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Critical Nonlinear Nonlocal Equations on a Half-Line

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Abstract. We study nonlinear nonlocal equations on a half-line in the critical case

$$\begin{cases} \partial_t u + \beta \left| u \right|^{\alpha} u + \mathbb{K} u = 0\,, & x > 0, t > 0\,, \\ u\left(0, x\right) = u_0\left(x\right)\,, & x > 0\,, \\ \partial_x^{j-1} u(0,t) = 0, j = 1, \ldots, M \end{cases} \tag{0.1}$$

where $\beta \in \mathbf{C}$. The linear operator $\mathbb{K}$ is a pseudodifferential operator defined by the inverse Laplace transform with dissipative symbol $K(p) = E_\alpha p^\alpha$, the number $M = [\frac{\alpha}{2}]$. The aim of this paper is to prove the global existence of solutions to the inital-boundary value problem (0.1) and to find the main term of the large time asymptotic representation of solutions in the critical case.

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1. Introduction

In this paper we study critical nonlinear nonlocal equations on a half-line

$$\begin{cases} \partial_t u + \beta \left| u \right|^{\alpha} u + \mathbb{K} u = 0\,, & x > 0, t > 0\,, \\ u\left(0, x\right) = u_0\left(x\right)\,, & x > 0\,, \\ \partial_x^{j-1} u(0,t) = 0\,, & j = 1, \ldots, M\,, \end{cases} \tag{1.1}$$

where $\beta \in \mathbf{C}, \alpha > 0$. The linear operator $\mathbb{K}$ is a pseudodifferential operator defined by the inverse Laplace transform as follows

$$\mathbb{K} u = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} K(p) \left(\hat{u}(p,t) - \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1} u(0,t)}{p^j} \right) dp\,, \tag{1.2}$$

where

$$\hat{u}(p) = \int_0^{+\infty} e^{-px} u(x) dx$$

denotes the Laplace transform of u. The symbol $K(p)$ is given by $K(p) = E_\alpha p^\alpha$, $E_\alpha \in \mathbf{C}$. We suppose that the operator $\mathbb{K}$ is dissipative, i.e. $\mathrm{Re}(K(p)) > 0$ for $\mathrm{Re}\,(p) = 0$. Here and below p^α is the main branch of the complex analytic function in the half-complex plane $\mathrm{Re}\, p \geq 0$, so that $1^\alpha = 1$ (we make a cut along the negative real axis $(-\infty, 0)$). The number of the boundary data $M = \left[\frac{\alpha}{2}\right]$ (see book [3]), $[\alpha]$ denotes the largest integer less then α.

The theory of nonlinear pseudodifferential equations plays an important role in the contemporary mathematical physics(see book [11]). Such equations serve as a basis for mathematical models to describe the different phenomena in modern Physics, Biology, Technology and others fields of science. As examples, we mention famous equations of Ott–Sudan–Ostrovskiy and Whitham (see [12–14]), which are picked out from complicated natural mathematical models via the separation of the basic mechanism responsible for interesting nonlinear effects. Another example is the equation

$$u_t + |u|^2 u + \int_0^{+\infty} q(x-y) u(s,t) ds = 0$$

encountered in the theory of the Langmuir waves in plasma [15].

In this paper we study initial-boundary value problem (1.1) in the critical case. Recently much attention was drown to the study of the global existence and large time asymptotic behavior of solutions to the Cauchy problems for nonlinear nonlocal equations in the critical and sub critical cases (see [2, 4–7, 9, 10, 16] and a literature cited therein). In the super critical case $\sigma > \alpha$ the global existence result for the initial-boundary value problem (1.1) was obtained in paper [1]. In the so-called sub critical case $\sigma \in (0, \alpha)$ initial-boundary value problem (1.1) was studied in the paper [8].

Here we give a general approach to study initial-boundary value problem in the critical nonconvective case. We now state our result.

We denote by $\mathbf{L}^p$ for $1 \leq p \leq \infty$, the usual Lebesgue space with a norm $\|\phi\|_p = (\int_0^{+\infty} |\phi(x)^p dx)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_\infty = \mathrm{ess.sup}_{x \geq 0} |\phi(x)|$. Define by $\mathbf{W}_p^{0,a} = \{\phi \in \mathbf{L}^p : \langle x \rangle^a \phi \in \mathbf{L}^p\}$ weighted Sobolev space with a norm $\|\phi\|_{\mathbf{W}_p^{0,a}} = \|\langle x \rangle^a \phi\|_{\mathbf{L}^p}$, where $\langle x \rangle = 1 + x$. By $\mathbf{C}(\mathbf{I}; \mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. Different positive constants could be denoted by the same letter C.

We define $\theta = \int_0^\infty u_0(x)\, dx > 0$. We denote

$$\eta = \mathrm{Re}\, \beta \int_0^{+\infty} |\Lambda(s)|^\alpha \Lambda(s) ds > 0\,, \tag{1.3}$$

where

$$\Lambda(s) = D\int_{-i\infty}^{i\infty} dz e^{sz} e^{-K(z)}\Gamma\left(\frac{\{\alpha\}}{\alpha}, -K(z)\right),$$

$$\Gamma(\nu, a) = \int_a^{+\infty} e^{-t}t^{\nu-1}dt\,, \quad \Gamma(\nu) = \int_0^{+\infty} e^{-t}t^{\nu-1}dt\,,$$

$$D = \frac{1}{2\pi^2 i}\frac{(-1)^{\frac{\{\alpha\}}{\alpha}}}{1+A}\sin\left(\frac{\{\alpha\}}{\alpha}\right)\Gamma\left(\frac{[\alpha]}{\alpha}\right),$$

and

$$A = (-1)^{\{\alpha\}}\{\alpha\}\Gamma\big(-\{\alpha\}\big)\Gamma\big(\{\alpha\}\big)2i\sin\pi\{\alpha\}\,.$$

Also we denote by

$$G(t,s) = (t+1)^{-\frac{1}{\alpha}}\Lambda(st^{-\frac{1}{\alpha}})$$

and

$$g(t) = 1 + \theta^\alpha \eta\log(1+t)\,.$$

In the present paper we prove the following result.

Theorem 1. *Let $\alpha, \beta \in \mathbf{C}$, $\operatorname{Re} E_\alpha e^{i\frac{\pi\alpha}{2}} > 0$. We assume that the initial data $u_0 \in \mathbf{W}_1^{0,a}$, $a \in (0,1)$ have a sufficiently small norm $\varepsilon = \|\langle x\rangle^a u_0\|_1$. Then there exists a unique solution $u(x,t) \in \mathbf{C}([0,\infty); \mathbf{W}_1^{0,a}) \cap \mathbf{C}((0,\infty); \mathbf{L}^\infty)$ of the initial-boundary value problem* (1.1)*, satisfying the following time decay estimate*

$$\left\|g^{\frac{1}{\alpha}}u\right\|_{\mathbf{L}^\infty} \le C\varepsilon\,.$$

Moreover the asymptotic is true

$$\left\|\left(u - \theta g^{-\frac{1}{\alpha}}e^{i\psi}G\right)g\right\|_{\mathbf{L}^\infty} \le C\,,$$

for $t \to \infty$, where $\psi(t)$ satisfies the asymptotic estimate

$$\psi(t) = \widetilde{\eta}\log\log t + O(1)$$

for $t \to \infty$, with some constant $\widetilde{\eta} \in \mathbf{R}$.

We organize our paper as follows. In Section 2 we give a general theory of the study critical initial-boundary value problem. Section 3 is devoted to the proof of preliminary estimates. In Section 4 we prove Theorem 1.

2. The General approach

Now we give a general approach for obtaining the large time asymptotic representation of solutions to the initial-boundary value problem

$$\begin{cases} u_t + \mathcal{N}(u) + \mathbb{K}u = 0\,, \quad x \in \mathbf{R}^+\,, \quad t > 0\,, \\ u(0,x) = u_0(x)\,, \quad x \in \mathbf{R}^+\,, \\ \partial_x^{j-1}u(0,t) = 0\,, \quad j = 1, \ldots, \left[\frac{\alpha}{2}\right] \end{cases} \tag{2.1}$$

in the case of critical nonlinearity $\mathcal{N}(u)$ of nonconvective type. We fix a metric space $\mathbf{Z}$ of functions defined on $\mathbf{R}^+$ and a complete metric space $\mathbf{X}$ of functions defined on $[0,\infty)\times\mathbf{R}^+$.

Definition 1. We call the function $G_0\in\mathbf{X}$ as the asymptotic kernel for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$ if there exists a continuous linear functional $f:\mathbf{Z}\to\mathbf{R}^+$ such that the estimate is true

$$\left\|\langle t\rangle^{\gamma}\left(\mathcal{G}(t)\phi-G_0(t)f(\phi)\right)\right\|_{\mathbf{X}}\le C\|\phi\|_{\mathbf{Z}}\,.$$

Definition 2. We call the nonlinearity $\mathcal{N}$ in equation (2.1) as critical nonconvective if the estimate is true

$$\int_0^t \operatorname{Re} f\Big(\mathcal{N}\big(\theta G_0\left(\tau\right)\big)\Big)d\tau\ge\varkappa\theta^{\sigma+1}\log\left(1+t\right) \tag{2.2}$$

for all $t>0$, $\theta>0$ with some positive constants $\varkappa$ and σ.

Now we prove global existence and obtain the large time asymptotic of small solutions to the problem (2.1) with a critical nonlinearity of nonconvective type. Define the function

$$g\left(t\right)=1+\eta\theta^{\sigma}\log\left(1+t\right)$$

with some positive constants η, θ and σ.

Theorem 2. *Assume that the linear operator* $\mathbb{K}$ *is such that* $f(\mathbb{K}u)=0$ *for any* $u\in\mathbf{X}$. *Let the nonlinearity* $\mathcal{N}(u)$ *in equation* (2.1) *be critical nonconvective. Assume that*

$$e^{z}\mathcal{N}(ue^{-z})=e^{-\sigma\operatorname{Re}z}\mathcal{N}(u) \tag{2.3}$$

for any $z\in\mathbf{C}$ *and* $u\in\mathbf{X}$, *where* $\sigma>0$. *Suppose that the estimates are valid*

$$\begin{aligned}&\Big|\log(2+t)f\Big(\mathcal{N}\big(v(t)\big)-\mathcal{N}\big(w(t)\big)\Big)\Big|\\&\qquad\le C\{t\}^{-\alpha}\langle t\rangle^{-1}\big\|\log(2+t)\big(v(t)-w(t)\big)\big\|_{\mathbf{X}}(\|v\|_{\mathbf{X}}^{\sigma}+\|w\|_{\mathbf{X}}^{\sigma}),\end{aligned} \tag{2.4}$$

for all $t>0$ *and for any* $v,w\in\mathbf{X}$, *where* $\alpha<1$, $\sigma>0$, *and*

$$\begin{aligned}&\left\|g(t)\int_0^t g^{-1}(\tau)\Big|\mathcal{G}(t-\tau)\Big(\mathcal{K}\big(v(\tau)\big)-\mathcal{K}\big(w(\tau)\big)\Big)\Big|d\tau\right\|_{\mathbf{X}}\\&\qquad\le C\|v-w\|_{\mathbf{X}}(\|v\|_{\mathbf{X}}+\|w\|_{\mathbf{X}})^{\sigma}\left(1+\frac{\|v\|_{\mathbf{X}}+\|w\|_{\mathbf{X}}}{\theta}\right)\end{aligned} \tag{2.5}$$

for any $v,w\in\mathbf{X}$ *such that* $f(v)=f(w)=\theta>0$, *where* $\sigma>0$ *and* $\mathcal{K}(v)=\mathcal{N}(v)-\frac{v}{\theta}f(\mathcal{N}(v))$. *Let the initial data* $u_0\in\mathbf{Z}$ *have a small norm norm* $\|u_0\|_{\mathbf{Z}}\le\varepsilon$ *and the mean value* $\theta\equiv|f(u_0)|\ge C\varepsilon>0$ *with some* $C>0$. *Then there exists a unique global solution* $u\in\mathbf{X}$ *to the initial problem* (2.1) *satisfying the time decay estimate*

$$\|g^{\frac{1}{\sigma}}u\|_{\mathbf{X}}\le C\varepsilon\,.$$

Moreover if we assume that

$$1+\frac{\sigma}{\theta}\int_0^t \operatorname{Re} f\Big(\mathcal{N}\big(\theta G_0(\tau)\big)\Big)d\tau=g(t)+O\big(\log g(t)\big) \tag{2.6}$$

for $t \to \infty$, then the asymptotic is true

$$\|(u - \theta g^{-\frac{1}{\sigma}} e^{i\psi} G_0) g\|_{\mathbf{X}} \le C\,,$$

where $\psi(t)$ satisfies the asymptotic estimate

$$\psi(t) = \widetilde{\eta} \log\log t + O(1)$$

for $t \to \infty$, with some constant $\widetilde{\eta} \in \mathbf{R}$.

Proof. We make a change of the dependent variable $u(t,x) = v(t,x)e^{-\varphi(t)+i\psi(t)}$ in equation (2.1). Then in view of condition (2.3) we get the following equation for the new unknown function $v(t,x)$

$$v_t + \mathbb{K}v + e^{-\sigma\varphi}\mathcal{N}(v) - (\varphi' - i\psi')v = 0\,.$$

We now choose the auxiliary functions $\varphi(t)$ and $\psi(t)$ by the following condition

$$f\big(e^{-\sigma\varphi}\mathcal{N}(v) - (\varphi' - i\psi')v\big) = 0\,,$$

then in view of the condition $f(\mathbb{K}v) = 0$ we obtain a conservation law

$$\frac{d}{dt} f\big(v(t)\big) = f\big(v_t(t)\big) = 0\,,$$

hence $f(v(t)) = f(v_0)$ for all $t > 0$. We also choose $\varphi(0) = 0$ and $\psi(0) = \arg f(u_0)$ so that

$$f\big(v(t)\big) = f(v_0) = |f(u_0)| = \theta > 0$$

and

$$\varphi' = \frac{1}{\theta} e^{-\sigma\varphi} \operatorname{Re} f\big(\mathcal{N}(v)\big)\,, \quad \psi' = -\frac{1}{\theta} e^{-\sigma\varphi} \operatorname{Im} f\big(\mathcal{N}(v)\big)\,.$$

Thus we obtain the initial-boundary value problem for the new dependent variable $v(t,x)$

$$\begin{cases} v_t + \mathbb{K}v = -e^{-\sigma\varphi}\left(\mathcal{N}(v) - \frac{v}{\theta} f\big(\mathcal{N}(v)\big)\right)\,, & t > 0\,, \quad x \in \mathbf{R}^n\,, \\ v(0,x) = v_0(x) \equiv u_0(x) e^{-i\arg f(u_0)}\,, & x \in \mathbf{R}^n\,. \end{cases} \tag{2.7}$$

We denote $h(t) = e^{\sigma\varphi(t)}$, then we get

$$h' = \frac{\sigma}{\theta} \operatorname{Re} f\big(\mathcal{N}(v)\big)\,, \quad h(0) = 1\,,$$

hence integration with respect to time yields

$$h(t) = 1 + \frac{\sigma}{\theta} \int_0^t \operatorname{Re} f\Big(\mathcal{N}\big(v(\tau)\big)\Big) d\tau\,.$$

Now the integral equation associated with (2.7) can be written as

$$v(t) = \mathcal{G}(t)v_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{K}\big(v(\tau)\big)\frac{d\tau}{h_v(\tau)}\,, \tag{2.8}$$

where the nonlinearity

$$\mathcal{K}\big(v(\tau)\big) = \mathcal{N}\big(v(\tau)\big) - \frac{v(\tau)}{\theta} f\Big(\mathcal{N}\big(v(\tau)\big)\Big)$$

and the functional

$$h_v(t) = 1 + \frac{\sigma}{\theta}\int_0^t \mathrm{Re}\, f\Big(\mathcal{N}\big(v(\tau)\big)\Big)d\tau\,.$$

We now prove the existence of the solution $v(t,x)$ for integral equation (2.8) by the contraction mapping principle. We define the transformation $\mathcal{M}(w)$ by the formula

$$\mathcal{M}(w) = \mathcal{G}(t)v_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{K}\big(w(\tau)\big)\frac{d\tau}{h_w(\tau)}\,, \tag{2.9}$$

for any $w \in \mathbf{B}$, where

$$\mathbf{B} = \Bigg\{ w \in \mathbf{X}:\ f(w)=\theta,\ \|w\|_{\mathbf{X}} \le C\varepsilon\,, \\ \big\|\log(2+t)\big(w-\theta G_0(t)\big)\big\|_{\mathbf{X}} \le C\varepsilon,\ \sup_{t>0}\frac{g(t)}{h_w(t)} \le 3\Bigg\}\,,$$

where $h_w(t) = 1 + \frac{\sigma}{\theta}\int_0^t \mathrm{Re}\, f(\mathcal{N}(w(\tau)))d\tau$. First we check that the mapping $\mathcal{M}$ transforms the set $\mathbf{B}$ into itself. Since

$$f\big(\mathcal{K}(w)\big) = f\big(\mathcal{N}(w)\big) - \frac{1}{\theta}f\big(\mathcal{N}(w)\big)f(w) = 0\,,$$

we see that

$$f\big(\mathcal{M}(w)\big) = f\big(\mathcal{G}(t)v_0\big) = f(v_0) = \theta\,.$$

By the definition of the asymptotic kernel we have

$$\begin{aligned}\big\|\log(2+t)\big(\mathcal{G}(t)v_0 - \theta G_0(t)\big)\big\|_{\mathbf{X}} &\le C\big\|\langle t\rangle^{\gamma}\big(\mathcal{G}(t)v_0 - \theta G_0(t)\big)\big\|_{\mathbf{X}} \\ &\le C\|v_0\|_{\mathbf{Z}} \le C\varepsilon\end{aligned}$$

and by the condition of the theorem we get

$$\begin{aligned}&\left\|\log(2+t)\ \left|\int_0^t \mathcal{G}(t-\tau)\mathbb{K}w(\tau)\frac{d\tau}{h_w(\tau)}\right|\right\|_{\mathbf{X}} \\ &\quad\le 3\left\|\log(2+t)\int_0^t g^{-1}(\tau)\big|\mathcal{G}(t-\tau)\mathcal{K}\big(w(\tau)\big)\big|d\tau\right\|_{\mathbf{X}} \\ &\quad\le \frac{C}{\eta\theta^{\sigma}}\|w\|_{\mathbf{X}}^{\sigma+1}\left(1+\frac{\|w\|_{\mathbf{X}}}{\theta}\right) \le C\varepsilon\,.\end{aligned}$$

In particular, we see that

$$\begin{aligned}\|\mathcal{M}(w)\|_{\mathbf{X}} &\le \|\mathcal{G}(t)v_0\|_{\mathbf{X}} + \left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{K}\big(w(\tau)\big)\frac{d\tau}{h_w(\tau)}\right\|_{\mathbf{X}} \\ &\le C\varepsilon + C\left\|\int_0^t g^{-1}(\tau)|\mathcal{G}(t-\tau)\mathcal{K}\big(w(\tau)\big)|d\tau\right\|_{\mathbf{X}} \le C\varepsilon + C\varepsilon^{\sigma+1} \le C\varepsilon\,,\end{aligned}$$

and

$$\begin{aligned}\big\| \log(2+t)\big(\mathcal{M}(w)-\theta G_0(t)\big)\big\|_{\mathbf{X}} &\le \big\| \log(2+t)\big(\mathcal{G}(t)v_0-\theta G_0(t)\big)\big\|_{\mathbf{X}} \\ &\quad + \left\|\log(2+t)\int_0^t \mathcal{G}(t-\tau)\mathcal{K}\big(w(\tau)\big)\frac{d\tau}{h_w(\tau)}\right\|_{\mathbf{X}} \\ &\le C\varepsilon\,.\end{aligned}$$

It remains to prove the estimate

$$h_{\mathcal{M}(w)}(t) = 1+\frac{\sigma}{\theta}\int_0^t \operatorname{Re} f\Big(\mathcal{N}\big(\mathcal{M}(w)\big)\Big)d\tau \ge \frac{1}{3}g(t)$$

for all $t>0$. We have by condition (2.2)

$$\begin{aligned}1+\frac{\sigma}{\theta}\int_0^t \operatorname{Re} f\Big(\mathcal{N}\big(\mathcal{M}(w)\big)\Big)d\tau &= 1+\frac{\sigma}{\theta}\int_0^t \operatorname{Re} f\Big(\mathcal{N}\big(\theta G_0(\tau)\big)\Big)d\tau \\ &\quad + \frac{\sigma}{\theta}\int_0^t \operatorname{Re} f\Big(\mathcal{N}\big(\mathcal{M}(w)\big) \\ &\quad - \mathcal{N}\big(\theta G_0(\tau)\big)\Big)d\tau \ge \frac{1}{2}g(t)+R(t)\,, \qquad (2.10)\end{aligned}$$

where

$$R(t) = \frac{\sigma}{\theta}\int_0^t \operatorname{Re} f\Big(\mathcal{N}\big(\mathcal{M}(w)\big)-\mathcal{N}\big(\theta G_0(\tau)\big)\Big)d\tau\,.$$

By estimate (2.4), we have

$$\begin{aligned}|R(t)| &\le C\big\| \log(2+t)\big(\mathcal{M}(w)-\theta G_0(t)\big)\big\|_{\mathbf{X}}\big(\|\mathcal{M}(w)\|_{\mathbf{X}}^{\sigma}+\|\theta G_0\|_{\mathbf{X}}^{\sigma}\big) \\ &\quad\times\int_0^t \{\tau\}^{-\alpha}\langle\tau\rangle^{-1}\frac{d\tau}{\log(2+\tau)} \le C\varepsilon^{\sigma+1}\log\log(4+\tau)\,.\end{aligned}$$

Therefore by virtue of (2.10) we find that

$$h_{\mathcal{M}(w)}(t) \ge \frac{1}{2}g(t) - C\varepsilon^{\sigma+1}\log\log(4+\tau) \ge \frac{1}{3}g(t)$$

for all $t>0$. Thus we see that $\mathcal{M}$ transforms $\mathbf{B}$ into itself. Now by virtue of (2.5) let us estimate the difference

$$\begin{aligned}&\|\mathcal{M}(v)-\mathcal{M}(w)\|_{\mathbf{X}} \\ &\quad= \left\|\int_0^t \mathcal{G}(t-\tau)\left(\mathcal{K}\big(v(\tau)\big)\frac{1}{h_v(\tau)} - \mathcal{K}\big(w(\tau)\big)\frac{1}{h_w(\tau)}\right)d\tau\right\|_{\mathbf{X}} \\ &\quad\le C\left\|\int_0^t g^{-1}(\tau)\Big|\mathcal{G}(t-\tau)\Big(\mathcal{K}\big(v(\tau)\big)-\mathcal{K}\big(w(\tau)\big)\Big)\Big|d\tau\right\|_{\mathbf{X}} \\ &\quad\quad + C\left\|\int_0^t g^{-1}(\tau)\big|\mathcal{G}(t-\tau)\mathcal{K}\big(w(\tau)\big)\big|\frac{|h_v(\tau)-h_w(\tau)|}{g(\tau)}d\tau\right\|_{\mathbf{X}} \\ &\quad\le C\|v-w\|_{\mathbf{X}}\left(\varepsilon^{\sigma}+\frac{1}{\theta}\left\|\int_0^t g^{-1}(\tau)\big|\mathcal{G}(t-\tau)\mathcal{K}\big(w(\tau)\big)\big|d\tau\right\|_{\mathbf{X}}\right) \\ &\quad\le C\varepsilon^{\sigma}\left(1+\frac{\varepsilon}{\theta}\right)\|v-w\|_{\mathbf{X}} \le \frac{1}{2}\|v-w\|_{\mathbf{X}}\,,\end{aligned}$$

where in view of (2.4) with $\nu(t) = 1$ we used the estimate

$$\begin{aligned}\frac{|h_v(\tau) - h_w(\tau)|}{g(\tau)} &\le \frac{C}{\theta g(t)} \left| \int_0^t f\Big(\mathcal{N}\big(v(\tau)\big) - \mathcal{N}\big(w(\tau)\big)\Big) d\tau \right| \\ &\le \frac{C\varepsilon^\sigma \log(1+t)}{\theta g(t)} \|v - w\|_{\mathbf{X}} \le \frac{C}{\theta} \|v - w\|_{\mathbf{X}} .\end{aligned}$$

Therefore $\mathcal{M}$ is a contraction mapping in the closed set $\mathbf{B}$ of a complete metric space $\mathbf{X}$. Hence there exists a unique global solution $v \in \mathbf{B}$ to the Cauchy problem (2.7) such that

$$\|v\|_{\mathbf{X}} \le C\varepsilon , \quad \big\| \log(2+t)\big(v - \theta G_0(t)\big)\big\|_{\mathbf{X}} \le C\varepsilon , \quad h_v(t) \ge \frac{1}{3} g(t) .$$

Using the relation $u(t, x) = v(t, x)e^{i\psi(t)} h_v^{-\frac{1}{\sigma}}(t)$ we obtain existence of the solution to the Cauchy problem (2.1), satisfying the following time decay estimates

$$\|g^{\frac{1}{\sigma}} u\|_{\mathbf{X}} \le C\varepsilon .$$

We now prove the asymptotic for the solution. By condition (2.6) and representation (2.10) we have

$$\begin{aligned}h_v(t) &= 1 + \frac{\sigma}{\theta} \int_0^t \operatorname{Re} f\big(\mathcal{N}(v)\big) d\tau \\ &= 1 + \frac{\sigma}{\theta} \int_0^t \operatorname{Re} f\Big(\mathcal{N}\big(\theta G_0(\tau)\big)\Big) d\tau \\ &\quad + \frac{\sigma}{\theta} \int_0^t \operatorname{Re} f\Big(\mathcal{N}(v) - \mathcal{N}\big(\theta G_0(\tau)\big)\Big) d\tau \\ &= g(t) + O\big(\log\log(4+t)\big) .\end{aligned}$$

Then via formulas $u(t, x) = e^{-\varphi(t)+i\psi(t)} v(t, x) = h_v^{-\frac{1}{\sigma}}(t) e^{i\psi(t)} v(t, x)$ we find the estimate

$$\left\| \log(2+t) \left(u - \theta G_0 e^{-i\psi} g^{-\frac{1}{\sigma}} \right) \right\|_{\mathbf{X}} \le C ,$$

since

$$\begin{aligned}&\left\| \frac{g^{1+\frac{1}{\sigma}}}{\log\log(4+t)} \left(\theta G_0 e^{-i\psi} \left(h_v^{-\frac{1}{\sigma}} - g^{-\frac{1}{\sigma}} \right) \right) \right\|_{\mathbf{X}} \\ &\qquad \le C \sup_{t>0} \left| \frac{g^{1+\frac{1}{\sigma}}}{\log\log(4+t)} \left(h_v^{-\frac{1}{\sigma}} - g^{-\frac{1}{\sigma}} \right) \right| \|G_0\|_{\mathbf{X}} \le C .\end{aligned}$$

Also we have

$$\begin{aligned}\psi(t) &= \arg f(u_0) - \frac{1}{\theta} \int_0^t h_v^{-1}(\tau) \operatorname{Im} f\Big(\mathcal{N}\big(v(\tau)\big)\Big) d\tau \\ &= \arg f(u_0) - \frac{1}{\theta} \int_0^t g^{-1}(\tau) \operatorname{Im} f\Big(\mathcal{N}\big(\theta G_0(\tau)\big)\Big) d\tau\end{aligned}$$

$$+ O\left(\int_0^t \{\tau\}^{-\alpha}\langle\tau\rangle^{-1}\frac{\log\log(4+\tau)d\tau}{\log^2(2+\tau)}\right)$$
$$= \widetilde{\eta}\log\log t + O(1)$$

for large $t \to \infty$, with some constant $\widetilde{\eta} \in \mathbf{R}$. This completes the proof of Theorem 2. □

3. Preliminaries

We consider the following linear initial-boundary value problem on half-line

$$\begin{cases} u_t + \mathbb{K}u = f\,, \quad (x,t) \in \mathbf{R}^+ \times \mathbf{R}^+\,, \\ u(x,0) = u_0(x)\,, \quad x \in \mathbf{R}^+\,, \\ \partial_x^{j-1} u(0,t) = 0 \quad \text{for} \quad j = 1,2,\ldots,M\,, \end{cases} \tag{3.1}$$

where $M = [\frac{\alpha}{2}]$. We have the following proposition (see book [3, 8].)

Proposition 1. *The solution $u(x,t)$ of the problem* (3.1) *has the following integral representation*

$$u(t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)f(\tau)d\tau\,,$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\phi = t^{-\frac{1}{\alpha}}\int_{\mathbf{0}}^{+\infty} G\big(xt^{-\frac{1}{\alpha}}, yt^{-\frac{1}{\alpha}}\big)\phi(y)dy \tag{3.2}$$

with a Green function $G(s,q)$

$$G(s,q) = \frac{1}{2\pi i}\int_{-\infty}^{i\infty} e^{zs-K(z)}dz$$
$$- \frac{1}{4\pi^2}(-E_\alpha)^{\frac{\{\alpha\}}{\alpha}}\int_{-i\infty}^{i\infty} dz e^{sz} z^{\{\alpha\}}\int_{-i\infty}^{i\infty} d\xi\; e^{\xi}\xi^{-\frac{\{\alpha\}}{\alpha}}\frac{f(\frac{z}{\phi_1(\xi)}, q)}{K(z)+\xi}\,,$$

where

$$f(s_1,q) = \sum_{k=1}^m \frac{r_k^{-[\alpha]}}{P_m'(r_k)} e^{-\phi_k(\xi)q}\prod_{l=1,l\neq k}^m (s_1 - r_l), r_k = \exp\left\{\frac{i2\pi k}{\alpha}\right\}$$

and

$$\phi_k(\xi) = \left(\frac{\xi}{E_\alpha}\exp\big(i\pi(2k-1)\big)\right)^{\frac{1}{\alpha}}\,, \quad k = 1\ldots,m\,,$$

and

$$P_m(u) = \prod_{j=1}^m (u - r_j)\,.$$

Denote by

$$\Lambda(s) = D\int_{-i\infty}^{i\infty} dze^{sz}e^{-K(z)}\Gamma\left(\frac{\{\alpha\}}{\alpha}, -K(z)\right), \tag{3.3}$$

$$D = \frac{1}{2\pi^2 i}\frac{(-1)^{\frac{\{\alpha\}}{\alpha}}}{1+A}\sin\left(\frac{\{\alpha\}}{\alpha}\right)\Gamma\left(\frac{[\alpha]}{\alpha}\right)$$

and

$$A = (-1)^{\{\alpha\}}\{\alpha\}\Gamma\big(-\{\alpha\}\big)\Gamma\big(\{\alpha\}\big)2i\sin\pi\{\alpha\}.$$

We first collect some preliminary estimates of the Green operator $\mathcal{G}(t)$ in the norms $\|\phi\|_{\mathbf{L}^p}$ and $\|\phi\|_{\mathbf{L}^{1,a}}$, where $a \in (0,1)$, $r = 1, \infty$.We have the following Lemma (see [10]).

Lemma 1. *The following estimates are true, provided that the right-hand sides are finite:*

$$\big\|\langle\cdot\rangle^b\mathcal{G}(t)\phi\big\|_{\mathbf{L}^p} \le C\big(t^{-\frac{1}{\alpha}(1-\frac{1}{p})}\langle t\rangle^{\frac{b}{\alpha}}\|\phi\|_{\mathbf{L}^1} + t^{-\frac{1}{\alpha}(\frac{1}{q}-\frac{1}{p})}\langle t\rangle^{\frac{b}{2}}\|\langle\cdot\rangle^b\phi\|_{\mathbf{L}^q}\big)$$

and

$$\Big\|(\cdot)^b\Big(\mathcal{G}(t)\phi - \vartheta(t+1)^{-\frac{1}{\alpha}}G\big((\cdot)(t+1)^{-\frac{1}{\alpha}}\big)\Big)\Big\|_{\mathbf{L}^p} \le C\langle t\rangle^{\frac{b-a}{\alpha}}t^{-\frac{1}{\alpha}(\frac{1}{q}-\frac{1}{p})}\|\langle\cdot\rangle^a\phi\|_q$$

for all $t > 0$, where $1 \le q \le p \le \infty$, $b \in [0,a]$, $a \in (0,1)$, $\vartheta = \int_{\mathbf{R}^+}\phi dx$.

Define the function

$$g(t) = 1 + k\log(1+t), \quad k > 0.$$

Now we prove the following results

Lemma 2. *Let the function $f(t,x)$ have the zero mean value $\hat{f}(t,0) = 0$ Then the following inequality is valid*

$$\left\|g(t)\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)f(\tau)d\tau\right\|_{\mathbf{X}} \le C\|\langle t\rangle f\|_{\mathbf{X}},$$

where

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\big(t^{\frac{1}{\alpha}}\|\phi(t)\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\frac{a}{\alpha}}\|(\cdot)^a\phi(t)\|_{\mathbf{L}_1} + \|\phi(t)\|_{\mathbf{L}^1}\big), \quad a \in [0,1).$$

Proof. In view of the estimate $g^{-1}(\tau) \le C$ and Lemma 1 we get

$$\begin{aligned}
\left\|\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)f(\tau)d\tau\right\|_{\mathbf{L}^\infty} + \left\|\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)f(\tau)d\tau\right\|_{\mathbf{L}^{1,a}} &\le C\int_0^4 (t-\tau)^{-\frac{1}{\alpha}}\|f(\tau)\|_{\mathbf{L}^1}d\tau \\
&\le C\|(1+t)f(t,x)\|_{\mathbf{X}}\int_0^4 (t-\tau)^{-\frac{1}{\alpha}}(1+\tau)^{-1}d\tau \\
&\le Cg^{-1}(t)\|(1+t)f(t,x)\|_{\mathbf{X}}
\end{aligned}$$

for all $0 \le t \le 4$. We now consider $t > 4$. Via the condition of the lemma for the function $g(t)$ we have the estimate $(1+t)^{-\frac{a}{2\rho}} \le Cg^{-1}(t)$ and

$$\sup_{\tau\in[\sqrt{t},t]} g^{-1}(\tau) \le C\left(1+\kappa\log\left(1+\sqrt{t}\right)\right)^{-1} \le C\left(1+\frac{\kappa}{2}\log(1+t)\right)^{-1} \le Cg^{-1}(t)\,.$$

Since $\hat{f}(t,0)=0$ we have $\mathcal{G}(t)f = \mathcal{G}(t)f - \hat{f}(0)G(t)$. Applying the second estimate of Lemma 1 we obtain

$$\begin{aligned}
&\left\|\int_0^t g^{-1}\ (\tau)\mathcal{G}(t-\tau)f(\tau)d\tau\right\|_{\mathbf{L}^p} \\
&\le C\int_0^{\sqrt{t}}(t-\tau)^{\frac{1}{\alpha}(\frac{1}{p}-1)-\frac{a}{\alpha}}\tau^{\frac{a}{\alpha}-1}d\tau\sup_{\tau>0}(1+\tau)^{-\frac{a}{\rho}}\|(1+\tau)f(\tau)\|_{\mathbf{L}^{1,a}} \\
&\quad + Cg^{-1}(t)\int_{\sqrt{t}}^{\frac{t}{2}}(t-\tau)^{\frac{1}{\alpha}(\frac{1}{p}-1)-\frac{a}{\alpha}}\tau^{\frac{a}{\alpha}-1}d\tau\sup_{\tau>0}(1+\tau)^{-\frac{a}{\alpha}}\|(1+\tau)f(\tau)\|_{\mathbf{L}^{1,a}} \\
&\quad + Cg^{-1}(t)\int_{\frac{t}{2}}^{t}\tau^{\frac{1}{\alpha}(\frac{1}{p}-1)-1}d\tau\sup_{\tau>0}(1+\tau)^{-\frac{1}{\alpha}(\frac{1}{p}-1)}\|(1+\tau)f(\tau)\|_{\mathbf{L}^p} \\
&\le C\left(t^{\frac{1}{\alpha}(\frac{1}{p}-1)-\frac{a}{2\alpha}} + g^{-1}(t)t^{\frac{1}{\alpha}(\frac{1}{p}-1)}\right)\|(1+t)f(t,x)\|_{\mathbf{X}} \\
&\le Cg^{-1}(t)t^{\frac{1}{\alpha}(\frac{1}{p}-1)}\|(1+t)f(t,x)\|_{\mathbf{X}}
\end{aligned}$$

for $1 \le p \le \infty$, and

$$\begin{aligned}
&\left\|\int_0^t g^{-1}\ (\tau)\mathcal{G}(t-\tau)f(\tau)d\tau\right\|_{\mathbf{L}^{1,a}} \\
&\le C\int_0^{\sqrt{t}}\tau^{\frac{a}{\alpha}-1}d\tau\sup_{\tau>0}(1+\tau)^{-\frac{a}{\alpha}}\|f(\tau)\|_{\mathbf{L}^{1,a}} \\
&\quad + Cg^{-1}(t)\int_{\sqrt{t}}^{t}\tau^{\frac{a}{\alpha}-1}d\tau\sup_{\tau>0}(1+\tau)^{-\frac{a}{\alpha}}\|f(\tau)\|_{\mathbf{L}^{1,a}} \\
&\le C\big(t^{\frac{a}{2\alpha}} + g^{-1}(t)t^{\frac{a}{\alpha}}\big)\|(1+t)f(t,x)\|_{\mathbf{X}} \\
&\le Cg^{-1}(t)t^{\frac{a}{\alpha}}\|(1+t)f(t,x)\|_{\mathbf{X}}
\end{aligned}$$

for all $t > 4$. Hence the result of the lemma follows. Lemma 2 is proved. □

4. Proof of Theorem 1

The local existence of solutions for the initial-value problem (1.1) can be obtained by the standard contraction mapping principle (for the proof see [1]).

Theorem 3. *Let the initial data* $u_0 \in \mathbf{W}_1^{0,a}$. *Then for some* $T > 0$ *there exists a unique solution* $u(t,x) \in \mathbf{C}([0,T);\mathbf{W}_1^{0,a}) \cap \mathbf{C}((0,T);\mathbf{L}^\infty)$ *of the initial-value* *problem* (1.1). *Moreover if the initial data are sufficiently small* $\|\langle\cdot\rangle^a u_0\|_1 \le \varepsilon$, *then there exists a time* $T \ge 1$ *such that the solution* $u(t,x)$ *of the problem* (1.1) *satisfy the estimates*

$$t^{\frac{1}{\alpha}}\|u(t)\|_{\mathbf{L}^\infty} \le 2\varepsilon\,.$$

By this theorem, it follows that the global solution (if it exists) is unique. Indeed, suppose that there exist two global solutions with the same initial data, which are different at some time $t > 0$. By virtue of the continuity of solutions with respect to time, we can find a maximal time segment $[0, T]$, where the solutions are equal, but for $t > T$ they are different. Now we apply the local existence theorem taking the initial time T and obtain that these solutions coincide on some interval $[T, T_1]$, which gives us a contradiction with the fact that T is a maximal time until which the solutions coincide. So our main purpose in the proof of Theorem 1 is to show the global in time existence of solutions.

To apply Theorem 2 we choose the space

$$\mathbf{Z} = \left\{\phi \in \mathbf{L}_1^{0,a}(\mathbf{R}^+)\right\}$$

with $a \in (0,1]$ and the space

$$\mathbf{X} = \left\{\phi \in \mathbf{C}\big([0,\infty);\mathbf{Z}\big) \cap \mathbf{C}\big((0,\infty);\mathbf{L}^\infty\big):\ \|\phi\|_{\mathbf{X}} < \infty\right\},$$

with the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\left(t^{\frac{1}{\alpha}}\|\phi(t)\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\frac{a}{\alpha}}\|(\cdot)^a\phi(t)\|_{\mathbf{L}^1} + \|\phi(t)\|_{\mathbf{L}^1}\right).$$

Denote

$$G(s,t) = D(t+1)^{-\frac{1}{\alpha}}\int_{-i\infty}^{i\infty} dz e^{s(t+1)^{\frac{1}{\alpha}}z}e^{-K(z)}\Gamma\left(\frac{\{\alpha\}}{\alpha}, -K(z)\right),$$

$$D = \frac{1}{2\pi^2 i}\frac{(-1)^{\frac{\{\alpha\}}{\alpha}}}{1+A}\sin\left(\frac{\{\alpha\}}{\alpha}\right)\Gamma\left(\frac{[\alpha]}{\alpha}\right)$$

and

$$A = (-1)^{\{\alpha\}}\{\alpha\}\Gamma\big(-\{\alpha\}\big)\Gamma\big(\{\alpha\}\big)2i\sin\pi\{\alpha\}\,.$$

From Lemma 1 we have that the function $G(s,t)$ is the asymptotic kernel for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$ with a continuous linear functional f :

$$f(\phi) = \int_{\mathbf{R}^+}\phi(x)dx\,.$$

Denote by

$$g(t) = 1 + \eta\theta^\alpha\log(1+t)\,,$$

where

$$\eta = \operatorname{Re} |\beta| \int_{\mathbf{R}^+} \Lambda^{\alpha+1}\big(x(\tau+1)^{-\frac{1}{\alpha}}\big) dx > 0\,,$$

$$\Lambda(s) = D \int_{-i\infty}^{i\infty} dz e^{sz} e^{-K(z)} \Gamma\left(\frac{\{\alpha\}}{\alpha}, -K(z)\right).$$

$$\mathcal{N}(\phi) = \beta |\phi|^{\alpha} \phi\,.$$

By a direct computation we have

$$1 + \frac{\alpha}{\theta} \int_0^t \operatorname{Re} f\Big(\mathcal{N}\big(\theta G(\tau)\big)\Big) d\tau$$

$$= 1 + \operatorname{Re} |\beta| \theta^{\alpha} \int_0^t \int_{\mathbf{R}^+} (\tau+1)^{-1-\frac{1}{\alpha}} \Lambda^{\alpha+1}\big(x(\tau+1)^{-\frac{1}{\alpha}}\big) dx d\tau = g(t)$$

and

$$e^z \mathcal{N}(ue^{-z}) = e^z |ue^{-z}|^{\alpha} ue^{-z} = e^{-\alpha z} \mathcal{N}(u)\,,$$

hence conditions (2.2), (2.3) and (2.6) are fulfilled. Since by interpolation inequality

$$\begin{aligned}\log(2+t) \big\|\mathcal{N}\big(v(t)\big) - \mathcal{N}\big(w(t)\big)\big\|_{\mathbf{L}^1} &\le C \log(2+t) \big(\|v(t)\|^{\alpha}_{\mathbf{L}^\infty} \\ &\quad + \|w(t)\|^{\alpha}_{\mathbf{L}^\infty}\big) \|v(t) - w(t)\|_{\mathbf{L}^1} \\ &\le C \langle t \rangle^{-1} \|\log(2+t)(v-w)\|_{\mathbf{X}} (\|v\|^{\sigma}_{\mathbf{X}} + \|w\|^{\sigma}_{\mathbf{X}})\,,\end{aligned}$$

hence condition (2.4) is true. From Lemma 2 we have if $\int_{\mathbf{R}^+} f(t,x) dx = 0$ then the following inequality is valid

$$\left\| g(t) \int_0^t g^{-1}(\tau) \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{X}} \le C \|f\|_{\mathbf{Y}}\,,$$

where $\|f\|_{\mathbf{Y}} = \|\langle t \rangle f\|_{\mathbf{X}}$. Also we have for $\mathcal{K}(\phi) = \mathcal{N}(\phi) - \frac{\phi}{\theta} \int_{\mathbf{R}^+} \mathcal{N}(\phi) dx$.

$$\begin{aligned}\|\mathcal{K}(v) - \mathcal{K}(w)\|_{\mathbf{Y}} &\le \|\mathcal{N}(v) - \mathcal{N}(w)\|_{\mathbf{Y}} \\ &\quad + \frac{1}{\theta} (\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}) \sup_{t>0} \{t\}^{\frac{1}{\alpha}} \langle t \rangle \big\|\mathcal{N}\big(v(t)\big) - \mathcal{N}\big(w(t)\big)\big\|_{\mathbf{L}^1} \\ &\quad + \frac{1}{\theta} \|v-w\|_{\mathbf{X}} \sup_{t>0} \{t\}^{\frac{1}{\alpha}} \langle t \rangle \Big(\big\|\mathcal{N}\big(v(t)\big)\big\|_{\mathbf{L}^1} + \big\|\mathcal{N}\big(w(t)\big)\big\|_{\mathbf{L}^1} \Big) \\ &\le C \|v-w\|_{\mathbf{X}} (\|v\|^{\alpha}_{\mathbf{X}} + \|w\|^{\alpha}_{\mathbf{X}}) \left(1 + \frac{1}{\theta} \|v\|_{\mathbf{X}} + \frac{1}{\theta} \|w\|_{\mathbf{X}}\right).\end{aligned}$$

Therefore we see that condition (2.5) is fulfilled. Now applying Theorem 2 we easily get the results of Theorem 1.

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