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On some nonlinear elliptic problems for $p-$Laplacian in $\mathbb{R}^N$

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Abstract. In this paper, we consider a nonlinear elliptic problem involving the p-Laplacian with perturbation terms in the whole $\mathbb{R}^N$. Via variational arguments, we obtain existence and regularity of nontrivial solutions.

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1. Introduction

We are interested in existence and regularity results for the following class of nonlinear elliptic problems in $\mathbb{R}^N$, $N \geq 2$

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = f(x)|u|^{\alpha-1}u \ \text{ in } \mathbb{R}^N \\ u > 0, \ \lim_{|x|\to+\infty} u(x) = 0 \end{cases} \tag{P}$$

where Δ_p denotes the p-Laplacian, that is, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$ and α is a real constant satisfying

$$0 < \alpha < p^* - 1. \tag{1.1}$$

Here, we will assume that f and the perturbation a are measurable functions satisfying some conditions which ensure the existence and regularity of solutions. We are interested in the existence of nontrivial solution of (P).

There is by now a large number of papers and an extensive literature for this kind of problems involving Laplacian and $p-$Laplacian operators both in bounded and unbounded domains, in particular since the appearance of the well-known paper by Ambrosetti and Rabinowitz [1]. There have been established many existence results via various variational methods such as min-max methods or by different faces of the celebrated Mountain Pass Theorem. Costa in [2], among others, obtained a nontrivial solution to the problem $-\Delta u + a(x)u = f(x,u)$ in $\mathbb{R}^N$, under the assumption that a is coercive and that the potential $F(x,u) = \int_0^u f(x,s)\,ds$ is nonquadratic at infinity.

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Other problems have been considered in this direction, see e.g. P. Drabek [5] and Lao Sen Yu [10].

The case of unbounded domains becomes more complicate, generally the main difficulty lies in the loss of Sobolev compact imbedding.

In this paper, by establishing sufficient condition, we give an extension and we complement some results for elliptic problems involving p-Laplacian operator. In particular, to treat variationally this class of problems, we assume a lower regularity condition on the function f (not necessary in L^∞, see assumption (H_2) below), so that a nontrivial solution can be obtained via Mountain Pass Theorem and local minimization of energy functional associated to our problem, we will distinguish two cases related to our study when $\alpha+1 > p$ and $\alpha+1 < p$. On the other hand, to overcome the lack of compactness that has arisen from the critical exponent and the unboundedness of the domain, we use a compact imbedding result essentially given by the coerciveness of the perturbation a (see assumption (H_1)).

In the second section, we establish a regularity result, more precisely, we prove that such solutions u belong to L^σ for all $\frac{Np}{N-p} \leq \sigma \leq \infty$.

Throughout this paper, we use standard notation:
$W^{1,p} := W^{1,p}(\mathbb{R}^N)$ is the ordinary Sobolev space, $L^p = L^p(\mathbb{R}^N)$ is the Lebesgue space equipped with the norm $|\cdot|_p$ and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, the Lebesgue integral in $\mathbb{R}^N$ will be denoted by the symbol $\int$ whenever the integration is carried out over all $\mathbb{R}^N$; $\mathcal{C}_0^\infty := \mathcal{C}_0^\infty(\mathbb{R}^N)$ is the space of all functions with compact support in $\mathbb{R}^N$ with continuous derivatives of arbitrary order.

Let us formulate the assumptions on the perturbation a and on the function $f = f(x)$.
(H_1) $a : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

$$a(x) \geq a_0 > 0 \text{ in } \mathbb{R}^N,$$

which is coercive, that is

$$\lim_{|x|\to+\infty} a(x) = +\infty,$$

(H_2) the function $f : \mathbb{R}^N \to \mathbb{R}$ is nonnegative and

$$f \in L^\omega \cap L^{\frac{\omega}{1-\delta}}, \text{ with } \omega = \frac{p^*}{p^* - (\alpha+1)},$$

for some $0 < \delta < 1$ small positive real.

We introduce the following function space

$$E = \left\{ u \in W^{1,p} \;\Big|\; \int |\nabla u|^p + a(x)|u|^p \, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_E = \left(\int |\nabla u|^p + a(x)|u|^p \, dx \right)^{\frac{1}{p}}$$

Since $a(x) \geq a_0 > 0$, we clearly see that the Banach space E is continuously embedded in $W^{1,p}$. We also conclude from Sobolev's Theorem the continuous imbedding $E \hookrightarrow L^{p_1}$, for all $p \leq p_1 \leq p^*$.

Definition 1.1. We say that $u \in E$ is a weak solution of (P) if

$$\int |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \int a(x)|u|^{p-2}u\varphi \, dx = \int f(x)|u|^{\alpha-1}u\varphi \, dx \qquad \text{(PV)}$$

holds for all $\varphi \in E$.

Let us remark that the assumptions (H_1) and (H_2) guarantee that integrals given in (PV) are well defined.

Now, we state the main results of this paper.

Theorem 1.1. *Suppose* $1 < p < N$, *(1.1),* (H_1) *and* (H_2) *are satisfied. Then the problem* (P) *has at least one nontrivial weak solution* $u \in E$.

Theorem 1.2. *Let* u *be a weak solution of (P). Then* $u \in L^{\sigma}$, *with* $p^* \leq \sigma \leq \infty$. *Moreover* $u > 0$ *in* $\mathbb{R}^N$ *and* u *decays uniformly as* $|x| \to \infty$.

Remark 1.1. Note that in Theorem 1.1, the existence of a nontrivial weak solution $u \in E$ is given in the sense of Definition 1.1, while the positivity and decay of the solution are proved in Theorem 2.1. However, these theorems guarantee the existence of solutions as stated in the definition of problem (P).

2. Preliminaries

Let us consider the functional $I : E \to \mathbb{R}$ given by

$$I(u) = \frac{\alpha+1}{p}\int |\nabla u|^p + a(x)|u|^p \, dx - \int f(x)|u|^{\alpha+1} \, dx.$$

By assumption (H_2) and Sobolev's inequality, we can see that the functional K defined by

$$K(u) = \int f(x)|u|^{\alpha+1} \, dx$$

is indeed well defined and of class $\mathcal{C}^1$ on the space E with

$$< K'(u); \varphi >= (\alpha+1)\int f(x)|u|^{\alpha-1}u\varphi \, dx,$$

for all $(u, \varphi) \in E$; where $<;>$ denotes the duality symbol from E to E^*.

Therefore, a weak solution of the problem (P) is a critical point u of I, i.e

$$< I'(u); \varphi >= 0 \qquad \forall \varphi \in E.$$

In order to prove our main result, we will use the following basic properties.

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$\int f(x)|u|^{\alpha+1}dx \le C|f|_{\frac{\omega}{1-\delta}}\|u\|_E^{\alpha+1}$$

for all $u \in E$.

Proof. For $m = \frac{p^*}{1+\frac{p^*-(\alpha+1)}{\alpha+1}\delta}$, we have $\frac{\alpha+1}{m} + \frac{1-\delta}{w} = 1$. Using Hölder's inequality, we obtain

$$\int f(x)|u|^{\alpha+1}dx \le |f|_{\frac{w}{1-\delta}}|u|_m^{\alpha+1}$$

for all $u \in E$

Then the lemma follows from the continuous imbedding $EC > L^m$ since $p < m < p^*$ for $0 < \delta < 1$ small enough. □

Lemma 2.2. *Suppose (H_1) and (H_2), then*
i) E is compactly embedded in L^p.
ii) K' is a compact map from E to E^*.

Proof. *i*) Proceeding as in Costa [2] for the case $p = 2$ and without loss of generality, we will show that for each sequence $\{u_n\}_{n=1}^{\infty} \subset E$ which converges weakly to 0 in $E, u_n \to 0$ strongly in L^p.

Indeed, we have $\|u_n\|_E \le C$ for some constant $C > 0$. From (H_1), for a given $\varepsilon > 0$ and $R > 0$ such that

$$a(x) \ge 2\frac{C^p}{\varepsilon} \quad \text{for all} \quad |x| \ge R,$$

we have

$$u_n \rightharpoonup 0 \text{ weakly in } W^{1,p}(B_R),$$

where B_R is the Ball of radius R centered at origin. By using the compact imbedding $W^{1,p}(B_R) \hookrightarrow L^p(B_R)$, we get

$$\int_{B_R} |u_n|^p\, dx \le \frac{\varepsilon}{2} \qquad \forall n \ge n_0 \tag{2.1}$$

for some $n_0 \in \mathbb{N}$. We also have

$$\frac{2}{\varepsilon}\int_{\mathbb{R}^N\setminus B_R} |u_n|^p\, dx \le \int_{\mathbb{R}^N\setminus B_R} \frac{a(x)}{C^p}|u_n|^p\, dx \le 1 \tag{2.2}$$

since $\|u_n\|_E^p \ge \int_{\mathbb{R}^N\setminus B_R} |u_n|^p\, dx$. Combining (2.1) and (2.2), we obtain that

$$\int_{\mathbb{R}^N} |u_n|^p\, dx \le \varepsilon, \qquad \forall n \ge n_0.$$

ii) Let $u_0 \in E$, the compactness of K' follows by estimating the quantity

$$J =< K'(u_n) - K'(u_0); \varphi > .$$

Explicitly

$$J = \int f(x)(|u_n|^{\alpha-1}u_n - |u_0|^{\alpha-1}u_0)\varphi\, dx.$$

The objective is to prove that $J \to 0$ whenever $u_n \to u_0$ weakly in E.

On one hand, by choosing δ sufficiently small such that $\theta = \frac{1}{\frac{\alpha}{p^*}+\frac{\delta}{\omega}} > 1$ and $p \leq \alpha\theta < p^*$, we obtain in view of (H_2) the following estimate

$$J \leq |f|_{\frac{\omega}{1-\delta}} \left| |u_n|^{\alpha-1}u_n - |u_0|^{\alpha-1}u_0 \right|_\theta |\varphi|_{p^*}.$$

On the other hand, since the imbedding $E \hookrightarrow L^p$ is compact, it follows from the interpolation inequality i.e.

$$|u|_t \leq |u|_p^\sigma |u|_{p^*}^{1-\sigma}, \qquad \forall u \in L^p \cap L^{p^*},$$

where $\frac{1}{t} = \frac{\sigma}{p} + \frac{1-\sigma}{p^*}$, that the imbedding $E \hookrightarrow L^{p_1}$ is compact for $p \leq p_1 < p^*$. Hence, we get $J \to 0$ (strongly) as n goes to infinity, since $p \leq \alpha\theta < p^*$. Therefore

$$K'(u_n) \to K'(u_0) \text{ strongly in } E^*.$$

as n tends to infinity.

This ends the proof of Lemma 2.2. □

Recall that $\{u_n\}_{n=1}^\infty \subset E$ is a Palais-Smale sequence if there exists $M > 0$ such that,

$$I(u_n) \leq M \text{ and } I'(u_n) \to 0 \text{ strongly in } E^* \text{ as } n \text{ goes to infinity.}$$

Remark 2.1. 1) Let us remark that for optimal values of δ, we may consider lower regularity condition on the function f.
2) Note that the assumption (H_1) gives a compact imbedding result which is used only to prove that the Palais-Smale sequence obtained by Mountain Pass type argument converges to a weak nontrivial solution.

Lemma 2.3. *Suppose $p < \alpha + 1$ and let u_n be a Palais-Smale sequence. Then u_n possesses a subsequence which converges strongly in E.*

Proof. Let $\{u_n\}_{n=1}^\infty \subset E$ be a Palais-Smale sequence. We have

$$I(u_n) - \frac{1}{p} < I'(u_n); u_n >= \left(-1 + \frac{\alpha+1}{p}\right) \int f(x)|u_n|^{\alpha+1}\, dx,$$

since

$$\frac{1}{p} < I'(u_n); u_n > = \frac{\alpha+1}{p} \int |\nabla u_n|^p + a(x)|u_n|^p \, dx$$
$$- \frac{\alpha+1}{p} \int f(x)|u_n|^{\alpha+1} \, dx.$$

Hence, we come to the conclusion that

$$\left(1 - \frac{1}{\frac{\alpha+1}{p}}\right)\left(\frac{\alpha+1}{p} \int |\nabla u_n|^p + a(x)|u_n|^p \, dx\right) \le M - \frac{1}{p} < I'(u_n); u_n > .$$

From this inequality, we easily deduce that u_n is a bounded sequence in E, since $p < \alpha + 1$. Consequently, there exists a subsequence still denoted by u_n such that u_n converges weakly in E.

Now, we claim that u_n converges strongly in E. Indeed, for any pair integer (i, j) we have

$$\int (|\nabla u_i|^{p-2}\nabla u_i - |\nabla u_j|^{p-2}\nabla u_j)(\nabla u_i - \nabla u_j)$$
$$+ (a(x)|u_i|^{p-2}u_i - a(x)|u_j|^{p-2}u_j)(u_i - u_j)$$
$$= I'(u_i) - < I'(u_j); (u_i - u_j) >$$
$$+ \int f(x)(|u_i|^{\alpha-1}u_i - |u_j|^{\alpha-1}u_j)(u_i - u_j) \, dx.$$

By Palais-Smale condition, it is easy to see that

$$I'(u_i) - < I'(u_j); (u_i - u_j) > \to 0$$

as i and j tend to infinity.

From the Lemma 2.2 (K' is compact), we have

$$\int f(x)(|u_i|^{\alpha-1}u_i - |u_j|^{\alpha-1}u_j)(u_i - u_j) \, dx \to 0,$$

as i and j tend to infinity. Finally, in virtue of the following algebraic relation

$$|\xi_1 - \xi_2|^r \le ((|\xi_1|^{r-2}\xi_1 - |\xi_2|^{r-2}\xi_2)(\xi_1 - \xi_2))^{s/2}(|\xi_1|^r + |\xi_2|^r)^{1-s/2}$$

with $s = r$, for $1 < r \le 2$ and $s = 2$ for $2 < r$, we deduce that (u_n) is a Cauchy sequence in E, therefore it converges strongly.

This concludes the proof of Lemma 2.3. □

3. Proof of the main results

In this section, we give the proof of the existence result, we apply Mountain Pass Lemma and minimization to find nontrivial solutions. For that, we will separately study the following cases: $\mathbf{p} < \alpha + \mathbf{1}$ and $\alpha + \mathbf{1} < \mathbf{p}$. After that, we use an iterative method to prove the regularity result.

Proof of Theorem 1.1

Lemma 3.1. *Suppose* $(H_1), (H_2)$ *and* $p < \alpha + 1$*, then*
1) There exist γ, ρ*, such that* $I(u) \geq \gamma$*, for* $\|u\|_E = \rho$.
2) $I(tu) \to -\infty$ *as* $t \to +\infty$.

Proof. 1) From lemma 2.1, we have

$$I(u) \geq \frac{\alpha+1}{p}\|u\|_E^p - C_0\|u\|_E^{\alpha+1},$$

for some positive constant C_0.

Denoting by θ the quantity $\|u\|_E$, we therefore obtain the following minoration of I for any $u \in E$

$$I(u) \geq \theta^p\left(\frac{\alpha+1}{p} - C_0\theta^{\alpha+1-p}\right).$$

Which implies that there exist γ, $\rho > 0$ such that $I(u) \geq \gamma > 0$ for all $\|u\|_E = \rho$.
2) From the expression

$$I(t^{1/p}u) = \frac{t(\alpha+1)}{p}\|u\|_E^p - t^{\frac{\alpha+1}{p}}\int f(x)|u|^{\alpha+1}\,dx,$$

it follows that

$$I(t^{1/p}u) \to -\infty \quad \text{as} \quad t \to +\infty$$

since $p < \alpha + 1$. □

Hence, in view of Lemmas 2.3 and 3.1, we can apply the Mountain-Pass Theorem (c.f. [1]) which guarantees the existence of nontrivial weak solutions of (p).

On the other hand, in the case $\alpha+1 < p$, we may use the local minimization of the functional I to prove the existence result. Indeed, by hypothesis (H_2), the functional I is weakly lower semicontinuous and differentiable. Moreover, I is bounded below. In fact, we have

$$I(u) \geq \frac{\alpha+1}{p}\|u\|_E^p - C|f|_\omega\|u\|_{p^*}^{\alpha+1},$$

which implies that

$$I(u) \geq \frac{\alpha+1}{p}\|u\|_E^p - C'|f|_\omega\|u\|_E^{\alpha+1}$$

since $|u|_{p^*} \leq D\|u\|_E$ and for some constant $D > 0$.

Then I is bounded below and consequently it has a critical point u (a minimizer)

$$I(u) = \inf\{I(v) : v \in E\},$$

which is a solution of the problem (P). We note that u must be nontrivial since

$$I(s\varphi) = s^p \frac{\alpha+1}{p} \|\varphi\|_E^p - s^{\alpha+1} \int f(x)|\varphi|^{\alpha+1}\, dx$$

for some $\varphi \in \mathcal{C}_0^\infty$. Hence, since $\alpha + 1 < p$, we get $I(s\varphi) < 0$ for small s.

This concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2 In this section, we may choose $u \geq 0$ since we can show that argument developed here is true for u^+ and u^-.

Let M be a nonnegative real number and define $u_M(x) = \min\{u(x), M\}$. For any real $i \geq 1$, $u_M^i \in E$. We have

$$\int |\nabla u|^{p-2}\nabla u \nabla\varphi + a(x)|u|^{p-2}u\varphi\, dx = \int f(x)|u|^{\alpha-1}u\varphi\, dx$$

for all $\varphi \in E$.

Substituting $\varphi = (u_M)^i$ in this equation, we obtain the following estimate

$$i \int (u_M)^{i-1}|\nabla u_M|^p \leq \int f(x)|u|^{\alpha+i}\, dx. \tag{3.1}$$

Due to the fact that $(u_M)^{i-1}|\nabla u_M|^p = \left(\frac{p}{i+p-1}\right)^p |\nabla(u_M)^{\frac{(i+p-1)}{p}}|^p$ and Sobolev's inequality, we get

$$\left(\int (u_M)^{\frac{N}{N-p}(i+p-1)}\right)^{\frac{N-p}{N}} \leq C \int f(x)|u|^{\alpha+i}\, dx$$

for some constant $C > 0$.

Setting $i = i_0 = 1 + p^* \frac{\delta}{\omega}$, $s_0 = \frac{N}{N-p}(i_0 + p - 1) = \frac{N}{N-p}(p + p^* \frac{\delta}{\omega})$. Letting $M \to \infty$, we conclude that $u \in L^{s_0}$ since

$$\frac{1-\delta}{\omega} + \frac{\alpha + i_0}{p^*} = 1.$$

Setting now $i_1 = 1 + p^* \frac{\delta}{\omega} + \frac{N}{N-p} p^* \frac{\delta}{\omega} = i_0 + \frac{N}{N-p} p^* \frac{\delta}{\omega}$, repeating the same argument, we get

$$u \in L^{s_1}, \quad \text{where} \quad s_1 = \frac{N}{N-p}(i_1 + p - 1),$$

since $\frac{\alpha+i_1}{p^*}+\left(\frac{1-\delta}{\omega}-\frac{N}{N-p}\frac{\delta}{\omega}\right)=1$ and $f\in L^{\frac{\omega}{1-\delta(1+\frac{N}{N-p})}}$ for δ small enough. Iterating this process gives

$$u\in L^{s_j} \quad \text{where} \quad s_j=\frac{N}{N-p}(i_j+p-1),$$

with $i_j=1+p^*\frac{\delta}{\omega}+\frac{N}{N-p}p^*\frac{\delta}{\omega}+\ldots+\left(\frac{N}{N-p}\right)^j p^*\frac{\delta}{\omega}$. Hence, it follows that

$$u\in L^{\sigma} \quad \text{for all} \quad \frac{Np}{N-p}\le\sigma<\infty.$$

To complete the proof of Theorem 1.2, we need the L^∞ estimate of solutions.

Indeed, set $i=kp+1$ in (3.1). From Sobolev's inequality, it follows that

$$\frac{kp+1}{C_1^p(k+1)^p}\Big(\int u_M^{(k+1)p^*}\,dx\Big)^{\frac{p}{p^*}}\le\int f(x)|u|^{\alpha+kp+1}\,dx$$

for some constant $C_1>0$.

On the other hand, fix $q>p^*$ and $t=\frac{p^*q}{\alpha(q-p^*)+q}$. Remark that $t<p^*$ and $\frac{1}{\omega}+\frac{\alpha}{q}+\frac{1}{t}=1$. Using ($H_2$) and Hölder's inequality we obtain

$$\frac{kp+1}{C_1^p(k+1)^p}\Big(\int u_M^{(k+1)p^*}\,dx\Big)^{\frac{p}{p^*}}\le\int(f(x))^\omega\,dx)^{\frac{1}{\omega}}\Big(\int u^q\,dx\Big)^{\frac{q}{\alpha}}\Big(\int u^{(k+1)t}\,dx\Big)^{\frac{p}{t}}.$$

Since $u\in L^q$ for $q>p^*$, there exists a constant $C_2>0$ independent of $M>0$ and $k>0$ such that

$$\Big(\int u_M^{(k+1)p^*}\,dx\Big)^{\frac{p}{p^*}}\le C_2\frac{(k+1)^p}{(kp+1)}\Big(\int u^{(k+1)t}\,dx\Big)^{\frac{p}{t}}$$

i.e.

$$|u_M|_{(k+1)p^*}\le C_3^{\frac{1}{k+1}}\left[\frac{k+1}{(kp+1)^{\frac{1}{p}}}\right]^{\frac{1}{k+1}}|u|_{(k+1)t} \tag{3.2}$$

with $C_3=C_2^{\frac{1}{p}}$. By choosing k_1 in (3.2) such that $(k_1+1)t=p^*$ i.e $k_1=\frac{p^*}{t}-1$, we obtain

$$|u_M|_{(k_1+1)p^*}\le C_3^{\frac{1}{k_1+1}}\left[\frac{k_1+1}{(k_1p+1)^{\frac{1}{p}}}\right]^{\frac{1}{k_1+1}}|u|_{p^*}\quad\forall M>0.$$

We have $\lim_{M\to\infty}u_M(x)=u(x)$, and Fatou's lemma implies

$$|u|_{(k_1+1)p^*}\le C_3^{\frac{1}{k_1+1}}\left[\frac{k_1+1}{(k_1p+1)^{\frac{1}{p}}}\right]^{\frac{1}{k_1+1}}|u|_{p^*}.$$

Then $u \in L^{(k_1+1)p^*}$. By the same argument we can choose k_2 in (3.2) such that $(k_2+1)t = (k_1+1)p^*$ i.e $k_2 = (\frac{p^*}{t})^2 - 1$. Then we have

$$|u|_{(k_2+1)p^*} \leq C_3^{\frac{1}{k_2+1}} \left[\frac{k_2+1}{(k_2 p+1)^{\frac{1}{p}}}\right]^{\frac{1}{k_2+1}} |u|_{(k_1+1)p^*}.$$

By iteration, we obtain $k_n = (\frac{p^*}{t})^n - 1$ such that

$$|u|_{(k_n+1)p^*} \leq C_3^{\frac{1}{k_n+1}} \left[\frac{k_n+1}{(k_n p+1)^{\frac{1}{p}}}\right]^{\frac{1}{k_n+1}} |u|_{(k_{n-1}+1)p^*} \quad \text{for all } \ n \in \mathbb{N}.$$

It follows

$$|u|_{(k_n+1)p^*} \leq C_3^{\sum_{i=1}^{n} \frac{1}{k_i+1}} \prod_{i=1}^{n} \left[\frac{k_i+1}{(k_i p+1)^{\frac{1}{p}}}\right]^{\frac{1}{k_i+1}} |u|_{p^*},$$

or equivalently

$$|u|_{(k_n+1)p^*} \leq C_3^{\sum_{i=1}^{n} \frac{1}{k_i+1}} \prod_{i=1}^{n} \left[\left[\frac{k_i+1}{(k_i p+1)^{\frac{1}{p}}}\right]^{\frac{1}{\sqrt{k_i+1}}}\right]^{\frac{1}{\sqrt{k_i+1}}} |u|_{p^*}.$$

Since

$$\left[\frac{a+1}{(ap+1)^{\frac{1}{p}}}\right]^{\frac{1}{\sqrt{a+1}}} > 1 \qquad \forall a > 0 \qquad \text{and} \qquad \lim_{a\to\infty} \left[\frac{a+1}{(ap+1)^{\frac{1}{p}}}\right]^{\frac{1}{\sqrt{a+1}}} = 1$$

there exists a constant $C_4 > 0$ independent of $n \in \mathbb{N}$ such that

$$|u|_{(k_n+1)p^*} \leq C_3^{\sum_{i=1}^{n} \frac{1}{k_i+1}} C_4^{\sum_{i=1}^{n} \frac{1}{\sqrt{k_i+1}}} |u|_{p^*},$$

where $\frac{1}{k_i+1} = (\frac{t}{p^*})^i$, $\frac{1}{\sqrt{k_i+1}} = (\sqrt{\frac{t}{p^*}})^i$ and $\frac{t}{p^*} < \sqrt{\frac{t}{p^*}} < 1$. Hence there exists a constant $C_5 > 0$ independent of $n \in \mathbb{N}$ such that

$$|u|_{(k_n+1)p^*} \leq C_5 |u|_{p^*} \quad \text{for all } \ n \in \mathbb{N}.$$

Letting n tend to infinity we get

$$|u|_{L^\infty} \leq C_5 |u|_{p^*}.$$

Therefore, it follows that $u \in L^\infty$.

This ends the proof of the L^∞-estimate of solutions of (P).

Letting $\varphi = u^-$ as a test function in (PV), implies $u \geq 0$ in $\mathbb{R}^N$. The positivity of solution follows immediately from the weak Harnack type inequality proved in Trudinger [15, Theorem 1.2]. Finally, the decay of u follows directly from the result of Serrin [13, Theorem 1], from which we obtain the estimate

$$|u|_{L^\infty(B_1(x))} \leq C_6 |u|_{p^*(B_2(x))}$$

independently of $x \in \mathbb{R}^N$, where $C_6 = C_6(N, p, \varepsilon)$. □

Corollary 3.1. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then* $f \in L^\infty_{loc}$, *implies that* $u \in \mathcal{C}^{1,\eta}(B_R(0))$ *for any* $R > 0$ *with some* $\eta(R) \in]0, 1[$.

The proof of this fact follows immediately from the regularity result of Tolksdorf [14].

Remark 3.1. For the case $p = \alpha + 1$, we can prove by the same argument used in the case $p > \alpha$ that there exists λ_* such that for all λ verifying $0 < \lambda < \lambda_*$, the following eigenvalue problem

$$\begin{cases} \Delta_p u + a(x)|u|^{p-2}u = \lambda f(x)|u|^{p-2}u \ \text{ in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases}$$

has at least one nontrivial solution in E.

Remark 3.2. We also obtain an existence result for problems of the form

$$-\Delta_p u + a(x)|u|^{p-2}u = \sum_I f_i(x)|u|^{\alpha_i - 1}u \qquad \text{in } \mathbb{R}^N. \tag{P'}$$

with $f_i \in L^{m_i}$ positive, $m_i \in [r_i, \frac{r_i}{1-\delta}]$ where

$$r_i = \frac{1}{1 - \frac{\alpha_i+1}{p^*}},$$

$\frac{\alpha_i+1}{p^*} < 1, \alpha_i > 0$ and $0 < \delta < 1$ is a small positive real.

Remark 3.3. We obtain the same result for the boundary value problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = f(x)|u|^{\alpha-1}u \qquad \text{in } \Omega \\ u > 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \qquad \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{P''}$$

where Ω is an exterior domain with $\mathcal{C}^{1,\eta}$ boundary, $0 < \alpha < 1$.

Here the result is related to that of Lao Sen Yu [10] but under rather more regularity conditions on the function f.

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