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A class of Adams–Fontana type inequalities and related functionals on manifolds

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Abstract. A class of Adams–Fontana type inequalities are established on compact Riemannian manifolds without boundary via the Young inequality together with the usual Adams–Fontana inequality (Comment Math Helv 68:415–454, 1993). As an application, a sequence of functionals are defined on manifolds, a sufficient condition on which the Palais–Smale condition holds is given and the existence of critical points of the functionals is also considered in the spirit of Adimurthi (Ann Scuola Norm Sup Pisa Cl Sci 17:393–413, 1990) and Adimurthi and Sandeep (Nonlinear Differ Equ Appl 13:585–603, 2007).

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1. Introduction and main results

Let (M, g) be a compact Riemannian manifold of dimension n $(n \geq 2)$ without boundary, and m be an integer such that $0 < m < n$. Recall that $W^{m,\frac{n}{m}}(M)$ is the usual Sobolev space, namely the completion of $C^\infty(M)$ under the norm

$$\|u\|_{m,n/m} = \left(\int_M \sum_{k=0}^{m} |\nabla^k u|^{n/m} dv_g \right)^{m/n}$$

or any of its equivalents, where

$$\nabla^k u = \begin{cases} \Delta_g^{k/2} u & \text{when } k \text{ is even} \\ \nabla \Delta_g^{\frac{k-1}{2}} u & \text{when } k \text{ is odd,} \end{cases}$$

∇ and Δ_g represent the gradient and the Laplace–Beltrami operator respectively with respect to (M, g). Precisely in local coordinates $\{x^i\}_{i=1}^n$, $g = g_{ij} dx^i dx^j$,

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}, \quad \Delta_g f = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{g} \frac{\partial f}{\partial x^i} \right)$$

for all $f \in C^\infty(M)$, where $\sqrt{g} = \det(g_{ij})$. Here we have used the repeated summation convention. In a celebrated paper [6], Fontana proved the following. For all $u \in W^{m,\frac{n}{m}}(M)$ with

$$\int_M u dv_g = 0, \quad \int_M |\nabla^m u|^{\frac{n}{m}} dv_g \le 1, \tag{1.1}$$

there holds

$$\int_M \exp\left\{\lambda(m,n)|u(x)|^{n/(n-m)}\right\} dv_g \le C \tag{1.2}$$

for some uniform constant $C = C(m, M)$, where

$$\lambda(m,n) = \begin{cases} \frac{n}{\omega_{n-1}} \left(\frac{\pi^{n/2} 2^m \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)} \right)^{\frac{n}{n-m}} & \text{if } m \text{ is odd} \\ \frac{n}{\omega_{n-1}} \left(\frac{\pi^{n/2} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)} \right)^{\frac{n}{n-m}} & \text{if } m \text{ is even.} \end{cases} \tag{1.3}$$

If $\lambda(m, n)$ is replaced by any larger number, the integral in (1.2) is still finite, but cannot be bounded uniformly by any constant.

Inequality (1.2) is a manifold version of the corresponding Euclidean case, the well-known Adams inequality [1]. It is also known as the Adams–Fontana inequality, which is an extension of the classical Trudinger–Moser inequality [14,16,19]. The main tool Fontana used to obtain (1.2) is the potential theory, which is the same as that of Adams.

One may ask the following question: Does the best constant $\lambda(m, n)$ change if we assume other conditions instead of (1.1) such that the Adams–Fontana inequality (1.2) holds? Regarding this topic, we have the following:

Theorem 1. *Let (M, g) be a compact Riemannian manifold of dimension n without boundary, m is an integer such that $n/2 \le m < n$, and $\beta : M \times \mathbb{R} \to \mathbb{R}$ is a nonnegative measurable function such that for some $c^* > 0$*

$$\beta(x,t) \ge c^* |t|^{\frac{n}{m}}. \tag{1.4}$$

Then for all $u \in W^{m,\frac{n}{m}}(M)$ with

$$\int_M \left(|\nabla^m u|^{\frac{n}{m}} + \beta(x,u)\right) dv_g \le 1, \tag{1.5}$$

there exists a uniform constant $C = C(m, M, c^*)$ such that (1.2) holds, where $\lambda(m, n)$ is defined by (1.3).

Theorem 1 follows from the Young inequality and the Adams–Fontana inequality. Here $m \ge n/2$ is a technical hypothesis. It should be remarked that for $0 < m < \frac{n}{2}$, one can easily get the subcritical Adams–Fontana type inequalities, namely for any $0 \le \alpha < \lambda(m, n)$ and all $u \in W^{m,\frac{n}{m}}(M)$ satisfying (1.4) and (1.5), there exists a uniform constant $C = C(m, n, \alpha)$ such that $\int_M e^{\alpha|u|^{\frac{n}{n-m}}} dv_g \le C$. This can also be deduced from the Young inequality and the Adams–Fontana inequality.

Next we consider a class of simple functionals related to the most interesting case of Theorem 1, namely $n = 2m$. In view of this special case, we have the following definition of critical growth for a function. This kind of definitions were originally proposed by Adimurthi [2].

Definition 2. Let (M, g) be a compact Riemannian manifold of even dimension. Let $h : M \times \mathbb{R} \to \mathbb{R}$ be a C^1-function and b be a positive real number. Let $f(x,t) = h(x,t)e^{bt^2}$. f is called a function of critical growth on M if f satisfies the following assumptions (H_1)–(H_4) for every $(x,t) \in M \times (0, +\infty)$:

(H_1) $f(x,t) > 0$, $f(x,-t) = -f(x,t)$;

(H_2) $f'(x,t) > \frac{f(x,t)}{t}$, where $f'(x,t) = \frac{\partial f}{\partial t}(x,t)$;

(H_3) $F(x,t) \le A(1 + t^\sigma f(x,t))$ for some $\sigma \in [0,1)$ and $A > 0$, where

$$F(x,t) = \int_0^t f(x,s)ds$$

is the primitive of f;

(H_4) For every $\epsilon > 0$,

$$\lim_{t\to+\infty} \sup_{x\in M} h(x,t)e^{-\epsilon t^2} = 0, \qquad \lim_{t\to+\infty} \inf_{x\in M} h(x,t)e^{\epsilon t^2} = +\infty.$$

Denote the set of all critical growth functions by $\mathcal{C}(M)$. Clearly $\mathcal{C}(M)$ is nonempty. A typical example is $f(x,t) = \lambda t e^{bt^2} \in \mathcal{C}(M)$ for some $\lambda > 0$. Let $f(x,t) = h(x,t)e^{bt^2} \in \mathcal{C}(M)$ and $F(x,t)$ be the primitive of f. We define a class of functionals on the space $W^{\frac{n}{2},2}(M)$ as following:

$$J_\tau(u) := \frac{1}{2}\int_M \left(|\nabla^{\frac{n}{2}} u|^2 + \tau u^2\right) dv_g - \int_M F(x,u)dv_g, \ \forall \tau > 0. \tag{1.6}$$

For any fixed $\tau > 0$, we define the modified first eigenvalue for $\Delta_g^{\frac{n}{2}}$, the Polyharmonic operator, by

$$\lambda_\tau(M) := \inf\left\{\int_M \left(|\nabla^{\frac{n}{2}} u|^2 + \tau u^2\right) dv_g : u \in W^{\frac{n}{2},2}(M), \int_M u^2 dv_g = 1\right\}. \tag{1.7}$$

Then we state the following:

Theorem 3. *Let (M, g) be a compact Riemannian manifold of dimension n without boundary, n be even, $f(x,t) = h(x,t)e^{bt^2} \in \mathcal{C}(M)$ be a function of critical growth, J_τ be defined by (1.6) and $\lambda_\tau(M)$ be defined by (1.7) for some $\tau > 0$. Then (i) the functional $J_\tau : W^{\frac{n}{2},2}(M) \to \mathbb{R}$ satisfies the Palais–Smale Condition on the interval $\left(-\infty, \frac{\beta_0}{2b}\right)$, where $\beta_0 = (4\pi)^n (n/2)!$; (ii) assume that*

$$\sup_{x\in M} f'(x,0) < \lambda_\tau(M), \quad \limsup_{t\to+\infty} \inf_{x\in M} th(x,t) = +\infty, \tag{1.8}$$

then there exists some $u_0 \in W^{\frac{n}{2},2}(M) \setminus \{0\}$ such that

$$\Delta_g^{\frac{n}{2}} u_0 + \tau u_0 = f(x,u_0) \quad \text{in } M. \tag{1.9}$$

Furthermore, for the case $n = 2$, u_0 can be chosen such that $u_0(x) > 0$ for all $x \in M$.

The method we adopt to prove Theorem 3 is the calculation of certain constrained variation, which has been originally used by Adimurthi [2], then by Lakkis [11], Adimurthi-Sandeep [3] and others. Precisely the Nehari manifold [15] with respect to the functional J_τ reads

$$\mathcal{N} = \left\{ u \in W^{\frac{n}{2},2}(M) : \int_M \left(|\nabla^{\frac{n}{2}} u|^2 + \tau u^2\right) dv_g = \int_M u f(x,u) dv_g, u \not\equiv 0 \right\}.$$

We will prove that there exists a minimizer of J_τ on $\mathcal{N}$, and it is indeed a critical point of J_τ. The key point in Adimurthi's compactness analysis is applying Lions' Lemma [10]. The reason we can use this method here is that a Lions' type lemma also holds in our case. Such kind of lemma has been used as a powerful tool in [20] for compact Riemannian surface without boundary and in [12] for bounded smooth domains in $\mathbb{R}^4$.

Let us interpret the relation between Theorems 1 and 3. Solutions of Eq. (1.9) are critical points of the functional J_τ. In view of the structure of J_τ, precisely its first part $\int_N (|\Delta^{\frac{n}{2}} u|^2 + \tau u^2) dv_g$, it is reasonable to use Theorem 1 to study the compactness of the Palais–Smale sequence of J_τ. This is the motivation of establishing Theorem 1.

For any $0 < \lambda < \lambda_\tau(M)$, if $f(x,t) = \lambda t e^{bt^2}$, the hypothesis (1.8) becomes simple. We deduce immediately from Theorem 3 the following:

Corollary 4. *Let (M, g) be a compact Riemannian manifold of dimension n without boundary, n is even. For any $\lambda : 0 < \lambda < \lambda_\tau(M)$, there exists a nontrivial solution of the following equation*

$$\begin{cases} \Delta_g^{\frac{n}{2}} u + \tau u = \lambda u e^{bu^2} & \text{in } M \\ u \in W^{\frac{n}{2},2}(M). \end{cases}$$

Furthermore, when $n = 2$, the solution can be chosen positive everywhere.

For related problems, a multi-bubble phenomenon for embedding of $W_0^{m,2}(\Omega)$ ($\Omega \subset \mathbb{R}^{2m}$) into Orlicz space has been investigated by Robert and Struwe [17] for $m = 2$, Druet [4] for $m = 1$, Struwe [18] for $m = 2$ and Martinazzi [13] for general m; Equations with critical growth nonlinearities in $\mathbb{R}^n$ was also considered by Ruf, etc. [5]; Brezis–Nirenberg problem and Corn problem for poly-harmonic operators were studied by Ge [7,8].

We organized this paper as follows: In Sect. 2 we prove the Adams–Fontana type inequality (Theorem 1). Section 3 contributes to compactness analysis and the Palais–Smale condition of the functional J_τ. Then we prove the remaining part of Theorem 3 in Sect. 4.

2. Proof of Theorem 1

Proof of Theorem 1: Let $0 < m < N$. Assume $u \in W^{m,\frac{n}{m}}(M)$ satisfies (1.5). Then we have by (1.4), $|\overline{u}| \leq C$ for some constant C, where $\overline{u} = \frac{1}{Vol(M)} \int_M u dv_g$. We distinguish two cases to estimate the integral in (1.2):
Case 1: u is a constant. Clearly $|u| = |\overline{u}| \leq C$, and then

$$\int_M e^{\lambda(m,n)|u|^{\frac{n}{n-m}}} dv_g \leq C$$

for some universal constant C. Here and throughout this paper we denote various constants by the same C.

Case 2: u is not a constant. It is easy to from (1.4) that

$$\int_M \beta(x,u) dv_g > 0. \tag{2.1}$$

Noting that $\int_M |\nabla^m u|^{\frac{n}{m}} dv_g = 0$ implies u is a constant, we get by (1.5) and (2.1)

$$0 < \int_M |\nabla^m u|^{\frac{n}{m}} dv_g < 1. \tag{2.2}$$

Define the set

$$E = \{x \in M : |u(x) - \overline{u}| > 2|\overline{u}|\}.$$

If $E = \varnothing$, then $|u| \leq 3|\overline{u}| \leq C$, whence $\int_M e^{\lambda(m,n)|u|^{\frac{n}{n-m}}} dv_g \leq C$ for some constant C. In the following, we assume $E \neq \varnothing$. When $x \in E$, we have $|u(x) - \overline{u}| > 0$, $\frac{|\overline{u}|}{|u(x)-\overline{u}|} \leq \frac{1}{2}$, and thus

$$\begin{aligned}
|u(x)|^{\frac{n}{n-m}} &= |u(x) - \overline{u} + \overline{u}|^{\frac{n}{n-m}} \\
&= |u(x) - \overline{u}|^{\frac{n}{n-m}} \left|1 + \frac{\overline{u}}{u(x) - \overline{u}}\right|^{\frac{n}{n-m}} \\
&\leq |u(x) - \overline{u}|^{\frac{n}{n-m}} \left(1 + C\frac{|\overline{u}|}{|u(x) - \overline{u}|}\right) \\
&\leq |u(x) - \overline{u}|^{\frac{n}{n-m}} + C|u(x) - \overline{u}|^{\frac{m}{n-m}} |\overline{u}| \\
&\leq (1+\delta)|u(x) - \overline{u}|^{\frac{n}{n-m}} + \frac{C}{\delta^{\frac{m}{n-m}}} |\overline{u}|^{\frac{n}{n-m}}
\end{aligned} \tag{2.3}$$

for all $\delta > 0$. Here we have used the mean value theorem in the above first inequality. In view of (2.2), we can take δ such that

$$1 + \delta = \left(\int_M |\nabla^m u|^{\frac{n}{m}} dv_g\right)^{-\frac{m}{n-m}}. \tag{2.4}$$

When $n/2 \leq m < n$, we have

$$\begin{aligned}\delta &\geq \left(1-\int_M \beta(x,u)dv_g\right)^{-\frac{m}{n-m}}-1\\ &\geq \left(1-\int_M \beta(x,u)dv_g\right)^{-1}-1\\ &= \frac{\int_M \beta(x,u)dv_g}{1-\int_M \beta(x,u)dv_g}.\end{aligned}$$

This together with the Hölder inequality and assumption (1.4) implies

$$\begin{aligned}|\overline{u}|^{\frac{n}{n-m}} &\leq C\left(\int_M |u|^{\frac{n}{m}}dv_g\right)^{\frac{m}{n-m}}\\ &\leq C\left(\int_M \beta(x,u)dv_g\right)^{\frac{m}{n-m}}\\ &\leq C\delta^{\frac{m}{n-m}}.\end{aligned} \tag{2.5}$$

Noting that $\int_M |\nabla^m u|^{\frac{n}{m}}dv_g = \int_M |\nabla^m(u-\overline{u})|^{\frac{n}{m}}dv_g$, we obtain by combining (2.3), (2.4) and (2.5),

$$\begin{aligned}\int_M e^{\lambda(m,n)|u|^{\frac{n}{n-m}}}dv_g &= \int_{M\setminus E} e^{\lambda(m,n)|u|^{\frac{n}{n-m}}}dv_g + \int_E e^{\lambda(m,n)|u|^{\frac{n}{n-m}}}dv_g\\ &\leq C + C\int_M \exp\left\{\lambda(m,n)\frac{|u-\overline{u}|^{\frac{n}{n-m}}}{\|\nabla^m u\|^{\frac{n}{n-m}}_{L^{\frac{n}{m}}(M)}}\right\}dv_g \leq C.\end{aligned}$$

Here we have used the Adams–Fontana inequality. Combining *Case 1* and *Case 2*, we completes the proof of the Theorem. □

3. Palais–Smale condition

In this section, we prove part (i) of Theorem 3. The method we used here was first used by Adimurthi [2]. Let n be the dimension of (M,g) and n is even. For all $u \in W^{\frac{n}{2},2}(M)$ and some $\tau > 0$, we define

$$\|u\|_g^2 = \int_M \left(|\nabla^{\frac{n}{2}}u|^2 + \tau u^2\right)dv_g.$$

Standard elliptic estimates (see for example [9, Chapter 8]) imply that $\|u\|_g$ is equivalent to the Sobolev norm

$$\|u\|_{\frac{n}{2},2} = \left(\int_M \sum_{k=0}^{n/2} |\nabla^k u|^2 dv_g \right)^{1/2}.$$

Define a functional

$$I(u) = \frac{1}{2}\int_M uf(x,u)dv_g - \int_M F(x,u)dv_g.$$

Let $\beta_0 = \lambda(\frac{n}{2},2) = (4\pi)^n (n/2)!$. Firstly we have the following:

Lemma 3.1. *Let $f \in \mathcal{C}(M)$. Then we have*

(i) *If $u \in W^{\frac{n}{2},2}(M)$, then $f(x,u) \in L^p(M)$ for all $p > 1$;*
(ii) $\frac{\beta_0}{b} = \sup\{c^2 : c > 0, \sup_{\|w\|_g \le 1} \int_M wf(x,cw)dv_g < +\infty\}$*;*
(iii) *Let $\{u_k\}$ and $\{v_k\}$ be bounded sequences in $W^{\frac{n}{2},2}(M)$ converging weakly and for almost every x in M to u and v respectively. Further assume that*

$$\limsup_{k\to+\infty} \|u_k\|_g^2 < \frac{\beta_0}{b}.$$

Then for every integer $m \ge 0$,

$$\lim_{k\to+\infty} \int_M \frac{f(x,u_k)}{u_k} v_k^m dv_g = \int_M \frac{f(x,u)}{u} v^m dv_g; \tag{3.1}$$

(iv) *Let $\{u_k\}$ be a sequence in $W^{\frac{n}{2},2}(M)$ converging weakly and for almost every x in M to u, such that*

$$\sup_k \int_M u_k f(x,u_k)dv_g < +\infty.$$

Then for any $\nu \in [0,1)$,

$$\lim_{k\to+\infty} \int_M |u_k|^\nu f(x,|u_k|)dv_g = \int_M |u|^\nu f(x,|u|)dv_g, \tag{3.2}$$

$$\lim_{k\to+\infty} \int_M F(x,u_k)dv_g = \int_M F(x,u)dv_g; \tag{3.3}$$

(v) *$I(u) \ge 0$ for all u, and $I(u) = 0$ if and only if $u \equiv 0$. Further there exists a constant $B > 0$ such that*

$$\int_M uf(x,u)dv_g \le B(1 + I(u)) \tag{3.4}$$

for all $u \in W^{\frac{n}{2},2}(M)$.

Proof. By the assumption (H_4), for any $\epsilon > 0$, there exist positive constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that

$$C_2(\epsilon)e^{b(1-\epsilon)t^2} \le tf(x,t) \le C_1(\epsilon)e^{b(1+\epsilon)t^2} \quad \text{for } t \ge 1. \tag{3.5}$$

This together with Theorem 1 gives (i) and (ii).

For (iii), since $\limsup_{k\to+\infty} \|u_k\|_g^2 < \beta_0/b$, from (3.5) and Theorem 1, there exists some $p > 1$ such that $f(x,u_k)$ is bounded in $L^p(M)$. On the other hand, Sobolev imbedding theorem implies that v_k (in fact a subsequence of v_k) converging to v in $L^s(M)$ for all $s > 1$. Then we have by using the Hölder inequality for any $N > 0$,

$$\int_{|u_k|>N} \frac{f(x,u_k)}{u_k} v_k^m dv_g \le \frac{1}{N}\left(\int_M |f(x,u_k)|^p dv_g\right)^{\frac{1}{p}} \left(\int_M |v_k|^{\frac{p}{p-1}} dv_g\right)^{1-\frac{1}{p}}.$$

It follows that

$$\int_M \frac{f(x,u_k)}{u_k} v_k^m dv_g = \int_{|u_k|\le N} \frac{f(x,u_k)}{u_k} v_k^m dv_g + O\left(\frac{1}{N}\right).$$

Passing to the limit $k \to +\infty$, then $N \to +\infty$, the dominated convergence theorem implies (3.1).

To prove (iv), noting that $tf(x,t) = |t|f(x,|t|)$, we have for any $N > 0$,

$$\int_{|u_k|>N} |u_k|^\nu f(x,|u_k|)dv_g \le \frac{1}{N^{1-\nu}} \int_M |u_k| f(x,|u_k|)dv_g = O\left(\frac{1}{N^{1-\nu}}\right).$$

The same argument in the proof of (iii) implies that (3.2) holds. Now from (H_3) we have $|F(x,t)| \le M(1 + |t|^\sigma |f(x,t)|)$ for some $\sigma \in [0,1)$. Again the dominated convergence theorem implies (3.3).

Finally we prove (v). For $t > 0$, we have by (H_2),

$$\frac{\partial}{\partial t}\{tf(x,t) - 2F(x,t)\} = tf'(x,t) - f(x,t) > 0,$$

which together with the assumption (H_1) implies that $f(x,t) - 2F(x,t)$ is an even nonnegative function and strictly increasing with respect to $t > 0$. Hence $I(u) \ge 0$ and $I(u) = 0$ if and only if $u \equiv 0$. By (H_3),

$$\begin{aligned} I(u) &= \frac{1}{2}\int_M uf(x,u)dv_g - \int_M F(x,u)dv_g \\ &\ge \frac{1}{2}\int_M uf(x,u)dv_g - \int_M A(1 + |u|^\sigma |f(x,u)|)dv_g \\ &\ge -C_1 + \frac{1}{4}\int_{|u|\ge C_2} uf(x,u)dv_g \end{aligned}$$

for some positive constants C_1 and C_2. This immediately leads to (3.4). □

Secondly we have the following Lions' type lemma:

Lemma 3.2. *Let* $\{u_k\} \subset W^{\frac{n}{2},2}(M)$ *be a sequence of functions such that* $\|u_k\|_g = 1$ *and* $u_k \rightharpoonup u$ *weakly in* $W^{\frac{n}{2},2}(M)$. *Then for any* $p < 1/(1-\|u\|_g^2)$,

$$\limsup_{k\to+\infty}\int_M e^{\beta_0 p u_k^2}dv_g < +\infty. \tag{3.6}$$

Proof. If $u \equiv 0$, then the conclusion is a easy corollary of Theorem 1. Hence we assume $u \not\equiv 0$. Since $u_k \rightharpoonup u$ weakly in $W^{\frac{n}{2},2}(M)$, it follows that

$$\|u_k - u\|_g^2 \to 1 - \|u\|_g^2 \quad \text{as}\, k \to +\infty,$$

thanks to the decomposition

$$\|u_k - u\|_g^2 = \|u_k\|_g^2 + \|u\|_g^2 - 2\int_M \left(\nabla^{\frac{n}{2}} u_k \nabla^{\frac{n}{2}} u + \tau u_k u\right) dv_g.$$

Noting that there exists some constant $C > 0$ such that

$$\beta_0 p u_k^2 \le \beta_0 p \frac{(u_k-u)^2}{1-\|u\|_g^2} + Cu^2,$$

we conclude (3.6) by using the Hölder inequality and Theorem 1. □

Thirdly we need the following compactness result:

Lemma 3.3. *Assume* $f \in \mathcal{C}(M)$, $\{u_k\} \subset W^{\frac{n}{2},2}(M)$ *satisfies* $u_k \rightharpoonup u$ *weakly in* $W^{\frac{n}{2},2}(M)$, $u_k \to u$ *a.e. in* M *and* $u \not\equiv 0$. *Further assume that*

(i) *there exists* $C \in \left(0, \frac{\beta_0}{2b}\right]$ *such that* $\lim_{k\to+\infty} J_\tau(u_k) = C$;
(ii) $\|u\|_g^2 \ge \int_M uf(x,u)dv_g$;
(iii) $\sup_k \int_M u_k f(x,u_k)dv_g < +\infty$,

then there holds

$$\lim_{k\to+\infty}\int_M u_k f(x,u_k)dv_g = \int_M uf(x,u)dv_g.$$

Proof. By (v) of Lemma 3.1, $I(u) > 0$. So (ii) implies that $J_\tau(u) \ge I(u) > 0$ and

$$J_\tau(u) \le \lim_{k\to+\infty} J_\tau(u_k) = C.$$

Let $\beta = \int_M F(x,u)dv_g$. Then we have by (iv) of Lemma 3.1,

$$\lim_{k\to+\infty}\|u_k\|_g^2 = 2(C+\beta).$$

Noting that

$$\frac{u_k}{\|u_k\|_g} \rightharpoonup \frac{u}{\sqrt{2(C+\beta)}} \quad \text{weakly in} \quad W^{\frac{n}{2},2}(M),$$

and for sufficiently large k,

$$\frac{b}{\beta_0}\|u_k\|_g^2 < \left(1 - \frac{\|u\|_g^2}{2(C+\beta)}\right)^{-1},$$

we obtain by using Lemma 3.2 that there exists some $p > 1$ such that

$$\sup_k \int_M e^{pbu_k^2} dx < +\infty. \tag{3.7}$$

The power of this inequality is evident. From (H_4) we know that

$$\Lambda := \sup_{(x,t)\in M\times\mathbb{R}} |th(x,t)|e^{-\frac{p-1}{2}bt^2} < +\infty,$$

which together with (3.7) implies for any $N > 0$ that

$$\begin{aligned}\int_{|u_k|\geq N} u_k f(x,u_k)dv_g &= \int_{|u_k|\geq N} u_k h(x,u_k)e^{bu_k^2}dv_g \\ &\leq \Lambda \int_{|u_k|\geq N} e^{-\frac{p-1}{2}bu_k^2}e^{pbu_k^2}dv_g \\ &= O\left(e^{-\frac{p-1}{2}bN^2}\right).\end{aligned}$$

With the same argument used in the proof of (iii) of Lemma 3.1, we get the desired result by employing the dominated convergence theorem. □

Finally we prove the part (i) of Theorem 3:

Proof of (i) of Theorem 3 (Palais–Smale condition): Let $C \in (-\infty, \frac{\beta_0}{2b})$, and $\{u_k\}$ be a sequence in $W^{\frac{n}{2},2}(M)$ such that

$$\lim_{k\to+\infty} J_\tau(u_k) = C, \quad \lim_{k\to+\infty} J_\tau'(u_k) = 0.$$

Noting that

$$J_\tau(u_k) - \frac{1}{2}\langle J_\tau'(u_k), u_k\rangle = I(u_k), \tag{3.8}$$

we have $I(u_k) = O(\|u_k\|_g)$ since $J(u_k)$ and $J'(u_k)$ are bounded. Then from (v) of Lemma 3.1, we have

$$\int_M u_k f(x,u_k)dv_g = O(\|u_k\|_g).$$

It follows from (H_3) that

$$\int_M F(x,u_k)dv_g = O(\|u_k\|_g),$$

and whence $\|u_k\|_g^2 = O(\|u_k\|_g)$ since $J(u_k)$ is bounded. It follows that

$$\sup_k \|u_k\|_g + \sup_k \int_M u_k f(x,u_k)dv_g < +\infty. \tag{3.9}$$

Up to a subsequence, we can assume $u_k \to u_0$ weakly in $W^{\frac{n}{2},2}(M)$, strongly in $L^2(M)$ and a.e. in M.

Case 1. $C \le 0$.

We have by (v) of Lemma 3.1 and Fatou's Lemma,

$$0 \le I(u_0) \le \liminf_{k\to+\infty} I(u_k) = \liminf_{k\to+\infty}\{J_\tau(u_k) - \frac{1}{2}\langle J'_\tau(u_k), u_k\rangle\} = C.$$

This leads to $u_0 = 0$. If $C < 0$, there is no Palais–Smale sequence. If $C = 0$, we have by (3.9) and (iv) of Lemma 3.1,

$$\lim_{k\to+\infty} \|u_k\|_g^2 = 2\lim_{k\to+\infty}\left\{J_\tau(u_k) + \int_M F(x,u_k)dv_g\right\} = 0.$$

Therefore $u_k \to 0$ strongly in $W^{\frac{n}{2},2}(M)$.

Case 2. $C \in (0, \beta_0/(2b))$.

Suppose $u_0 = 0$. Then from (3.9) and (iv) of Lemma 3.1, we have

$$\lim_{k\to+\infty} \|u_k\|_g^2 = \lim_{k\to+\infty} 2\left\{J_\tau(u_k) + \int_M F(x,u_k)dv_g\right\} < \frac{\beta_0}{b}.$$

This together with (iii) of Lemma 3.1 leads to

$$\lim_{k\to+\infty}\int_M u_k f(x,u_k)dv_g = \int_M u_0 f(x,u_0)dv_g = 0.$$

Hence $\lim_{k\to+\infty} I(u_k) = 0$. From (3.8) we obtain

$$0 < C = \lim_{k\to+\infty} J_\tau(u_k) = \lim_{k\to+\infty}\left\{I(u_k) + \frac{1}{2}\langle J'_\tau(u_k), u_k\rangle\right\} = 0,$$

and a contradiction. Hence $u_0 \neq 0$.

By $J'_\tau(u_k) \to 0$, (3.9) and (iv) of Lemma 3.1, we have

$$\|u_0\|_g^2 = \int_M u_0 f(x,u_0)dv_g.$$

Hence $u_0 \in \mathcal{N}$. Then applying Lemma 3.3 to u_k and u_0, we obtain

$$\begin{aligned}
\|u_0\|_g^2 &\le \lim_{k\to+\infty}\|u_k\|_g^2 \\
&= 2\lim_{k\to+\infty}\left\{J_\tau(u_k) + \int_M F(x,u_k)dv_g\right\} \\
&= 2\lim_{k\to+\infty}\left\{I(u_k) + \frac{1}{2}\langle J'_\tau(u_k), u_k\rangle + \int_M F(x,u_k)dv_g\right\} \\
&= 2\lim_{k\to+\infty}\left\{\int_M u_k f(x,u_k)dv_g + \langle J'_\tau(u_k), u_k\rangle\right\} \\
&= \int_M u_0 f(x,u_0)dv_g = \|u_0\|_g^2.
\end{aligned}$$

This implies $u_k \to u_0$ strongly in $W^{\frac{n}{2},2}(M)$, and the Palais–Smale condition. $\square$

4. Existence result

In this section, we will prove part (ii) of Theorem 3. Firstly we need several technical lemmas.

Lemma 4.1. *Let* $f(x,t) = h(x,t)e^{bt^2} \in \mathcal{C}(M)$ *and define* $h_0(t) = \inf_{x\in M} h(x,t)$. *Suppose that* $\limsup_{t\geq 0} th_0(t) = +\infty$, *and* $a \geq 0$ *is such that*

$$\sup_{\|w\|_g \leq 1} \int_M wf(x,aw)dv_g < +\infty, \tag{4.1}$$

then $a^2 < \frac{\beta_0}{b}$.

Proof. By (ii) of Lemma 3.1, we have $a^2 \leq \frac{\beta_0}{b}$. Suppose $a^2 = \frac{\beta_0}{b}$. We employ a function sequence $\widetilde{\phi}_\delta$ introduced by Adams ([1], pp. 393–395) such that $\|\widetilde{\phi}_\delta\|_g = 1$ and on the geodesic ball $B_p(\delta)$,

$$|\widetilde{\phi}_\delta|^2 \geq \left(\log \frac{1}{\delta^n}\right) \frac{1}{\beta_0} \left(1 - C\epsilon + O\left(\log \frac{1}{\delta}\right)^{-1}\right).$$

This implies

$$\int_M \widetilde{\phi}_\delta f(x, \widetilde{\phi}_\delta)dv_g \geq \int_{B_p(\delta)} \widetilde{\phi}_\delta f(x, \widetilde{\phi}_\delta)dv_g \to +\infty$$

as $\delta \to 0$, provided that $\epsilon > 0$ is sufficiently small. This contradicts (4.1). Hence $a^2 < \frac{\beta_0}{b}$. $\square$

Lemma 4.2. *Let* $f(x,t) \in \mathcal{C}(M)$ *and assume that* (i) $\limsup_{t\to+\infty} h_0(t)t = +\infty$, *where* $h_0(t) = \inf_{x\in M} h(x,t)$; (ii) $\sup_{x\in M} f'(x,0) < \lambda_\tau(M)$; *then for any* $u \in W^{\frac{n}{2},2}(M) \setminus \{0\}$, *there exists a constant* $\gamma > 0$ *such that* $\gamma u \in \mathcal{N}$. *Moreover, if* $\|u\|_g^2 \leq \int_M uf(x,u)dv_g$, *then* $\gamma \leq 1$ *and* $\gamma = 1$ *if and only if* $u \in \mathcal{N}$.

Proof. For any $u \in W^{\frac{n}{2},2}(M) \setminus \{0\}$, $\gamma > 0$, we define

$$\psi(\gamma) = \frac{1}{\gamma} \int_M uf(x, \gamma u)dv_g.$$

Thanks to the monotonicity of the function $\frac{f(x,t)}{t}$ with respect to $t > 0$, we have by using the Levi's theorem and our assumption (ii),

$$\lim_{\gamma\to 0} \psi(\gamma) = \int_M u^2 f'(x,0)dv_g < \|u\|_g^2.$$

On the other hand, from $u \not\equiv 0$ we have $\lim_{\gamma\to+\infty} \psi(\gamma) = +\infty$. Hence there exists some $\gamma > 0$ such that $\psi(\gamma) = \|u\|^2$, which implies that $\gamma u \in \mathcal{N}$.

From (H_1) and (H_2), it follows that $f(x,tu)u/t$ is increasing for $t>0$. It follows that, if $\|u\|_g^2 \leq \int_M uf(x,u)dv_g$, then $\gamma \leq 1$ and $\gamma = 1$ if and only if $u \in \mathcal{N}$. □

By (v) of Lemma 3.1, $I(u) \geq 0$. Hence $J_\tau(u) \geq 0$ on $\mathcal{N}$. Let

$$S = \sqrt{2 \inf_{u\in\mathcal{N}} J_\tau(u)}.$$

Lemma 4.3. *Under the assumption of Lemma 4.2, we have* $0 < S^2 < \frac{\beta_0}{b}$.

Proof. Firstly we prove $S^2 > 0$. Suppose not. There exists a sequence $\{u_k\} \subset \mathcal{N}$ such that $J_\tau(u_k) \to 0$ as $k \to +\infty$. Since $J_\tau(u_k) = I(u_k)$, we have by using (v) of Lemma 3.1,

$$\sup_k \int_M u_k f(x,u_k)dv_g < +\infty; \quad \sup_k \|u_k\|_g^2 < +\infty. \tag{4.2}$$

Then we can assume without loss of generality that $u_k \to u$ weakly in $W^{\frac{n}{2},2}(M)$, strongly in $L^2(M)$, and for almost all $x \in M$. Thanks to Fatou's lemma,

$$0 \leq I(u) \leq \liminf_{k\to+\infty} I(u_k) \leq \liminf_{k\to+\infty} J_\tau(u_k) = 0.$$

Thus $u = 0$. We obtain by using (4.2) and (iv) of Lemma 3.1,

$$\lim_{k\to+\infty} \|u_k\|_g^2 = 2\lim_{k\to+\infty}\left(J_\tau(u_k) + \int_M F(x,u_k)dv_g\right) = 0.$$

Let $v_k = u_k/\|u_k\|_g$ and assume $v_k \rightharpoonup v$ weakly in $W^{\frac{n}{2},2}(M)$. The above equality together with (iii) of Lemma 3.1 and our assumption (ii) leads to

$$\begin{aligned} 1 &= \lim_{k\to+\infty}\int_M \frac{f(x,u_k)}{u_k} v_k^2 dv_g \\ &= \int_M f'(x,0)v^2 dv_g < \lambda_\tau(M)\int_M v^2 dv_g \leq 1, \end{aligned}$$

and a contradiction. Therefore $S^2 > 0$.

Secondly we prove $S^2 < \frac{\beta_0}{b}$. Let $w \in W^{\frac{n}{2},2}(M)$ such that $\|w\|_g = 1$. By Lemma 4.2, there exists a $\gamma > 0$ such that $\gamma w \in \mathcal{N}$. Hence

$$\frac{S^2}{2} \leq J_\tau(\gamma w) \leq \frac{\gamma^2}{2}\|w\|_g^2 = \frac{\gamma^2}{2}.$$

This implies $S \leq \gamma$. Using the fact that $\frac{f(x,tw)}{t}w$ is an increasing function of t in $(0,+\infty)$ and $\gamma w \in \mathcal{N}$, we have

$$\int_M \frac{f(x,Sw)}{S} w dv_g \leq \int_M \frac{f(x,\gamma w)}{\gamma} w dv_g = \frac{1}{\gamma^2}\|\gamma w\|_g^2 = 1.$$

Hence

$$\sup_{\|w\|_g\le 1}\int_M f(x,Sw)w dv_g \le S,$$

this together with Lemma 4.1 leads to $S^2 < \frac{\beta_0}{b}$. □

Lemma 4.4. *Let $f(x,t)\in\mathcal{C}(M)$ and $u_0\in\mathcal{N}$ such that $J_\tau(u_0)=\inf\{J_\tau(u):u\in\mathcal{N}\}$. Then $J'_\tau(u_0)=0$, i.e., u_0 is a critical point of J_τ.*

Proof. Suppose $J'_\tau(u_0)\neq 0$, then there exists $v\in W^{\frac{n}{2},2}(M)$ such that $\langle J'_\tau(u_0),v\rangle=1$. For α, $t\in\mathbb{R}$, we define $\sigma_t(\alpha)=\alpha u_0-tv$. Passing to the limit $t\to 0$ and $\alpha\to 1$, we have

$$\frac{d}{dt}J_\tau(\sigma_t(\alpha))\to-\langle J'_\tau(u_0),v\rangle=-1.$$

Thus we can choose two positive numbers ϵ and δ such that for all $\alpha\in[1-\epsilon,1+\epsilon]$ and $t\in(0,\delta)$,

$$J_\tau(\sigma_t(\alpha))<J_\tau(\sigma_0(\alpha))=J_\tau(\alpha u_0). \tag{4.3}$$

We want to show that $\sigma_t(\alpha)\in\mathcal{N}$ for some t and α. For this purpose we set

$$\rho_t(\alpha)=\|\sigma_t(\alpha)\|_g^2-\int_M\sigma_t(\alpha)f(x,\sigma_t(\alpha))dv_g.$$

Since $u_0\in\mathcal{N}$, we have $\rho_0(1)$. Since $u_0f(x,\alpha u_0)/\alpha$ is increasing with respect to $\alpha>0$, we have $\rho_0(1-\epsilon)>0$ and $\rho_0(1+\epsilon)<0$. This implies that $\rho_t(1-\epsilon)>0$ and $\rho_t(1+\epsilon)<0$ for sufficiently small δ. Hence $\rho_t(\alpha)=0$ for some $t\in(0,\delta)$ and $\alpha\in[1-\epsilon,1+\epsilon]$. Hence from (4.3) we have

$$\begin{aligned}\inf\{J_\tau(u):\,u\in\mathcal{N}\}&\le J_\tau(\sigma_t(\alpha))\\&<J_\tau(\alpha u_0)\le\sup_{t\in\mathbb{R}^+}J_\tau(tu_0)=J_\tau(u_0).\end{aligned}$$

This contradicts our assumption $J_\tau(u_0)=\inf\{J_\tau(u):u\in\mathcal{N}\}$, hence $J'_\tau(u_0)=0$. □

Now we can prove the second part of Theorem 3.

Proof of Part (ii) of Theorem 3 (existence result): Since the critical points of J_τ are solutions of Eq. (1.9). In view of Lemma 4.4, we only need to prove the existence of $u_0\neq 0$ such that

$$J_\tau(u_0)=\inf\{J_\tau(u):\,u\in\mathcal{N}\}=\frac{S^2}{2}.$$

Let $u_k\in\mathcal{N}$ be such that $\lim_{k\to+\infty}J_\tau(u_k)=\frac{S^2}{2}$. Since $J_\tau(u_k)=I(u_k)$, we have by (v) of Lemma 3.1,

$$\sup_k\left\{\|u_k\|_g^2+\int_M u_kf(x,u_k)dv_g\right\}<+\infty. \tag{4.4}$$

Hence, up to a subsequence, $u_k \to u$ weakly in $W^{\frac{n}{2},2}(M)$, strongly in $L^2(M)$, and for almost all $x \in M$. We *claim* that $u_0 \neq 0$ and

$$\|u_0\|_g^2 \leq \int_M u_0 f(x,u_0)dv_g. \tag{4.5}$$

Suppose $u_0 = 0$, then from (4.4) and (iv) of Lemma 3.1

$$\lim_{k\to+\infty} \|u_k\|_g^2 = 2 \lim_{k\to+\infty} \left\{ J_\tau(u_k) + \int_M F(x,u_k)dv_g \right\} = S^2.$$

This together with Lemma 4.3 and (iii) of Lemma 3.1 leads to

$$\lim_{k\to+\infty} \int_M u_k f(x,u_k)dv_g = 0.$$

It follows that

$$0 < \frac{S^2}{2} = \lim_{k\to+\infty} J_\tau(u_k) = \lim_{k\to+\infty} I(u_k) = 0,$$

a contradiction. Hence $u_0 \neq 0$. Suppose (4.5) is false,

$$\|u_0\|_g^2 > \int_M u_0 f(x,u_0)dv_g. \tag{4.6}$$

Then we obtain by using Lemma 3.3,

$$\lim_{k\to+\infty} \int_M u_k f(x,u_k)dv_g = \int_M u_0 f(x,u_0)dv_g.$$

This leads to

$$\|u_0\|_g^2 \leq \lim_{k\to+\infty} \|u_k\|_g^2 = \lim_{k\to+\infty} \int_M u_k f(x,u_k)dv_g = \int_M u_0 f(x,u_0)dv_g,$$

which contradicts (4.6). We have proved our claim.

By (4.5) and Lemma 4.2, there exists $0 < \gamma \leq 1$ such that $\gamma u_0 \in \mathcal{N}$. Hence

$$\begin{aligned} \frac{S^2}{2} &\leq J_\tau(\gamma u_0) = I(\gamma u_0) \leq I(u_0) \\ &\leq \lim_{k\to+\infty} I(u_k) = \lim_{k\to+\infty} J_\tau(u_k) = \frac{S^2}{2}. \end{aligned}$$

This implies $\gamma = 1$ and $u_0 \in \mathcal{N}$. Hence $J_\tau(u_0) = \frac{S^2}{2}$, and the proof of Part *(ii)* of Theorem 3 is finished.

2 dimensional case. Finally we consider the case $dimM = n = 2$. Noting that $|u_0| \in \mathcal{N}$ if $u_0 \in \mathcal{N}$, and $J_\tau(|u|) \leq J_\tau(u)$ for all $u \in \mathcal{N}$. Hence $J_\tau(|u_0|) = \inf_{u\in\mathcal{N}} J_\tau(u)$ if $J_\tau(u_0) = \inf_{u\in\mathcal{N}} J_\tau(u)$. Therefore the nontrivial solution u_0 can be chosen such that $u_0 \geq 0$. Hence the fact $u_0(x) > 0$ for all $x \in M$ follows immediately from the maximum principle. □

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