NoDEA
Nonlinear differ. equ. appl. 15 (2008) 147—158
1021–9722/08/0200147–12
DOI 10.1007/s00030-007-7005-9

Nonlinear Differential Equations
and Applications NoDEA

Periodic solutions of singular nonautonomous second order differential equations

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Abstract. We consider a class of singular nonautonomous second order differential equations. We prove the existence of at least one periodic solution. We use a variational approach based on the mountain pass theorem.

2000 *Mathematics Subject Classification*: 34B15 - 34C15 - 34C25 - 58E05.
Key words: Periodic solutions, nonautonomous differential equations, singular nonlinearities, mountain pass theory.

1 Introduction

This paper is devoted to the study of the existence of solutions of the following periodic boundary value problem

$$\begin{cases} u''(t) + f(t, u(t)) = g(t) \quad 0 < t < 2\pi \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0 \end{cases} \tag{1.1}$$

where g is a bounded measurable function, and $f : [0, 2\pi] \times (u_0, +\infty) \to \mathbb{R}$ is continuous, $2\pi-$periodic in t and $f(t, .)$ is singular at $u_0 \in \mathbb{R}$; the singularity consists in the fact that the nonlinearity $f(t, .)$ is unbounded from below in a neighborhood of u_0.

Nonlinear second order differential equations with singular restoring forces describe, among other things, the dynamics of particules under the action of Newtonian type forces caused by compressed gazes. Singular two-point boundary value problems have received a great deal of attention. For a good account on recent works we refer the reader to the monographs [1], [15] and the memoire [2].

The problem of existence of periodic solutions of autonomous and nonautonomous singular second order differential equations, with or without a friction term, have been investigated by many authors using a topological method based on the topological degree theory and upper-lower solutions method. See for instance [3-10], [12], [14], [17], [21] and [22]. In almost all these papers the differential equation contains a friction term that does not allow the application of a variational method. Problem (1.1) is treated as a particular case. See for example [3] and [12]. However, our assumptions are different, and moreover, the novelty in our study consists in the application of a variational approach based on the mountain pass theorem. The advantage, here, is that we are able to consider the case when the singular nonlinearity $f(t, .)$ is bounded from above that does not meet the conditions to apply the topological approach, see for instance the assumption (H4) in [4] and the assumptions of Theorem 4.1 in [3]. More precisely, we state sufficient conditions on f such that solutions of (1.1) will be sought as critical points of some functional in an appropriate Sobolev space. Moreover, this functional is shown to have a mountain pass geometry. In fact, we shall use a variant of the mountain pass theorem as stated in [18]. For definitions and results on critical point theory, we refer to [11], [13], [18]. Other variants of the mountain pass theorem can be found in [16], [19] and [20].

Finally, we should point out that our main result cannot be deduced trivially from all the above cited works. It complementes quite well the results in [3], [4] and [12].

2 Preliminaries

Let $F(t, u) := \int_{1+u_0}^{u} f(t, s)ds$. Then problem (1.1) has a variational structure with corresponding functional φ given by

$$\varphi(u) := \int_0^{2\pi} \left[\frac{1}{2}u'(t)^2 - F(t, u(t)) + g(t)u(t)\right] dt$$

and is defined on the Banach space (in fact it is a Hilbert space)

$$H^1_{2\pi} := \left\{u : [0, 2\pi] \to \mathbb{R} \text{ absolutely continuous}; u(0) = u(2\pi), u' \in L^2([0, 2\pi]; \mathbb{R})\right\},$$

equipped with the norm

$$\|u\| = \left(\int_0^{2\pi} |u(t)|^2 dt + \int_0^{2\pi} |u'(t)|^2 dt \right)^{\frac{1}{2}},$$

for $u \in H^1_{2\pi}$.

It is well known (see for instance [13]) that φ is well defined on $H^1_{2\pi}$, continuously differentiable and weakly lower semicontinuous. Moreover, the critical points of φ (i.e. $u \in H^1_{2\pi}$ such that $\varphi'(u) = 0$) are solutions of (1.1).

In our work, we shall use a variant of the mountain pass theorem (see [18] for details) to prove our main result.

3 Main Result

Consider problem (1.1) with $g : [0, 2\pi] \to \mathbb{R}$ is a bounded measurable function, and $f : [0, 2\pi] \times (u_0, +\infty) \longrightarrow \mathbb{R}$ is continuous and satisfies

(H1) $\lim\limits_{u \to u_0^+} f(t, u) = -\infty$ (uniformly in t)

(H2) $\lim_{u \to u_0^+} F(t, u) = +\infty$ (uniformly in t)

(H3) $M(t) := sup\{f(t, s);\ u_0 < s < +\infty\}$ is bounded

(H4) $\lim\limits_{u \to +\infty} \int_0^{2\pi} [F(t, u) - g(t)u]dt = +\infty$

(H5) $D_1F(t, u) := \dfrac{\partial F}{\partial t}(t, u)$ exists and is nonnegative

The above problem was considered in [4] in the case f unbounded above and satisfying (H1), (H2), (H5) and $\lim\limits_{u \to +\infty} 2\dfrac{F(t, u)}{u^2} = \mu(t)$, $\mu(t) > 0$ such that $\mu_0 = \sup_t \mu(t) < \frac{1}{4}$, using a topological approach based on the topological degree theory and upper-lower solutions method. The fact that the nonlinearity is unbounded from above and from below allows the author in [4] to exhibit a constant upper solution and a constant lower solution, and then construct a set Ω, which is admissible for the use of the topological degree. In the present paper, we consider problem (1.1) when f is bounded above, i.e. we suppose that $M(t) := sup\{f(t, s);\ u_0 < s < +\infty\} \leq M < +\infty$. In this case we cannot use the approach of [4].

Our main result reads as follows.

Theorem A. Assume (H1), (H2), (H3), (H4) and (H5) are satisfied. Then problem (1.1) has at least one solution.

Proof. The proof will be based on several claims.

For $\lambda \in (u_0, u_0+1)$ we consider the following modified problem

$$\begin{cases} u''(t) + f_\lambda(t, u(t)) = g(t) \quad 0 < t < 2\pi \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0 \end{cases} \tag{3.1}$$

where $f_\lambda : [0, 2\pi] \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$f_\lambda(t,u) = \begin{cases} f(t,u) & u \geq \lambda \\ f(t,\lambda) & u < \lambda \end{cases}$$

Let $F_\lambda(t,u) = \displaystyle\int_{u_0+1}^{u} f_\lambda(t,s)ds$ and consider the functional

$$\varphi_\lambda : H^1_{2\pi} \longrightarrow \mathbb{R}$$

defined by

$$\varphi_\lambda(u) = \int_0^{2\pi} \left[\frac{1}{2}u'(t)^2 - F_\lambda(t, u(t)) + g(t)u(t)\right] dt. \tag{3.2}$$

It is well known (see for instance [13]) that φ_λ is well defined on $H^1_{2\pi}$, continuously differentiable and weakly lower semicontinuous. Moreover, the critical points of φ_λ are solutions of (3.1).

Claim 1. φ_λ satisfies the Palais-Smale condition.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H^1_{2\pi}$ such that $\{\varphi_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\varphi'_\lambda(u_n) \longrightarrow 0$ as $n \to +\infty$ weakly in $H^1_{2\pi}$; i.e. there exist a constant $c_1 > 0$ and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $\epsilon_n \to 0$ as $n \to +\infty$ such that

$$\left|\int_0^{2\pi} \left[\frac{1}{2}u_n'(t)^2 - F_\lambda(t, u_n(t)) + g(t)u_n(t)\right] dt\right| \leq c_1 \text{ for all } n, \tag{3.3}$$

and for every $v \,\epsilon\, H^1_{2\pi}$,

$$\left|\int_0^{2\pi} \left[u_n'(t)v'(t) - f_\lambda(t, u_n(t))v(t) + g(t)v(t)\right] dt\right| \leq \epsilon_n \|v\|_{H^1_{2\pi}}. \tag{3.4}$$

We show that $\{u_n\}_{n\in\mathbb{N}}$ has a bounded subsequence in $H^1_{2\pi}$, and this will be enough to derive the Palais-Smale condition.

Taking $v(t) \equiv -1$ in (3.4) we obtain

$$\left|\int_0^{2\pi} \left[f_\lambda(t, u_n(t)) - g(t)\right] dt\right| \leq \epsilon_n\sqrt{2\pi} \text{ for all } n.$$

So that

$$\left|\int_0^{2\pi} f_\lambda(t,u_n(t))dt\right| \leq \epsilon_n\sqrt{2\pi} + \left|\int_0^{2\pi} g(t)dt\right| := c_2. \tag{3.5}$$

Let

$$I_{1,n} := \{t\in[0,2\pi];\ \ f_\lambda(t,u_n(t)) \geq 0\},$$

and

$$I_{2,n} := \{t\in[0,2\pi];\ \ f_\lambda(t,u_n(t)) < 0\}.$$

It follows from (3.5) that

$$\left|\int_{I_{2,n}} f_\lambda(t,u_n(t))dt\right| \leq c_2 + \int_{I_{1,n}} f_\lambda(t,u_n(t))dt \leq c_2 + 2\pi M,$$

where M is such that $M(t)\leq M$ for all $t\in[0,2\pi]$.

Hence, there exists $c_3>0$ such that

$$\int_0^{2\pi} |f_\lambda(t,u_n(t))|\,dt \leq c_3 \ \text{ for all } \ n. \tag{3.6}$$

On the other hand, if we take, in (3.4), $v(t)\equiv w_n(t) := u_n(t)-\bar{u}_n$,where $\bar{u}_n$ is the average of u_n over the interval $[0,2\pi]$,we get (taking into account (3.6))

$$\begin{aligned} c_4\|w_n\|_{H^1_{2\pi}} &\geq \int_0^{2\pi}\left[\frac{1}{2}w_n'(t)^2 - f_\lambda(t,u_n(t))w_n(t) + g(t)w_n(t)\right]dt \\ &\geq \frac{1}{2}\|w_n'\|^2_{L^2} - (c_3 + ||g||_{L^1})\,||w_n||_{L^\infty} \\ &\geq \frac{1}{2}\|w_n'\|^2_{L^2} - c_5\|w_n\|_{H^1_{2\pi}}. \end{aligned}$$

Consequently, using the Poincaré-Wirtinger inequality for zero mean functions in the Sobolev space $H^1_{2\pi}$, there exists $c_6>0$ such that

$$\|u_n'\|_{L^2} \leq \|w_n\|_{H^1_{2\pi}} \leq c_6. \tag{3.7}$$

Suppose, now, that

$$\|u_n\|_{H^1_{2\pi}} \to +\infty \ \text{ as } \ n\to+\infty.$$

Since (3.7) holds, we have, passing to subsequences if necessary, that either

$$\begin{aligned} m_n &:= \min u_n \to -\infty \ \text{ as } \ n\to+\infty, \ \text{ or} \\ M_n &:= \max u_n \to +\infty \ \text{ as } \ n\to+\infty. \end{aligned}$$

(i) Assume that the second possibility occurs. We have

$$\begin{aligned}
&\int_0^{2\pi} [F_\lambda(t,u_n(t)) - g(t)u_n(t)]\,dt \\
&\quad = \int_0^{2\pi} \left[\left(\int_{1+u_0}^{u_n(t)} f_\lambda(t,s)ds\right) - g(t)u_n(t)\right] dt \\
&\quad = \int_0^{2\pi} \left[\left(\int_{1+u_0}^{M_n} f_\lambda(t,s)ds - \int_{u_n(t)}^{M_n} f_\lambda(t,s)ds\right) - g(t)u_n(t)\right] dt \\
&\quad = \int_0^{2\pi} [F_\lambda(t,M_n) - M_n g(t)]\,dt - \int_0^{2\pi}\left[\int_{u_n(t)}^{M_n} (f_\lambda(t,s) - g(t))\,ds\right] dt \\
&\quad \geq \int_0^{2\pi} [F_\lambda(t,M_n) - M_n g(t)]\,dt - \|M_\lambda - g\|_{L^1}\|M_n - u_n\|_C.
\end{aligned}$$

(Here $M_\lambda(t) = \sup\{f_\lambda(t,s);\ \ u_0 < s < +\infty\}$).

Thus, applying Sobolev and Poincaré's inequalities to $M_n - u_n(\cdot)$,

$$\int_0^{2\pi} [F_\lambda(t,M_n) - M_n\, g(t)]\,dt \leq \int_0^{2\pi} [F_\lambda(t,u_n(t)) - g(t)u_n(t)]\,dt$$

$$+\|M_\lambda - g\|_{L^1}\sqrt{2\pi}\, c_6 \ \text{ for all } \ n.$$

Using (3.3) and (3.7) we see that the sequence

$$\int_0^{2\pi} [F_\lambda(t,M_n) - M_n\, g(t)]\,dt \ \text{ is bounded.}$$

This contradicts (H4).

(ii) Assume the first possibility occurs; i.e. $m_n \to -\infty$ as $n \to +\infty$. We replace M_n by $-m_n$ in the preceeding arguments, and we also arrive at a contradiction.

Therefore φ_λ satisfies the Palais-Smale condition. This completes the proof of the claim.

Let

$$\Omega := \{u \in H^1_{2\pi}; \min u > 1 + u_0\},$$

and

$$\begin{aligned}
\partial\Omega = \big\{&u \in H^1_{2\pi}; u(t) \geq 1 + u_0 \,\text{for every}\, t \in (0,2\pi), \\
&\exists t_u \in (0,2\pi) : u(t_u) = 1 + u_0\big\}.
\end{aligned}$$

We now proceed to show that φ_λ has a mountain pass geometry.

Claim 2. There exists $m > 0$ such that $\inf_{u\in\partial\Omega} \varphi_\lambda(u) \geq -m$ whenever $\lambda \in (u_0, u_0 + 1)$.

For $u \in \partial\Omega$, we have $\min u = u(t_u) = 1 + u_0$ for some t_u. Extending the functions by 2π-periodicity, we can write

$$\begin{aligned}\varphi_\lambda(u) &= \int_{t_u}^{t_u+2\pi} \left[\frac{1}{2}u'(t)^2 - F_\lambda(t, u(t)) + g(t)u(t)\right] dt \\ &\geq \int_{t_u}^{t_u+2\pi} \frac{1}{2}u'(t)^2 dt - \left[\int_{t_u}^{t_u+2\pi} (M_\lambda(t) - g(t))(u(t) - u_0 - 1)dt\right. \\ &\quad \left. - \int_{t_u}^{t_u+2\pi} g(t)(u_0 + 1)dt\right].\end{aligned}$$

Schwarz inequality and the fact that $u'(t) = (u(\cdot) - u_0 - 1)'(t)$ imply

$$\varphi_\lambda(u) \geq \frac{1}{2} ||u(\cdot) - u_0 - 1)'||_{L^2}^2 - \|M_\lambda - g\|_{L^2} \cdot \|u(.) - u_0 - 1\|_{L^2} + (1 + u_0)\|g\|_{L^1}.$$

Applying Poincaré's inequality to $u(\cdot) - u_0 - 1$, we get

$$\varphi_\lambda(u) \geq \frac{1}{2}\|u'\|_{L^2}^2 - \gamma\|M_\lambda - g\|_{L^2}\|u'\|_{L^2} + (1 + u_0)\|g\|_{L^1},$$

where $\gamma = \gamma(t_u)$.

The above inequality shows that

$$\varphi_\lambda(u) \to +\infty \text{ as } \|u'\|_{L^2} \to +\infty.$$

When $\min u = 1 + u_0$, we have that $\|u(\cdot) - u_0 - 1\|_{H^1_{2\pi}} \to +\infty$ is equivalent to $\|u'\|_{L^2} \to +\infty$.

- Hence $\varphi_\lambda(u) \to +\infty$ as $\|u\|_{H^1_{2\pi}} \to +\infty$, $u \in \partial\Omega$. We infer that φ_λ is coercive, and so it has a minimizing sequence. The weak lower semicontinuity of φ_λ yields

$$\inf_{u\in\partial\Omega} \varphi_\lambda(u) > -\infty.$$

It follows that there exists $m > 0$ such that $\inf_{u\in\partial\Omega} \varphi_\lambda(u) \geq -m$, and this is true for all $\lambda \in (u_0, u_0 + 1)$.

The proof of the claim is complete.

Claim 3. There exists $\lambda_0 \in (u_0, u_0 + 1)$ with the property that for every $\lambda \in (u_0, \lambda_0)$, any solution u of (3.1) satisfying $\varphi_\lambda(u) \geq -m$ is such that $\min u \geq \lambda_0$, and hence u is a solution of (1.1).

For, assume on the contrary that there are sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that

(i) $\lambda_n \leq u_0 + \frac{1}{n}$

(ii) u_n is a solution of (3.1) with $\lambda = \lambda_n$

(iii) $\varphi_{\lambda_n}(u_n) \geq -m$

(iv) $\min u_n < u_0 + \dfrac{1}{n}$

Since f is bounded above by M and $\displaystyle\int_0^{2\pi} [f_{\lambda_n}(t, u_n(t)) - g(t)]\, dt = 0$, we have

$$\|f_{\lambda_n}(\cdot, u_n(\cdot))\|_{L^1} \leq c_7, \text{for some constant}\, c_7 > 0.$$

Hence

$$\|u'_n\|_{L^\infty} \leq c_8,\ \text{for some constant}\, c_8 > 0.$$

Since $\varphi_{\lambda_n}(u_n) \geq -m$ it follows that there must exist two constants R_1 and R_2, with $u_0 < R_1 < R_2$ such that

$$\max\{u_n(t);\ \ t \in [0, 2\pi]\} \in [R_1, R_2],$$

otherwise, u_n would tend uniformly to u_0 or $+\infty$, and in this case $\varphi_{\lambda_n}(u_n)$ would go to $-\infty$, (because of (H4) and $\|u'_n\|_{L^\infty} \leq c_8$), which contradicts $\varphi_{\lambda_n}(u_n) \geq -m$.

Let τ_n^1, τ_n^2 be such that, for n large enough

$$u_n(\tau_n^1) = u_0 + \frac{1}{n} < R_1 = u_n(\tau_n^2).$$

Multiplying the differential equation in (3.1) by u'_n and integrating the resulting equation on $[\tau_n^1, \tau_n^2]$, or on $[\tau_n^2, \tau_n^1]$, we get

$$J := \int_{\tau_n^1}^{\tau_n^2} u''_n(t) u'_n(t) dt + \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(t, u_n(t)) u'_n(t) dt$$

$$= \int_{\tau_n^1}^{\tau_n^2} g(t) u'_n(t) dt.$$

It is clear that

$$J = J_1 + \frac{1}{2}[{u'}^2_n(\tau_n^2) - {u'}^2_n(\tau_n^1)],$$

where

$$J_1 = \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(t, u_n(t) u'_n(t) dt.$$

Since g is integrable and $\|u_n'\|_{L^\infty} \leq c_8$, it follows that J is bounded, and consequently J_1 is bounded On the other hand, we have

$$f_{\lambda_n}(t, u_n(t))u_n'(t) = \frac{d}{dt}\left[F_{\lambda_n}(t, u_n(t))\right] - D_1 F_{\lambda_n}(t, u_n(t)).$$

Thus

$$J_1 = F_{\lambda_n}(\tau_n^2, R_1) - F_{\lambda_n}\left(\tau_n^1, u_0 + \frac{1}{n}\right) - \int_{\tau_n^1}^{\tau_n^2} D_1 F_{\lambda_n}(t, u_n(t))dt.$$

The assumption (H5) implies that

$$J_1 \leq F_{\lambda_n}(\tau_n^2, R_1) - F_{\lambda_n}\left(\tau_n^1, u_0 + \frac{1}{n}\right).$$

It follows from (H2) that J_1 is not bounded. This is a contradiction.

Claim 4. φ_λ has a mountain-pass geometry for $\lambda \leq \lambda_0$.

Fix $\lambda \in (u_0, \lambda_0]$ such that $f(t, \lambda) < 0$ for any $t \in [0, 2\pi]$. This is possible because of (H1).

$$\begin{aligned} F_\lambda(t, u_0) &= \int_{1+u_0}^{u_0} f_\lambda(t, s)ds = -\int_{u_0}^{u_0+1} f_\lambda(t, s)ds \\ &= -\int_{u_0}^{\lambda} f_\lambda(t, s)ds - \int_{\lambda}^{u_0+1} f_\lambda(t, s)ds \\ &= -\int_{u_0}^{\lambda} f(t, \lambda)ds - \int_{\lambda}^{u_0+1} f_\lambda(t, s)ds \\ &= -(\lambda - u_0)f(t, \lambda) - \int_{\lambda}^{u_0+1} f_\lambda(t, s)ds. \end{aligned}$$

This implies that

$$F_\lambda(t, u_0) > -\int_{\lambda}^{u_0+1} f(t, s)ds = \int_{u_0+1}^{\lambda} f(t, s)ds = F_\lambda(t, \lambda).$$

Hence

$$\begin{aligned} \varphi_\lambda(u_0) &= -\int_0^{2\pi} F_\lambda(t, u_0)dt + \int_0^{2\pi} g(t)u_0\, dt \\ &< -\int_0^{2\pi} F_\lambda(t, \lambda)dt + u_0\|g\|_{L^1}. \end{aligned}$$

Consider $\lambda \in (u_0, \lambda_0]$ such that

$$F_\lambda(t, \lambda) > \frac{m + \|g\|_{L^1}u_0}{2\pi} \text{ for all } t \in [0, 2\pi].$$

This is possible by (H2).

It follows that $\varphi_\lambda(u_0) < -m - u_0\|g\|_{L^1} + u_0\|g\|_{L^1}$ or $\varphi_\lambda(u_0) < -m$.

Also, using (H4) we can find R, sufficiently large so that $R > 1 + u_0$ and

$$F_\lambda(t, R) > \frac{m + R\|g\|_{L^1}}{2\pi} \text{ for all } t\epsilon[0, 2\pi].$$

This implies that

$$\varphi_\lambda(R) < -m.$$

Since Ω is a neighborhood of R, $u_0 \notin \Omega$ and

$$\max\{\varphi_\lambda(u_0), \varphi_\lambda(R)\} < \inf_{u\in\partial\Omega} \varphi_\lambda(u),$$

we are in the situation of the mountain-pass theorem (see [18]).

Claim 1 and Claim 4 imply that φ_λ has a critical point u_λ such that

$$\varphi_\lambda(u_\lambda) = \inf_{\eta\in\Gamma} \max_{0\le s\le 1} \varphi_\lambda(\eta(s)) \ge \inf_{u\in\partial\Omega} \varphi_\lambda(u),$$

where $\Gamma := \{\eta \in C([0, 1]; H^1_{2\pi}); \eta(0) = u_0, \eta(1) = R\}$.

Since $\inf\limits_{u\in\partial\Omega} \varphi_\lambda(u) \ge -m$, it follows from claim 3 that u_λ is a solution of (1.1).

This completes the proof of the main result.

4 Acknowledgement

The authors wish to thank an anonymous referee for suggestions and comments that led to the improvement of the manuscript. Also, A. Boucherif is grateful to KFUPM for its constant support.

References

[1] R. P. AGARWAL and D. O'REGAN, Singular Differential and Integral Equatons with Applications, *Kluwer Acad. Publ.*, Dordrecht, 2003.

[2] D. BONHEURE, Second Order Periodic Boundary Value Problems with Singularities, Memoire de D.E.A., Louvain-La-Neuve, 2001.

[3] D. BONHEURE and C. DE COSTER, Forced singular oscillatiors and the method of lower and upper solutions, *Topological Methods in Nonl. Anal.* **22** (2003), 297–317

[4] N. DAOUDI-MERZAGUI, Periodic Solutions of Nonautonomous Second Order Differential Equations with a Singularity, *Appl. Anal.* **73**(3–4) (1999), 449–462.

[5] M. A. DEL PINO and R. MANASEVICH, Infinitely many T-periodic Solutions for a Problem Arising in Nonlinear Elasticity, *J. Diff. Equations* **103**(2) (1993), 260–277.

[6] R. FAURE, Solutions periodiques d'equations admettant des poles; etude par la methode de Leray-Schauder et par un theoreme de point fixe, C.R. Acad. Sci. Paris **283** A(1975), 481–484.

[7] A. FONDA, Periodic solutions of scalar second order differential equations with a singularity, memoire de la Classe des Sciences, Acad. Royale Belgique, Tome **4** (1993), 5–39.

[8] A. FONDA, R. MANASEVICH and F. ZANOLIN, Subharmonic solutions for some second order differential equations with singularities, *S.I.A.M. J. Math. Anal.* **24**(5) (1993), 1294–1311.

[9] P. HABETS and L. SANCHEZ, Periodic solutions of some Lineard equations with singularities, *Proc. Amer. Math. Soc.* **109**(4) (1990), 1035–1044.

[10] A. C. LAZER and S. SOLIMINI, On periodic solutions of nonlinear differential equations with singulatiries, *Proc. Amer. Math. Soc.* **99**(1) (1987), 109–114.

[11] P. MAJER, Ljusternik-Schnirelmann theory with local Palais-Smale conditions and singular dynamical systems, *Ann. Inst. H. Poincare, Anal. Non Lineaire* **8** (1991), 459–476.

[12] R. F. MARTINS, Existence of periodic solutions for second order differential equations with singularities and the strong force condition, *J. Math. Anal. Appl.* **317** (2006), 1–13

[13] J. MAWHIN and M. WILLEM, Critical Point Theory and Hamiltonian Systems, Springer Verlag, New York, 1989.

[14] P. OMARI and W. YE, Necessary and sufficient conditions for the existence of periodic solutions of second order ordinary differential equations with singular nonlinearities, Differential and Integral Eq. **8**(7) (1995), 1843–1858.

[15] D. O'REGAN, Theory of Singular Boundary Value Problems, World Sc. Publ. Co., River Edge, New Jersey, 1994.

[16] R. PRECUP, On the Palais-Smale condition for Hammerstein integral equations in Hilbert spaces, *Nonl. Anal.* **47** (2001), 1233–1244.

[17] R. PRECUP, Periodic solutions of superlinear singular ordinary differential equations, *Differential Integral Eq.* To appear.

[18] P. H. RABINOWITZ, Minimax Methods in Critical Point Theory with Applications to Differential Equations, *C.B.M.S. Math. 65 Amer. Math. Soc.*, Providence, R.I., 1986.

[19] M. SCHECHTER, An Introduction to Nonlinear Analysis, Cambridge University Press, 2004

[20] K. TINTAREV, Mountain pass and impasse for nonsmooth functionals with constraints, Nonl. Anal. **27**(9) (1996), 1049–1054.

[21] J. Y. WANG and D. Q. JIANG, A singular nonlinear second order periodic boundary value problem, *Tohoku Math. J.* **50** (1998), 203–210.

[22] M. ZHANG, Periodic solutions of damped differential systems with repulsive singular forces, *Proc. Amer. Math. Soc.* **127**(2) (1999), 401–407.

Received 20 January 2007; accepted 20 January 2007;