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Multivalued evolution equations with nonlocal initial conditions in Banach spaces

Sergiu AIZICOVICI
Department of Mathematics, Ohio University
Athens, OH 45701, USA
e-mail: aizicovi@math.ohiou.edu

Vasile STAICU
Department of Mathematics, Aveiro University
3810-193 Aveiro, Portugal
e-mail: vasile@ua.pt

Abstract. We prove the existence of integral solutions to the nonlocal Cauchy problem

$$u'(t) \in -Au(t) + F(t, u(t)), 0 \le t \le T; u(0) = g(u)$$

in a Banach space X, where $A : D(A) \subset X \to X$ is m-accretive and such that $-A$ generates a compact semigroup, $F : [0,T] \times X \to 2^X$ has nonempty, closed and convex values, and is strongly-weakly upper semicontinuous with respect to its second variable, and $g : C\left([0,T]; \overline{D(A)}\right) \to \overline{D(A)}$. The case when A depends on time is also considered.

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1 Introduction

This paper is concerned with the existence of solutions to the nonlocal Cauchy problem

$$u'(t) \in -Au(t) + F(t, u(t)),\ t \in I := [0,T];\ u(0) = g(u), \tag{1.1}$$

in a real reflexive, separable Banach space X. Here $A : D(A) \subset X \to X$ is a nonlinear (possibly multivalued) operator on X, such that $-A$ generates a compact semigroup of contractions on $\overline{D(A)}$, $F : I \times X \to 2^X \setminus \{\varnothing\}$ denotes a closed and convex valued multifunction, which is strongly-weakly upper semicontinuous with respect to its second variable, and $g : C\left(I; \overline{D(A)}\right) \to \overline{D(A)}$. The time dependent counterpart of (1.1), namely

$$u'(t) \in -A(t)\, u(t) + F(t, u(t)), \ t \in I := [0,T]; \ u(0) = g(u), \tag{1.2}$$

where $\{A(t), t \in I\}$ are m-accretive operators on X and generate a compact evolution operator, is also discussed.

The study of nonlocal initial-value problems in Banach spaces was initiated by Byszewski [15], who considered an equation of the form (1.1) with A linear, F single valued, and g of a special structure. Results on fully nonlinear abstract nonlocal Cauchy problem have been obtained in [2], [3], [4]. While we were concluding the writing up of this article, Xue's very recent work [28] has been brought to our attention. These papers are primarily concerned with equations governed by accretive operators and single-valued perturbations. To our knowledge, the only existing result for (1.1) with A nonlinear and F multivalued is Theorem 3.8 in [4], where F is supposed to be closed-valued and lower semicontinuous in its second variable. On the other hand, finite dimensional versions of (1.1) (with $A = 0$) appear in [13], [21], while abstract semilinear evolution inclusions with nonlocal initial conditions have been considered in [1], [9], [10], [11]. In particular, in [1], the problem (1.1) is analyzed under the assumption that $-A$ is the infinitesimal generator of a linear C_0-semigroup on X, F is closed, convex valued and upper semicontinuous in its second argument, and g is an integral operator.

The present work complements [4] by allowing $F(.,.)$ to be upper semicontinuous in its second variable (as opposed to lower semicontinuous) and also generalizes the theory of [1] to the case when A is fully nonlinear.

The plan of the paper is as follows. Section 2 contains background material on multifunctions, m-accretive operators and evolution equations. The main results for the problems (1.1) and (1.2) are stated in Section 3, while the proofs are carried out in Section 4. Finally, Section 5 contains an example to which our abstract theory applies.

2 Preliminaries

For further background and details pertaining to this section, we refer the reader to [5], [6], [8], [14], [22], [24], [25], [26] and [27].

Let X be a real Banach space with norm $\|.\|$ and dual $(X^*, \|.\|_*)$. As usual, 2^X stands for the family of all nonempty subsets of X and $\overline{\Omega}$ denotes the closure of a set $\Omega \in 2^X$. The collection of all nonempty closed (resp., nonempty, closed and convex) subsets of X is denoted by $\mathcal{P}(X)$ (resp., $\mathcal{P}_c(X)$). Let $\sigma(X, X^*)$ be the weak topology on X, which is known to be a Hausdorff topology. The space X endowed with the $\sigma(X, X^*)$ topology will be denoted by X_w. The duality mapping

$J : X \to 2^{X^*}$ is defined by

$$J(x) = \left\{ x^* \in X^* : x^*(x) = \|x\|^2 = \|x^*\|_*^2 \right\}, \ \forall x \in X.$$

If X^* is uniformly convex, then J is single-valued and uniformly continuous on bounded subsets of X. The so called upper semi-inner product on X, denoted by $\langle ., . \rangle_+$ is given by

$$\langle y, x \rangle_+ = \sup \{ x^*(y) : x^* \in J(x) \}.$$

Let $I = [0, T]$, where $0 < T < \infty$. We denote by $C(I, X)$ (resp. $L^1(I, X)$) the Banach space of all continuous (resp. Bochner integrable) functions $u : I \to X$ with norm $\|u\|_\infty = \sup_{t \in I} \|u(t)\|$, (resp. $\|u\|_1 = \int_0^T \|u(t)\| \, dt$).

Let A be a (possibly multivalued) operator in X. We define the *domain* and respectively, the *range* of A, by

$$D(A) := \{ x \in X : Ax \neq \varnothing \}, \ R(A) := \bigcup_{x \in D(A)} Ax.$$

The operator A is called *accretive* if $\langle y' - y, x' - x \rangle_+ \geq 0$ for all x, $x' \in D(A)$, $y \in Ax$ and $y' \in Ax'$. If also $R(Id + \lambda A) = X$, for each $\lambda > 0$, where Id is the identity map on X, then A is said to be *m-accretive.*

By a celebrated result of Crandall and Liggett [17], if A is m-accretive, then $-A$ generates a semigroup of contractions $\{S(t) : t \geq 0\}$ on $\overline{D(A)}$. The respective semigroup is said to be compact if $S(t)$ maps bounded subsets of $\overline{D(A)}$ into precompact subsets of $\overline{D(A)}$, for each $t > 0$.

Let A be m-accretive in X. For $u_0 \in \overline{D(A)}$ and $f \in L^1(I, X)$ we consider the initial value problem:

$$u'(t) \in -Au(t) + f(t), \ t \in I; \ u(0) = u_0, \tag{2.1}$$

whose solutions are meant in the sense of the following definition that is due to Bénilan [12]:

Definition 1 An *integral solution* to (2.1) is a continuous function $u : I \to \overline{D(A)}$ with $u(0) = u_0$, such that, for all $x \in D(A)$, $y \in Ax$ and all $0 \leq s \leq t \leq T$,

$$\|u(t) - x\|^2 \leq \|u(s) - x\|^2 + 2 \int_s^t \langle f(\tau) - y, u(\tau) - x \rangle_+ \, d\tau. \tag{2.2}$$

It is well known that equation (2.1) has a unique solution $u \in C\left(I, \overline{D(A)}\right)$. The following proposition summarizes an important property of integral solutions:

Proposition 2 *Let u and v be integral solutions of (2.1) that correspond to (u_0, f) and (v_0, g) respectively (where $u_0, v_0 \in \overline{D(A)}$ and f, $g \in L^1(I, X)$). Then*

$$\|u(t) - v(t)\| \leq \|u_0 - v_0\| + \int_0^t \|f(s) - g(s)\| \, ds \tag{2.3}$$

for all $t \in I$.

Next, let $\{A(t), t \in I\}$ be a family of (possibly multivalued) operators on X, of domains $D(A(t))$, with $\overline{D(A(t))} = \overline{D}$ (independent of t) which satisfy the assumption:

$(H_{A(t)})$ (i) $R(Id + \lambda A(t)) = X$, for all $\lambda > 0$ and $t \in I$,

(ii) there exists a continuous function $m_1 : I \to X$ and a continuous nondecreasing function $m_2 : \mathbb{R}^+ \to \mathbb{R}^+$ ($\mathbb{R}^+ := [0, \infty)$) such that

$$\langle y_1 - y_2, x_1 - x_2 \rangle_+ \geq \\ -\|m_1(t) - m_1(s)\| \|x_1 - x_2\| m_2(\max\{\|x_1\|, \|x\|_2\}),$$

for all $x_1 \in D(A(t))$, $y_1 \in A(t)x_1$, $x_2 \in D(A(s))$, $y_2 \in A(s)x_2$, $0 \leq s \leq t \leq T$.

In particular, for each $t \in I$, the operator $A(t)$ is m-accretive. If $(H_{A(t)})$ holds, then (see, e.g., [26]) the family $\{A(t), t \in I\}$ generates a so-called evolution operator $U(t,s)$ on $\overline{D}$ via the formula

$$U(t,s)x = \lim_{n \to \infty} \prod_{i=1}^{n} \left(Id + \frac{t-s}{n} A\left(s + i\frac{t-s}{n} \right) \right)^{-1} x,$$

for all $x \in \overline{D}$ and all $0 \leq s \leq t \leq T$. The evolution operator U is said to be compact if $U(t,s)$ maps bounded subsets of $\overline{D}$ into precompact subsets of $\overline{D}$, for all $0 \leq s < t \leq T$.

Consider the nonautonomous Cauchy problem

$$u'(t) \in -A(t)u(t) + f(t), \ t \in I; \ u(0) = u_0, \tag{2.4}$$

where $A(t)$ satisfies $(H_{A(t)})$, $f \in L^1(I,X)$ and $u_0 \in \overline{D}$.

Definition 3 An integral solution of (2.4) is a function $u \in C(I, \overline{D})$ satisfying $u(0) = u_0$ and the inequality

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \leq 2\int_s^t [\langle f(\tau) - y, u(\tau) - x\rangle_+ \\ + C\|u(\tau) - x\| \|m_1(\tau) - m_1(\theta)\|] d\tau$$

for all $0 \leq s \leq t \leq T$, $\theta \in I$, $x \in D(A(\theta))$, $y \in A(\theta)x$, and $C = m_2(\max\{\|x\|, \|u\|_\infty\})$, with m_1 and m_2 as in $(H_{A(t)})$ (ii).

Recall that (2.4) has an unique integral solution for each $u_0 \in \overline{D}$ and $f \in L^1(I,X)$, provided that $(H_{A(t)})$ is satisfied. Moreover, the following analog of Proposition 2 is true:

Proposition 4 *Let $\left(H_{A(t)}\right)$ be satisfied and let u and v be integral solutions of* (2.4) *corresponding to (u_0, f) and (v_0, g), respectively (with u_0, $v_0 \in \overline{D}$ and f, $g \in L^1(I, X)$). Then the inequality* (2.3) *holds for all $t \in I$.*

The remainder of this section is devoted to a brief review of multifunctions. In what follows, the Banach space X will be assumed separable.

Let (Ω, Σ, μ) be a measure space and $\Phi : \Omega \to 2^X$. We say that Φ is measurable if $\{\omega \in \Omega : \Phi(\omega) \cap C \neq \varnothing\} \in \Sigma$ for any closed subset C of X. If $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, then Σ is the $\sigma-$algebra of Lebesgue measurable subsets of Ω.

A function $\varphi : \Omega \to X$ that satisfies $\varphi(\omega) \in \Phi(\omega)$, $\mu - a.e.$ on Ω, is called a selection of Φ. We define $\mathcal{S}^1_\Phi$ as the set of all selections of Φ that belong to $L^1(\Omega, X)$.

Let Y and Z be Hausdorff topological spaces and let $\Psi : Y \to 2^Z$. For $A \in 2^Z$ we set

$$\Psi^-(A) := \{y \in Y : \Psi(y) \cap A \neq \varnothing\}, \quad \Psi^+(A) := \{y \in Y : \Psi(y) \subset A\}.$$

The multifunction Ψ is said to be *upper semi-continuous on* Y (u.s.c., for short) if the set $\Psi^+(A)$ is open in Y for any open subset of A of Z.(Equivalently, Ψ is u.s.c. if $\Psi^-(C)$ is closed in Y for each closed subset C of Z).

If $Z = X_w$ and $\Psi : Y \to 2^Z$ is u.s.c. and closed valued, then, by ([22], Chapter 1, Proposition 2.17) Ψ is closed, that is,

$$Gr(\Psi) := \{(y, z) \in Y \times Z : z \in \Psi(y)\}$$

is closed in $Y \times Z$. Conversely, if $\Psi : Y \to \mathcal{P}(Z)$ is closed and locally compact (i.e., for each $y \in Y$, there exists a neighborhood U of y such that $\Psi(U)$ is precompact), then Ψ is u.s.c. (see ([22], Chapter 1, Proposition 2.23).

We will be mainly concerned with multifunctions $F : I \times X \to 2^X$ satisfying

$\left(H^1_F\right)$ For each $x \in X$, $F(., x)$ is measurable,

$\left(H^2_F\right)$ For a.a. $t \in I$, $F(t, .)$ is upper semicontinuous from X into X_w.

The following special form of the Convergence Theorem in [5], p.60, as given in [27], p.120, will be used in the sequel:

Proposition 5 *Let $F : I \times X \to \mathcal{P}_c(X)$ satisfy assumptions $\left(H^1_F\right)$ and $\left(H^2_F\right)$. Let u_n, $f_n : I \to X$ $(n \in \mathbb{N})$ be measurable functions such that u_n converges almost everywhere on I to a function $u : I \to X$ and f_n converges weakly in $L^1(I, X)$ to $f : I \to X$. If $f_n(t) \in F(t, u_n(t))$, for all $n \in \mathbb{N}$ and almost all $t \in I$, then $f(t) \in F(t, u(t))$, a.e. on I.*

The final result of this section is the following:

Lemma 6 *Let $F : I \times X \to \mathcal{P}(X)$ satisfy assumptions $\left(H_F^1\right)$ and $\left(H_F^2\right)$, and let $u : I \to X$ be a measurable function.*

Then the multifunction $t \to F(t, u(t))$ is measurable. If also there exists $\varphi \in L^1\left(I, \mathbb{R}^+\right)$ such that

$$|F(t, u(t))| := \sup\{\|w\| : w \in F(t, u(t))\} \leq \varphi(t), \text{ a.e. on } I, \tag{2.5}$$

then $S^1_{F(.,u(.))}$ is a nonempty and closed subset of $L^1(I, X)$. Moreover $S^1_{F(.,u(.))}$ is convex, provided that F is convex valued.

Proof. Since $u : I \to X$ is measurable, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of simple functions from I to X, such that u_n converges to u, a.e. on I, as $n \to \infty$. By $\left(H_F^1\right)$, the multifunction $t \to F(t, u_n(t))$ is measurable for each $n \in \mathbb{N}$. Using $\left(H_F^2\right)$, Propositions 2.13 and 2.17 in [22], Chapter 1, and Theorem 8.2.4 in [6] (cf., also [6], p.41, 313) we conclude that

$$F(t, u(t)) = \bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} F(t, u_k(t))}$$

and that $t \to F(t, u(t))$ is measurable, as a multifunction from I to X. Then, by the Kuratowski-Ryll Nardzewski Theorem (see, e.g. [22], p.154), there exists a measurable function $v : I \to X$ such that $v(t) \in F(t, u(t))$, $t \in I$. If (2.5) holds, then $v \in L^1(I, X)$, hence $S^1_{F(.,u(.))} \neq \varnothing$. Recalling that F is closed valued, it is easily verified that $S^1_{F(.,u(.))}$ is closed in $L^1(I, X)$. Finally, if F also has convex values, then it is immediate that $S^1_{F(.,u(.))}$ is convex, as well. □

3 Statement of results

Throughout this section, X is supposed to be a real separable Banach space with uniformly convex dual X^*, $I := [0, T]$, with $0 < T < \infty$, and $\mathbb{R}^+ := [0, \infty)$. We study the nonlocal initial value problem (1.1) under the following assumptions:

(H_A) A is an m-accretive operator in X, such that $-A$ generates a compact semigroup $\{S(t) : t \geq 0\}$ on $\overline{D(A)}$.

(H_g) $g : C\left(I, \overline{D(A)}\right) \to \overline{D(A)}$ is such that

$$\|g(u) - g(v)\| \leq m \|u - v\|_\infty, \forall u, v \in C\left(I, \overline{D(A)}\right),$$

for some m with $0 < m < 1$.

(H_F) $F : I \times X \to \mathcal{P}_c(X)$ satisfies:

(i) $F(.,x)$ is measurable for each $x \in X$,

(ii) $F(t,.)$ is upper semicontinuous from X to X_w for a.a. $t \in I$,

(iii) there exists a function $\gamma : I \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma(.,r) \in L^1(I,\mathbb{R})$ for every $r \in \mathbb{R}^+$, $\gamma(t,.)$ is continuous and nondecreasing for a.a. $t \in I$ and

$$\limsup_{r\to\infty} \frac{1}{r}\int_0^T \gamma(t,r)\,dt < 1-m, \tag{3.1}$$

where m is the same as in condition (H_g), with the additional property that

$$|F(t,x)| := \sup\{\|w\| : w \in F(t,x)\} \le \gamma(t,\|x\|), \tag{3.2}$$

for a.a. $t \in I$, and all $x \in \overline{D(A)}$.

Definition 7 A function $u \in C\left(I,\overline{D(A)}\right)$ is called an integral solution of the problem (1.1) if there exists $f \in L^1(I,X)$ with $f(t) \in F(t,u(t))$, a.e. on I, such that u is an integral solution in the sense of of Definition 1, of (2.1) with $g(u)$ in place of u_0.

Our basic existence result is the following:

Theorem 8 *Let assumptions* (H_A), (H_g) *and* (H_F) *be satisfied. Then the set of integral solutions of the problem* (1.1) *is a nonempty, compact subset of* $C(I,X)$.

Next we are concerned with the existence of integral solutions to the problem (1.2), where the operators $A(t)$ satisfy $\left(H_{A(t)}\right)$, while F and g are subject to conditions (H_F) and (H_g), respectively, with the mention that in (H_g), $\overline{D(A)}$ is to be replaced by $\overline{D}$. The corresponding modified assumption (H_g) will be denoted by $\left(H_g'\right)$.

Definition 9 A function $u \in C\left(I,\overline{D}\right)$ is said to be an integral solution of the problem (1.2) if there exists $f \in L^1(I,X)$ with $f(t) \in F(t,u(t))$, a.e. on I, such that u is an integral solution, in the sense of of Definition 3, of (2.4), where u_0 is replaced by $g(u)$.

The following counterpart of Theorem 8 is valid:

Theorem 10 *Let assumptions* $\left(H_{A(t)}\right)$, $\left(H_g'\right)$ *and* (H_F) *be satisfied. Then the set of integral solutions of the problem* (1.2) *is a nonempty, compact subset of* $C(I,X)$.

4 Proofs

Proof of Theorem 8. Without loss of generality, we assume that $0 \in D(A)$. We start with the initial value problem

$$u'(t) \in -Au(t) + f(t),\ t \in I;\ u(0) = g(u), \tag{4.1}$$

where $f \in L^1(I, X)$, A is m-accretive in X, and g satisfies (H_g). By an integral solution to (4.1), we mean a function $u \in C\left(I, \overline{D(A)}\right)$ that satisfies the requirements of Definition 1, with $g(u)$ in place of u_0.

We claim that the problem (4.1) has a unique integral solution that will be denoted by u_f. Indeed, for each $v \in C\left(I, \overline{D(A)}\right)$, there exists a unique integral solution u_v of the initial value problem

$$u'(t) \in -Au(t) + f(t),\ t \in I;\ u(0) = g(v), \tag{4.2}$$

By (2.3) and (H_g), we have

$$\|u_v(t) - u_w(t)\| \leq \|g(v) - g(w)\| \leq m\|v - w\|_\infty, \forall t \in I,$$

hence

$$\|u_v - u_w\|_\infty \leq m\|v - w\|_\infty, \tag{4.3}$$

for all $v, w \in C\left(I, \overline{D(A)}\right)$. Since $0 < m < 1$, (4.3) implies that $v \to u_v$ is a strict contraction in $C\left(I, \overline{D(A)}\right)$. Therefore, by the Contraction Mapping Principle, the map $v \to u_v$ has a unique fixed point in $C\left(I, \overline{D(A)}\right)$. Obviously, this is the unique integral solution of (4.1).

Next, we obtain an a-priori bound for all possible solutions of (1.1). Let u be an integral solution of (1.1), in the sense of Definition 7. Accordingly, there exists $f \in L^1(I, X)$ with $f(t) \in F(t, u(t))$, *a.e.* on I, such that u is an integral solution of the related problem (4.1).

Noting that each $x \in D(A)$ can be viewed as the integral solution of (2.1) with $f(t) = y \in Ax$ and $u_0 = x$, we obtain with the help of Proposition 2 that

$$\|u(t) - x\| \leq \|g(u) - x\| + \int_0^t \|f(s) - y\|\, ds, t \in I, \tag{4.4}$$

for a fixed $x \in D(A)$ and $y \in Ax$. From (4.4) it follows that

$$\|u(t)\| \leq 2\|x\| + \|g(0)\| + T\|y\| + \|g(u) - g(0)\| + \int_0^T \|f(s)\|\, ds. \tag{4.5}$$

Using (H_g) and $(H_F)\,(iii)$ in (4.5) we arrive at

$$(1-m)\,\|u\|_\infty \leq C + \int_0^T \gamma\left(s, \|u\|_\infty\right) ds, \tag{4.6}$$

where $C = 2\,\|x\| + \|g\,(0)\| + T\,\|y\|$. This in conjunction with (3.1) implies that there exists a finite positive constant M (which is independent of u) such that

$$\|u\|_\infty \leq M. \tag{4.7}$$

Indeed, if this is not the case, one can construct a sequence $(u_n)_{n\in N}$ of integral solutions of (1.1), such that $\|u_n\|_\infty \to \infty$, as $n \to \infty$. Then, (4.6) leads to

$$(1-m) \leq \lim\sup_{n\to\infty} \frac{\int_0^T \gamma\left(s, \|u_n\|_\infty\right) ds}{\|u_n\|_\infty} \leq \lim\sup_{r\to\infty} \frac{1}{r}\int_0^T \gamma\,(t,r)\,dt,$$

in contradiction with (3.1). Hence, a constant M must exist such that (4.7) is satisfied by all integral solutions u of (1.1). Let

$$\varphi\,(t) := \gamma\,(t, M) \tag{4.8}$$

and remark (cf. $(H_F)\,(iii)$) that $\varphi \in L^1\,(I, \mathbb{R})$. Combining (3.2), (4.7) and (4.8) we get

$$|F\,(t, u\,(t))| \leq \varphi\,(t)\,,\ \ a.e.\ on\ I, \tag{4.9}$$

for each integral solution u of (1.1). In view of (4.9), we may assume without lost of generality that

$$|F\,(t,x)| \leq \varphi\,(t)\,,\ \forall x \in X,\ \ a.e.\text{ on } I. \tag{4.10}$$

Otherwise, we replace $F\,(t,x)$ by $\widetilde{F}\,(t,x) = F\,(t, p_M\,(x))$, where $p_M : X \to X$ is given by

$$p_M\,(x) = \begin{cases} x & if\ \ \|x\| \leq M \\ M\frac{x}{\|x\|} & if\ \ \|x\| > M \end{cases}, \tag{4.11}$$

with M as in (4.7). It is easily seen that all conditions in (H_F) are satisfied with $\widetilde{F}$ in place of F. In particular, one uses the strong-weak upper semicontinuity of F and the continuity of p_M to verify (ii) in (H_F). As regards to (iii), simply remark that $\|p_M\,(x)\| \leq \|x\|$, $\forall x \in X$, so that

$$\left|\widetilde{F}\,(t,x)\right| \leq \gamma\,(t, \|p_M\,(x)\|) \leq \gamma\,(t, \|x\|)\,,$$

since γ is nondecreasing in its second variable.

Assume that we have found an integral solution $\widetilde{u}$ of the problem (1.1) where F has been replaced by $\widetilde{F}$. By (4.7) it follows that

$$\|\widetilde{u}(t)\| \leq M, \ \forall t \in I,$$

and therefore (cf. (4.11)), $p_M(\widetilde{u}(t)) = \widetilde{u}(t)$. As a result, $\widetilde{F}(t, \widetilde{u}(t)) = F(t, \widetilde{u}(t))$, so that $\widetilde{u}(.)$ is an integral solution of our original problem (1.1). Summarizing, we will henceforth suppose that (4.10) holds.

We now introduce the set $K \subset L^1(I, X)$ by

$$K = \left\{ f \in L^1(I, X) : \|f(t)\| \leq \varphi(t), \ a.e. \ on \ I \right\}. \tag{4.12}$$

Clearly, K is nonempty, closed and convex. In addition, K is uniformly integrable, hence, by [18], p.101, it is compact in $L^1_w(I, X)$, where $L^1_w(I, X)$ denotes the space $L^1(I, X)$ equipped with its weak topology. We also note (see Theorem V.6.3 in [19]) that K, endowed with the relative $L^1_w(I, X)$ topology is a metric space. Define the map $\mathcal{F} : K \to 2^{L^1(I,X)}$ by

$$\mathcal{F}(f) := S^1_{F(.,u_f(.))} = \left\{ v \in L^1(I, X) : v(t) \in F(t, u_f(t)) \ a.e. \ on \ I \right\}, \tag{4.13}$$

where $u_f(.)$ denotes the integral solution of (4.1), for a given $f \in K$.

By (H_F) and Lemma 6, we conclude that $\mathcal{F}(f)$ has nonempty, closed and convex values. Moreover, in view of (4.9), $\mathcal{F}(K) \subset K$. We regard K as a compact convex subset, denoted as K_w, of $L^1_w(I, X)$ and show that $\mathcal{F}$ is u.s.c. from K_w into 2^{K_w}. By Proposition 2.23 in [22], Chapter 1 (cf., also Theorem 2.5 in [25]) it is sufficient to prove that its graph $Gr(\mathcal{F})$ is sequentially closed in $K_w \times K_w$.

Let $(f_n, v_n) \in Gr(\mathcal{F})$ with $f_n \to f$ and $v_n \to v$ in $L^1_w(I, X)$, as $n \to \infty$. From (4.12), (4.4), (4.5) and (H_g), it follows that $\{u_{f_n}\}_{n\in\mathbb{N}}$ is bounded in $C(I, X)$, and consequently, $\{g(u_{f_n})\}_{n\in\mathbb{N}}$ is bounded in X.

We now show that $\{u_{f_n}\}_{n\in\mathbb{N}}$ is precompact in $C(I, X)$. To accomplish this, we reason as follows. (A comparable argument was independently used by Xue [28], in a different context.)

Let $\mathcal{X} := C\left(I, \overline{D(A)}\right)$ and define the operator $L : \mathcal{X} \to \mathcal{X}_0$ by

$$(Lw)(t) = w(t) - S(t) g(w), \forall t \in I, w \in \mathcal{X}, \tag{4.14}$$

where $\mathcal{X}_0 = \{u \in C(I, X) : u(t) = w(t) - S(t) g(w), \ t \in I, \text{ for some } w \in \mathcal{X}\}$ and $\{S(t) : t \geq 0\}$ denotes the contraction semigroup generated by $-A$. By(H_g), it is easily verified that L is one-to-one and onto, and that L^{-1} is continuous on $\mathcal{X}_0$, see [28], Lemma 2.5. Accordingly, we write

$$u_{f_n}(t) = L^{-1}(w_{f_n})(t), t \in I, n \in \mathbb{N} \tag{4.15}$$

where (cf. (4.14)),

$$w_{f_n}(t) = u_{f_n}(t) - S(t) g(u_{f_n}). \tag{4.16}$$

On account of (H_A), one can invoke [7] (see also [16], Theorem 2.1) to infer that $\{w_{f_n}(.)\}_{n\in\mathbb{N}}$ is precompact in $C([\delta,T],X)$ for any $0<\delta<T$. Next, since by (4.16),

$$w_{f_n}(0)=g(u_{f_n})-g(u_{f_n})=0, \forall n\in\mathbb{N},$$

the set $\{w_{f_n}(0)\}_{n\in\mathbb{N}}$ is trivially compact in X. To prove the equicontinuity of $\{w_{f_n}(.)\}_{n\in\mathbb{N}}$ at $t=0$, remark that $z_n(.)=S(.)g(u_{f_n})$ can be viewed as an integral solution of

$$z'(t)\in -Az(t),\ t\in I;\ z(0)=g(u_{f_n}).$$

Applying Proposition 2, we obtain

$$\|w_{f_n}(t)-w_{f_n}(0)\|=\|u_{f_n}(t)-z_n(t)\|\le\int_0^t\|f_n(s)\|\,ds, \forall t\in I.$$

Recalling that $f_n\in K$ (see (4.12)), we conclude that

$$\|w_{f_n}(t)-w_{f_n}(0)\|\le\int_0^t\varphi(s)\,ds, \forall t\in I.$$

This implies that $\{w_{f_n}(.)\}_{n\in\mathbb{N}}$ is equicontinuous at $t=0$, as desired. In summary, by the Ascoli-Arzelà Theorem (cf., e.g., [27], Theorem 1.3.1), it follows that $\{w_{f_n}(.)\}_{n\in\mathbb{N}}$ is precompact in $\mathcal{X}_o$. This, in conjunction with (4.15) and the continuity of L^{-1} on $\mathcal{X}$, implies that $\{u_{f_n}(.)\}_{n\in\mathbb{N}}$ in precompact is $\mathcal{X}$ as claimed.

Without loss of generality, we can assume that

$$u_{f_n}\to u \textit{ in } C(I,X),\ \textit{as } n\to\infty, \tag{4.17}$$

for some $u\in C(I,X)$. In view of (4.17) and the fact that $v_n\to v$ weakly in $L^1(I,X)$ with $v_n\in\mathcal{F}(f_n)$ (that is (cf. (4.13)), $v_n(t)\in F(t,u_{f_n}(t))$, *a.e. on* I), we can apply Proposition 5 to conclude that

$$v(t)\in F(t,u(t)),\ \textit{a.e. on } I. \tag{4.18}$$

Finally, since X^* is uniformly convex, the duality map $J:X\to X^*$ is single valued and continuous. By (2.2) and the definition of $\langle .,.\rangle_+$ (see Section 2), we have

$$\|u_{f_n}(t)-x\|^2\le\|u_{f_n}(s)-x\|^2+2\int_s^t J(u_{f_n}(t)-x)(f_n(\tau)-y)\,d\tau, \tag{4.19}$$

for all $x\in D(A)$, $y\in Ax$ and all $0\le s\le t\le T$. Using (4.17), the fact that $f_n\to f$ weakly in $L^1(I,X)$, and the uniform continuity of J, we can pass to the limit as $n\to\infty$ in (4.19) to obtain

$$\|u(t)-x\|^2\le\|u(s)-x\|^2+2\int_s^t\langle f(\tau)-y,u(\tau)-x\rangle_+\,d\tau, \tag{4.20}$$

for all $x \in D(A)$, $y \in Ax$ and all $0 \le s \le t \le T$. Recalling that $u_{f_n}(0) = g(u_{f_n})$, and employing (H_g) and (4.17), we infer that

$$u(0) = g(u). \tag{4.21}$$

From (4.20) and (4.21), it follows that $u = u_f$. This, together with (4.18), enables us to conclude that $v \in \mathcal{F}(f)$, which completes the proof of the the sequential closedness of $Gr(\mathcal{F})$.

We can now invoke the Kakutani-Ky Fan Fixed Point Theorem [20] to deduce that there exists $\widehat{f} \in K$ such that $\widehat{f} \in \mathcal{F}\left(\widehat{f}\right)$. By (4.13), it is obvious that the corresponding integral solution of the problem (4.1), denoted by $u_{\widehat{f}}$, is an integral solution of the problem (1.1), in the sense of Definition 7. This shows that the set of integral solutions of the problem (1.1) is a nonempty subset of $C(I, X)$, which will be denoted by $\mathcal{S}$.

It remains to show that $\mathcal{S}$ is compact in $C(I, X)$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}$, hence $u_n = u_{f_n}$ for some $f_n \in K$, with $f_n(t) \in F(t, u_n(t))$, a.e. on I. Recalling that K is compact in $L^1_w(I, X)$ and arguing as before, we may assume (without changing the notation for subsequences) that $u_n \to u$ in $C(I, X)$, $f_n \to f$ weakly in $L^1(I, X)$, as $n \to \infty$. We then conclude (compare to (4.17)-(4.20)) that $u = u_f$, with $f(t) \in F(t, u(t))$, a.e. on I. In other words, $u \in \mathcal{S}$ and the proof of Theorem 8 is complete. □

Proof of Theorem 10. One follows, step by step, the proof of Theorem 8, with the mention that $A(t)$, $\overline{D}$, $\left(H_{A(t)}\right)$, Definition 3 and Proposition 4 are now used in place of A, $\overline{D(A)}$, (H_A), Definition 1 and Proposition 2, respectively. In addition, in (4.14), (4.16) and the definition of $z_n(t)$, $S(t)$ is to be replaced by $U(t, 0)$. One must also adapt the theory in [24] (see in particular Theorems 2, 3, 4 and Lemma 4) to obtain the needed compactness property of the solution map $(u_0, f) \to u$, where u_0 varies in a bounded subset of $\overline{D}$, f lies in a uniformly integrable subset of $L^1(I, X)$ and u denotes the corresponding integral solution of (2.4), and justify the passage to the limit in the time-dependent counterpart of (4.19) (cf. Definition 3). The details are left to the reader. □

5 An example

In this section we illustrate the applicability of our abstract theory, by discussing the obstacle type problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = v \ \ on \ I \times \Omega, \\ -\frac{\partial u}{\partial \nu} \in \beta(u) \ \ on \ I \times \Gamma, \\ u(0, x) = \int_0^T h(s, u(s, x))\, ds \ on \ \Omega, \\ v \in L^1\left(I, L^2(\Omega)\right), \ 0 \le v(t, x) \le q(t, x, u(t, x)) \ \ on \ I \times \Omega. \end{cases} \tag{5.1}$$

Here $I = [0, T]$, Ω is a bounded domain in $\mathbb{R}^n$ $(n \ge 1)$ with a smooth boundary Γ, $x = (x_1, ..., x_n) \in \Omega$, β is a multivalued operator in $\mathbb{R}$, $h : I \times \mathbb{R} \to \mathbb{R}$, $q : I \times \Omega \times \mathbb{R} \to \mathbb{R}^+$ and $\partial/\partial\nu$ denotes the outward normal derivative to Γ.

We impose the following restrictions on the data of the problem (5.1) :

(H_β) β is m-accretive in $\mathbb{R}$ with $0 \in \beta(0)$;

(H_h) $h : I \times \mathbb{R} \to \mathbb{R}$ satisfies:

- (i) $h(.,r)$ is measurable for each $r \in \mathbb{R}$ and $h(.,0) \in L^1(I,\mathbb{R})$,
- (ii) $h(t,.)$ is continuous for a.a. $t \in I$,
- (iii) there exists $k \in L^1(I,\mathbb{R}^+)$ such that

$$|h(t,r) - h(t,\overline{r})| \leq k(t)|r - \overline{r}|$$

for almost all $t \in I$ and all $r, \overline{r} \in \mathbb{R}$;

(H_q) $q : I \times \Omega \times \mathbb{R} \to \mathbb{R}^+$ satisfies

- (i) $q(.,.,r)$ is measurable for each $r \in \mathbb{R}$,
- (ii) $q(t,x,.)$ is continuous for a.a. $(t,x) \in I \times \Omega$,
- (iii) there exist two functions $a \in L^1(I, L^2(\Omega))$ and $b \in L^1(I,\mathbb{R})$, such that, for almost all $(t,x) \in I \times \Omega$ and all $r \in \mathbb{R}$, one has $a(t,x)$, $b(t) \geq 0$ and

$$q(t,x,r) \leq a(t,x) + b(t)|r|.$$

Let $X = L^2(\Omega,\mathbb{R})$, equipped with the norm $\|.\|_2$ (defined by $\|u\|_2 = \left(\int_\Omega u^2(x)\,dx\right)^{\frac{1}{2}}$, $\forall u \in L^2(\Omega,\mathbb{R})$). This is a separable Hilbert space. Define the operator A in X by

$$Au = -\triangle u,\ D(A) = \left\{u \in H^2(\Omega) : -\frac{\partial u}{\partial \nu} \in \beta(u),\ a.e.\ on\ \Gamma\right\}. \tag{5.2}$$

By (H_β) and [27], Remark 2.6, it follows that A is m-accretive in X, with $\overline{D(A)} = X$, and $-A$ generates a compact semigroup on X. Next we introduce the functional $g : C(I,X) \to X$ by

$$g(u)(x) = \int_0^T h(s,u(s,x))\,ds,\ \forall u \in C(I,X),\ a.e.\ on\ \Omega. \tag{5.3}$$

From (H_h) it follows that g is well defined and

$$\|g(u) - g(\overline{u})\|_2 \leq \|k\|_1 \|u - \overline{u}\|_\infty,\ \forall u, \overline{u} \in C(I,X). \tag{5.4}$$

Finally, let $F : I \times X \to 2^X$ be given by

$$F(t,u) = \{v \in X : 0 \leq v(x) \leq q(t,x,u(x)),\ a.e.\ on\ \Omega\}. \tag{5.5}$$

Using (H_q), we conclude that F has nonempty, closed and convex values and $F(., u)$ is measurable for all $u \in X$; see [23], p.191 and [6], Theorem 8.14. Also

$$|F(t,u)| = \sup\{\|w\|_2 : w \in F(t,u)\} \leq \|a(t,.)\|_2 + b(t)\|u\|_2 . \tag{5.6}$$

Moreover, Proposition 2.2.3 in [22], Chapter 1, and the analysis in [23], p.191 imply that $F(t,.)$ is u.s.c. from X into X_w for a.a. $t \in I$. See also [25], Theorem 2.5.

It is now obvious that with the identifications (5.2), (5.3), and (5.5), we can rewrite (5.1), in the form (1.1), in the space $X = L^2(\Omega, \mathbb{R})$. On account of the above discussion (cf. in particular (5.4) and (5.6)), we conclude that assumptions (H_A), (H_F) and (H_g) are satisfied, provided that

$$\|k\|_1 + \|b\|_1 < 1. \tag{5.7}$$

(Note that, by (5.4), (5.7) one can take $m = \|k\|_1$ in (H_g), and that (5.6) and (5.7) imply that (3.1) and (3.2) hold with $\gamma(t,r) = \|a(t,.)\|_2 + b(t)r$.

By a generalized solution of (5.1), we mean an integral solution of its abstract equivalent (1.1) in the space $X = L^2(\Omega, \mathbb{R})$, with the identifications (5.2), (5.3), and (5.5), in the sense of Definition 7. A direct application of Theorem 8 yields:

Theorem 11 *Let conditions* (H_β), (H_h), (H_q) *and* (5.7) *be satisfied. Then the set of generalized solutions of the problem* (5.1) *is a nonempty, compact subset of* $C(I, L^2(\Omega, \mathbb{R}))$.

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