end{cases}$$

$f \in L^2(\mathbb{R}^N) \cap L^{\frac{2(1-\theta)}{1-2\theta}}(\mathbb{R}^N),\ N \geq 3,\ f \geq 0,\ f \neq 0$ and $0 < \theta < \frac{1}{2}$ We prove that this problem has at least two solutions via variational methods, one of them is nonnegative. Also, we study the continuity of the nonnegative solution in the perturbation parameter f at 0.

Mathematics Subject Classification (2000). 34, 35, 49.

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1. Introduction

In this paper, we study the existence and the multiplicity of solutions for the following nonhomogeneous elliptic problem

$$\begin{cases} -\Delta u = u - |u|^{-2\theta} u + f \\ u \in H^1(\mathbb{R}^N) \cap L^{2(1-\theta)}(\mathbb{R}^N) \end{cases} \tag{1}$$

$$f \in L^2(\mathbb{R}^N) \cap L^{\frac{2(1-\theta)}{1-2\theta}}(\mathbb{R}^N),\ f \geq 0,\ f \neq 0,\ N \geq 3 \text{ and } \theta \in (0, \tfrac{1}{2}) \tag{H}$$

The problem (1) can be considered as a perturbation of the following homogeneous problem

$$\begin{cases} -\Delta u = u - |u|^{-2\theta} u \\ u \in H^1(\mathbb{R}^N) \cap L^{2(1-\theta)}(\mathbb{R}^N) \end{cases} \tag{2}$$

The problem (2) has been intensively studied [8–10,13,25]. In [10], Balabane et al. proved that for each integer k, there exists a radial compactly supported solution of (2) which has k-zeros in it's support. In [25], Ounaies proved the existence of the ground state of (2) which is non-negative radial compactly supported.

In recent years, many authors have studied the existence of solutions of the following problem

$$\begin{cases} -\Delta u + V(x)u = g(x,u) + f \\ u > 0 \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{3}$$

where $g(x,\cdot)$ is superlinear. See [3–7,15,18–20,29] and the references therein.

In most of problems studied in papers cited in the references above, the linear part is $-\Delta u + V(x)u$ with $V > 0$.

In the case $V < 0$, various difficulties arise. On this subject Ghimenti and Micheletti [17] considered the following equation

$$-\Delta u + V(x)u = f'(u) + g(x) \tag{4}$$

under the assumptions:

$$V \leq 0, \ \lim_{|x|\to\infty} V(x) = 0, \ f \text{ is } C^3 \text{ with double power behaviour} \tag{GM}$$

They proved the existence of two solutions when $\|g\|_{\frac{2N}{N+2}}$ is sufficiently small.

It seems to us that very few results are known on the perturbation of sublinear elliptic equation in $\mathbb{R}^N$. While in bounded domains, the sublinear problem has been intensively studied by many authors [13,21,22]. Aside from the papers [12,16,28], to the best of our knowledge there are only a few works dealing with this kind of problems in $\mathbb{R}^N$. In [12], Benrhouma and Ounaies proved the existence and uniqueness of positive solution of the equation:

$$-\Delta u + u = |u|^p \ sgn(u) + f \tag{5}$$

where $0 < p < 1$ and $f \in L^2(\mathbb{R}^N)$, $f > 0$ a.e in $\mathbb{R}^N$.

In [28], Tehrani proved the existence of at least one solution of the equation:

$$-\Delta u + V(x)u = g(x,u) \tag{6}$$

such that g is a sublinear function and V satisfying:

$$V \in L^\infty, v_\infty = \liminf_{|x|\to\infty} V(x) > 0, \int_{\mathbb{R}^N} |\nabla\varphi|^2 + V(x)\varphi^2 < 0 \text{ for some } \varphi \in C_c^\infty \tag{T}$$

In the present study, we treated the case of $V = -1$ ($-\Delta - I$ is not coercive) and g is a sublinear function, where the assumptions (T) of (6) and (GM) of (4) are not satisfied. We prove the existence of at least two solutions for the nonhomogeneous problem (1).

Our approach is based on the mountain-pass theorem and the concentration compactness principle. One of the main difficulties in this type of problems is the break down of Palais–Smale condition. To prove that a Palais–Smale sequence (u_n) of the functional associated to problem (1) is bounded, we have developed a technic based on the control of $\|u_n\|_{2(1-\theta)}$ by $\|\nabla u_n\|_2$. Moreover, for the existence of a second solution we use a refined technic that exploits the properties of the ground state of (2). Our main result is the following.

Theorem 1.1. *Assume* (H)*, then there exists* $c_\theta > 0$ *such that if* $\|f\|_2^2 \le c_\theta$*, the problem* (1) *admits at least two solutions. One of these solutions is nonnegative*

We organize this paper into four sections. In Sect. 2, we give some preliminaries and useful results. In Sect. 3, we prove the existence of a first solution by minimizing methods and proving that this solution tend to zero in E as f tend to zero in L^2. In the last section, we show the existence of second solution by using mountain-pass type argument.

2. Preliminaries

Let

$$E = H^1(\mathbb{R}^N) \cap L^{2(1-\theta)}(\mathbb{R}^N)$$

we endow E with the norm

$$\|u\| = \|\nabla u\|_2 + \|u\|_{2(1-\theta)}$$

$((E, \| \ \|)$ is a Banach space). We define the functionals I^∞ and I on E by

$$I^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 - |u|^2)dx + \frac{1}{2(1-\theta)}\int_{\mathbb{R}^N} |u|^{2(1-\theta)}\, dx$$

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 - |u|^2)dx + \frac{1}{2(1-\theta)}\int_{\mathbb{R}^N} |u|^{2(1-\theta)}\, dx - \int_{\mathbb{R}^N} fudx$$

The functionals I^∞, $I \in C^1$ on E. It is well known that the critical points of I are weak solutions of the problem (1). So, we need some properties of critical values for I and I^∞.

Now, we give a useful result

Lemma 2.1. *There exists* $c_0 > 0$ *such that for any* $u \in E$ *we have*

$$\frac{3}{2}\|u\|_2^2 < \frac{1}{4(1-\theta)}\|u\|_{2(1-\theta)}^{2(1-\theta)} + c_0\,\|\nabla u\|_2^{2^*}$$

Proof. For each $\alpha > 0$, there exists $c_\alpha > 0$ such that

$$\frac{3}{2}s^2 < \frac{\alpha}{2(1-\theta)}|s|^{2(1-\theta)} + c_\alpha|s|^{2^*}$$

for $\alpha = \frac{1}{2}$, we have

$$\frac{3}{2}s^2 < \frac{1}{4(1-\theta)}|s|^{2(1-\theta)} + c|s|^{2^*}$$

it follows that

$$\frac{3}{2}\|u\|_2^2 < \frac{1}{4(1-\theta)}\|u\|_{2(1-\theta)}^{2(1-\theta)} + c\|u\|_{2^*}^{2^*}$$

thus

$$\frac{3}{2}\|u\|_2^2 < \frac{1}{4(1-\theta)}\|u\|_{2(1-\theta)}^{2(1-\theta)} + c_0\,\|\nabla u\|_2^{2^*}$$

□

We need the following result to study the existence of local minimum for the functional I.

Lemma 2.2. *There exists* $c_\theta > 0$, $\rho > 0$ *and* $a > 0$ *such that if* $\|f\|_2^2 < c_\theta$ *and* $\|u\| = \rho$, *then* $I(u) \geq a$

Proof. First, we observe that if $x \geq 0$ and $0 \leq y \leq 1$, then

$$x^2 + y^{2(1-\theta)} \geq \frac{1}{4}(x+y)^2$$

hence, for $\|u\|_{2(1-\theta)} \leq 1$ and $\|\nabla u\|_2 < r$ (r small enough) we get

$$\begin{aligned} I(u) &\geq \frac{1}{2}(\|\nabla u\|_2^2 - \|u\|_2^2) + \frac{1}{2(1-\theta)}\|u\|_{2(1-\theta)}^{2(1-\theta)} - \|f\|_2 \|u\|_2 \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2(1-\theta)}\|u\|_{2(1-\theta)}^{2(1-\theta)} - \frac{3}{2}\|u\|_2^2 - \|f\|_2^2 \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 - c_0\|\nabla u\|_2^{2^*} + \frac{1}{4(1-\theta)}\|u\|_{2(1-\theta)}^{2(1-\theta)} - \|f\|_2^2 \\ &\geq \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{4(1-\theta)}\|u\|_{2(1-\theta)}^{2(1-\theta)} - \|f\|_2^2 \\ &\geq \frac{1}{4}(\|\nabla u\|_2^2 + \|u\|_{2(1-\theta)}^{2(1-\theta)}) - \|f\|_2^2 \\ &\geq \frac{1}{16}\|u\|^2 - \|f\|_2^2 \end{aligned}$$

for $\|u\| = \rho = \ min(r,1)$ and $\|f\|_2^2 < \frac{1}{16}\rho^2 = c_\theta$ we obtain $I(u) \geq \frac{1}{16}\rho^2 - \|f\|_2^2 = a > 0$ □

Lemma 2.3.

$$c^\infty = \inf_{u \in B_\rho} I^\infty(u) = I^\infty(0) = 0 \tag{7}$$

where $B_\rho = \{u \in E, \|u\| \leq \rho\}$ *and* ρ *is fixed in lemma 2.2*

Proof. $c^\infty \leq I^\infty(0) = 0$. Suppose $c^\infty < 0$, let $(v_n) \in B_\rho$ a minimizing sequence of problem (7), $I^\infty(v_n) < 0$ for n large enough then

$$\frac{1}{2}\|\nabla v_n\|_2^2 + \frac{1}{2(1-\theta)}\|v_n\|_{2(1-\theta)}^{2(1-\theta)} < \frac{1}{2}\|v_n\|_2^2 \leq \frac{3}{2}\|v_n\|_2^2$$

by lemma 2.1, we obtain

$$\frac{1}{2}\|\nabla v_n\|_2^2 + \frac{1}{2(1-\theta)}\|v_n\|_{2(1-\theta)}^{2(1-\theta)} < \frac{1}{4(1-\theta)}\|v_n\|_{2(1-\theta)}^{2(1-\theta)} + c_0\|\nabla v_n\|_2^{2^*}$$

and

$$\begin{aligned} \frac{1}{4}\|\nabla v_n\|_2^2 + \frac{1}{4(1-\theta)}\|v_n\|_{2(1-\theta)}^{2(1-\theta)} &\leq \frac{1}{2}\|\nabla v_n\|_2^2 - c_0\|\nabla v_n\|_2^{2^*} \\ &\quad + \frac{1}{4(1-\theta)}\|v_n\|_{2(1-\theta)}^{2(1-\theta)} < 0 \end{aligned}$$

this is a contradiction, therefore $c^\infty = 0$ □

At the end of this section we give the following useful remark.

Remark 2.4. There exists $c > 0$ such that for any $a, b \in \mathbb{R}$ we have

$$\begin{aligned}
\text{i)}\quad & |a|^{2(1-\theta)} + |b|^{2(1-\theta)} - c|a|^{1-\theta}|b|^{1-\theta} \le |a+b|^{2(1-\theta)} \\
& \qquad \le |a|^{2(1-\theta)} + |b|^{2(1-\theta)} + c|a|^{1-\theta}|b|^{1-\theta} \\
\text{ii)}\quad & |a|^{2(1-\theta)} + |b|^{2(1-\theta)} - c|a|^{1-2\theta}|b| \le |a+b|^{2(1-\theta)} \\
& \qquad \le |a|^{2(1-\theta)} + |b|^{2(1-\theta)} + c|a|^{1-2\theta}|b| \\
\text{iii)}\quad & |a+b|^{1-2\theta} \le |a|^{1-2\theta} + |b|^{1-2\theta}
\end{aligned}$$

For the proof of this remark see appendix.

3. Existence of the first solution for problem (1)

Let

$$c = \inf_{B_\rho} I(u) \tag{8}$$

where ρ is fixed in lemma 2.3

Lemma 3.1. $-\infty < c < 0$

Proof. 1) $c > -\infty$: by the same argument used in lemma 2.3, we have

$$I(u) \ge -\|f\|_2^2, \quad \text{for any } u \in B_\rho$$

therefore $c > -\infty$.

2) $c < 0$: since $f \ne 0$, $f \ge 0$ then we can choose a function $\varphi \in E$ such that $\int_{\mathbb{R}^N} f\varphi dx > 0$. let $\lambda > 0$, we have

$$I(\lambda\varphi) = \frac{\lambda^2}{2}\int_{\mathbb{R}^N} (|\nabla\varphi|^2 - \varphi^2) + \frac{\lambda^{2(1-\theta)}}{2(1-\theta)}\int_{\mathbb{R}^N} |\varphi|^{2(1-\theta)} - \lambda\int_{\mathbb{R}^N} f\varphi$$

for λ small enough, $\lambda\varphi \in B_\rho$ and $I(\lambda\varphi) < 0$, thus $c < 0$ □

Now, we state the main result of this section.

Theorem 3.2. *There exists $U_0 \in B_\rho$, such that $c = I(U_0)$ and U_0 is a nonnegative solution of problem* (1)

Proof. Let (u_n) a minimizing sequence of problem (8) in B_ρ, $I(u_n) \to c$. Since $c \le I(|u_n|) \le I(u_n)$, then $I(|u_n|) \to c$; we can choose $u_n \ge 0$. (u_n) is bounded on E, we extract a subsequence of (u_n), also denoted (u_n) such that $u_n \rightharpoonup U_0$ in E, $u_n \to U_0$ in $L^q_{loc}(\mathbb{R}^N)$, $\forall 1 \le q < 2^*$ and $u_n \to U_0$ a.e in $\mathbb{R}^N$. It follows that $U_0 \ge 0$ a.e on $\mathbb{R}^N$.

$u_n \to U_0$ in E indeed, put $v_n = u_n - U_0 \rightharpoonup 0$ in E. By remark 2.4, we have

$$\begin{aligned} I(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_n|^2 - u_n^2)dx + \frac{1}{2(1-\theta)}\int_{\mathbb{R}^N}|u_n|^{2(1-\theta)}dx - \int_{\mathbb{R}^N} f u_n dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v_n|^2 + |\nabla U_0|^2 + 2\nabla v_n \nabla U_0 - |v_n|^2 - |U_0|^2 - 2v_n U_0)dx \\ &\quad + \frac{1}{2(1-\theta)}\int_{\mathbb{R}^N}(|v_n|^{2(1-\theta)} + |U_0|^{2(1-\theta)})dx - c\int_{\mathbb{R}^N}|v_n||U_0|^{1-2\theta}dx \\ &\quad - \int_{\mathbb{R}^N} f v_n dx - \int_{\mathbb{R}^N} f U_0 dx \end{aligned}$$

$v_n \rightharpoonup 0$ in E, so

$$\int_{\mathbb{R}^N}(\nabla U_0 \nabla v_n - U_0 v_n)dx \to 0,\ \int_{\mathbb{R}^N} f v_n dx \to 0,\ \int_{\mathbb{R}^N}|v_n||U_0|^{1-2\theta}dx \to 0$$

therefore

$$c \geq I(U_0) + \lim_{n\to\infty}\frac{1}{2}(\|\nabla v_n\|_2^2 - \|v_n\|_2^2) + \frac{1}{2(1-\theta)}\|v_n\|_{2(1-\theta)}^{2(1-\theta)}$$

we have $\|\nabla v_n\|_2 \leq \rho$ indeed,

$$\|\nabla u_n\|_2^2 = \|\nabla v_n\|_2^2 + \|\nabla U_0\|_2^2 + 2\int_{\mathbb{R}^N}\nabla v_n \nabla U_0 dx < \rho^2$$

since $\int_{\mathbb{R}^N}\nabla v_n \nabla U_0 dx \to 0$, then for n large enough $\|\nabla v_n\|_2 \leq \rho$ and

$$\frac{1}{2}(\|\nabla v_n\|_2^2 - \|v_n\|_2^2) + \frac{1}{2(1-\theta)}\|v_n\|_{2(1-\theta)}^{2(1-\theta)} \geq \frac{1}{4}\|\nabla v_n\|_2^2 + \frac{1}{4(1-\theta)}\|v_n\|_{2(1-\theta)}^{2(1-\theta)}$$

it follows that

$$c \geq I(U_0) + \lim_{n\to\infty}\frac{1}{4}\|\nabla v_n\|_2^2 + \frac{1}{4(1-\theta)}\|v_n\|_{2(1-\theta)}^{2(1-\theta)} \geq c$$

thus $u_n \to U_0$ in E and $I(u_n) \to I(U_0) = c < 0$ so $U_0 \in B_\rho,\ I'(U_0) = 0$ and U_0 is a nonnegative solution of problem (1). □

Now, we give a continuity result about the first solution.

Theorem 3.3. *If $f \to 0$ in L^2 then $U_0(f) \to 0$ in E*

Proof. (k) $U_0 \to 0$ in $L^{2(1-\theta)}$, indeed: U_0 is a solution of s(1), $I'(U_0) = 0$. and

$$\|\nabla U_0\|_2^2 = \|U_0\|_2^2 - \|U_0\|_{2(1-\theta)}^{2(1-\theta)} + \int_{\mathbb{R}^N} f(x)U_0(x)dx$$

we replace in $I(U_0)$ expression's, we obtain

$$I(U_0) = \frac{\theta}{2(1-\theta)}\|U_0\|_{2(1-\theta)}^{2(1-\theta)} - \frac{1}{2}\int_{\mathbb{R}^N} f U_0 dx$$

$$\text{since}\quad I(U_0) < 0\ \text{ then }\ \frac{\theta}{2(1-\theta)}\|U_0\|_{2(1-\theta)}^{2(1-\theta)} < \frac{1}{2}\int_{\mathbb{R}^N} f U_0 dx \to 0$$

therefore $U_0 \to 0$ in $L^{2(1-\theta)}$. (kk) $\|\nabla U_0\|_2 \to 0$, indeed: there exists $r \in]0,1[$ such that

$$\|U_0\|_2 \le \|U_0\|_{2(1-\theta)}^r \|U_0\|_{2^*}^{1-r} \le c \|U_0\|_{2(1-\theta)}^r \|\nabla U_0\|_2^{1-r} \le c_1 \|U_0\|_{2(1-\theta)}^r$$

this gives

$$\|\nabla U_0\|_2^2 = \|U_0\|_2^2 - \|U_0\|_{2(1-\theta)}^{2(1-\theta)} + \int_{\mathbb{R}^N} fU_0 dx \to 0$$

it leads to $U_0 \to 0$ in E. □

4. Existence of a second solution for problem (1)

Seeking a second solution for problem (1), we need a mountain-pass structure, for that we give the following result.

Lemma 4.1. *Let w the ground state of the homogeneous problem (2) [25], then*

$$I(U_0 + tw) < I(U_0) + I^\infty(w) \quad \forall t \ge 0$$

where U_0 is the first solution of problem (1) given by therem 3.2

Proof. First, we observe that if $a,\ b$ are arbitrary positive reals then

$$(a+b)^{2(1-\theta)} < a^{2(1-\theta)} + b^{2(1-\theta)} + 2(1-\theta)a^{1-2\theta}b$$

let $t > 0$, we have

$$\begin{aligned} I(U_0 + tw) = &\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla(U_0+tw)|^2 - (U_0+tw)^2)dx + \frac{1}{2(1-\theta)}\int_{\mathbb{R}^N} |U_0+tw|^{2(1-\theta)} dx \\ &- \int_{\mathbb{R}^N} f(x)(U_0(x)+tw(x))dx < \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla U_0|^2 - U_0^2)dx + \frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 - w^2)dx \\ &+ t\int_{\mathbb{R}^N}(\nabla U_0 \nabla w - U_0 w)dx + \frac{1}{2(1-\theta)}\int_{\mathbb{R}^N}|U_0|^{2(1-\theta)}dx + \frac{t^{2(1-\theta)}}{2(1-\theta)}\int_{\mathbb{R}^N}|w|^{2(1-\theta)}dx \\ &+ t\int_{\mathbb{R}^N} |U_0|^{1-2\theta} w dx - \int_{\mathbb{R}^N} fU_0 dx - t\int_{\mathbb{R}^N} fw dx \end{aligned}$$

U_0 is a critical point of I and w is a critical point of I^∞, then

$$\int_{\mathbb{R}^N} \nabla U_0 \nabla w = \int_{\mathbb{R}^N} U_0 w - \int_{\mathbb{R}^N} |U_0|^{-2\theta} U_0 w + \int_{\mathbb{R}^N} fw dx$$

and

$$\int_{\mathbb{R}^N} (|\nabla w|^2 - w^2)dx = -\int_{\mathbb{R}^N} |w|^{2(1-\theta)} dx$$

this gives

$$I^\infty(w) = \left(\frac{1}{2(1-\theta)} - \frac{1}{2}\right)\|w\|_{2(1-\theta)}^{2(1-\theta)} > 0$$

and

$$I(U_0+tw) < I(U_0) + \left(\frac{t^{2(1-\theta)}}{2(1-\theta)} - \frac{t^2}{2}\right)\|w\|_{2(1-\theta)}^{2(1-\theta)} \le I(U_0) + I^\infty(w)$$

it holds that

$$I(U_0 + tw) < I(U_0) + I^\infty(w) \qquad \forall t > 0$$

□

Lemma 4.2. *There exists $\varphi_0 \in E \backslash B_\rho$, $\varphi_0 \geq 0$ such that $I(\varphi_0) < 0$*

Proof

$$I(U_0 + tw) < I(U_0) + \left(\frac{t^{2(1-\theta)}}{2(1-\theta)} - \frac{t^2}{2} \right) \|w\|_{2(1-\theta)}^{2(1-\theta)}$$

for t_0 large enough, $I(U_0 + t_0 w) < 0$. We put $\varphi_0 = U_0 + t_0 w$. □

Consider the problem

$$d = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \tag{9}$$

where

$$\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = U_0,\ \gamma(1) = \varphi_0\} \neq \emptyset$$

Lemma 4.3. $a \leq d < I(U_0) + I^\infty(w)$ (*a fixed by lemma* 2.2)

Proof. 1. $d < I(U_0) + I^\infty(w)$ indeed: let $\gamma_0 :\ [0,1] \to\ E$ defined by $\gamma_0(t) = U_0 + t t_0 w$, where t_0 fixed in the proof of lemma 4.2, $\gamma_0 \in \Gamma$. There exists $T \in [0,1]$ such that

$$\begin{aligned} d = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) &\leq \max_{t \in [0,1]} I(\gamma_0(t)) \\ &= I(\gamma_0(T)) = I(U_0 + T t_0 w) < I(U_0) + I^\infty(w) \end{aligned}$$

2. $d \geq a$ indeed: let $\gamma \in \Gamma$, $\gamma(0) \in B_\rho$ and $\gamma(1) \notin B_\rho$. There exists $s \in [0,1]$ such that $\gamma(s) \in \partial B_\rho$, then

$$\sup_{t \in [0,1]} I(\gamma(t)) \geq \inf_{u \in \partial B_\rho} I(u) \geq a \ \text{ thus } d \geq a$$

□

Lemma 4.4. *Let $(u_n) \subset E$ such that $(I(u_n))$ is bounded and $I'(u_n) \to 0$ in E', then (u_n) is bounded in E.*

Proof. There exists $M > 0$ such that for n large enough, $|I(u_n)| < M$ and $|I'(u_n).u_n| \leq \|u_n\|$.

$$\begin{aligned} I(u_n) - \frac{1}{2} I'(u_n).u_n &= \frac{\theta}{2(1-\theta)} \|u_n\|_{2(1-\theta)}^{2(1-\theta)} - \frac{1}{2} \int_{\mathbb{R}^N} f u_n dx \\ &\geq \frac{\theta}{2(1-\theta)} \|u_n\|_{2(1-\theta)}^{2(1-\theta)} - \frac{1}{2} \|f\|_{\frac{2(1-\theta)}{1-2\theta}} \|u_n\|_{2(1-\theta)} \\ &\geq \frac{\theta}{4(1-\theta)} \|u_n\|_{2(1-\theta)}^{2(1-\theta)} - c \|f\|_{\frac{2(1-\theta)}{1-2\theta}}^{\frac{2(1-\theta)}{1-2\theta}} \ \text{(by Young inequality)} \end{aligned}$$

we have

$$\frac{\theta}{4(1-\theta)} \|u_n\|_{2(1-\theta)}^{2(1-\theta)} - c_1 < M + \frac{1}{2} \|u_n\|$$

and

$$\frac{\theta}{4(1-\theta)} \|u_n\|_{2(1-\theta)}^{2(1-\theta)} - \frac{1}{2} \|u_n\|_{2(1-\theta)} < \frac{1}{2} \|\nabla u_n\|_2 + c_2$$

we suppose by contrary $\|u_n\|_{2(1-\theta)} \to +\infty$, let $1 < p < 2(1-\theta)$, for n large enough we get

$$\|u_n\|_{2(1-\theta)}^p < \frac{\theta}{4(1-\theta)} \|u_n\|_{2(1-\theta)}^{2(1-\theta)} - \frac{1}{2} \|u_n\|_{2(1-\theta)} < \frac{1}{2} \|\nabla u_n\|_2 + c_2$$

this leads to

$$\|u_n\|_{2(1-\theta)} < c \|\nabla u_n\|_2^{\frac{1}{p}} + c_3 \qquad (*)$$

on the other hand, by Gagliardo–Nirenberg–Sobolev and Holder inequalities we obtain

$$\begin{aligned} \|u_n\|_2^2 \leq c_4 \|\nabla u_n\|_2^{2r} \|u_n\|_{2(1-\theta)}^{2(1-r)} &\leq c_4 \|\nabla u_n\|_2^{2r} (c \|\nabla u_n\|_2^{\frac{1}{p}} + c_3)^{2(1-r)} \\ &\leq c_5 \|\nabla u_n\|_2^{2r+\frac{2(1-r)}{p}} + c_6 \|\nabla u_n\|_2^{2r} \\ &\quad + c_7 \|\nabla u_n\|_2^{2r+\frac{1-r}{p}} \end{aligned}$$

with $r = \dfrac{2^* \ \theta}{2^* - 2(1-\theta)}$. It yields that

$$\begin{aligned} M > I(u_n) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 - u_n^2)dx + \frac{1}{2(1-\theta)} \int_{\mathbb{R}^N} |u_n|^{2(1-\theta)} dx - \int_{\mathbb{R}^N} f u_n dx \\ &\geq \frac{1}{2} \|\nabla u_n\|_2^2 - \|u_n\|_2^2 - \frac{1}{2} \|f\|_2^2 \\ &\geq \frac{1}{2} \|\nabla u_n\|_2^2 - c_5 \|\nabla u_n\|_2^{2r+\frac{2(1-r)}{p}} - c_6 \|\nabla u_n\|_2^{2r} \\ &\quad - c_7 \|\nabla u_n\|_2^{2r+\frac{1-r}{p}} - \frac{1}{2} \|f\|_2^2 \end{aligned}$$

it follows that $\|\nabla u_n\|_2$ is bounded, that contradicts $(*)$. So (u_n) is bounded in $L^{2(1-\theta)}(\mathbb{R}^N)$. We obtain

$$\|u_n\|_2^2 \leq c \|\nabla u_n\|_2^{2r} \|u_n\|_{2(1-\theta)}^{2(1-r)} \leq c' \|\nabla u_n\|_2^{2r}$$

and

$$M > I(u_n) \geq \frac{1}{2} \|\nabla u_n\|_2^2 - c' \|\nabla u_n\|_2^{2r} - \frac{1}{2} \|f\|_2^2$$

therefore $\|\nabla u_n\|_2$ is bounded and (u_n) is bounded in E. □

Theorem 4.5. *There exists V_0 a second solution of problem* (1)

Proof. By the mountain-pass theorem of Ambrosetti Rabinowitz [2] that there exists a (PS) sequence (u_n) in E such that

$$I(u_n) \to d \quad \text{and} \quad I'(u_n) \to 0 \ \text{ in } E'$$

by lemma 4.4 (u_n) is bounded in E then up to a subsequence $u_n \rightharpoonup V_0$ in E, $u_n \to V_0$ in L^q_{loc} $\forall 1 \leq q < 2^*$ and $u_n \to V_0$ a.e in $\mathbb{R}^N$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$

$$I'(u_n).\varphi = \int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi - u_n \varphi) dx + \int_{\mathbb{R}^N} |u_n|^{-2\theta} u_n \varphi dx - \int_{\mathbb{R}^N} f \varphi dx \to 0$$

since $u_n \rightharpoonup V_0$ then, $\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi - u_n \varphi) dx \to \int_{\mathbb{R}^N} (\nabla V_0 \nabla \varphi - V_0 \varphi) dx$ let us show that:

$$\int_{\mathbb{R}^N} |u_n(x)|^{-2\theta} u_n(x) \varphi(x) dx \to \int_{\mathbb{R}^N} |V_0(x)|^{-2\theta} V_0(x)\ \varphi(x) dx$$

since $u_n \to V_0$ in $L^{2(1-\theta)}_{loc}(\mathbb{R}^N)$ then, there exists a subsequence denoted by (u_n) and $h \in L^{2(1-\theta)}(\mathbb{R}^N)$ such that

$$\begin{cases} |u_n|^{-2\theta} u_n\ \varphi \to |V_0|^{-2\theta} V_0\ \varphi \quad \text{a.e} \\ |u_n|^{1-2\theta} |\varphi| \leq |h|^{1-2\theta} |\varphi| \ \text{ in } L^1(\mathbb{R}^N) \end{cases}$$

then by dominated convergence theorem

$$\int_{\mathbb{R}^N} |u_n(x)|^{-2\theta} u_n(x) \varphi(x) dx \to \int_{\mathbb{R}^N} |V_0(x)|^{-2\theta} V_0(x) \varphi(x) dx$$

it follows that

$$I'(u_n).\varphi \to I'(V_0).\varphi = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N), \quad V_0 \text{ is a weak solution of problem (1)}$$

Now, we prove that $V_0 \neq U_0$, where U_0 the first solution of (1) given by theorem 3.2.

We have $u_n \rightharpoonup V_0$ in E up to a subsequences, then $\|V_0\|_E \leq \underline{\lim} \|u_n\| = \lim_{n\to\infty} \|u_n\|$. We distinguish two cases:

($\bullet$) (Compactness) $\|V_0\|_E = \lim_{n\to\infty} \|u_n\|$ then $u_n \to V_0$ in E, in fact

$$\begin{aligned} \overline{\lim}_{n\to\infty} \|u_n\|_{2(1-\theta)} &= \|V_0\|_{2(1-\theta)} + \|\nabla V_0\|_2 - \overline{\lim}\, \|\nabla u_n\|_2 \\ &\leq \|V_0\|_{2(1-\theta)} + \|\nabla V_0\|_2 - \underline{\lim}\, \|\nabla u_n\|_2 \end{aligned}$$

and

$$\|\nabla V_0\|_2 \leq \underline{\lim}\, \|\nabla u_n\|_2\,, \ \|V_0\|_{2(1-\theta)} \leq \underline{\lim}\, \|u_n\|_{2(1-\theta)}$$

we obtain

$$\|V_0\|_{2(1-\theta)} \leq \underline{\lim}\, \|u_n\|_{2(1-\theta)} \leq \overline{\lim}\, \|u_n\|_{2(1-\theta)} \leq \|V_0\|_{2(1-\theta)}$$

thus

$$\begin{cases} u_n \to V_0 \quad \text{ae in } \mathbb{R}^N \\ \|u_n\|_{2(1-\theta)} \to \|V_0\|_{2(1-\theta)} \end{cases}$$

this gives

(i)

$$u_n \to V_0 \quad \text{in} \quad L^{2(1-\theta)}(\mathbb{R}^N)$$

therefore

$$\|\nabla u_n\|_2 \to \|\nabla V_0\|_2$$

$$\int_{\mathbb{R}^N} |\nabla u_n - \nabla V_0|^2 \, dx = \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} |\nabla V_0|^2 \, dx - 2\int_{\mathbb{R}^N} \nabla u_n \nabla V_0 dx$$

else

$$\int_{\mathbb{R}^N} \nabla u_n \nabla V_0 dx \to \int_{\mathbb{R}^N} |\nabla V_0|^2 \, dx$$

so

(ii)

$$\|\nabla u_n - \nabla V_0\|_2 \to 0$$

(i) and (ii) give $u_n \to V_0$ in E and then $I(u_n) \to d = I(V_0) > 0$. So $V_0 \neq U_0$ $(I(U_0) < 0)$

□

(••) (Dichotomy) $\|V_0\|_E < \lim_{n\to\infty} \|u_n\|$. Set $v_n(x) = u_n(x) - V_0(x)$, $v_n \rightharpoonup 0$ in E.

Step 1: there exists $(y_n^1) \subset \mathbb{R}^N$ such that $v_n(\cdot + y_n^1) \rightharpoonup V_1 \neq 0$ in E

Proof. Suppose that, $\forall (y_n) \subset \mathbb{R}^N$, $v_n(\cdot + y_n) \rightharpoonup 0$ in E, then

$$\forall R > 0 \qquad \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |v_n|^{2(1-\theta)} dx \to 0$$

by the argument of P.L. Lions [24],

$$v_n \to 0 \text{ in } L^q(\mathbb{R}^N), \quad \forall \quad 2(1-\theta) < q < 2^*.$$

On the other hand, $u_n(x) = v_n(x) + V_0(x)$

$$\begin{aligned} I'(u_n)\cdot u_n &= \int_{\mathbb{R}^N} (|\nabla u_n|^2 - u_n^2)dx + \int_{\mathbb{R}^N} |u_n|^{2(1-\theta)} dx - \int_{\mathbb{R}^N} f u_n dx \\ &= \int_{\mathbb{R}^N} (|\nabla v_n|^2 - v_n^2 + |\nabla V_0|^2 - |V_0|^2 + \nabla v_n \nabla V_0 - v_n V_0)dx \\ &\quad + \int_{\mathbb{R}^N} |u_n|^{2(1-\theta)} dx - \int_{\mathbb{R}^N} f v_n - \int_{\mathbb{R}^N} f V_0 \end{aligned}$$

so by remark 2.4 we infer that

$$0 = \lim_{n\to\infty} I'(u_n)\cdot u_n = I'(V_0)\cdot V_0 + \lim_{n\to\infty}\left(\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} |v_n|^{2(1-\theta)} dx\right)$$

then $v_n \to 0$ in E, that is a contradiction and there exist $(y_n^1) \subset \mathbb{R}^N$ such that $v_n(\cdot + y_n^1) \to V_1 \neq 0$ in E. □

Step 2: (y_n^1) is not bounded.

Proof. suppose (y_n^1) is bounded, we extract a subsequence of (y_n^1), also denoted by (y_n^1) such that $y_n^1 \to y$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, since $y_n^1 \to y$ and $v_n \rightharpoonup 0$ in E, then

$$\int_{\mathbb{R}^N} \varphi(x - y_n^1) v_n(x) dx \to 0$$

$v_n(\cdot + y_n^1) \rightharpoonup V_1$ in E, so

$$\int_{\mathbb{R}^N} \varphi(x - y_n^1) v_n(x) dx = \int_{\mathbb{R}^N} \varphi(x) v_n(x + y_n^1) dx \to \int_{\mathbb{R}^N} \varphi(x) V_1(x) dx$$

it yields

$$\int_{\mathbb{R}^N} \varphi(x) V_1(x) dx = 0, \ \forall \varphi \in C_c^\infty(\mathbb{R}^N)$$

Hence $V_1 = 0$ ae in $\mathbb{R}^N$, that is a contradiction. Thus (y_n^1) is not bounded. □

Conclusion There exists $(y_n^1) \subset \mathbb{R}^N$ such that (y_n^1) is not bounded, $v_n(\cdot + y_n^1) \rightharpoonup V_1 \neq 0$ in E, and V_1 is a solution of the homogeneous problem (2) (to be proved later). Also

$$\|V_1\|_E \leq \lim_{n\to\infty} \left\|v_n(\cdot + y_n^1)\right\|_E$$

(j) if $\|V_1\|_E = \lim_{n\to\infty} \|v_n(\cdot + y_n^1)\|_E$ then, $u_n - V_0 - V_1(\cdot - y_n^1) \to 0$ in E thus

$$I(u_n) \to d = I(V_0) + I^\infty(V_1)$$

Since w is the ground state of (2), then

$$I(V_0) + I^\infty(w) \leq I(V_0) + I^\infty(V_1) = d < I(U_0) + I^\infty(w) \quad \text{(by lemma 4.3)}$$

Hence $U_0 \neq V_0$ $(jj)\|V_1\|_E < \lim_{n\to\infty} \|v_n(\cdot + y_n^1)\|_E$, put $v_n^1 = u_n - V_0 - V_1(\cdot - y_n^1)$. We restart the analysis with the sequence (v_n^1) while reiterating the process as many time as necessary, we have a general decomposition of the form:

$$u_n(x) - (V_0(x) + \sum_{k=1}^{m} V_k(x - y_n^k)) \to 0 \text{ in E} \quad (\text{eventually } m = +\infty)$$

$$\forall k \geq 1, \ \left|y_n^{(k)}\right| \to +\infty, \ \left|y_n^{(k)} - y_n^{(l)}\right| \to +\infty \text{ if } k \neq l$$

and

$$I(u_n) \to d = I(V_0) + \sum_{k=1}^{m} I^\infty(V_k)$$

even, $(V_k)_{1\leq k\leq m}$ are solutions of the homogeneous problem (2) and $I^\infty(V_k) > 0, \ \forall 1 \leq k \leq m$. Since w is the ground state of (2), then

$$I(U_0) + I^\infty(w) > d > I(V_0) + I^\infty(V_1) \geq I(V_0) + I^\infty(w) \quad \text{therefore} \quad U_0 \neq V_0$$ □

Now, we prove that $(V_k)_{1\le k\le m}$ are solutions of the homogeneous problem (2)

Proposition 4.6. *$(V_k)_{1\le k\le m}$ are solutions of the homogeneous problem (2).*

Proof. V_1 is a solution of problem (2)s, indeed: $u_n(x+y_n^1) = V_0(x+y_n^1) + v_n(x+y_n^1) \rightharpoonup V_1$ in E. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\langle I'(u_n), \varphi(\cdot - y_n^1)\rangle &= \int_{\mathbb{R}^N} (\nabla u_n \nabla\varphi(x-y_n^1) - u_n(x)\varphi(x-y_n^1))dx \\ &\quad + \int_{\mathbb{R}^N} |u_n(x)|^{-2\theta} u_n(x)\varphi(x-y_n^1)dx - \int_{\mathbb{R}^N} f(x)\varphi(x-y_n^1)dx \\ &= \int_{\mathbb{R}^N} (\nabla u_n(x+y_n^1)\nabla\varphi(x) - u_n(x+y_n^1)\varphi(x))dx \\ &\quad + \int_{\mathbb{R}^N} |u_n(x+y_n^1)|^{-2\theta} u_n(x+y_n^1)\varphi(x)dx - \int_{\mathbb{R}^N} f(x)\varphi(x-y_n^1)dx\end{aligned}$$

since $|y_n^1| \to +\infty$ then, $\int_{\mathbb{R}^N} f(x)\varphi(x-y_n^1)dx \to 0$
$u_n(\cdot + y_n^1) \rightharpoonup V_1$ in E, so

$$\int_{\mathbb{R}^N} \nabla u_n(x+y_n^1)\nabla\varphi(x) - u_n(x+y_n^1)\varphi(x) \to \int_{\mathbb{R}^N} \nabla V_1 \nabla\varphi - V_1\varphi$$

also

$$\langle I'(u_n), \varphi(\cdot - y_n^1)\rangle \to 0$$

by the same argument used above, we get

$$\int_{\mathbb{R}^N} |u_n(x+y_n^1)|^{-2\theta} u_n(x+y_n^1)\varphi(x)dx \to \int_{\mathbb{R}^N} |V_1(x)|^{-2\theta} V_1(x)\ \varphi(x)dx$$

It follows that, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (\nabla V_1 \nabla\varphi - V_1\varphi + |V_1|^{-2\theta} V_1\varphi)dx = 0$$

then V_1 is a weak solution of problem (2). by the same argument we show that $(V_k)_{1\le k\le m}$ are solutions of (2).

This achieves the proof. □

Appendix

We give here the proof of remark 2.4.

Proof of Remark 2.4. It easy to check (i), (ii) and (iii) for $a = 0$. Suppose $a \neq 0,\ b \in \mathbb{R}$.

1. Proof of (i) Consider the continuous function φ defined on $\mathbb{R}\backslash\{0\}$ by

$$\varphi(x) = \frac{|1+x|^{2(1-\theta)} - |x|^{2(1-\theta)} - 1}{|x|^{1-\theta}}$$

We have $\lim_{|x|\to+\infty} \varphi(x) = 0$ and $\lim_{|x|\to 0} \varphi(x) = 0$. Then, there exists $c > 0$ such that $|\varphi(x)| < c$ for any $x \in \mathbb{R}$. $|\varphi(\frac{b}{a})| < c$ leads to

$$\left|\left|1+\frac{b}{a}\right|^{2(1-\theta)} - \left|\frac{b}{a}\right|^{2(1-\theta)} - 1\right| < c\left|\frac{b}{a}\right|^{1-\theta}$$

and multiplying by $|a|^{2(1-\theta)}$, we obtain the desire result.

2. Proof of (ii) Consider the continuous function ψ defined on $\mathbb{R}\backslash\{0\}$ by

$$\psi(x) = \frac{|1+x|^{2(1-\theta)} - |x|^{2(1-\theta)} - 1}{|x|}$$

We have $\lim_{|x|\to+\infty} \psi(x) = 0$ and $\lim_{x\to 0^\pm} \psi(x) = \pm 2(1-\theta)$. Then, there exists $c > 0$ such that $|\psi(x)| < c$ for any $x \in \mathbb{R}$. $\left|\psi\left(\frac{b}{a}\right)\right| < c$ yield

$$\left|\left|1+\frac{b}{a}\right|^{2(1-\theta)} - \left|\frac{b}{a}\right|^{2(1-\theta)} - 1\right| < c\left|\frac{b}{a}\right|$$

and multiplying by $|a|^{2(1-\theta)}$, we obtain the desire result.

3. Proof of (iii) Consider the continuous function ϕ defined on $\mathbb{R}$ by

$$\phi(x) = |1+x|^{1-2\theta} - |x|^{1-2\theta} - 1$$

we have $\phi(x) \leq \phi(0) = 0$ for any $x \in [-1, +\infty[$, ϕ is decreasing on $]-\infty, -1[$ and $\lim\limits_{x\to-\infty} \phi(x) = -1$. Consequently for any x in $\mathbb{R}$, we have $\phi(x) \leq 0$.

Then $\phi\left(\frac{b}{a}\right) < 0$ and

$$\left|1+\frac{b}{a}\right|^{1-2\theta} \leq \left|\frac{b}{a}\right|^{1-2\theta} + 1$$

and multiplying by $|a|^{1-2\theta}$, we obtain the desire result.

This achieves the proof of Remark 2.4. □

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