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Existence of Periodic Solutions for the Quasi-Static Thermoelastic Thermistor Problem

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Abstract. We deal with a new model for the thermistor problem formulated as a coupled system of PDE's involving nonlinear energy heat equation, stationary charge conservation equation of electrical current and thermoelastic equations of displacement. We establish the existence of weak periodic solutions rewriting our system as an abstract problem in order to utilize the maximal monotone mappings theory and a fixed point argument for a suitable operator equation.

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1. Introduction

The aim of this paper is to study the existence of weak periodic solutions for a new model for the thermistor problem. The model consists of a nonlinear coupled system of partial differential equations which includes the evolution equation for the temperature in a conductor body with the Joule heating $\sigma(\theta) \mid \nabla\varphi \mid^2$as a source, the stationary charge conservation equation of electrical current with temperature dependent electric conductivity $\sigma(\theta)$ and a system of quasi-static linear thermoelasticity equations of displacement subject to external forces and heat sources.

In what follows, the reference configuration of the thermistor Ω is an open, bounded and regular set of R^n $(n \geq 2)$ with boundary $\partial\Omega$ and the unit outward normal $\nu = (\nu_1, \dots, \nu_n)$ is defined at points of $\partial\Omega$. We assume that the smooth boundary $\partial\Omega$ is divided into two relatively open parts Γ_D and Γ_N such that $\Gamma_D \cap \Gamma_N = \emptyset$ and $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$. We denote by $P := R/\omega Z$ the period interval $[0, \omega]$, so the functions defined in Q are automatically time ω-periodic.

We may write the system as

$$\left\{\begin{array}{c} \theta_t - \sum_{i,j=1}^n \frac{\partial}{\partial x_j}(k_{ij}\frac{\partial\theta}{\partial x_i}) = \sigma(\theta) \mid \nabla\varphi \mid^2 - \sum_{i,j=1}^n m_{ij}\frac{\partial^2 u_i}{\partial t\partial x_j} \text{ in } Q := \Omega\times P, \\ div(\sigma(\theta)\nabla\varphi) = 0 \text{ in } Q, \\ -\sum_{i,j,k,l=1}^n \frac{\partial}{\partial x_j}(a_{ijkl}\frac{\partial u_k}{\partial x_l}) + \sum_{i,j=1}^n \frac{\partial}{\partial x_j}(m_{ij}\theta) = f_i \text{ in } Q, \end{array}\right\} \tag{1.1}$$

The k_{ij} and m_{ij} are the components of the heat conduction tensor and the thermal expansion tensor respectively, both assumed to be symmetric. The a_{ijkl} are the components of the elasticity tensor, the symmetries of which are indicated below. Finally, $\mathbf{f} = (f_1, \ldots, f_n)$ is the density of the applied body forces. The boundary and periodic conditions for the problem are

$$-\sum_{i,j=1}^n k_{ij}\frac{\partial\theta}{\partial x_i}\nu_j = h(\theta-\theta_a) \quad \text{on} \quad \Gamma := \partial\Omega\times P, \tag{1.2}$$

$$\varphi(x,t) = \varphi_0(x,t) \quad \text{on} \quad \Sigma_D := \Gamma_D \times P\,,$$

$$\frac{\partial\varphi(x,t)}{\partial\nu} = 0 \quad \text{on} \quad \Sigma_N := \Gamma_N\times P\,, \tag{1.3}$$

$$\mathbf{u}(x,t) = 0 \quad \text{on} \quad \Sigma_D\,,$$

$$\frac{\partial\mathbf{u}(x,t)}{\partial\nu} = 0 \quad \text{on} \quad \Sigma_N, \tag{1.4}$$

$$\theta(x,t+\omega) = \theta(x,t)\,, \quad \varphi(x,t+\omega) = \varphi(x,t)\,,$$

$$\mathbf{u}(x,t+\omega) = \mathbf{u}(x,t) \quad \text{in} \quad Q,\, \omega > 0\,. \tag{1.5}$$

The coefficient of heat exchange h is positive and the temperature near $\partial\Omega$ is denoted by θ_a. Moreover, we assume that the body is held fixed on Σ_D i.e. $\mathbf{u}=\mathbf{0}$ while on Σ_N, the body is free. The conditions on the boundary hold in the trace sense.

We observe that if

$$\varphi \in L^2\big(P; W^{1,2}(\Omega)\big) \cap L^\infty(Q)\,,$$

the equation $(1.1)_2$ allows to compute

$$\sigma(\theta) \mid \nabla\varphi \mid^2 = div\big(\sigma(\theta)\varphi\nabla\varphi\big)$$

in the sense of distributions.

Therefore, we will establish the existence of periodic solutions for the problem

$$\left\{\begin{array}{c} \theta_t - \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\Big(k_{ij}\frac{\partial\theta}{\partial x_i}\Big) = div\big(\sigma(\theta)\varphi\nabla\varphi\big) - \sum_{i,j=1}^n m_{ij}\frac{\partial^2 u_i}{\partial t\partial x_j} \text{ in } Q, \\ div\big(\sigma(\theta)\nabla\varphi\big) = 0 \text{ in } Q, \\ -\sum_{i,j,k,l=1}^n \frac{\partial}{\partial x_j}\Big(a_{ijkl}\frac{\partial u_k}{\partial x_l}\Big) + \sum_{i,j=1}^n \frac{\partial}{\partial x_j}(m_{ij}\theta) = f_i \text{ in } Q, \end{array}\right\} \tag{1.1'}$$

with boundary and periodic conditions (1.2)–(1.5).

The unknown of system are the temperature $\theta(x,t)$ inside the conductor Ω, the electrical potential $\varphi(x,t)$ and the displacement vector field $\mathbf{u}(x,t) = (u_1(x,t), \ldots, u_n(x,t))$.

Thermistors are electrical devices of thermally sensitive resistors extensively utilized in industrial world. They have received much attention in the mathematical literature (see, e.g. [1, 2, 5–11, 18, 19]). In [20], is showed the existence of "capacity solutions" for the thermistor problem with the addition of dynamic thermoelastic equations containing terms of the form $\rho\frac{\partial^2 u_i}{\partial t^2}$. When $\rho\frac{\partial^2 u_i}{\partial t^2}$ may be omitted, the resulting problem is quasi-static. Quasi-static thermoelastic systems have been widely considered in literature. For more details we refer to the works [12, 16, 17] and the references therein. However, in all these references the periodic case was not studied. The motivation for considering the quasi-static thermoelastic thermistor, arises from the often cracked and broken for these devices because of the thermal stresses.

In this paper, the periodic solutions for the new thermistor problem are seeked in suitable spaces of ω-periodic functions without to look for fixed points to the periodic Poincaré map. Methodologically, we employ the functional frame of some results on the maximal monotone mappings.

To carry out this kind of program, the starting point relies on the possibility to reformulate our system as an abstract problem to which one applies some techniques deriving from the maximal monotone mappings theory and to get periodic solutions as a fixed point of an operator equation.

The plan of paper is the following: In Sections 2 and 3 we give some preliminaries and resolve the stationary charge equation of electrical current when we substitute $\sigma(\theta)$ by $\sigma(\zeta)$, for $\zeta \in L^2(Q)$. Next, we pass to treat the thermoelastic equations and the nonlinear heat equation in Sections 4 and 5. Finally, Section 6 is devoted to prove the existence and uniqueness of weak periodic solutions θ_n, φ_n, $\mathbf{u}_n$ for the approximating problems obtained replacing $\sigma(\zeta)$ by $\sigma(\zeta_n)$ and $\zeta_n \in L^2(Q)$ in place of θ in (2.3) below. Futhermore, are deduced necessary uniform a priori estimates which will play a foundamental role joint to a fixed point argument, to infer the existence of weak periodic solutions for system (1.1')–(1.5) as limit of the approximating solutions θ_n, φ_n, $\mathbf{u}_n$.

2. Preliminaries

Next, let us introduce the functional framework for the periodic solutions of problem. To this end, we consider the functional spaces

$$W_D^{1,2}(\Omega) := \left\{v \in W^{1,2}(\Omega) : v = 0 \text{ on } \Gamma_D\right\},$$
$$\left(W_D^{1,2}(\Omega)\right)^n := \left\{v \in \left(W^{1,2}(\Omega)\right)^n : v = 0 \text{ on } \Gamma_D\right\}$$

and a triplet of Hilbert spaces

$$V_D := L^2\left(P; W_D^{1,2}(\Omega)\right),$$
$$V := L^2\left(P; W^{1,2}(\Omega)\right),$$
$$W := L^2\left(P; \left(W_D^{1,2}(\Omega)\right)^n\right),$$

endowed with the respective norms

$$\|v\|_{V_D} := \left(\int_Q |\, v(x,t)\,|^2\, dxdt + \int_Q |\, \nabla v(x,t)\,|^2\, dxdt\right)^{1/2}$$
$$\|v\|_{V} := \left(\int_Q |\, \nabla v(x,t)\,|^2\, dxdt + \int_\Gamma |\, v(x,t)\,|^2\, dsdt\right)^{1/2}$$

and

$$\|\mathbf{u}\|_W := \left(\sum_{i=1}^n \int_Q |\, u_i(x,t)\,|^2\, dxdt + \sum_{i=1}^n \int_Q |\, \nabla u_i(x,t)\,|^2\, dxdt\right)^{1/2}.$$

The topological dual spaces of V_D, V and W shall be denoted by V_D^*, V^* and W^* respectively, with $\|.\|_*$ norm. The pairing of duality between these spaces shall be written as $\langle .,. \rangle$.

Remark 2.1. We note that the norm in V is equivalent to

$$\left(\int_Q |\, v(x,t)\,|^2\, dxdt + \int_Q |\, \nabla v(x,t)\,|^2\, dxdt\right)^{1/2}.$$

To study our system we will do the following structural assumptions on the data

H$_\sigma$) $\sigma \in C(R)$ such that $0 < \sigma_* \le \sigma(s) \le \sigma^*$, $\forall s \in R$;

H$_K$) $k_{ij} \in L^\infty(Q)$ and $\exists\ \mu > 0 : \sum_{i,j=1}^n k_{ij}\xi_i\xi_j \ge \mu \mid \xi \mid^2, \forall \xi \in R^n$;

H$_M$) m_{ij}, $\frac{\partial m_{ij}}{\partial t} \in W^{1,\infty}(Q)$, $m_{ij} = m_{ji} > 0$;

H$_A$) $\left\{ \begin{array}{c} a_{ijkl} \in L^\infty(Q),\ a_{ijkl} = a_{jikl} = a_{klij},\ \forall i,j,k,l \text{ and} \\ \exists \lambda > 0 : \sum_{i,j,k,l=1}^n a_{ijkl}\eta_{ij}\eta_{kl} \ge \lambda \sum_{i,j=1}^n \mid \eta_{ij} \mid^2,\ \forall \eta = (\eta_{ij}),\ \eta_{ij} = \eta_{ji}; \end{array} \right\}$

H$_\mathbf{f}$) $\mathbf{f} \in (L^2(Q))^n$;

H$_0$) φ_0 is a t-periodic bounded function with an extention in Q, $\widetilde{\varphi}_0 \in L^\infty(P; W^{1,\infty}(\Omega))$ and $\frac{\partial \widetilde{\varphi}_0}{\partial \nu} = 0$ on Γ_D.

To complete the presentation of the quasi-static problem, we give the notion of weak solution.

Definition 2.2. A triplet $\{\theta, \varphi, \mathbf{u}\}$ is said to be a weak periodic solution of problem (1.1')–(1.5) if satisfies

$$\theta \in L^2\big(P; W^{1,2}(\Omega)\big), \quad \theta_t \in L^2\Big(P; \big(W^{1,2}(\Omega)\big)^*\Big)$$
$$\varphi - \varphi_0 \in L^2\big(P; W_D^{1,2}(\Omega)\big)$$
$$\mathbf{u} \in L^2\Big(P; \big(W_D^{1,2}(\Omega)\big)^n\Big), \quad \mathbf{u}_t \in L^2\Big(P; \big(W^{-1,2}(\Omega)\big)^n\Big)$$

and

$$\int_Q \theta_t w dxdt + \sum_{i,j=1}^{n} \int_Q k_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial w}{\partial x_j} dxdt + \int_\Gamma h(\theta - \theta_a) w dsdt \qquad (2.1)$$

$$= -\int_Q \sigma(\theta)\varphi \nabla\varphi \nabla w dxdt + \sum_{i,j=1}^{n} \int_Q \frac{\partial m_{ij}}{\partial t} \frac{\partial u_i}{\partial x_j} w dxdt\,, \quad \forall w \in V$$

$$\int_Q \sigma(\theta)\nabla\varphi\nabla\xi dxdt = 0\,, \quad \forall \xi \in V_D \qquad (2.2)$$

$$\sum_{i,j,k,l=1}^{n} \int_Q a_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \rho_i}{\partial x_j} dxdt - \sum_{i,j=1}^{n} \int_Q m_{ij}\theta \frac{\partial \rho_i}{\partial x_j} dxdt$$

$$= \int_Q f_i \rho_i dxdt\,, \quad \forall \rho \in W\,. \qquad (2.3)$$

3. The stationary conservation of electrical current problem

We begin showing the existence of weak periodic solutions for the stationary conservation of electrical current problem. Fixed $\zeta \in L^2(Q)$ and defined $z = \varphi - \varphi_0$, then this function should satisfy

$$div\big(\sigma(\zeta)(\nabla z + \nabla\varphi_0)\big) = 0 \quad \text{in} \quad Q\,, \qquad (3.1)$$

$$z(x,t) = 0 \quad \text{on} \quad \Sigma_D\,, \qquad (3.2)$$

$$\frac{\partial z(x,t)}{\partial \nu} = 0 \quad \text{on} \quad \Sigma_N\,, \qquad (3.3)$$

$$z(x, t+\omega) = z(x,t) \quad \text{in} \quad Q\,, \quad \omega > 0\,. \qquad (3.4)$$

Definition 3.1. A weak periodic solution to (3.1)–(3.4) is a function

$$z \in L^2\big(P; W_D^{1,2}(\Omega)\big)$$

such that

$$\int_Q \sigma(\zeta)(\nabla z + \nabla\varphi_0)\nabla\xi dxdt = 0\,, \quad \forall \xi \in V_D\,. \qquad (3.5)$$

In our setting, the existence of solutions for (3.5) can be deduced from the next result.

Theorem 3.1 ([3,4,14]). *. If A is a monotone, hemicontinuous mapping from V_D to V_D^* such that A is coercitive, then $Range(A) = V_D^*$.*

To proceed, we define the mapping A by setting

$$\langle Az, \xi\rangle := \int_Q \sigma(\zeta)(\nabla z + \nabla\varphi_0)\nabla\xi dxdt,\ \forall z\,, \quad \xi \in V_D\,.$$

The properties of the mapping A are given in the following result.

Proposition 3.2. *Under assumptions H_σ) and H_0), the mapping A is:*

i) *hemicontinuous;*
ii) *monotone;*
iii) *coercive.*

Proof. i) By the Hölder inequality, we infer that

$$| \langle Az, \xi \rangle | \leq \sigma^* \left(\int_Q | \nabla z + \nabla \varphi_0 |^2 \, dxdt \right)^{1/2} \left(\int_Q | \nabla \xi |^2 \, dxdt \right)^{1/2}$$
$$\leq \sigma^* \sqrt{2} (\|z\|_{V_D} + \|\nabla \varphi_0\|_{L^2(Q)}) \|\xi\|_{V_D} \,,$$

by which

$$\|Az\|_* \leq \sigma^* \sqrt{2} (\|z\|_{V_D} + \|\nabla \varphi_0\|_{L^2(Q)}) \,.$$

The hemicontinuity is deduced from a result of [13, Theorems 2.1 and 2.3].

ii) $\langle Az_1 - Az_2, \, z_1 - z_2 \rangle = \int_Q \sigma(\zeta) \mid \nabla(z_1 - z_2) \mid^2 dxdt \geq 0$.

iii) By the Poincaré inequality, for some constant c, one has

$$\langle Az, \; z \rangle = \int_Q \sigma(\zeta) \mid \nabla z \mid^2 dxdt \geq \sigma_* c \|z\|^2_{V_D} - \sigma^* \|\nabla \varphi_0\|_{L^2(Q)} \|z\|_{V_D}$$

so that

$$\frac{\langle Az, \; z \rangle}{\|z\|_{V_D}} \geq \sigma_* c \|z\|_{V_D} - \sigma^* \|\nabla \varphi_0\|_{L^2(Q)} \to +\infty \quad \text{as} \quad \|z\|_{V_D} \to +\infty \,. \qquad \square$$

Thus, the equivalence between (3.5) and the abstract problem

$$Az = 0 \,. \tag{3.6}$$

is readly understood.

Hence, we can state the main result for the stationary consevation of electrical current problem.

Theorem 3.3. *Let $H_\sigma)$ and $H_0)$ be satisfied, there exists a unique periodic solution to* (3.6)*.*

Proof. The existence of solutions is a consequence of Theorem 3.1. The uniqueness descends from the strict monotonicity of the mapping A. $\square$

Therefore, in correspondence of a fixed $\zeta \in L^2(Q)$, has been proved the existence and uniqueness of the weak periodic solution φ for the problem

$$div\big(\sigma(\zeta)\nabla\varphi\big) = 0 \quad \text{in} \quad Q \,, \tag{3.7}$$
$$\varphi(x,t) = \varphi_0(x,t) \quad \text{on} \quad \Sigma_D := \Gamma_D \times P \,, \tag{3.8}$$
$$\frac{\partial \varphi(x,t)}{\partial \nu} = 0 \quad \text{on} \quad \Sigma_N := \Gamma_N \times P \,, \tag{3.9}$$
$$\varphi(x, t+\omega) = \varphi(x,t) \,, \quad \text{in} \quad Q \,, \omega > 0 \,, \tag{3.10}$$

Since $\widetilde{\varphi}_0 \in L^\infty(P; W^{1,\infty}(\Omega))$, by the weak maximum principle one obtains

$$\|\varphi\|_{L^\infty(Q)} \leq ess \sup_Q \mid \widetilde{\varphi}_0(x,t) \mid \tag{3.11}$$

that is

$$\varphi \in L^2\big(P; W^{1,2}(\Omega)\big) \cap L^\infty(Q)\,.$$

To establish the classical energy estimate, we multiply (3.7) by $\varphi - \varphi_0$ and integrate over Q. This yields

$$\begin{aligned}
\sigma_* \int_Q \mid \nabla\varphi(x,t) \mid^2 dxdt &\le \int_Q \sigma(\zeta) \mid \nabla\varphi(x,t) \mid^2 dxdt \\
&\le \int_Q \sigma(\zeta) \nabla\varphi_0 \nabla\varphi dxdt \\
&\le \sigma^* \left(\int_Q \mid \nabla\varphi(x,t) \mid^2 dxdt \right)^{1/2} \\
&\quad \times \left(\int_Q \mid \nabla\varphi_0(x,t) \mid^2 dxdt \right)^{1/2}
\end{aligned}$$

hence

$$\int_Q \mid \nabla\varphi(x,t) \mid^2 dxdt \le \left(\frac{\sigma^*}{\sigma_*} \right)^2 \|\nabla\varphi_0\|^2_{L^2(Q)}\,. \tag{3.12}$$

Combining (3.11) and (3.12) one deduces that

$$\int_Q \mid \varphi(x,t) \mid^2 dxdt + \int_Q \mid \nabla\varphi(x,t) \mid^2 dxdt \le N\,.$$

Here and in the sequel, $N > 0$ denotes various different constants.

4. Thermoelastic systems

Now, turn our interest to investigate the problem

$$\sum_{i,j,k,l=1}^{n} \int_Q a_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \rho_i}{\partial x_j} dxdt - \sum_{i,j=1}^{n} \int_Q m_{ij} \zeta \frac{\partial \rho_i}{\partial x_j} dxdt = \int_Q f_i \rho_i dxdt\,, \quad \forall \rho \in W\,. \tag{4.1}$$

According to Theorem 3.1, let define the mapping $B : W \to W^*$ for any $\mathbf{u} \in W$ as

$$\langle B\mathbf{u},\ \rho\rangle := \sum_{i,j,k,l=1}^{n} \int_Q a_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \rho_i}{\partial x_j} dxdt\,, \quad \forall \rho \in W\,.$$

It is easy to check that B verifies the same statements i)–iii) in Proposition 3.2. Indeed,

Proposition 4.1. *If H_A)$-H_{\mathbf{f}}$) are fulfilled then, the mapping B satisfies the properties* i)–iii) *of Proposition* 3.2.

Proof. i) One has

$$| \langle B\mathbf{u},\ \rho\rangle| \le (ess\sup_Q |a_{ijkl}(x,t)|)\left(\int_Q \left|\frac{\partial u_k}{\partial x_l}(x,t)\right|^2 dxdt\right)^{1/2}$$
$$\times\left(\int_Q \left|\frac{\partial \rho_i}{\partial x_j}(x,t)\right|^2 dxdt\right)^{1/2}$$
$$\le \left(ess\sup_Q |a_{ijkl}(x,t)|\right)\|\mathbf{u}\|_W\|\rho\|_W\,,$$

hence

$$\|B\mathbf{u}\|_* \le \left(ess\sup_Q \mid a_{ijkl}(x,t)\mid\right)\|\mathbf{u}\|_W\,.$$

From [13, Theorems 2.1 and 2.3] we obtain the hemicontinuity.

ii) From H$_A$), we have

$$\langle B\mathbf{u}_1 - B\mathbf{u}_2,\ \mathbf{u}_1-\mathbf{u}_2\rangle = \sum_{i,j,k,l=1}^{n}\int_Q a_{ijkl}\frac{\partial(u_k^1-u_k^2)}{\partial x_l}\frac{\partial(u_i^1-u_i^2)}{\partial x_j}dxdt \ge 0\,.$$

iii) On account of the inequality of Poincaré, we infer that

$$\langle B\mathbf{u},\ \mathbf{u}\rangle = \sum_{i,j,k,l=1}^{n}\int_Q a_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial x_j}dxdt$$
$$\ge \lambda\sum_{i,j=1}^{\infty}\int_Q\left|\frac{\partial u_i}{\partial x_j}\right|^2 dxdt \ge \lambda c\|\mathbf{u}\|_W^2\,,$$

therefore

$$\frac{\langle B\mathbf{u},\ \mathbf{u}\rangle}{\|\mathbf{u}\|_W} \ge \lambda c\|\mathbf{u}\|_W \to +\infty\,,\quad \text{as}\quad \|\mathbf{u}\|_W\to+\infty\,. \qquad \square$$

A solution of (4.1) will be obtained by studying the equivalent abstract problem.

$$B\mathbf{u} = H \tag{4.2}$$

where $H \in W^*$ is so defined

$$\langle H,\ \rho\rangle := \sum_{i,j=1}^{n}\int_Q m_{ij}\zeta\frac{\partial\rho_i}{\partial x_j}dxdt + \int_Q f_i\rho_i dxdt\,,\qquad \forall\rho\in W\,.$$

Because of Theorem 3.1 and the strict monotonicity of the mapping B, we are able to state the main result for the quasi-static thermoelastic problem affirming that

Proposition 4.2. *Given a $\zeta\in L^2(Q)$, assuming H_A)–$H_{\mathbf{f}}$) the problem* (4.2) *admits a unique periodic solution.*

Taking as a test function $\mathbf{u}$ in (4.1), we get

$$\lambda \int_Q \mid \nabla \mathbf{u}(x,t) \mid^2 dxdt \leq \sum_{i,j,k,l=1}^{n} \int_Q a_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial u_i}{\partial x_j} dxdt$$

$$= \sum_{i,j=1}^{n} \int_Q m_{ij} \zeta \frac{\partial u_i}{\partial x_j} dxdt + \int_Q f_i u_i dxdt$$

and by Poincaré's inequality one checks that

$$\lambda \int_Q \mid \nabla \mathbf{u}(x,t) \mid^2 dxdt \leq \left(\left(ess \sup_Q m_{ij} \right) \|\zeta\|_{L^2(Q)} + c\|\mathbf{f}\|_{L^2(Q)}\| \right) \|\nabla \mathbf{u}\|_{L^2(Q)} .$$

Hence,

$$\int_Q \mid \nabla \mathbf{u}(x,t) \mid^2 dxdt \leq \left(\frac{((ess \sup_Q m_{ij})\|\zeta\|_{L^2(Q)} + c\|\mathbf{f}\|_{L^2(Q)}\|)}{\lambda} \right)^2$$

and the classical energy estimate is established.

$$\int_Q \mid \mathbf{u}(x,t) \mid^2 dxdt + \int_Q \mid \nabla \mathbf{u}(x,t) \mid^2 dxdt \leq N .$$

5. Nonlinear energy heat flow

Remain to take into consideration the heat conduction problem

$$\int_Q \theta_t w dxdt + \sum_{i,j=1}^{n} \int_Q k_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial w}{\partial x_j} dxdt + \int_\Gamma h(\theta - \theta_a) w ds dt$$

$$= - \int_Q \sigma(\zeta) \varphi \nabla \varphi \nabla w dxdt + \sum_{i,j=1}^{n} \int_Q \frac{\partial m_{ij}}{\partial t} \frac{\partial u_i}{\partial x_j} w dxdt , \quad \forall w \in V . \quad (5.1)$$

The existence of weak periodic solutions to (5.1) is based on the following result

Theorem 5.1 ([3, 4, 14]). . *Let L be a linear closed densely defined operator from the reflexive Banach space V to V^*, L maximal monotone and let C be a bounded, hemicontinuous monotone mapping from V into V^*. Moreover, if $L + C$ is coercive then the Range$(L + C) = V^*$.*

In order to use Theorem 5.1, we must define the operators L and C.

The set

$$D := \left\{ \theta \in L^2\big(P; W^{1,2}(\Omega)\big) \ ; \ \theta_t \in L^2\big(P; W^{1,2}(\Omega)^*\big) \right\}$$

is dense in V because of the density of $C(\overline{Q}) \subset D$ in V.

Let

$$L : D \to V^*$$

be the linear defined by

$$\langle L\theta,\ w\rangle := \int_Q \theta_t w dxdt\,, \qquad \forall w \in V\,.$$

This operator L is closed, skew-adjoint (i.e. $L = -L^*$) and maximal monotone (see [14, Lemma 1.1, p. 313]).

As operator C we take

$$\langle C\theta,\ w\rangle := \sum_{i,j=1}^{n} \int_Q k_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial w}{\partial x_j} dxdt + \int_\Gamma h\theta w dsdt\,, \qquad \forall w \in V\,.$$

The properties of C are resumed in the next result

Proposition 5.2. *If assumptions H_σ)–H_M) are satisfied, the operator C is*

i) *hemicontinuous;*
ii) *monotone;*
iii) *coercive.*

Proof. i) Applying Hölder's inequality one proves that

$$\begin{aligned} \mid \langle C\theta,\ w\rangle \mid \le & \left(ess\sup_Q \mid k_{ij}(x,t) \mid \right) \\ & \times \left(\int_Q \left| \frac{\partial \theta}{\partial x_i} \right|^2 dxdt \right)^{1/2} \left(\int_Q \left| \frac{\partial w}{\partial x_j} \right|^2 dxdt \right)^{1/2} \\ & + h \left(\int_\Gamma \mid \theta \mid^2 dsdt \right)^{1/2} \left(\int_\Gamma \mid w \mid^2 dsdt \right)^{1/2} \\ \le & \left(ess\sup_Q \mid k_{ij}(x,t) \mid + h \right) \|\theta\|_V \|w\|_V \end{aligned}$$

thus,

$$\|C\theta\|_* \le \left(ess\sup_Q \mid k_{ij}(x,t) \mid + h \right) \|\theta\|_V$$

and the hemicontinuity is a consequence of a result of [13, Theorems 2.1 and 2.3].

ii)

$$\begin{aligned} \langle C\theta_1 - C\theta_2,\ \theta_1 - \theta_2\rangle = & \sum_{i,j=1}^{n} \int_Q k_{ij} \frac{\partial(\theta_1 - \theta_2)}{\partial x_i} \frac{\partial(\theta_1 - \theta_2)}{\partial x_j} dxdt \\ & + \int_\Gamma h(\theta_1 - \theta_2)^2 dsdt \\ \ge & \ \mu \int_Q \mid \nabla(\theta_1 - \theta_2) \mid^2 dxdt \\ & + \int_\Gamma h(\theta_1 - \theta_2)^2 dsdt \ge 0\,. \end{aligned}$$

iii)

$$\begin{aligned}\langle C\theta,\ \theta\rangle &= \sum_{i,j=1}^{n}\int_Q k_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}dxdt + h\int_\Gamma \mid\theta\mid^2 dsdt \\ &\geq \mu\int_Q \mid\nabla\theta\mid^2 dxdt + h\int_\Gamma \mid\theta\mid^2 dsdt \min\{\mu,\ h\}\|\theta\|_V^2 .\end{aligned}$$

Consequently,

$$\frac{\langle C\theta,\ \theta\rangle}{\|\theta\|_V} \geq \min\{\mu,\ h\}\|\theta\|_V \to +\infty , \quad \text{as} \quad \|\theta\|_V \to +\infty . \qquad \square$$

We point out that to apply Theorem 5.1, we must reformulate problem (5.1) in abstract form. Let define $G \in V^*$ by setting

$$\begin{aligned}\langle G,\ w\rangle = &-\int_Q \sigma(\zeta)\varphi\nabla\varphi\nabla w dxdt + \sum_{i,j=1}^{n}\int_Q \frac{\partial m_{ij}}{\partial t}\frac{\partial u_i}{\partial x_j} w dxdt \\ &+ \int_\Gamma h\theta_a w dsdt ,\end{aligned}$$

then, problem (5.1) can be handled as the abstract problem

$$L\theta + C\theta = G \tag{5.2}$$

that admits a unique periodic solution, thanks to Theorem 5.1 and the strict monotonicity of C.

Now, we derive a priori estimate for θ. The chosen of θ as a test function in (5.1) gives us

$$\begin{aligned}\mu\int_Q \mid\nabla\theta\mid^2 dxdt + \int_\Gamma h(\theta-\theta_a)\theta dsdt &\leq \sum_{i,j=1}^{n}\int_Q k_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}dxdt \\ &\quad + \int_\Gamma h(\theta-\theta_a)\theta dsdt \\ &= -\int_Q \sigma(\zeta)\varphi\nabla\varphi\nabla\theta dxdt \\ &\quad + \sum_{i,j=1}^{n}\int_Q \frac{\partial m_{ij}}{\partial t}\frac{\partial u_i}{\partial x_j}\theta dxdt\end{aligned}$$

which leads to

$$\mu \int_Q |\nabla\theta|^2\, dxdt + h \int_\Gamma |\theta|^2\, dsdt$$

$$\leq \sigma^* \|\varphi\|_{L^\infty(Q)} \left(\int_Q |\nabla\varphi(x,t)|^2\, dxdt\right)^{1/2} \left(\int_Q |\nabla\theta|^2\, dxdt\right)^{1/2}$$

$$+ h \left(\int_\Gamma |\theta_a|^2\, dsdt\right)^{1/2} \left(\int_\Gamma |\theta|^2\, dsdt\right)^{1/2}$$

$$+ ess \sup_Q \left|\frac{\partial m_{ij}}{\partial t}\right| \left(\int_Q |\nabla\mathbf{u}(x,t)|^2\, dxdt\right)^{1/2} \left(\int_Q |\theta|^2\, dxdt\right)^{1/2}.$$

The equivalence of the norm in V, yields

$$\int_Q |\nabla\theta|^2\, dxdt + \int_\Gamma |\theta|^2\, dsdt \leq N\,.$$

6. A fixed point argument

The periodicity of weak solutions to (1.1')–(1.5) is proved by means of a topological argument which is the appropriate tool to research fixed points for a suitable operator equation. To this end, let

$$\Phi : L^2(Q) \to L^2(Q)$$

be the nonlinear mapping defined by

$$\Phi(\zeta) = \theta$$

where θ is the unique weak periodic solution to (5.1). The mapping Φ is well-defined. In order to prove its continuity, we prove some crucial estimates and convergences, needed to utilize the Schauder fixed point theorem. Let $\zeta_n \in L^2(Q)$ be a sequence such that $\zeta_n \to \zeta$ and $\sigma(\zeta_n) \to \sigma(\zeta)$ strongly in $L^2(Q)$, we denote with θ_n, φ_n, $\mathbf{u}_n$ respectively, the weak periodic solutions of

$$\int_Q \theta_{nt} w dxdt + \sum_{i,j=1}^n \int_Q k_{ij} \frac{\partial\theta_n}{\partial x_i}\frac{\partial w}{\partial x_j} dxdt + \int_\Gamma h(\theta_n - \theta_a) w dsdt \tag{6.1}$$

$$= -\int_Q \sigma(\zeta_n)\varphi_n \nabla\varphi_n \nabla w dxdt + \sum_{i,j=1}^n \int_Q \frac{\partial m_{ij}}{\partial t}\frac{\partial u_{ni}}{\partial x_j} w dxdt\,, \quad \forall w \in V\,.$$

$$\int_Q \sigma(\zeta_n)\nabla\varphi_n \nabla\xi dxdt = 0\,, \quad \forall \xi \in V_0 \tag{6.2}$$

and

$$\sum_{i,j,k,l=1}^{n}\int_Q a_{ijkl}\frac{\partial u_{nk}}{\partial x_l}\frac{\partial \rho_i}{\partial x_j}dxdt-\sum_{i,j=1}^{n}\int_Q m_{ij}\zeta_n\frac{\partial \rho_i}{\partial x_j}dxdt$$
$$=\int_Q f_i\rho_i dxdt\,,\quad \forall\rho\in W. \quad (6.3)$$

Arguing as previously, we get the estimates

$$\|\varphi_n\|_{L^\infty(Q)}\leq ess\sup_Q \mid \widetilde{\varphi}_0(x,t)\mid \quad (6.4)$$

$$\int_Q \mid \varphi_n(x,t)\mid^2 dxdt+\int_Q \mid \nabla\varphi_n(x,t)\mid^2 dxdt\leq N \quad (6.5)$$

$$\int_Q \mid \nabla\theta_n\mid^2 dxdt+\int_\Gamma \mid \theta_n\mid^2 dsdt\leq N \quad (6.6)$$

$$\int_Q \mid \mathbf{u}_n(x,t)\mid^2 dxdt+\int_Q \mid \nabla\mathbf{u}_n(x,t)\mid^2 dxdt\leq N\,. \quad (6.7)$$

In this section, $N>0$ denotes various different constants independent of n.

Moreover,

Lemma 6.1. *The sequence $\nabla\varphi_n$ converges strongly to $\nabla\varphi$ in $L^2(P;(L^2(\Omega))^n)$.*

Proof. Choosing $\varphi_n-\varphi$ as test function in (6.2), we have

$$\int_Q \sigma(\zeta_n)\mid\nabla(\varphi_n-\varphi)\mid^2 dxdt=\int_Q\sigma(\zeta_n)\nabla\varphi\nabla(\varphi_n-\varphi)dxdt$$

so

$$\sigma_*\int_Q \mid\nabla(\varphi_n-\varphi)\mid^2 dxdt\leq\int_Q\sigma(\zeta_n)\nabla\varphi\nabla(\varphi_n-\varphi)dxdt\,.$$

The weak convergence of $\sigma(\zeta_n)\nabla(\varphi_n-\varphi)$ to zero as $n\to+\infty$, leads to the conclusion. □

Lemma 6.1, allows to obtain the convergence

$$\varphi_n\to\varphi\quad\text{in}\quad L^2\Big(P;\big(L^2(\Omega)\big)^n\Big)\quad\text{and a.e. in}\quad Q\,.$$

By virtue of (6.1) and (6.7), θ_{nt} is bounded with respect to the norm of V^*. This ensures that θ_n belongs to a bounded set of D

$$\|\theta_n\|_D\leq C\,.$$

Thus, we can select a subsequence, still denoted by θ_n, such that

$$\theta_n\rightharpoonup\theta\quad\text{in}\quad D\,.$$

By a result of [14, Theorem 5.1], the sequence θ_n is precompact in $L^2(Q)$ that is

$$\theta_n\to\theta\quad\text{in}\quad L^2(Q)\quad\text{and a.e. in}\quad Q\,.$$

Lemma 6.2. *The mapping Φ is continuous.*

Proof. The previous results, imply the following convergences for a suitable chosen subsequence

$$\nabla\theta_n \rightharpoonup \nabla\theta \quad \text{in} \quad L^2\Big(P; \big(L^2(\Omega)\big)^n\Big)$$

$$\nabla u_{ni} \rightharpoonup \nabla u_i \quad \text{in} \quad L^2\Big(P; \big(L^2(\Omega)\big)^n\Big)$$

$$\nabla\varphi_n \to \nabla\varphi \quad \text{in} \quad L^2\Big(P; \big(L^2(\Omega)\big)^n\Big)$$

$$\zeta_n \to \zeta \quad \text{in} \quad L^2(Q)$$

$$\sigma(\zeta_n) \to \sigma(\zeta) \quad \text{in} \quad L^2(Q)$$

and according to the trace theorem (see [15, Theorem 3.4.5]), one obtains

$$\theta_n \rightharpoonup \theta \quad \text{in} \quad L^2\big(P; L^2(\partial\Omega)\big)\,.$$

Therefore, the assertion of lemma descends from the above convergences since $\Phi(\zeta_n) = \theta_n$ converges strongly to $\Phi(\zeta) = \theta$ in $L^2(Q)$. □

Lemma 6.3. *There exists a constant $R > 0$ such that*

$$\|\Phi(\zeta)\|_{L^2(Q)} \le R\,, \quad \forall\zeta \in L^2(Q)\,.$$

Proof. Letting the limit in (6.6), we have the conclusion. □

Since $\Phi(L^2(Q)) \subset D$ and the embedding of $D \hookrightarrow L^2(Q)$ is compact, Φ is a compact mapping from $L^2(Q)$ into itself.

Now, only remains to state the main result of paper

Theorem 6.4. *Let H_σ)–H_0) be satisfied, there is at least a weak periodic solution to system* (1.1')–(1.5).

Proof. As a consequence of Lemmas 6.2 and 6.3 the mapping Φ is both continuous and compact. Hence, by the Schauder fixed point theorem Φ has as a fixed point which is a weak periodic solution for (1.1')–(1.5). This conclude the proof. □

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