Nonlinear Differ. Equ. Appl. 17 (2010), 323–336

1021-9722/10/030323-14
*published online* January 30, 2010
DOI 10.1007/s00030-010-0056-3

**Nonlinear Differential Equations and Applications NoDEA**

# Global bifurcation results for semilinear elliptic boundary value problems with indefinite weights and nonlinear boundary conditions

Kenichiro Umezu

**Abstract.** We investigate the global nature of bifurcation components of positive solutions of a general class of semilinear elliptic boundary value problems with nonlinear boundary conditions and having linear terms with sign-changing coefficients. We first show that there exists a *subcontinuum*, i.e., a maximal closed and connected component, emanating from the line of trivial solutions at a simple principal eigenvalue of a linearized eigenvalue problem. We next consider sufficient conditions such that the subcontinuum is unbounded in some space for a semilinear elliptic problem arising from population dynamics. Our approach to establishing the existence of the subcontinuum is based on the global bifurcation theory proposed by López-Gómez. We also discuss an a priori bound of solutions and deduce from it some results on the multiplicity of positive solutions.

**Mathematics Subject Classification (2000).** Primary: 35P15, 35J20; Secondary: 92D25.

**Keywords.** Semilinear elliptic problem, Indefinite weight, Positive solution, Bifurcation, Global subcontinuum, A priori bound, Multiplicity, Population dynamics.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. This paper is devoted to a general class of semilinear indefinite weighted elliptic boundary value problems with nonlinear boundary conditions of the following type:

$$\begin{cases} -\Delta u = \lambda(m(x)u + g_1(x,u)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda(\sigma(x)u + g_2(x,u)) & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here,

(1) $\Delta$ denotes the usual Laplacian in $\mathbb{R}^N$;
(2) $\lambda \geq 0$ is a parameter;
(3) $m \in C^\theta(\overline{\Omega})$ and $\sigma \in C^{1+\theta}(\partial\Omega)$ are Hölder continuous functions with some exponent $0 < \theta < 1$, which *cannot* satisfy the condition that $m(x) \leq 0$ in $\overline{\Omega}$ and $\sigma(x) \leq 0$ on $\partial\Omega$;
(4) $g_1 \in C^\theta(\overline{\Omega} \times [0,\rho])$ and $g_2 \in C^{1+\theta}(\partial\Omega \times [0,\rho])$ for every $\rho > 0$, and it holds that

$$\lim_{u \to +0} \frac{g_1(x,u)}{u} = 0 \quad \text{uniformly in } x \in \overline{\Omega}, \tag{1.2}$$

$$\lim_{u \to +0} \frac{g_2(x,u)}{u} = 0 \quad \text{uniformly in } x \in \partial\Omega; \quad \text{and} \tag{1.3}$$

(5) $\mathbf{n}$ is the unit outer normal to $\partial\Omega$.

A *solution* $u$ of (1.1) implies that $u \in C^2(\overline{\Omega})$ satisfies (1.1). A solution $u$ of (1.1) is called *positive* if it is positive in $\overline{\Omega}$. The aim of this paper is to investigate the set of $(\lambda, u)$ where $u$ is a positive solution of (1.1) for some $\lambda > 0$. It is clear that any positive constant $c$ is a positive solution of (1.1) for $\lambda = 0$. It is also clear from (1.2) and (1.3) that $u \equiv 0$ is a solution of (1.1) for all $\lambda \geq 0$. Thus, problem (1.1) possesses the following *two* trivial lines of solutions: $\Gamma_1 := \{(\lambda, 0) : \lambda \geq 0\}$ and $\Gamma_2 := \{(0, c) : c \geq 0 \text{ is a constant}\}$.

In this paper, we concentrate on the following case:

$$\int_\Omega m(x)dx + \int_{\partial\Omega} \sigma(x)ds < 0, \tag{1.4}$$

where $ds$ denotes the surface element of $\partial\Omega$. It is known ([19, Theorem 2.2]) that if (1.4) is satisfied, then there exists a unique positive principal eigenvalue $\lambda_1(m,\sigma)$ of the linearized eigenvalue problem

$$\begin{cases} -\Delta \varphi = \lambda m(x)\varphi & \text{in } \Omega, \\ \dfrac{\partial \varphi}{\partial \mathbf{n}} = \lambda\sigma(x)\varphi & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

which is simple, and it has no positive principal eigenvalue otherwise. We refer to the classical result of Brown and Lin [4] for the case when $\sigma \equiv 0$ on $\partial\Omega$. An eigenvalue of (1.5) is called *principal* if it has positive eigenfunctions in $\overline{\Omega}$. By $\varphi_1(m,\sigma)$ we denote the positive eigenfunction associated with $\lambda_1(m,\sigma)$ for (1.4). In the case when (1.4) does not hold, 0 is a unique non-negative principal eigenvalue of (1.5), for which all the eigenfunctions are constants ([19, Theorem 2.2]).

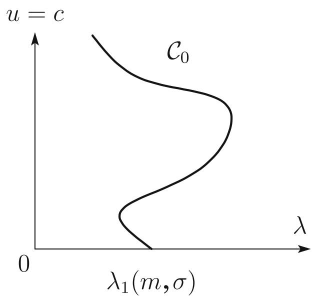


FIGURE 1. Unbounded subcontinuum in $\mathbb{R}_+ \times C(\overline{\Omega})$

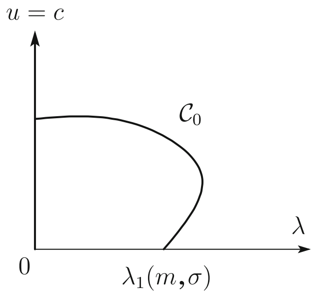


FIGURE 2. Subcontinuum containing $\Gamma_2$

By $C(\overline{\Omega})$ we denote the space of continuous functions in $\overline{\Omega}$ with norm $\|\cdot\|_{C(\overline{\Omega})}$. The main aim of this study is to find the closed and connected components consisting of positive solutions for (1.1) and emanating from $\Gamma_1 \setminus \{(0, 0)\}$, which can be stated as follows.

**Theorem 1.1.** *Assume* (1.4), (1.2), *and* (1.3). *Then, the following two assertions hold true:*

(I) *The closure of the positive solution set* $\{(\lambda, u) \in (\mathbb{R}_+ \cup \{0\}) \times C(\overline{\Omega}) : u \textit{ is a positive solution of } (1.1) \textit{forsome}\lambda\}$ *contains a subcontinuum* $\mathcal{C}_0$ *(i.e., a maximal (for the inclusion) closed and connected subset) bifurcating from the trivial line* $\Gamma_1$ *at* $(\lambda_1(m, \sigma), 0)$, *where* $\mathbb{R}_+ = \{\lambda \in \mathbb{R} : \lambda > 0\}$. *Moreover,* $\lambda_1(m, \sigma)$ *is the unique positive value of* $\lambda$ *for which the bifurcation of positive solutions, emanating from* $\Gamma_1 \setminus \{(0, 0)\}$, *can occur.*

(II) *The subcontinuum* $\mathcal{C}_0$ *cannot contain a point* $(\lambda, 0) \in \Gamma_1 \setminus \{(0, 0), (\lambda_1(m, \sigma), 0)\}$, *and it satisfies the following alternative conditions:*
  - (i) $\mathcal{C}_0$ *is unbounded in* $\mathbb{R}_+ \times C(\overline{\Omega})$ *(Fig. 1), or*
  - (ii) $\mathcal{C}_0$ *contains* $\Gamma_2$ *and consists of points belonging to* $\Gamma_2$ *in some neighborhood of* $(0, 0)$ *(Fig. 2).*

*Remark 1.2.* Using the local bifurcation theory proposed by Crandall and Rabinowitz [8], we can prove that the subcontinuum $(\lambda, u) \in \mathcal{C}_0$ is parametrized by $s \geq 0$ as $\lambda = \lambda(s)$ and $u = u(s)$ in a neighborhood of $(\lambda_1(m, \sigma), 0)$

such that $(\lambda(0), u(0)) = (\lambda_1(m, \sigma), 0)$ and $u(s)$ is a positive solution of (1.1) for $\lambda = \lambda(s)$ with $s > 0$.

Morales-Rodrigo and Suárez [15, Theorem 3.1] considered a problem similar to (1.1), where the linear terms in $u$ with non-negative coefficients appear in $\Omega$ and on $\partial\Omega$ both, at least one of which is nontrivial. The proof of Theorem 1.1 is based on the abstract result of the global bifurcation theory proposed by López-Gómez [13, Theorem 6.4.3]. For more recent works on the global bifurcation analysis, we refer to López-Gómez and Molina-Meyer [14], Dancer [9], Cano-Casanova [5], and Brown [3].

Next, we consider only the case when

$$\begin{aligned} g_1(x,u) &= -u^2, \\ \sigma(x) &\equiv 0, \\ g_2(x,u) &= h(x)u^p, \quad p > 1, \end{aligned}$$

and study the problem

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda h(x) u^p & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Problem (1.6) arises from population dynamics [6] when one studies the steady states of the following initial-boundary value problem with $d = 1/\lambda$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot d\nabla u + (m(x)u - u^2) & \text{in } (0,\infty) \times \Omega, \\ u(0,x) = u_0(x) \geq 0 & \text{in } \Omega, \\ d\nabla u \cdot \mathbf{n} = h(x)u^p & \text{in } (0,\infty) \times \partial\Omega, \end{cases}$$

where $u$ denotes the population density of some species diffusing in the region $\Omega$ at the rate $d > 0$, obeying the logistic growth law $m(x) - u$, and taking into account the nonlinear flux $h(x)u^p$ with flux rate $h(x)$ on the boundary $\partial\Omega$.

Our main interest here is the case when the subcontinuum $\mathcal{C}_0$ of positive solutions of (1.6) is unbounded in $\mathbb{R}_+ \times C(\overline{\Omega})$, that is, when bifurcation from infinity can occur in $\lambda \geq 0$. We can easily state a sufficient condition for the unboundedness in $\mathbb{R}_+ \times C(\overline{\Omega})$, that is, $h \leq 0$ on $\partial\Omega$. This is based on the fact that problem (1.6) has no positive solution for any $0 < \lambda \leq \lambda_1(m, 0)$.

Now, we consider

$$\phi(c) := \int_\Omega m\,dx - c|\Omega| + c^{p-1} \int_{\partial\Omega} h\,ds, \quad c > 0, \tag{1.7}$$

where $|\Omega|$ denotes the volume of $\Omega$. Then, we can state a sufficient condition for the subcontinuum $\mathcal{C}_0$ to be unbounded in $\mathbb{R}_+ \times C(\overline{\Omega})$ in the case when $h$ takes a positive value, too, as discussed below.

**Theorem 1.3.** *Assume* (1.4) *with $\sigma \equiv 0$, and assume that there exists $x_0 \in \partial\Omega$ satisfying the condition that $h(x_0) > 0$. Then, the subcontinuum $\mathcal{C}_0$ of positive solutions of* (1.6) *bifurcating from $\Gamma_1$ at $(\lambda_1(m, 0), 0)$, which is guaranteed by Theorem 1.1, is unbounded in $\mathbb{R}_+ \times C(\overline{\Omega})$, provided that the function $\phi$ defined in* (1.7) *cannot admit a positive zero.*

The following result shows sufficient conditions such that the function $\phi$ cannot admit a positive zero.

**Corollary 1.4.** *Under the hypotheses of Theorem 1.3, the subcontinuum $\mathcal{C}_0$ is unbounded in $\mathbb{R}_+ \times C(\overline{\Omega})$ if at least one of the following three conditions is satisfied:*

$$\int_{\partial\Omega} h ds \leq 0, \tag{1.8}$$

$$p = 2 \quad \textit{and} \quad |\Omega| \geq \int_{\partial\Omega} h ds, \tag{1.9}$$

$$1 < p < 2 \quad \textit{and} \quad |\Omega| > m_p \int_{\partial\Omega} h ds. \tag{1.10}$$

*Here, $m_p$ denotes a positive constant given by*

$$m_p = \frac{(p-1)^{p-1}(2-p)^{2-p}}{(-\overline{m})^{2-p}} \quad \textit{with} \quad \overline{m} = \frac{1}{|\Omega|}\int_\Omega m dx.$$

*Moreover, we define $\Lambda_0 := \inf\{\lambda \geq 0 : (\lambda, u) \in \mathcal{C}_0\}$, and then we have*

$$\begin{cases} 0 < \Lambda_0 \leq \lambda_1(m,0), \\ \{\lambda \geq 0 : (\lambda, u) \in \mathcal{C}_0\} = [\Lambda_0, \infty) \end{cases} \tag{1.11}$$

*if we suppose, in addition to (1.10), that $2 \leq N \leq 5$ and $1 < p < p_0(N)$, where*

$$p_0(N) = 1 + \frac{4}{N + 3 + \sqrt{N^2 - 2N + 25}}.$$

*Remark 1.5.* By considering the condition $1 < p < 2$, we can prove in the same manner as shown in [18, Theorem 2.2(1)] that $\lambda(s)$, as stated in Remark *1.2*, satisfies the following condition:

$$\lambda(s) = \lambda_1(m,0) - \frac{\lambda_1(m,0)\int_{\partial\Omega} h\varphi_1(m,0)^{p+1} ds}{\int_\Omega m\varphi_1(m,0)^2 dx} s^{p-1} + \mathrm{o}(s^{p-1}), \quad s \to +0.$$

Since $\int_\Omega m\varphi_1(m,0)^2 dx > 0$, we can see that $\mathcal{C}_0$ bifurcates to the left and right at $(\lambda_1(m,0), 0)$ when $\int_{\partial\Omega} h\varphi_1(m,0)^{p+1} ds$ is positive and negative, respectively. Therefore, when $\int_{\partial\Omega} h\varphi_1(m,0)^{p+1} ds$ is positive, assertion (1.11) implies that $0 < \Lambda_0 < \lambda_1(m,0)$, so that there exist at least two positive solutions of (1.6) for each $\Lambda_0 < \lambda < \lambda_1(m,0)$ and one positive solution for $\lambda = \Lambda_0$ and for $\lambda \geq \lambda_1(m,0)$, as shown in Fig. 3.

In order to show that the subcontinuum $\mathcal{C}_0$ is unbounded in $\mathbb{R}_+ \times C(\overline{\Omega})$, we investigate the set of positive solutions of (1.6) for $\lambda > 0$ small. In particular, we discuss the nonexistence of secondary bifurcation points in $\Gamma_2 \setminus \{(0,0)\}$. In addition, an *a priori* bound for solutions of (1.6) in $C(\overline{\Omega})$, uniformly in $0 < \lambda \leq \lambda^*$ for fixed $\lambda^* > 0$, is established for obtaining the unbounded component of positive solutions, as shown in Fig. 3. For the global nature of bifurcation components of semilinear elliptic problems with nonlinear boundary conditions arising from population dynamics, we refer to Morales-Rodrigo and Suárez [15], García-Melián, Morales-Rodrigo, Rossi, and Suárez [11], and

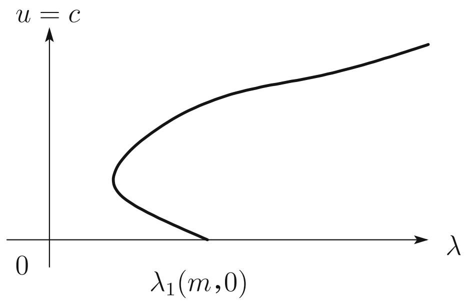


FIGURE 3. Bifurcation diagram when (1.11) holds with $0 < \Lambda_0 < \lambda_1(m, 0)$

Cantrell, Cosner, and Martinez [7], where linear terms with constant coefficients are considered.

The rest of this paper is organized as follows. In Sect. 2, we prove Theorem 1.1, and in Sect. 3, we prove Theorem 1.3 and Corollary 1.4.

## 2. Proof of Theorem 1.1

In this section, we apply [13, Theorem 6.4.3] to prove Theorem 1.1. In this direction, we first reduce (1.1) to a fixed point equation for continuous and compact nonlinear mapping in $C(\overline{\Omega})$. For constants $M_1 > 0$ and $M_2 > 0$, we consider the following two boundary value problems: given $f \in C^{\alpha}(\overline{\Omega})$ and $\psi \in C^{1+\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$, find $v, w \in C^{2+\alpha}(\overline{\Omega})$ such that

$$\begin{cases} (-\Delta + M_1)v = f(x) & \text{in } \Omega, \\ \left(\dfrac{\partial}{\partial \mathbf{n}} + M_2\right) v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

$$\begin{cases} (-\Delta + M_1)w = 0 & \text{in } \Omega, \\ \left(\dfrac{\partial}{\partial \mathbf{n}} + M_2\right) w = \psi(x) & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

We introduce resolvents $\mathcal{K}_\Omega : C^{\alpha}(\overline{\Omega}) \to C^{2+\alpha}(\overline{\Omega})$ and $\mathcal{K}_{\partial\Omega} : C^{1+\alpha}(\partial\Omega) \to C^{2+\alpha}(\overline{\Omega})$ for (2.1) and (2.2), respectively, and both are bijective and homeomorphic ([12]). It is well-known (see Amann [2]) that $\mathcal{K}_\Omega$ and $\mathcal{K}_{\partial\Omega}$ can be extended to continuous linear operators, denoted by the same notation, of $C(\overline{\Omega})$ into $W^{2,r}(\Omega)$ and of $C(\partial\Omega)$ into $W^{1,r}(\Omega)$ for any $1 < r < \infty$, respectively. Here, $W^{m,r}(\Omega)$ ($m = 0, 1, 2, \ldots,\ 1 < r < \infty$) denotes the usual Sobolev space. By the Sobolev imbedding theorem and Ascoli-Arzéla theorem, $\mathcal{K}_\Omega$ and $\mathcal{K}_{\partial\Omega} \circ i$, where $i : C(\overline{\Omega}) \to C(\partial\Omega)$ denotes the usual trace operator, are both compact in $C(\overline{\Omega})$. By virtue of the strong maximum principle and boundary point lemma ([16]), $\mathcal{K}_\Omega$ is *strongly positive*, meaning that, for any $u \in P \setminus \{0\}$, $\mathcal{K}_\Omega u$ is an interior point of $P$, where $P$ denotes the positive cone of $C(\overline{\Omega})$ defined as $P := \{u \in C(\overline{\Omega}) : u \geq 0 \text{ in } \overline{\Omega}\}$. In a similar manner, we find that, for any $u \in C(\partial\Omega)$ satisfying the condition that $u$ is non-negative and nontrivial, $\mathcal{K}_{\partial\Omega} u$ is an interior point of $P$.

Now, we prove the following:

**Lemma 2.1.** *Let $u \in C(\overline{\Omega})$ be non-negative. If $u$ satisfies the condition that*

$$u = \mathcal{K}_\Omega(M_1 u + \lambda(mu + g_1(\cdot, u))) + \mathcal{K}_{\partial\Omega}(M_2 u + \lambda(\sigma u + g_2(\cdot, u))),$$

*then $u$ is a solution of* (1.1).

*Proof.* The proof is obtained by considering a bootstrap argument and using the elliptic regularity. Let $u = v + w$, where

$$v = \mathcal{K}_\Omega(M_1 u + \lambda(mu + g_1(\cdot, u))),$$
$$w = \mathcal{K}_{\partial\Omega}(M_2 u + \lambda(\sigma u + g_2(\cdot, u))).$$

Note that $M_1 u + \lambda(mu + g_1(\cdot, u)) \in C(\overline{\Omega})$ and $M_2 u + \lambda(\sigma u + g_2(\cdot, u)) \in C(\partial\Omega)$. Since $\mathcal{K}_\Omega : C(\overline{\Omega}) \to W^{2,r}(\Omega)$ and $\mathcal{K}_{\partial\Omega} : C(\partial\Omega) \to W^{1,r}(\Omega)$, $r > N$, are both continuous, we have $v \in W^{2,r}(\Omega)$ and $w \in W^{1,r}(\Omega)$ for all $r > N$. It follows that $u = v + w \in W^{1,r}(\Omega)$ for all $r > N$. By virtue of elliptic regularity, we obtain $u \in W^{2,r}(\Omega)$ for all $r > N$. By the Sobolev imbedding theorem, $u \in C^{1+\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Finally, a Hölder estimate provides the desired conclusion. □

If we consider

$$\mathcal{L}(\lambda)u = u - \{\mathcal{K}_\Omega((M_1 + \lambda m)u) + \mathcal{K}_{\partial\Omega}((M_2 + \lambda\sigma)u)\},$$
$$\mathcal{B}(\lambda, u) = -\lambda\{\mathcal{K}_\Omega(g_1(\cdot, u)) + \mathcal{K}_{\partial\Omega}(g_2(\cdot, u))\},$$

then we have the following operator equation, as desired:

$$\mathcal{L}(\lambda)u + \mathcal{B}(\lambda, u) = 0 \quad \text{in } C(\overline{\Omega}). \tag{2.3}$$

Now, we verify the conditions of [13, Theorem 6.4.3]. In the rest of this section, $\lambda_1(m, \sigma)$ and $\varphi_1(m, \sigma)$ from (1.5) are written simply as $\lambda_1$ and $\varphi_1$, respectively. First, denoting by $\mathcal{N}[\lambda]$ the null space of $\mathcal{L}(\lambda)$, we have $\dim \mathcal{N}[\lambda_1] = 1$ and $\mathcal{N}[\lambda_1] = \operatorname{span}\langle\varphi_1\rangle$, as stated in Section 1.

Next, we define

$$\mathcal{A}(\lambda)u := u - \mathcal{L}(\lambda)u = \mathcal{K}_\Omega((M_1 + \lambda m)u) + \mathcal{K}_{\partial\Omega}((M_2 + \lambda\sigma)u).$$

If we assume $M_1$ and $M_2$ to be sufficiently large, then we have

$$M_1 + \lambda_1 m(x) > 0, \quad x \in \overline{\Omega},$$
$$M_2 + \lambda_1 \sigma(x) > 0, \quad x \in \partial\Omega.$$

Hence, the strong positivity of $\mathcal{K}_\Omega$ and $\mathcal{K}_{\partial\Omega}$ shows that $\mathcal{A}(\lambda_1)$ is also strongly positive. In the same way, if $u \in P \setminus \{0\}$ satisfies (2.3), then $u$ is an interior point of $P$.

Finally, we verify that $\lambda_1$ satisfies the transversality condition of Crandall and Rabinowitz [8]. For this, we only have to check that

$$\mathcal{L}'(\lambda_1)\varphi_1 \notin \mathcal{R}[\lambda_1], \tag{2.4}$$

where $\mathcal{R}[\lambda]$ denotes the range of $\mathcal{L}(\lambda)$. Indeed, we can see that

$$\mathcal{L}'(\lambda_1)\varphi_1 = -\{\mathcal{K}_\Omega(m\varphi_1) + \mathcal{K}_{\partial\Omega}(\sigma\varphi_1)\}.$$

Assume to the contrary that there exists $u \in C(\overline{\Omega})$ such that $\mathcal{L}(\lambda_1)u = \mathcal{L}'(\lambda_1)\varphi_1$. Then, by virtue of elliptic regularity, we obtain $u \in C^2(\overline{\Omega})$ and

$$\begin{cases} -\Delta u = \lambda_1 m u - m\varphi_1 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda_1 \sigma u - \sigma\varphi_1 & \text{on } \partial\Omega. \end{cases}$$

By using Green's formula,

$$-\int_\Omega m\varphi_1^2 dx = \int_\Omega (-\Delta u \varphi_1 + u\Delta\varphi_1)dx = \int_{\partial\Omega} \left(-\frac{\partial u}{\partial \mathbf{n}}\varphi_1 + u\frac{\partial \varphi_1}{\partial \mathbf{n}}\right) ds = \int_{\partial\Omega} \sigma\varphi_1^2 ds.$$

Meanwhile, we can derive from (1.5) that

$$\begin{aligned} \lambda_1 \int_\Omega m\varphi_1^2 dx &= \int_\Omega -\Delta\varphi_1\varphi_1 \\ &= \int_\Omega |\nabla\varphi_1|^2 dx - \int_{\partial\Omega} \frac{\partial\varphi_1}{\partial\mathbf{n}}\varphi_1 ds = \int_\Omega |\nabla\varphi_1|^2 dx - \lambda_1 \int_{\partial\Omega} \sigma\varphi_1^2 ds. \end{aligned}$$

It follows that $\int_\Omega |\nabla\varphi_1|^2 dx = 0$. This contradicts the fact that $\varphi_1$ is not a constant. Assertion (2.4) now follows.

After the above verifications, we can apply the result of [13, Theorem 6.4.3] to (2.3) at $(\lambda_1, 0)$ to obtain a subcontinuum, denoted by $\mathcal{C}_0$, emanating from $(\lambda_1, 0)$ such that, for some $\delta_1 > 0$, $u$ is a positive solution of (2.3) if $(\lambda, u) \in (\mathcal{C}_0 \cap B_{\delta_1}((\lambda_1, 0))) \setminus \{(\lambda_1, 0)\}$, where $B_\delta((\lambda_0, u_0)) = \{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) : |\lambda - \lambda_0| + \|u - u_0\|_{C(\overline{\Omega})} < \delta\}$, $\delta > 0$. Moreover, the subcontinuum $\mathcal{C}_0$ consists of positive solutions, it cannot contain a point $(\lambda, 0) \in \Gamma_1 \setminus \{(0,0), (\lambda_1(m,\sigma), 0)\}$, and it satisfies the following alternative conditions: Either assertion (i) of Theorem 1.1 holds, or $\mathcal{C}_0$ contains $\Gamma_2$. These alternative conditions are obtained in the same manner as in the proof of [14, Theorem 1.1] by virtue of the strong positivity of non-negative, nontrivial solutions of (2.3), the strong positivity of mapping $\mathcal{A}(\lambda_1)$, and the fact that $\lambda_1$ is the unique positive principal eigenvalue of (1.5).

However, if $\mathcal{C}_0$ contains $\Gamma_2$, then it is impossible that there exists $(\lambda_j, u_j) \in \mathcal{C}_0$ with $\lambda_j > 0$ such that $(\lambda_j, u_j) \to (0, 0)$ in $\mathbb{R} \times C(\overline{\Omega})$ because the eigenvalue 0 of (1.5) satisfies the transversality condition, as stated in (2.4). Namely, from (1.4), it is verified in the same way that $\dim \mathcal{N}[0] = 1$, $\mathcal{N}[0] = \mathrm{span}\langle 1 \rangle$, and $\mathcal{L}'(0)1 \notin \mathcal{R}[0]$. Thus, the local bifurcation theory [8] is applicable at $(0, 0)$. On the other hand, we have already seen that the vertical axis $\Gamma_2$ consisting of positive constants emanates from $\Gamma_1$ at $(\lambda, u) = (0, 0)$. Hence, problem (2.3) cannot admit a nonconstant positive solution in some neighborhood of $(0, 0)$.

Finally, no other bifurcation of positive solutions from $\Gamma_1 \setminus \{(0,0)\}$ exists because of the unique positive principal eigenvalue $\lambda_1$ of (1.5). The proof of Theorem 1.1 is complete. □

## 3. Proofs of Theorem 1.3 and Corollary 1.4

First, we prove Theorem 1.3. From Theorem 1.1, it suffices to show that problem (1.6) cannot admit a secondary bifurcation point of positive solutions in $\Gamma_2 \setminus \{(0,0)\}$. Indeed, if $(0,c)$ is the point where $c$ is a positive constant, then it is a positive zero of the function $\phi$ defined in (1.7). For verifying this assertion, we let $u_j$ denote positive solutions of (1.6) with $\lambda = \lambda_j > 0$ such that $(\lambda_j, u_j) \to (0,c)$ in $\mathbb{R} \times C(\overline{\Omega})$. Then, it follows that

$$\int_\Omega (mu_j - u_j^2)dx + \int_{\partial\Omega} hu_j^p ds = 0,$$

and, by passing to the limit, we verify the above assertion. Hence, the subcontinuum $\mathcal{C}_0$ cannot contain a point $(0,c) \in \Gamma_2$. The proof of Theorem 1.3 is complete. □

Next, we prove Corollary 1.4. We show that each of the three conditions (1.8), (1.9), and (1.10) are sufficient such that the function $\phi$ cannot admit a positive zero. If either conditions (1.8) or (1.9) is satisfied, then it is easily observed from (1.7) that $\phi(c) < 0$ for all $c > 0$, because $\int_\Omega mdx < 0$. Considering condition (1.10), we discuss $\phi'(t)$. The function $\phi'$ has a unique positive zero $c^*$ given by

$$c^* = \left(\frac{(p-1)\int_{\partial\Omega} hds}{|\Omega|}\right)^{\frac{1}{2-p}}.$$

By a direct computation, we have

$$\phi(c^*) = \int_\Omega mdx + c^* \left(\frac{2-p}{p-1}\right)|\Omega|.$$

Since the condition $|\Omega| > m_p \int_{\partial\Omega} hds$ implies that $\phi(c^*) < 0$ and the condition $1 < p < 2$ implies that $\phi$ is concave, we obtain $\phi(c) \le \phi(c^*) < 0$ for all $c > 0$. The first assertion of Corollary 1.4 follows.

Finally, we prove (1.11). We denote by $\|\cdot\|_{1,2}$ the usual norm of the Sobolev space $W^{1,2}(\Omega)$. Assertion (1.11) is a direct consequence of the following *a priori* bound for positive solutions of (1.6):

**Lemma 3.1.** *Let $m \in C^\theta(\overline{\Omega})$ and $\sigma \in C^{1+\theta}(\partial\Omega)$ be arbitrary, and let $2 \le N \le 5$ and $1 < p < p_0(N)$. Then, for any $\lambda^* > 0$, there exists a constant $C_0 > 0$ such that any positive solution $u$ of (1.6) with $0 < \lambda \le \lambda^*$ satisfies the condition that $\|u\|_{C(\overline{\Omega})} \le C_0$.*

*Remark 3.2.* The a priori bound is lost at $\lambda = 0$, because there is a vertical bifurcation there.

*Proof.* Let $\lambda^* > 0$ be fixed, and let $u$ be a positive solution of (1.6) for $0 < \lambda \le \lambda^*$. We denote the generic constants independent of $u$ and $\lambda$ by $C$. First, we prove that $\|u\|_{1,2} \le C$. For this, we use the following integral inequality derived by Filo and Kačur [10, Proposition 2]:

**Proposition 3.3.** *For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\int_{\partial\Omega} |u|^{q+1} ds \le \varepsilon \|u\|_{1,2}^2 + C_\varepsilon \left(\int_\Omega |u|^{q+1} dx\right)^r \quad \textit{for all } u \in W^{1,2}(\Omega), \tag{3.1}$$

*where $1 < q < \frac{N+1}{N-1}$ and $r > \frac{N-q(N-2)}{N+1-q(N-1)}$.*

Green's formula shows that

$$\int_\Omega \lambda(mu-u^2)udx = \int_\Omega(-\Delta u)udx$$
$$= \int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}uds = \int_\Omega |\nabla u|^2dx - \int_{\partial\Omega}\lambda h u^{p+1}ds.$$

It follows that

$$\int_\Omega |\nabla u|^2dx = \lambda\int_\Omega mu^2dx - \lambda\int_\Omega u^3dx + \lambda\int_{\partial\Omega} hu^{p+1}ds$$

and

$$\|u\|_{1,2}^2 \le C\int_\Omega u^2dx + \lambda\left\{C\int_{\partial\Omega}u^{p+1}ds - \int_\Omega u^3dx\right\}.$$

Since $1 < p < p_0(N) < \frac{N+1}{N-1}$, we apply (3.1) to obtain

$$\|u\|_{1,2}^2 \le C\int_\Omega u^2dx + \lambda\left\{C\left(\varepsilon\|u\|_{1,2}^2 + C\left(\int_\Omega u^{p+1}dx\right)^r\right) - \int_\Omega u^3dx\right\},$$

where $r$ will be determined later to be as large as $r > \frac{N-p(N-2)}{N+1-p(N-1)}$. By assuming $\varepsilon$ to be very small at $\lambda^*$ and using the Hölder inequality, it follows that

$$\|u\|_{1,2}^2 \le C\int_\Omega u^2dx + \lambda\left\{C\left(\int_\Omega u^3dx\right)^{\frac{(p+1)r}{3}} - \int_\Omega u^3dx\right\}. \tag{3.2}$$

We claim here that the constant $r$ can be chosen such that

$$\frac{(p+1)r}{3} < 1. \tag{3.3}$$

Indeed, it suffices to show that

$$\frac{p+1}{3}\cdot\frac{N-p(N-2)}{N+1-p(N-1)} < 1. \tag{3.4}$$

Condition (3.4) is equivalent to the assertion

$$-(N-2)p^2 + (3N-1)p - (2N+3) < 0.$$

It is easy to verify this inequality if $1 < p < p_0(N)$. Claim (3.3) now follows. As a consequence of (3.3), we derive from (3.2) that

$$\|u\|_{1,2}^2 \le C\left(1 + \int_\Omega u^2dx\right). \tag{3.5}$$

On the other hand, we have from Green's formula

$$\int_\Omega \lambda(mu-u^2)dx = \int_\Omega -\Delta udx = -\int_{\partial\Omega}\frac{\partial u}{\partial\mathbf{n}}ds = -\int_{\partial\Omega}\lambda hu^pds.$$

Since $\lambda > 0$, it follows that

$$\int_\Omega u^2dx = \int_\Omega mudx + \int_{\partial\Omega}hu^pds$$

and by the Hölder inequality,

$$\int_\Omega u^2 dx \le C\left\{\left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} + \left(\int_{\partial\Omega} u^2 ds\right)^{\frac{p}{2}}\right\}. \tag{3.6}$$

To substantiate our claim, we use the following integral inequality of Afrouzi and Brown [1, Lemma 1]:

**Proposition 3.4.** *For any $\varepsilon > 0$, there exists a constant $\tilde{C}_\varepsilon > 0$ such that*

$$\int_{\partial\Omega} u^2 ds \le \varepsilon \int_\Omega |\nabla u|^2 dx + \tilde{C}_\varepsilon \int_\Omega u^2 dx \quad \textit{for all } u \in W^{1,2}(\Omega). \tag{3.7}$$

By using (3.7) and (3.5), inequality (3.6) implies that

$$\int_\Omega u^2 dx \le C\left\{\left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} + \left(1 + \int_\Omega u^2 dx\right)^{\frac{p}{2}}\right\}.$$

If $u \neq 0$, then it follows that

$$1 \le C\left\{\left(\int_\Omega u^2 dx\right)^{-\frac{1}{2}} + \left(1 + \left(\int_\Omega u^2 dx\right)^{-1}\right)^{\frac{p}{2}} \left(\int_\Omega u^2 dx\right)^{-\frac{2-p}{2}}\right\}.$$

Since $1 < p < 2$, we have

$$\int_\Omega u^2 dx \le C.$$

From (3.5), we obtain

$$\|u\|_{1,2} \le C. \tag{3.8}$$

Next, we consider the uniform bound in $\overline{\Omega}$ for solutions of (1.6) in $0 < \lambda \le \lambda^*$. We see from (1.6) that

$$\begin{cases} (-\Delta + 1)u = u + \lambda(m(x)u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda h(x) u^p & \text{on } \partial\Omega, \end{cases}$$

where we have a constant $\delta > 0$ such that

$$|\lambda h(x) u^p| \le C(1 + u^{p_0(N)}) \le C(1 + u^{\frac{N}{N-2} - \delta})$$

and also such that

$$|u + \lambda(m(x)u - u^2)| \le C(1 + u^2) \le C\left(1 + u^{\frac{N+2}{N-2} - \delta}\right),$$

by using the assumptions $2 \le N \le 5$ and $1 < p < p_0(N)$. For the subcritical case that $\frac{N+2}{N-2} - \delta$ and $\frac{N}{N-2} - \delta$, we can derive from (3.8) that $\|u\|_{C^\alpha(\overline{\Omega})} \le C$ at some $0 < \alpha < 1$, using the same bootstrap argument as in the proof of [17, Theorem 2.2]. We end the proof of Lemma 3.1. □

Lemma 3.1 and the nonexistence of secondary bifurcation points in $\Gamma_2 \setminus \{(0, 0)\}$, as discussed above, show that problem (1.6) has no positive solutions at $0 < \lambda \le \overline{\lambda}$ for sufficiently small $\overline{\lambda} > 0$ and it has no bifurcation from infinity of positive solutions in $0 < \lambda < \lambda^*$ for arbitrary fixed $\lambda^* > 0$. Assertion (1.11) has been verified. The proof of Corollary 1.4 is now complete. □

*Remark 3.5.* As a concluding remark, we define the limiting cases of (1.10) as $p \to 2-0$ and $p \to 1+0$. When we consider (1.9), the limiting case of (1.10) is $p \to 2-0$. The limiting case of (1.10) as $p \to 1+0$ is given as

$$\int_\Omega m dx + \int_{\partial\Omega} h ds \leq 0. \tag{3.9}$$

We consider the limiting problem

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda h(x) u & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

It is known ([20]) that problem (3.10) has a unique positive solution for all $\lambda > \lambda_1(m,h)$ and no positive solution for any $0 < \lambda \leq \lambda_1(m,h)$. Here, it is understood that $\lambda_1(m,h) = 0$ when $\int_\Omega m dx + \int_{\partial\Omega} h ds \geq 0$. It is easy to verify that the implicit function theorem can be applied to the unique positive solution $u_\lambda$, $\lambda > \lambda_1(m,h)$, which implies that the positive solution set consists of a smooth curve parametrized by $\lambda$. For the limiting case of the unique positive solution, i.e., $\lambda \to \lambda_1(m,h)+0$, we can prove that $u_\lambda \to 0$ in $C(\overline{\Omega})$ if we assume (3.9), and otherwise, $u_\lambda \to (\int_\Omega m dx + \int_{\partial\Omega} h ds)/|\Omega|$ in $C(\overline{\Omega})$.

## Acknowledgements

Partly supported by a grant-in-aid for Scientific Research (C), No. 19540192, Japan Society for the Promotion of Science.

## References

[1] Afrouzi, G.A., Brown, K.J.: On principal eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions. Proc. Am. Math. Soc. **127**, 125–130 (1999)

[2] Amann, H.: Nonlinear elliptic equations with nonlinear boundary conditions. New developments in differential equations. In: Proceedings of the 2nd Scheveningen Conference, Scheveningen, 1975. North-Holland Mathematical Studies, vol. 21, pp. 43–63. North-Holland, Amsterdam (1976)

[3] Brown, K.J.: Local and global bifurcation results for a semilinear boundary value problem. J. Differ. Equ. **239**, 296–310 (2007)

[4] Brown, K.J., Lin, S.S.: On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function. J. Math. Anal. Appl. **75**, 112–120 (1980)

[5] Cano-Casanova, S.: Compact components of positive solutions for superlinear indefinite elliptic problems of mixed type. Topol. Methods Nonlinear Anal. **23**, 45–72 (2004)

[6] Cantrell, R.S., Cosner, C.: Spatial Ecology via Reaction-Diffusion Equations, Wiley Series in Mathematical and Computational Biology. Wiley, Chichester (2003)

[7] Cantrell, R.S., Cosner, C., Martinez, S.: Global bifurcation of solutions to diffusive logistic equations on bounded domains subject to nonlinear boundary conditions. Proc. R. Soc. Edinb. **139A**, 45–56 (2009)

[8] Crandall, M.G., Rabinowitz, P.H.: Bifurcation from simple eigenvalues. J. Funct. Anal. **8**, 321–340 (1971)

[9] Dancer, E.N.: Bifurcation from simple eigenvalues and eigenvalues of geometric multiplicity one. Bull. Lond. Math. Soc. **34**, 533–538 (2002)

[10] Filo, J., Kačur, J.: Local existence of general nonlinear parabolic systems. Nonlinear Anal. TMA **24**, 1597–1618 (1995)

[11] García-Melián, J., Morales-Rodrigo, C., Rossi, J.D., Suárez, A.: Nonnegative solutions to an elliptic problem with nonlinear absorption and a nonlinear incoming flux on the boundary. Ann. Mat. Pura Appl. **187**, 459–486 (2008)

[12] Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Classics in Mathematics. Springer, Berlin (2001). Reprint of the 1998 edition

[13] López-Gómez, J.: Spectral Theory and Nonlinear Functional Analysis. Research Notes in Mathematics 426. Chapman & Hall/CRC, Boca Raton (2001)

[14] López-Gómez, J., Molina-Meyer, M.: Bounded components of positive solutions of abstract fixed point equations: mushrooms, loops and isolas. J. Differ. Equ. **209**, 416–441 (2005)

[15] Morales-Rodrigo, C., Suárez, A.: Some elliptic problems with nonlinear boundary conditions. In: Cano-Casanova, S., López-Gómez, J., Mora-Corral, C. (eds.) Spectral Theory and Nonlinear Analysis with Applications to Spatial Ecology, pp. 175–199. World Scientific Publishing Company, Hackensack (2005)

[16] Protter, M.H., Weinberger, H.F.: Maximum Principles in Differential Equations. Prentice-Hall, Englewood Cliffs (1967)

[17] Rossi, J.D.: Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem. Stationary Partial Differential Equations, vol. II, pp. 311–406, Handbook of the Differential Equations. Elsevier/North-Holland, Amsterdam (2005)

[18] Umezu, K.: One parameter-dependent nonlinear elliptic boundary value problems arising in population dynamics. Advances in Analysis, pp. 177–186. World Scientific Publishing Company, Hackensack (2005)

[19] Umezu, K.: On eigenvalue problems with Robin type boundary conditions having indefinite coefficients. Appl. Anal. **85**, 1313–1325 (2006)

[20] Umezu, K.: Positive solutions of semilinear elliptic eigenvalue problems with concave nonlinearities. Adv. Differ. Equ. **12**, 1415–1436 (2007)

K. Umezu
Department of Mathematics,
Faculty of Education,
Ibaraki University,
Mito 310-8512, Japan
e-mail: uken@mx.ibaraki.ac.jp

Received: 27 February 2009.
Accepted: 07 January 2010.