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Steiner symmetrization: a weighted version of Pólya-Szegö principle

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1 Introduction

Let us consider a Dirichlet integral of the type

$$\mathcal{F}(u) = \int_\Omega f(Du)\,dx$$

where u is a nonnegative Sobolev function with compact support in Ω. The well known Pólya-Szegö principle states that if u^s denotes the Steiner symmetrand of u, then:

$$\mathcal{F}(u^s) \le \mathcal{F}(u). \tag{1.1}$$

Our aim is to study the case when the integrand f depends also on x and u. The only result in this direction have been obtained in [3] and [10]. Here we extend the results contained in the papers quoted above and discuss the equality case in (1.1). In particular we characterize the functions for which from the equality

$$\int_{\Omega} f(x,u,Du)\,dx = \int_{\Omega^s} f(x,u^s,Du^s)\,dx$$

one can one deduce that $u = u^s$ up to translation.

This problem has been completely solved in [5] for an integrand independent of x and u, in the general setting of BV functions.

It is clear that without any assumption on the dependence of the integrand f with respect to the x variable, the extremals in Pólya - Szegö principle could be not symmetric, and also inequality (1.1) could be false. Indeed introducing the functionals

$$\mathcal{G}_\alpha(u) = \int_{\mathbb{R}} (\mid y\mid^{\alpha/2}\mid u'(y)\mid)^2 dy, \quad \alpha \geq 0, \tag{1.2}$$

we show (see Section 4) that if $\alpha < 2$ then $\mathcal{G}_\alpha(u)$ may even increase under symmetrization, while for $\alpha = 2$ the Pólya-Szegö inequality holds but the extremals are not necessarely symmetric. This phenomenon is due to the fact that in the integral $\mathcal{G}_\alpha$ which is of the type

$$\int_{\mathbb{R}} f(g(y)u'(y))\,dy$$

the weight g is not convex for $\alpha < 2$ and not strictly convex for $\alpha = 2$. In this paper we show that for a multiple integral of the type

$$\int_{\mathbb{R}^n} f(x', u(x), \nabla_{x'}u(x), g(y)\nabla_y u(x))dx,$$

where $x = (x', y)$, $x' \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, the convexity of f with respect to the gradient and the stricty convexity of the weight g, not only ensure the validity of Pólya-Szegö inequality, but also force the extremals to be Steiner symmetric.

2 Preliminary results and definitions

This section is essentially devoted to recall some basic properties of Steiner symmetrization and sets of finite perimeter. In the following we denote the generic point $x \in \mathbb{R}^n$ $(n \geq 2)$ by $x \equiv (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}$, where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$ and the generic point $x \in \mathbb{R}^{n+1}$ by $x \equiv (x', y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}_y \times \mathbb{R}_t$. For any measurable set $E \subset \mathbb{R}^n$, we define, for $x' \in \mathbb{R}^{n-1}$, the slice of E through x' in the y direction as :

$$E_{x'} = \{y \in \mathbb{R} : (x', y) \in E\} \tag{2.1}$$

and

$$l_E(x') = \mathcal{L}^1(E_{x'}), \tag{2.2}$$

where $\mathcal{L}^1$ denotes the Lebesgue outer measure in $\mathbb{R}$.

In the sequel π will denote the orthogonal projection over the first $(n-1)$ variables. Moreover for every $E \subset \mathbb{R}^n$ we will denote $\pi(E)^+$ the essential projection of E onto $\mathbb{R}^{n-1}$ that is

$$\pi(E)^+ = \{x' \in \mathbb{R}^{n-1} : l_E(x') > 0\}.$$

The Steiner symmetral E^s of E about the hyperplane $\{y = 0\}$ is defined as

$$E^s = \{(x', y) \in \mathbb{R}^n :| \, y \, |< l_E(x')/2\}.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $u : \Omega \to \mathbb{R}$ be a nonnegative measurable function. We still denote by u the function

$$u(x) = \begin{cases} u(x) & \text{for } x \in \Omega \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

The Steiner rearrangement u^s of u is defined as follows

$$u^s(x', y) = \inf\{t > 0 : \mu_u(x', t) \le 2|y|\} \qquad (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}_y,$$

where

$$\mu_u(x', t) = \mathcal{L}^1(\{y \in \mathbb{R} : u(x', y) > t\}),$$

is the distribution function of $u(x', \cdot)$.

We define

$$M(x') = \inf\{t > 0 : \mu_u(x', t) = 0\} \quad \text{for } x' \in \pi(\Omega).$$

The measurable function $M(x')$ agrees with $\operatorname{esssup}\{u(x', y) \; : \; y \; \in \; \Omega_{x'}\}$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(\Omega)$.

Remark 2.1 It is easy to prove that for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, we have $u^s(x', y) > t$ for some $y \in \mathbb{R}$ and $t \in \mathbb{R}^+$ if and only if $\mu_u(x', t) > 2|y|$. This obviously implies

$$\{(x', y) : u(x', y) > t\}^s \text{ is equivalent to } \quad \{(x', y) : u^s(x', y) > t\} \quad \forall \, t > 0.$$

Hence $u(x', \cdot)$ and $u^s(x', \cdot)$ are equidistributed for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, and thus u and u^s are equidistributed.

If $E \subset \mathbb{R}^n$ is a measurable set, and $x \in \mathbb{R}^n$, the density of E at x is defined as follows

$$D(E, x) = \lim_{r \to 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))},$$

and the essential boundary of E is the Borel set

$$\partial^* E = \mathbb{R}^n \setminus \{x \in \mathbb{R}^n \ : D(E; x) = 0 \quad \text{or } D(E; x) = 1\}.$$

We recall that an open set $E \subset \mathbb{R}^n$ is a set of finite perimeter if $\mathcal{H}^{n-1}(\partial^* E) < +\infty$. By Theorem 3.61 in [1] and Theorem 4.5.11 in [8] this definition is equivalent to the one given in [6] and for every open set of $\mathbb{R}^n$, $\mathcal{H}^{n-1}(\partial^* E \cap \Omega)$, coincides with the perimeter of E in Ω, $P(E, \Omega)$, as defined in [6].

If $u : \Omega \to \mathbb{R}$ is a measurable function the approximate upper and lower limit of u at a point x are defined as the functions

$$u_+(x) = \inf\{t : D(\{u > t\}, x) = 0\} \quad \text{and} \quad u_-(x) = \sup\{t : D(\{u < t\}, x) = 0\},$$

respectively. The function u is approximately continuous at a point $x \in \Omega$ if $u^-(x) = u^+(x) < \infty$. Finally we will denote by C^u the Borel set of all points where u is approximately continuous, and by $S^u = \Omega \setminus C^u$.

We define the precise representative u^* of u

$$u^*(x) = \begin{cases} \dfrac{u_-(x) + u_+(x)}{2} & \text{if } u_-(x) \text{ and } u_+(x) \text{ are both finite} \\ 0 & \text{otherwise} \end{cases}$$

Remark 2.2 Observe that, if $u \in W^{1,1}(\Omega)$, then

$$u_+(x) = u_-(x) = u^*(x) \qquad \text{for } \mathcal{H}^{n-1} - \text{a.e. } x \in \Omega$$

From the above definition it follows that

$$\partial^*\{u > t\} \subset \{u^- \leq t \leq u^+\} \qquad \text{for every } t \in \mathbb{R} \tag{2.4}$$

Moreover (see (2.20) in [4]) for $\mathcal{L}^1$-a.e. t

$$\mathcal{H}^{n-1}(\{u^- \leq t \leq u^+\} \setminus \partial^*\{u > t\}) = 0$$

We also recall the coarea formula (see for example [1] pg 159)

Proposition 2.1 *If $u \in W^{1,1}(\Omega)$ and $g : \Omega \to [0, +\infty]$ is a Borel function, then*

$$\begin{aligned} \int_\Omega g(x) \mid \nabla u \mid \ dx &= \int_{-\infty}^{+\infty} dt \int_{\partial^*\{u > t\} \cap \Omega} g(x) \ d\mathcal{H}^{n-1} \\ &= \int_{-\infty}^{+\infty} dt \int_{\{u^* = t\} \cap \Omega} g(x) \ d\mathcal{H}^{n-1}. \end{aligned} \tag{2.5}$$

The next lemma gives the representation formulas for the approximate gradient of the distribution function of a Sobolev function. For the proof see [5] and [9].

Lemma 2.1 *Let Ω be an open bounded subset of $\mathbb{R}^n$, and let u be a nonnegative function from $W_0^{1,1}(\Omega)$. Then $\mu_u \in BV(\omega \times \mathbb{R}_t^+)$, for any open set $\omega \subset\subset \pi_{n-1}(\Omega)$, and, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(\Omega)$,*

$$\nabla_t \mu_u(x', t) = -\int_{\{y:u^*(x',y)=t\}} \frac{1}{\mid \nabla_y u \mid} \, d\mathcal{H}^0 \tag{2.6}$$

$$\nabla_i \mu_u(x', t) = \int_{\{y:u^*(x',y)=t\}} \frac{\nabla_i u}{\mid \nabla_y u \mid} \, d\mathcal{H}^0 \qquad i = 1, \ldots, n-1, \tag{2.7}$$

for $\mathcal{L}^1$-a.e. $t \in (0, M(x'))$.

3 A weighted version of Pólya - Szegö inequality

In this section $g : \mathbb{R} \to [0, +\infty[$ will be a non zero, even, convex function. It is known (see [2] for the general n-dimensinal case) that if $E \subset \mathbb{R}$ is a bounded set of finite perimeter, E^s is the open interval centered at the origin having the same measure as E then

$$\int_{\partial^* E} g(y) \, d\mathcal{H}^0 \geq \int_{\partial E^s} g(y) \, d\mathcal{H}^0. \tag{3.1}$$

Moreover the following result holds true.

Proposition 3.1 *Let $E \subset \mathbb{R}$ be a bounded set of finite perimeter. Assume that*

$$\int_{\partial^* E} g(y) \, d\mathcal{H}^0 = \int_{\partial E^s} g(y) \, d\mathcal{H}^0,$$

then E is equivalent to an interval; if the function g is strictly convex then E is equivalent to an interval centered at the origin.

Proof. Since E is a bounded set of finite perimeter, then by Proposition 3.52 in [1] we may assume with no loss of generality that E is the union of a finite number of pairwise disjoint intervals. Let us set $\alpha = \min\{y \in E\}$ and $\beta = \max\{y \in E\}$. Obviously $E \subset [\alpha, \beta]$. If $|E| = \beta - \alpha$ then $E = [\alpha, \beta]$ and the claim follows. If $|E| < \beta - \alpha$ we consider $\gamma > 0$ be such that

$$[-\gamma, \gamma] \subset \left[\frac{\alpha - \beta}{2}, \frac{\beta - \alpha}{2}\right]; \quad 2\gamma = |E|.$$

Hence, recalling that g is increasing in $[0, +\infty[$, even and convex, we have

$$\int_{\partial E^s} g(y)\, d\mathcal{H}^0 = \int_{\partial[-\gamma,\gamma]} g(y)\, d\mathcal{H}^0 \leq \int_{\partial[\frac{\alpha-\beta}{2},\frac{\beta-\alpha}{2}]} g(y)\, d\mathcal{H}^0$$
$$\leq \int_{\partial[\alpha,\beta]} g(y)\, d\mathcal{H}^0 < \int_{\partial E} g(y)\, d\mathcal{H}^0,$$

thus getting a contradiction. Hence E is an interval. If g is strictly convex the claim follows since

$$g(\alpha) + g(\beta) > 2g(\frac{\alpha-\beta}{2}). \tag{3.2}$$

□

Remark 3.1 Let us mention that one can deduce that the set E is an interval centered at the origin, also in the case that g is just strictly convex in some subinterval $(c, d) \subset [\alpha, \beta] \equiv E$ since in this case (3.2) is still satisfied.

Here and in the following we assume $u \in W_0^{1,1}(\Omega)$ and we still denote by u its zero extension outside of Ω defined in (2.3).

We consider $f : (x', u, \xi', z) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \to [0, +\infty[$ a continuous function such that

$$f : (\xi', z) \in \mathbb{R}^{n-1} \times \mathbb{R} \to f(x', u, \xi', z) \in \mathbb{R} \text{ is convex for every } x' \in \mathbb{R}^{n-1}, u \in \mathbb{R}; \tag{3.3}$$

$$f : z \in \mathbb{R} \to f(x', u, \xi', z) \in \mathbb{R} \text{ is even and increasing for every } x', \tag{3.4}$$
$\xi' \in \mathbb{R}^{n-1}, u \in \mathbb{R}, y > 0,;$

$$f(x', u, 0) = 0 \text{ for every } x' \in \mathbb{R}^{n-1}, u \in \mathbb{R}, \tag{3.5}$$

and we define the following integral functional:

$$\mathcal{F} : u \in W_0^{1,1}(\Omega) \to \int_\Omega f\,(x', u, \nabla_1 u, \ldots, \nabla_{n-1} u, g(y)\nabla_y u)\, dx. \tag{3.6}$$

Our aim is to prove that the functional $\mathcal{F}$ decreases under Steiner symmetrization.

Theorem 3.1 *Let $\mathcal{F}$ be the functional defined in (3.6) with f satisfying (3.3)–(3.5). The following inequality holds:*

$$\int_{\Omega^s} f\,(x', u^s(x), \nabla_1 u^s, \ldots, \nabla_{n-1} u^s, g(y)\nabla_y u^s)\ dx$$
$$\leq \int_\Omega f\,(x', u(x), \nabla_1 u, \ldots, \nabla_{n-1} u, g(y)\nabla_y u)\ dx \tag{3.7}$$

for every nonnegative function $u \in W_0^{1,1}(\Omega)$.

Proof. Step 1. We first assume that u satisfies

$$\mathcal{L}^1(\{y : \nabla_y u(x', y) = 0\} \cap \{y : 0 < u(x', y) < M(x')\}) = 0 \tag{3.8}$$

for a.e. $x' \in \pi(\Omega)$ such that $M(x') > 0$. This technical assumption will be removed by means of an approximation argument in the next step. Let us recall that if $u \in W^{1,1}(\Omega)$, then for $\mathcal{L}^{n-1}$ a.e. $x' \in M(\Omega)$, $u^*(x', \cdot)$ is an absolutely continuous function in $\Omega_{x'}$ and

$$\nabla_y u(x', y) = \frac{d}{dy}[u^*(x', \cdot)](y) \quad \text{for } \mathcal{L}^1 \text{ a.e } y \in \Omega_{x'}. \tag{3.9}$$

Let us choose a point x' such that (3.8) and (3.9) hold and such that $u^*(x', \cdot)$ is absolutely continuous. Notice that for such a point (see Lemma 3.3 in [4]) we have also that

$$\mathcal{L}^1(\{y : \nabla_y u^s(x', y) = 0\} \cap \{y : 0 < u^s(x', y) < M(x')\}) = 0. \tag{3.10}$$

In order to simplify the notation in the remaining part of the proof, we shall still denote by u the precise representative of the Sobolev funtion u. The same convention will be applied to u^s. Coarea formula ensures that

$$\begin{aligned}
&\int_{\partial^*\{y:u^s(x',y)>0\}} f(x', u^s, \nabla u^s(x', y))\, dy \\
&= \int_0^{M(x')} dt \int_{\{y:u^s(x',y)=t\}} \frac{1}{|\nabla_y u^s|} f(x', u^s, \nabla u^s(x', y))\, d\mathcal{H}^0_y
\end{aligned} \tag{3.11}$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(\Omega)$ such that $M(x') > 0$

Using formulas (2.6), (2.7) and (3.1) we get

$$\begin{aligned}
&\int_{\{y:u^s(x',y)=t\}} \frac{1}{\mid \nabla_y u^s \mid} f(x', u^s(x), \nabla_1 u^s, \dots, \nabla_{n-1} u^s, g(y)\nabla_y u^s) d\mathcal{H}^0_y \\
&= \int_{\{y:u^s(x',y)=t\}} \frac{1}{\mid \nabla_y u^s \mid} f(x', t, \nabla_1 u^s, \dots, \nabla_{n-1} u^s, g(y) \mid \nabla_y u^s \mid) d\mathcal{H}^0_y \\
&= -\nabla_t \mu_u(x', t) f\left(x', t, \frac{\nabla_1 \mu_u(x', t)}{-\nabla_t \mu_u(x', t)}, \dots, \frac{\nabla_{n-1} \mu_u(x', t)}{-\nabla_t \mu_u(x', t)}, \frac{\int_{\{u^s=t\}} g(y) d\mathcal{H}^0_y}{-\nabla_t \mu_u(x', t)}\right) \\
&\le -\nabla_t \mu_u(x', t) f\left(x', t, \frac{\nabla_1 \mu_u(x', t)}{-\nabla_t \mu_u(x', t)}, \dots, \frac{\nabla_{n-1} \mu_u(x', t)}{-\nabla_t \mu_u(x', t)}, \frac{\int_{\{u=t\}} g(y) d\mathcal{H}^0_y}{-\nabla_t \mu_u(x', t)}\right) \\
&= f\left(x', t, \frac{\int_{\{y:u=t\}} \frac{\nabla_1 u}{|\nabla u_y|} d\mathcal{H}^0}{\int_{\{y:u=t\}} \frac{d\mathcal{H}^0_y}{|\nabla u_y|}}, \dots, \frac{\int_{\{y:u=t\}} \frac{\nabla_{n-1} u}{|\nabla u_y|} d\mathcal{H}^0}{\int_{\{y:u=t\}} \frac{d\mathcal{H}^0_y}{|\nabla u_y|}}, \frac{\int_{\{y:u=t\}} g(y) d\mathcal{H}^0_y}{\int_{\{y:u=t\}} \frac{d\mathcal{H}^0}{|\nabla u_y|}}\right) \\
&\quad \cdot \int_{\{y:u(x',y)=t\}} \frac{d\mathcal{H}^0_y}{\mid \nabla u_y \mid} \\
&\le \int_{\{y:u(x',y)=t\}} \frac{1}{\mid \nabla_y u \mid} f(x', u(x), \nabla_1 u, \dots, \nabla_{n-1} u, g(y)\nabla_y u) d\mathcal{H}^0_y
\end{aligned} \tag{3.12}$$

where in the last line we have used the Jensen inequality.

Now let B any Borel subset of $\pi(\Omega)$ and let $\tilde{B} = B \cap \{M(x') > 0\}$. Integrating the last estimate with respect to t and to x' respectively over $(0, M(x'))$ and over $\tilde{B}$ we get

$$\begin{aligned}
&\int_{B\times\mathbb{R}_y} f\left(x', u^s(x), \nabla_1 u^s, \ldots, \nabla_{n-1} u^s, g(y)\nabla_y u^s\right) dx \\
&\quad = \int_{\tilde{B}} dx' \int_{\{u^s(x',y)>0\}} f\left(x', u^s(x), \nabla_1 u^s, \ldots, \nabla_{n-1} u^s, g(y)\nabla_y u^s\right) \\
&\quad = \int_{\tilde{B}} dx' \int_0^{M(x')} dt \int_{\{u^s(x',y)=t\}} \frac{1}{\mid \nabla u^s_y \mid} \\
&\qquad \times f\left(x', u^s(x), \nabla_1 u^s, \ldots, \nabla_{n-1} u^s, g(y)\nabla_y u^s\right) \\
&\quad \le \int_{\tilde{B}} dx' \int_0^{M(x')} dt \int_{\{u(x',y)=t\}} \frac{1}{\mid \nabla u_y \mid} f\left(x', u(x), \nabla_1 u, \ldots, \nabla_{n-1} u, g(y)\nabla_y u\right) \\
&\quad = \int_{B\times\mathbb{R}_y} f\left(x', u(x), \nabla_1 u, \ldots, \nabla_{n-1} u, g(y)\nabla_y u\right) dx.
\end{aligned}$$

Hence the desired estimate

$$\begin{aligned}
&\int_{B\times\mathbb{R}_y} f\left(x', u^s(x), \nabla_1 u^s, \ldots, \nabla_{n-1} u^s, g(y)\nabla_y u^s\right) dx \\
&\qquad \le \int_{B\times\mathbb{R}_y} f\left(x', u(x), \nabla_1 u, \ldots, \nabla_{n-1} u, g(y)\nabla_y u\right) dx
\end{aligned}$$

follows for every Borel set $B \subset \pi_{n-1}(\Omega)$.

Step 2. We now remove assumption (3.8). Let ω be an open set such that $\omega \subset\subset \pi(\Omega)$. We preliminarly observe that there exists a sequence u_h of nonnegative Lipschitz functions satisfying (3.8) and strongly converging to u in $W^{1,1}(\omega \times \mathbb{R}_y)$.

Without loss of generality we can assume that u belongs to $W^{1,1}(\mathbb{R}^n)$. Hence, given $h \in \mathbb{N}$, there exists $v_h \in C^1_0(\mathbb{R}^n)$ such that $v_h \ge 0$ and $||v_h - u||_{W^{1,1}(\mathbb{R}^n)} < 1/h$. By standard arguments one can find a polynomial p_h such that $||v_h - p_h||_{C^1(\overline{B}_{2r(0)})} < 1/h$ where r is such that the support of v_h is contained in $B_r(0)$. Replacing p_h by $p_h + 1/h + \delta y^2$, with $\delta > 0$ sufficiently small, we can also assume that $p_h > 0$ in $B_{2r(0)}$ and that $\mathcal{L}^1(\{\nabla_y p_h(x', y) = 0\}) = 0$ for $\mathcal{L}^{n-1}$a.e x' in $\mathbb{R}^{n-1}$. Let η_r be the cut-off function between B_{2r} and B_r defined as $\eta_r(x) = 1$ if $|x| \le r$, $\eta_r(x) = 0$ if $|x| \ge 2r$ and $\eta_r(x) = \frac{4r^2 - |x|^2}{3r^2}$ if $r < |x| < 2r$. Setting $u_h = \eta_r p_h$ one can check that

$$||u - u_h||_{W^{1,p}(\mathbb{R}^n)} < c/h$$

for some positive constant c. Finally, (3.8) follows from the fact that $u_h(x) > 0$ only if $x \in B_{2r}$ and that u_h agrees with the polynomial p_h in B_r and with the polynomial $p_h \eta_r$ in $B_{2r} \setminus \overline{B}_r$. Now if the function f verifies

$$0 \le f(x, \xi) \le C(1 + |\xi|) \tag{3.13}$$

then by Lebesgue dominated convergence Theorem we have that $f(x, \nabla u_h)$ converges strongly in $L^1(\omega \times \mathbb{R}_y)$ to $f(x, \nabla u)$. Recalling that Steiner symmetrization is continuous in L^1, by semicontinuity we have

$$\begin{aligned}\int_{\omega\times\mathbb{R}_y} f(x, \nabla u^s)dx &\leq \liminf_{h\to+\infty} \int_{\omega\times\mathbb{R}_y} f(x, \nabla u_h^s)dx \\ &\leq \liminf_{h\to+\infty} \int_{\omega\times\mathbb{R}_y} f(x, \nabla u_h)dx = \int_{\omega\times\mathbb{R}_y} f(x, \nabla u)dx.\end{aligned}$$

Then the claim follows. Assumption (3.13) can be easily removed observing that

$$f(x, \xi) = \sup_{j\in\mathbb{N}}\{< a_j(x), \xi > + b_j(x)\} = \sup_{j\in\mathbb{N}}\{< a_j(x), \xi > + b_j(x)\}^+.$$

Moreover since f is even with respect to ξ_n it results that

$$\sup_{j\in\mathbb{N}}\{\{< a_j(x), \xi > + b_j(x)\}^+, \{< \overline{a}_j(x), \xi > + b_j(x)\}^+\}$$

where $\overline{a}_j(x) = ((a_j)_1, \dots, (a_j)_{n-1}, -(a_j)_n)$. Finally setting for $N \in \mathbb{N}$

$$f_N(x, \xi) = \sup_{1\leq j\leq N}\{\{< a_j(x), \xi > + b_j(x)\}^+, \{< \overline{a}_j(x), \xi > + b_j(x)\}^+\}.$$

Since $f_N(x, \xi)$ converges to $f(x, \xi)$ and is even with respect to ξ_n and satisfies (3.13) by monotone convergence the claim follows. □

Now we are able to investigate what happens when equality holds in (3.7). We will prove that extremals functions u for which equality holds in (3.7) are Steiner symmetric. To do this we must assume that the set where $| \nabla_y u |= 0$ has zero Lebesgue measure, otherwise the result could be false (see [5]). We will assume that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi_{n-1}(\Omega)$, $M(x') > 0$, and the derivative of the restriction $u(x', \cdot)$ is $\mathcal{L}^1$-a.e. different from 0 in the set where $u(x', \cdot) < M(x')$. The last condition is equivalent to the following one

$$\begin{aligned}&\mathcal{L}^n(\{(x', y) \in \Omega : \nabla_y u(x', y) = 0\}) \\ &\quad \cap \{(x', y) \in \Omega : M(x') = 0 \text{ or } u(x', y) < M(x')\} = 0. \end{aligned} \tag{3.14}$$

Theorem 3.2 *Assume that g is strictly convex, f satisfies (3.3)–(3.5), $f(x', u, \xi', \cdot)$ is strictly increasing and $u \in W_0^{1,1}(\Omega)$ satisfies (3.14). If*

$$\begin{aligned}&\int_{\Omega^s} f\left(x', u^s, \frac{\partial u^s}{\partial x_1}, \dots, \frac{\partial u^s}{\partial x_{n-1}}, g(y)\frac{\partial u^s}{\partial y}\right)dx \\ &\quad = \int_{\Omega} f\left(x', u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_{n-1}}, g(y)\frac{\partial u}{\partial y}\right)dx \end{aligned} \tag{3.15}$$

then $u = u^s$ up to translations along the y axis.

Proof. Since (3.15) holds, one deduces that in (3.12) the inequalities become equalities. Hence for $\mathcal{L}^{n-1}$-a.e. $x' \in M(\Omega)$ and $\mathcal{L}^1$-a.e $0 < t < M(x')$,

$$- \nabla_t \mu_u(x',t) f\left(x', u, \frac{\nabla_1 \mu_u(x',t)}{-\nabla_t \mu_u(x',t)}, \dots, \frac{\nabla_{n-1} \mu_u(x',t)}{-\nabla_t \mu_u(x',t)}, \frac{\int_{\partial^*\{u^s>t\}} g(y) d\mathcal{H}^0}{-\nabla_t \mu_u(x',t)}\right)$$
$$= -\nabla_t \mu_u(x',t) f\left(x', u, \frac{\nabla_1 \mu_u(x',t)}{-\nabla_t \mu_u(x',t)}, \dots, \frac{\nabla_{n-1} \mu_u(x',t)}{-\nabla_t \mu_u(x',t)}, \frac{\int_{\partial^*\{u>t\}} g(y) d\mathcal{H}^0}{-\nabla_t \mu_u(x',t)}\right).$$

Being $f(x', \cdot)$ strictly increasing one deduces that

$$\int_{\partial^*\{y: u^s(x',y)>t\}} g(y) d\mathcal{H}^0 = \int_{\partial^*\{y: u(x',y)>t\}} g(y) d\mathcal{H}^0.$$

By Proposition 3.1 the claim follows. □

4 Functionals nondecreasing under Steiner symmetrization

In this section we exhibit a functional of the type (3.6) that may increase under Steiner symmetrization. This counterexample shows that the convexity of the weight $g(y)$ is necessary to prove the desired Pólya - Szëgo inequality. We also observe that the same counterexample proves that for some weight $g(y)$ which is just convex, non constant, but not strictly convex, equality can hold true in (3.7) also for non symmetric function in the y direction.

Let us consider the following integral functional defined on $W_0^{1,1}(\mathbb{R})$

$$\mathcal{G}_\alpha(u) = \int_{\mathbb{R}} (\mid y \mid^{\alpha/2} \mid u'(y) \mid)^2 dy, \quad \alpha > 1 \tag{4.1}$$

and let us define the following Lipschitz function

$$u_\lambda(y) = \begin{cases} (y+\lambda)/(2\lambda) & \text{for } \mid y \mid \le \lambda \\ (y+\lambda-2)/(2\lambda-2) & \text{for } \lambda \le y \le 2-\lambda \\ 0 & \text{otherwise ,} \end{cases}$$

for some $0 < \lambda < 1$. The Steiner symmetrand of u_λ is

$$u_\lambda^s(y) = \begin{cases} 1- \mid y \mid & \text{for } \mid y \mid \le 1 \\ 0 & \text{otherwise .} \end{cases}$$

For $\alpha < 2$ the weight $\mid y \mid^{\alpha/2}$ is not convex and so the assumptions of Theorem 3.1 are violated for the functional $\mathcal{G}_\alpha$. In fact it is easy to check that for $\alpha < 2$

and $0 < \lambda < 1$

$$\mathcal{G}_\alpha(u_\lambda^s) = \frac{2}{\alpha+1} > \frac{1}{\alpha+1}(\frac{\lambda^{\alpha-1}}{2} + \frac{(2-\lambda)^{\alpha+1} - \lambda^{\alpha+1}}{4(\lambda-1)^2}) = \mathcal{G}_\alpha(u_\lambda)$$

The critical exponent for the functional $\mathcal{G}$ is $\alpha = 2$, in this case the weight $g(y) =| y |$ is just convex but not strictly convex then we consider the translated of u_λ

$$v_\lambda(y) = \begin{cases} (y+\lambda)/(\lambda) & \text{for } -\lambda < y < 0 \\ (y+\lambda-2)/(\lambda-2) & \text{for } 0 \le y \le 2-\lambda \\ 0 & \text{otherwise }, \end{cases}$$

and $v_\lambda^s = u_\lambda^s$. We have

$$\mathcal{G}_2(v_\lambda) = \mathcal{G}_2(v_\lambda^s) = 2/3 \quad \forall \lambda < 2$$

but v_λ is not symmetric.

Nevertheless we will remark in the next section that the symmetry of extremals in (3.7) can be obtained also for a special class of non strictly convex weights.

5 When the weight is not strictly convex

In this last section we remark that Theorem 3.2 can be proved for a special class of non strictly convex weights. Here we allow, as a particular case, that $g(y)$ is constant, so we must assume as in [5] some kind of geometric property for the open set Ω. We will assume that

$$\pi(\Omega) \text{ is connected,} \tag{5.1}$$

and that the reduced boundary $\partial^*\Omega$ of Ω is almost nowhere parallel to the y-axis inside the cylinder $\pi(\Omega) \times \mathbb{R}_y$, that is

$$\begin{aligned} &\Omega \text{ has locally finite perimeter in } \pi(\Omega) \times \mathbb{R}_y \text{ and} \\ &\mathcal{H}^{n-1}(\{(x',y) \in \partial^*\Omega : \nu_y^\Omega(x',y) = 0\} \cap (\pi(\Omega) \times \mathbb{R}_y)) = 0, \end{aligned} \tag{5.2}$$

where $\mathcal{H}^k$ stands for $k-$ dimensional Hausdorff measure, and ν_y^Ω denotes the component along the y- axis of the generalized inner normal ν^Ω to Ω. Moreover in this section we assume that (3.14) holds. We remark that assumption (3.14) can be relaxed by replacing u with u^s, by means of the following proposition

Proposition 5.1 *Let Ω be an open bounded subset of $\mathbb{R}^n$ and let u be a nonnegative function from $W_0^{1,1}(\Omega)$. Then, for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(\Omega)$,*

$$\begin{aligned} &\mathcal{L}^1(\{y : \nabla_y u(x',y) = 0, t < u(x',y) < M(x')\}) \\ &\quad = \mathcal{L}^1(\{y : \nabla_y u^s(x',y) = 0, t < u^s(x',y) < M(x')\}) \end{aligned}$$

for every $t \in (0, M(x'))$.

Proof. See [5] Proposition 2.3. □

Actually Theorem 3.2 still holds true when the weight is $g(y) \equiv 1$ as proved in [5]. The counterexample of section 4 is built using the fact that the weight g looses his strict convexity and is not constant. Nevertheless the weight $g(y)$ can be not strictly convex where it is constant and Theorem 3.2 still holds true. In fact let us consider an even convex real function $g : \mathbb{R}^+ \to \mathbb{R}$ such that, $g(t) \equiv c > 0$ $\forall t \in [-\rho, \rho]$ and g is strictly convex in $\mathbb{R} \setminus [-\rho, \rho]$, for some constant $\rho, c > 0$. The following theorem holds

Theorem 5.1 *Assume that f satisfies (3.3)–(3.5), $f(x', u, \cdot)$ is strictly convex, and $f(x', u, \xi', \cdot)$ is strictly increasing. If g is defined as above and if (3.15) holds then $u = u^s$ up to translation along the y axis.*

Proof. Let us preliminarily observe that as in Theorem 3.1 we can obtain (3.12) via Jensen inequality. Since equality holds, and since f is strictly convex then

$$\nabla_i u_{|\partial^*\{y:u(x',y)>t\}} = c_i \qquad i = 1, \ldots, n-1 \tag{5.3}$$

and

$$g(y) \mid \nabla_y u \mid_{|\partial^*\{y:u(x',y)>t\}} = c_n \tag{5.4}$$

for some constant $c_1, \ldots, c_{n-1}, c_n$. Arguing as in Theorem 3.2 we obtain also

$$\int_{\partial^*\{y:u^s(x',y)>t\}} g(y) d\mathcal{H}^0 = \int_{\partial^*\{y:u(x',y)>t\}} g(y) d\mathcal{H}^0,$$

and this implies, using Proposition 3.1, that $\{y : u(x', y) > t\}$ is equivalent to some interval $(y_1(x', t), y_2(x', t))$. In the case that $\{y : u(x', y) > t\}$ is not included in $[-\rho, \rho]$, by Remark 3.1 $\{y : u(x', y) > t\}$ is centered at the origin and being g even, using (5.4) we obtain

$$\nabla_y u(x', y_1(x', t)) = -\nabla_y u(x', y_2(x', t)). \tag{5.5}$$

In the case that $\{y : u(x', y) > t\} \subset [-\rho, \rho]$, g is constant in $\{y : u(x', y) > t\}$, then (5.5) follows as in Lemma 4.7 of [5]. In any case we obtain (5.3) and (5.5); from these symmetry properties of the gradient the proof follows exactly as in [5] (see Lemmas 4.7–4.10). □

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