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Locating the peaks of the least energy solutions to an elliptic system with Dirichlet boundary conditions

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Abstract. We consider a system of the form $-\varepsilon^2\Delta u + u = g(v)$, $-\varepsilon^2\Delta v + v = f(u)$ in Ω with Dirichlet boundary condition on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N\geqslant 3$ and f, g are power-type nonlinearities having superlinear and subcritical growth at infinity. We prove that the least energy solutions to such a system concentrate, as ε goes to zero, at a point of Ω which maximizes the distance to the boundary of Ω.

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1 Introduction

The aim of this paper is to extend to the following system of elliptic equations with Dirichlet boundary conditions:

$$-\varepsilon^2\Delta u + u = g(v), \quad -\varepsilon^2\Delta v + v = f(u) \ \text{ in } \Omega, \quad u = v = 0 \ \text{ on } \partial\Omega, \tag{1.1}$$

the results obtained in [15] for the corresponding problem with Neumann boundary conditions. Here $\varepsilon > 0$ is a small parameter and Ω is a bounded open domain of $\mathbb{R}^N$ ($N\geqslant 3$) of class $C^{2,\alpha}$ ($0 < \alpha < 1$). Our framework is a typical "superlinear", "subcritical" and "convex" one, namely we assume, as in [15], that the following holds.

(H) $f, g \in C^1(\mathbb{R})$, $f(0) = 0 = f'(0)$, $g(0) = 0 = g'(0)$ and there exist real numbers ℓ_1, $\ell_2 > 0$ and $p, q > 2$ such that $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$ and

$$\lim_{|s|\to\infty} \frac{f'(s)}{|s|^{p-2}} = \ell_1, \qquad \lim_{|s|\to\infty} \frac{g'(s)}{|s|^{q-2}} = \ell_2. \tag{1.2}$$

Moreover, for some $\delta > 0$ and every $s \in \mathbb{R}$, $s \neq 0$,

$$f(s)s \geqslant (2+\delta)\, F(s) > 0 \quad \text{and} \quad f^2(s) \leqslant 2f'(s)F(s), \tag{1.3}$$

and

$$g(s)s \geqslant (2+\delta)\, G(s) > 0 \quad \text{and} \quad g^2(s) \leqslant 2g'(s)G(s), \tag{1.4}$$

where $F(s) := \int_0^s f(\sigma)d\sigma$ and $G(s) := \int_0^s g(\sigma)d\sigma$. We look for positive solutions of (1.1), and therefore we let $f(s) = g(s) = 0$ for $s \leqslant 0$.

Examples of nonlinearities satisfying (H) are given in [15]. Our motivation for the study of such a problem goes back to the pioneering work of Ni and Takagi (see e.g. [13]) concerning the single equation

$$-\varepsilon^2 \Delta u + u = f(u) \qquad \text{in } \Omega. \tag{1.5}$$

These authors proved that the least energy solutions of equation (1.5) together with Neumann boundary conditions concentrate, as $\varepsilon \to 0$, at points of the boundary of Ω where the mean curvature is maximized. Subsequently, Ni and Wey ([14]) considered equation (1.5) together with Dirichlet boundary conditions and showed that in this case the least energy solutions concentrate at points of Ω which maximize the distance to the boundary of Ω. Related results were proved e.g. in [12], [18], [9] (where a "local" version of Ni-Wei's result is proved) and in [2], [6], [11] (where a linear term $V(x)u$ is added to equation (1.5) while $\Omega = \mathbb{R}^N$). The subject was revisited by Del Pino and Felmer, for both Neumann and Dirichlet boundary conditions, in [5], where shorter and more elementay arguments were introduced, with respect to those in [13, 14]. We refer the reader to the nice Introduction in [5] for further references and a developed discussion on the subject.

It is known that the extension of these results to systems such as (1.1) presents some difficulties, both technical and structural. Roughly, they are due to the strongly indefinite character of the energy functional associated to the system, that is

$$I(u, v) = \int_\Omega \Big(\varepsilon^2 \langle \nabla u, \nabla v\rangle + uv - F(u) - G(v)\Big)\, dx.$$

Other difficulties have to do with our assumption that $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$ (which is quite more general than to assume that $p, q < 2N/(N-2)$) and to the unclear picture of the "limit problem" associated to (1.1). We refer the reader to the Introductions in [3, 15] for more details on this.

A first approach to the singularly perturbed system in (1.1) together with Neumann boundary conditions appeared in [3] by means of a dual variational formulation of the problem (restricted to the case where $f(s) = s^{p-1}$, $g(s) = s^{q-1}$, $2 < p, q < 2N/(N-2)$). A more direct approach was proposed in [16] and was subsequently developed in [15]; in these papers the authors extend to system (1.1) (with Neumann boundary conditions) the elementary point of view of the paper [5] for the single equation (1.5).

In the present paper our goal is to prove the following.

Theorem 1.1 *Under assumptions* (H) *and for any sufficiently small* $\varepsilon > 0$*, the system admits a ground-state solution* $(u_\varepsilon, v_\varepsilon) \in C^2(\Omega) \cap C^1(\overline{\Omega})$*,* u_ε*,* $v_\varepsilon > 0$ *with the following properties:*

(i) both functions u_ε *and* v_ε *attain their maximum value at some unique and commom point* $x_\varepsilon \in \Omega$*;*

(ii) $d_\varepsilon := \operatorname{dist}(x_\varepsilon, \partial\Omega) \to \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$ *as* $\varepsilon \to 0$*;*

(iii) the associated critical value $c_\varepsilon(\Omega)$ *of the energy functional can be estimated as*

$$c_\varepsilon(\Omega) = \varepsilon^N \left(c(\mathbb{R}^N) + e^{-2(1+\mathrm{o}(1))\frac{d_\varepsilon}{\varepsilon}} \right), \qquad \mathrm{o}(1) \to 0 \text{ as } \varepsilon \to 0.$$

We have denoted by $c(\mathbb{R}^N)$ the ground-state critical level of the energy for the limit problem associated to (1.1) (set $\varepsilon = 1$ and $\Omega = \mathbb{R}^N$ in (1.1); see Section 2 for details). By a *ground-state solution* we mean a solution of (1.1) which minimizes the energy amongst all nonzero solutions of the system.

Comparing Theorem 1.1 with Theorem 0.2 in [5] we see that the former is the natural extension of the latter to system (1.1) (except that in [5] the authors work with somewhat more general conditions on the nonlinearity).

The rest of the paper is devoted to the proof of Theorem 1.1. Section 2 contains the heart and the main novelty of our paper. Indeed, contrarily to what happens for the single equation case, it does not seem straighforward to realize that $c_\varepsilon(\Omega)$ is monotonic with respect to the domain (i.e. that $c_\varepsilon(\Omega') \leqslant c_\varepsilon(\Omega)$ whenever $\Omega \subset \Omega'$); this is due to the fact that the "natural" constraint minimization problem associated to $c_\varepsilon(\Omega)$ (see Proposition 2.1) is not stable by zero extension of the functions. However, by means of a continuation argument we are able to essentially prove this monotonic dependence with respect to the domain (cf. Proposition 2.2).

In Section 4 we combine the arguments of Section 2 with some ideas introduced in [5, 16] and we give a proof of Theorem 1.1. We mention that, due to the presence of the strongly indefinite quadratic part of the energy functional, and contrarily to [5, Theorem 0.2], we do not rely on Schwarz's symmetrization arguments. Some preliminary estimates are stated in Section 3; here we use classical arguments on the maximum principle and elliptic estimates but we also rely on the information of the Morse index of the ground-state solutions in order to bypass the delicate question of proving the *a priori* decay of u_ε and v_ε near the maximum point x_ε. Some of the arguments (specially in the Appendix) are merely sketched and the reader is refered to [15, 16] for details.

2 Dependence of the ground-state critical level with respect to the domain

Given $f, g \in C^1(\mathbb{R};\mathbb{R})$ and a $C^{2,\alpha}$ bounded domain $\Omega \subset \mathbb{R}^N$ $(0 < \alpha < 1)$, we consider the system

$$-\Delta u + u = g(v), \quad -\Delta v + v = f(u), \quad u, v > 0, \quad u, v \in H_0^1(\Omega). \tag{2.1}$$

We assume that the conditions (H) stated at the Introduction hold for f and g. As explained in [15, 16], we may assume without loss of generality that the numbers p, q in (H) are such that $2 < p = q < 2N/(N-2)$. This is due to the fact that in the present paper we will always work with ground-states having a bounded Morse index (see the proof of Proposition 2.1 below); these solutions are *a priori* bounded for the L^∞ norm and this bound is unchanged whenever we truncate f and g in a neighborhood of infinity so that (H) is still satisfied for the modified functions and moreover (1.2) holds with $p = q$ (see Theorem 1.16 or Theorem 3.2 in [16] for details).

Taking the preceding remark into account, we can introduce the energy functional associated to system (2.1), namely

$$I(u,v) = \langle u, v\rangle - \int_\Omega (F(u) + G(v)), \qquad (u,v) \in E := H_0^1(\Omega) \times H_0^1(\Omega),$$

where

$$\langle u, v\rangle := \int_\Omega (\langle \nabla u, \nabla v\rangle + uv).$$

It is clear that $I \in C^2(E;\mathbb{R})$ and that

$$I'(u,v)(\phi,\psi) = \langle u, \psi\rangle + \langle v, \phi\rangle - \int_\Omega (f(u)\phi + g(v)\psi), \qquad \forall (\phi,\psi) \in E,$$

so that critical points of I correspond to solutions of problem (2.1). It can also be proved that I admits a ground-state critical level $c(\Omega)$, that is

$$c(\Omega) := \min\{I(u,v) : (u,v) \in E \setminus \{(0,0)\},\ I'(u,v) = 0\}.$$

Proposition 2.1 *Under assumptions* (H)*, we have that*

$$c(\Omega) = \min\{I(u,v) : (u,v) \in N, v \neq -u\},$$

where N consists of the functions $(u,v) \in E \setminus \{(0,0)\}$ satisfying

$$I'(u,v)(u,v) = 0 \quad \textit{and} \quad I'(u,v)(\phi,-\phi) = 0,\ \forall \phi \in H_0^1(\Omega).$$

Proof. This is essentially proved in [16]. For the reader's convenience and for further reference in the paper, we sketch the argument of the proof. First, it can be proved that I satisfies the Palais-Smale condition in E. Then one splits the space as

$$E = E_- \oplus E_+, \qquad \text{where } E_\pm := \{(\phi, \pm\phi), \phi \in H_0^1(\Omega)\},$$

and apply Benci-Rabinowitz's linking theorem, according to which, given $(u, v) \in E \setminus E_-$, I admits a critical point (u_1, v_1) such that

$$0 < I(u_1, v_1) \leqslant \sup\{I(s(u, v) + (\phi, -\phi)) : s \in \mathbb{R}^+, \phi \in H_0^1(\Omega)\}.$$

In particular,

$$c(\Omega) \leqslant \sup\{I(s(u, v) + (\phi, -\phi)) : s \in \mathbb{R}^+, \phi \in H_0^1(\Omega)\}. \tag{2.2}$$

On the other hand, if moreover $(u, v) \in N$ then it is proved in Theorem 3.1 of [16] that the expression in the right-hand side of (2.2) equals $I(u, v)$, and the proposition follows. We point out that in [16, Theorem 3.1] it is assumed that $I'(u, v)(\phi, \psi) = 0$ for every $\phi, \psi \in H_0^1(\Omega)$ but that an inspection of its proof immediately shows that it is sufficient to take any (u, v) such that $I'(u, v)(u, v) = 0$ and $I'(u, v)(\phi, -\phi) = 0$ for every $\phi \in H_0^1(\Omega)$. We also recall from [1] that $c(\Omega)$ is attained at some (u, v) and (u, v) can be chosen as to have Morse index not greater than 1. □

Given bounded open sets $\Omega \subset \Omega' \subset \mathbb{R}^N$, we would like to prove that $c(\Omega') \leqslant c(\Omega)$. We will prove a weaker version of this property which is sufficient to our purposes in Section 4. For any $\mu > 0$ we denote $\Omega_\mu := \{x : \operatorname{dist}(x, \Omega) < \mu\}$.

Proposition 2.2 *Under assumptions* (H)*, the map $\mu \mapsto c(\Omega_\mu)$ is continuous and decreasing.*

Proof. 1) Let $\Omega' \supset \Omega$ be a smooth neighborhood of Ω. We first observe that it is possible to construct a superharmonic function $h(x)$ satisfying, for some $\nu > 0$,

$$h \in W^{1,\infty} \cap C(\Omega'), \quad -\Delta h \geqslant 0 \text{ in } \Omega', \quad h = 0 \text{ and } \frac{\partial h}{\partial n} \leqslant -\nu < 0 \text{ on } \partial\Omega.$$

Here $\partial h(x)/\partial n$ stands for the unit outward normal of Ω at the point $x \in \partial\Omega$. For example, one can define $h(x)$ as:

$$h(x) = h_1(x) \quad \text{if } x \in \Omega, \qquad h(x) = h_2(x) + M h_3(x) \quad \text{if } x \in \Omega' \setminus \Omega,$$

where h_1, h_2, h_3 satisfy

$$-\Delta h_3 = 0 \quad \text{in } \Omega' \setminus \overline{\Omega}, \qquad h_3 = 0 \ \text{ on } \partial\Omega, \qquad h_3 = -1 \ \text{ on } \partial\Omega'$$

(so that $\partial h_3/\partial n \leqslant -\delta < 0$ on $\partial\Omega$, by Hopf lemma),

$$-\Delta h_2 = 1 \quad \text{in } \Omega' \setminus \overline{\Omega}, \qquad h_2 = 0 \ \text{ on } \partial\Omega, \qquad h_2 = -1 \ \text{ on } \partial\Omega',$$

and

$$-\Delta h_1 = 1 \quad \text{in } \Omega, \qquad h_1 = 0 \quad \text{on } \partial\Omega.$$

Then, if M is large enough, we have, for every $\varphi \in \mathcal{D}(\Omega')$, $\varphi \geqslant 0$,

$$\begin{aligned}\int_{\Omega'} \langle \nabla h, \nabla\varphi\rangle &= \int_{\Omega} \langle \nabla h_1, \nabla\varphi\rangle + \int_{\Omega'\setminus\overline{\Omega}} \langle \nabla h_2, \nabla\varphi\rangle + M\int_{\Omega'\setminus\overline{\Omega}} \langle \nabla h_3, \nabla\varphi\rangle \\ &= \int_{\Omega'} \varphi + \int_{\partial\Omega} (\frac{\partial h_1}{\partial n} - \frac{\partial h_2}{\partial n} - M\,\frac{\partial h_3}{\partial n})\varphi \\ &\geqslant \int_{\partial\Omega} (\frac{\partial h_1}{\partial n} - \frac{\partial h_2}{\partial n} + M\delta)\varphi \geqslant 0,\end{aligned}$$

provided $M\delta \geqslant ||\nabla h_1||_\infty + ||\nabla h_2||_\infty$.

2) Now, let $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be a ground-state solution for the energy functional, i.e. $I'(u, v) = 0$ and $I(u, v) = c(\Omega)$. Then $u, v \in C^{2,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$, $u, v > 0$ in Ω and, by Hopf lemma, the map $x \mapsto \frac{\partial v/\partial n}{\partial u/\partial n}(x) > 0$ is well-defined and of class $C^{1,\alpha}(\partial\Omega)$. Similarly as before, let z_1 and z_2 solve the problems

$$-\Delta z_1 = 0 \quad \text{in } \Omega, \qquad z_1 = \frac{\partial v/\partial n}{\partial u/\partial n} \quad \text{on } \partial\Omega \tag{2.3}$$

and

$$-\Delta z_2 = 0 \quad \text{in } \Omega' \setminus \overline{\Omega}, \qquad z_2 = \frac{\partial v/\partial n}{\partial u/\partial n} \ \text{ on } \partial\Omega, \qquad z_2 = 1 \ \text{ on } \partial\Omega'. \tag{2.4}$$

Then, if $M > 0$ is large enough (namely, $M\nu \geqslant ||\nabla z_1||_\infty + ||\nabla z_2||_\infty$) then the map $\lambda \in W^{1,\infty} \cap C(\Omega')$ given by

$$\lambda(x) = z_1(x) \quad \text{if } x \in \Omega, \qquad \lambda(x) = z_2(x) + Mh(x) \quad \text{if } x \in \Omega' \setminus \Omega \tag{2.5}$$

is such that

$$\lambda(x) = \frac{\partial v/\partial n}{\partial u/\partial n}(x) \qquad \forall x \in \partial\Omega \tag{2.6}$$

and

$$-\Delta\lambda \geqslant 0 \quad \text{in } \Omega'. \tag{2.7}$$

We also have that $\inf_\Omega \lambda \geqslant \min_{\partial\Omega} z_1 > 0$, and so we can restrict further the neighborhood Ω' in such a way that

$$\inf_{\Omega'} \lambda > 0. \tag{2.8}$$

3) We extend u and v by zero to Ω' and from now on we work in the Hilbert space $E' = H_0^1(\Omega') \times H_0^1(\Omega')$. We still denote by I the energy functional in the space E'. We can split the space in a direct sum

$$E = E'_- \oplus E'_+ \text{ with } E'_- = \{(\phi, -\lambda\phi) : \phi \in H_0^1(\Omega')\},\ E'_+ = \{(\lambda\phi, \phi) : \phi \in H_0^1(\Omega')\}.$$

We observe that any $(\psi_1, \psi_2) \in E'$ can be written as

$$(\psi_1, \psi_2) = (\lambda\phi_1, \phi_1) + (\phi_2, -\lambda\phi_2),$$

with $\phi_1 = (\psi_2 + \lambda\psi_1)/(1+\lambda^2)$ and $\phi_2 = (\psi_1 - \lambda\psi_2)/(1+\lambda^2)$. Moreover, we have that

$$\langle \lambda\phi, \phi\rangle = \int_{\Omega'} \langle \nabla(\lambda\phi), \nabla\phi\rangle + \int_{\Omega'} \lambda\phi^2 \geqslant \int_{\Omega'} \lambda(|\nabla\phi|^2 + \phi^2) \geqslant c \int_{\Omega'} (|\nabla\phi|^2 + \phi^2),$$

as follows from (2.7) and (2.8), and, of course, $\langle \lambda\phi, \phi\rangle \leqslant C \int_{\Omega'} (|\nabla\phi|^2 + \phi^2)$. As a consequence, we can apply Benci-Rabinowitz's linking theorem to deduce that

$$c(\Omega') \leqslant \sup\{I(s(u,v) + (\phi, -\lambda\phi)) : s \in \mathbb{R}^+,\ \phi \in H_0^1(\Omega')\}. \tag{2.9}$$

On the other hand, since, by construction (cf. (2.6)),

$$I'(u,v)(u,v) = 0 \quad \text{and} \quad I'(u,v)(\phi, -\lambda\phi) = 0 \quad \forall\phi \in H_0^1(\Omega'),$$

it follows as in [16, Theorem 3.1] that the right-hand side of (2.9) equals $I(u,v)$, as explained in the proof of Proposition 2.1 above. This shows that

$$c(\Omega') \leqslant c(\Omega).$$

4) Now, let $\mu_n \to \mu \geqslant 0$, $\mu_n > \mu$. Then $\Omega_\mu \subset \Omega_{\mu_n}$ and it follows from the preceding argument that $c(\Omega_{\mu_n}) \leqslant c(\Omega_\mu)$ if n is large enough. On the other hand, by using the Palais-Smale condition and elliptic estimates we easily see that, up to a subsequence, the ground-state solutions (u_n, v_n) associated to Ω_{μ_n} converge in the C^1-norm to a nonzero critical point of the energy $(u_0, v_0) \in H_0^1(\Omega_\mu) \times H_0^1(\Omega_\mu)$. Thus

$$c(\Omega_\mu) \leqslant \liminf_{n\to\infty} c(\Omega_{\mu_n}) \leqslant \limsup_{n\to\infty} c(\Omega_{\mu_n}) \leqslant c(\Omega_\mu),$$

and this proves that the map $\mu \mapsto c(\Omega_\mu)$ is right-continuous.

5) Suppose that $\mu_n \to \mu$, $\mu_n < \mu$. As before, $c(\Omega_\mu) \leqslant \liminf_{n\to\infty} c(\Omega_{\mu_n})$. Now, let (u,v) be a ground-state solution associated to Ω_μ and, for any $\varepsilon > 0$, let u_ε, $v_\varepsilon \in \mathcal{D}(\Omega_\mu)$ be such that $||u - u_\varepsilon||_{H^1(\Omega_\mu)} \leqslant \varepsilon$ and $||v - v_\varepsilon||_{H^1(\Omega_\mu)} \leqslant \varepsilon$. Since u_ε, $v_\varepsilon \in \mathcal{D}(\Omega_{\mu_n})$ for large n, by Benci-Rabinowitz's linking theorem we have that

$$c(\Omega_{\mu_n}) \leqslant \sup\{I(s(u_\varepsilon, v_\varepsilon) + (\phi, -\phi)),\ s \in \mathbb{R}^+,\ \phi \in H_0^1(\Omega_{\mu_n})\},$$

so that, since $\Omega_{\mu_n} \subset \Omega_\mu$,

$$\limsup_{n\to\infty} c(\Omega_{\mu_n}) \leqslant \sup\{I(s(u_\varepsilon, v_\varepsilon) + (\phi, -\phi)),\ s \in \mathbb{R}^+,\ \phi \in H_0^1(\Omega_\mu)\}. \tag{2.10}$$

On the hand, since $I'(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) = \mathrm{o}(1)$ as $\varepsilon \to 0$ and also $I'(u_\varepsilon, v_\varepsilon)(\phi_\varepsilon, -\phi_\varepsilon) = \mathrm{o}(1)$ as $\varepsilon \to 0$ for any $\phi_\varepsilon \in H_0^1(\Omega_{\mu_n})$ such that $||\phi_\varepsilon||_{H^1(\Omega_{\mu_n})} \leqslant 1$, it is shown in [16, Theorem 3.5] and [15, Lemma 3.3] that the right-hand side of (2.10) is $I(u_\varepsilon, v_\varepsilon) + \mathrm{o}(1)$, thus also $I(u, v) + \mathrm{o}(1)$ as $\varepsilon \to 0$. By passing to the limit in (2.10) we conclude that

$$\limsup_{n\to\infty} c(\Omega_{\mu_n}) \leqslant I(u, v) = c(\Omega_\mu)$$

This shows that the map $\mu \mapsto c(\Omega_\mu)$ is left-continuous.

6) In steps 3, 4 and 5 we have proved that, as a function of μ, $c(\Omega_\mu)$ is a continuous and locally decreasing function. This implies that $\mu \mapsto c(\Omega_\mu)$ is a decreasing function in $\mathbb{R}^+$. □

Corollary 2.3 *Under assumptions* (H)*, for any smooth bounded set* $\Omega \subset \mathbb{R}^N$,

$$c(\mathbb{R}^N) \leqslant c(\Omega).$$

Proof. We first observe that the system in (2.1) indeed admits a ground-state solution in the case $\Omega = \mathbb{R}^N$; the main point here is that the Palais-Smale condition is satisfied, thanks to the invariance by translation of the domain, see e.g. [17] for details. Now, if $\mu_n \to \infty$ it is easy to see that the ground-state solutions $(u_n, v_n) \in H_0^1(\Omega_{\mu_n}) \times H_0^1(\Omega_{\mu_n})$ of the system will converge in the H^1-norm to a nonzero solution $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of the system (see e.g. [16, Proposition 1.6]). Thus $c(\mathbb{R}^N) \leqslant \lim_{n\to\infty} c(\Omega_{\mu_n})$ and the conclusion follows. (In fact, by using the argument in step 5 of the preceding proof we see that $c(\mathbb{R}^N) = \lim_{\mu\to\infty} c(\Omega_\mu)$.) □

3 Preliminary estimates

In this section we assume that the conditions (H) displayed in the Introduction hold (without loss of generality we may take $p = q \in]2, 2N/(N-2)[$, as mentioned in Section 2), and we prove some basic estimates that will be needed in the proof of Theorem 1.1.

Let $(u_\varepsilon, v_\varepsilon) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be any ground-state solutions for system (1.1) chosen such that their with Morse index is not greater than 1. Then $u_\varepsilon > 0, v_\varepsilon > 0$ and we let $x_\varepsilon \in \Omega$ be such that

$$\max_{\overline{\Omega}} u_\varepsilon = u(x_\varepsilon).$$

We fix a sequence $\varepsilon_j \to 0$ in such a way that $x_j := x_{\varepsilon_j} \to x_0 \in \overline{\Omega}$ and also $y_j \to y_0 \in \partial\Omega$, where $y_j \in \partial\Omega$ are such that

$$d_j := \mathrm{dist}(x_j, \partial\Omega) = |x_j - y_j|. \tag{3.1}$$

Let us recall some basic facts proved in [15, 16]. We denote $u_j := u_{\varepsilon_j}$ and $v_j := v_{\varepsilon_j}$. We have that

$$0 < \liminf_{j\to\infty} \min\{||u_j||_\infty, ||v_j||_\infty\}, \quad \limsup_{j\to\infty} \max(\{||u_j||_\infty, ||v_j||_\infty\} < +\infty$$

and the rescaled solutions

$$\widetilde{u}_j(x) = u_j(\varepsilon_j x + x_j), \quad \widetilde{v}_j(x) = v_j(\varepsilon_j x + x_j), \quad x \in \Omega_j := \frac{1}{\varepsilon_j}(\Omega - x_j), \tag{3.2}$$

which satisfy

$$-\Delta\widetilde{u}_j + \widetilde{u}_j = g(\widetilde{v}_j), \quad -\Delta\widetilde{v}_j + \widetilde{v}_j = f(\widetilde{u}_j) \quad \text{in } \Omega_j, \tag{3.3}$$

converge in H^1 and in C^1_{loc} to a nonzero solution of the limit system

$$-\Delta u + u = g(v), \qquad -\Delta v + v = f(u) \tag{3.4}$$

defined in the open set $\omega = \{x \in \mathbb{R}^N : \langle x, n(y_0)\rangle < d_0\}$, where

$$\rho_0 := \lim_{j\to\infty} \rho_j, \qquad \rho_j := d_j/\varepsilon_j$$

and $n(y_0)$ is the unit outward normal of Ω at y_0. Our first lemma states that in fact $\omega = \mathbb{R}^N$.

Lemma 3.1 *We have that $\rho_j \to \infty$ as $j \to \infty$.*

Proof. Assume by contradiction that d_0 is finite. We argue similarly to [7, Proposition 1.2]. Since $u = v = 0$ on $\partial\omega$, by conservation of energy, i.e. by integrating in ω the identity

$$\mathrm{div}(\langle\nabla v, y\rangle\nabla u + \langle\nabla u, y\rangle\nabla v - \langle\nabla u, \nabla v\rangle y + (G(v) + F(u) - uv)y)) = 0$$

with $y \in \mathbb{R}^N$ arbitrary, we see that

$$\int_{\partial\omega} \frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = 0.$$

This contradicts the fact that, by Hopf lemma, $\partial u/\partial n < 0$ and $\partial v/\partial n < 0$ on $\partial\omega$. □

The following result follows easily from the previous lemma.

Proposition 3.2 *There exists j_0 such that, for every $j \geqslant j_0$, x_j is the unique maximum point of u_j and also of v_j.*

Proof. This follows similarly as in [16, Theorem 2.1] and therefore we omit the proof. We just mention that the conclusion relies on the fact that every positive solutions u, v of the system (3.4) lying in $H^1(\mathbb{R}^N)$ are radially symmetric with respect to some point $z_0 \in \mathbb{R}^N$ (see [4, Theorem 2]). The proof also uses an argument based on the information on the Morse index of the solutions (u_j, v_j), similar to the one in Lemma 3.3 below. □

We now concentrate on the (uniform in j) decay of the solutions.

Lemma 3.3 *For some $q = q(N)$, we have that*

$$u_j(x) + v_j(x) + \varepsilon_j^q |\nabla u_j(x)| + \varepsilon_j^q |\nabla v_j(x)| \leqslant e^{-(1+o(1))\frac{|x-x_j|}{\varepsilon_j}} \quad \forall x \in \overline{\Omega}.$$

Proof. 1) By considering the rescaled solutions $\widetilde{u}_j$, $\widetilde{v}_j$ (cf. (3.2)), we must prove that given $a < 1$ there exist $C > 0$ and $j_0 \in \mathbb{N}$ such that

$$\widetilde{u}_j(x) + \widetilde{v}_j(x) + \varepsilon_j^q |\nabla \widetilde{u}_j(x)| + \varepsilon_j^q |\nabla \widetilde{v}_j(x)| \leqslant C\, e^{-a|x|} \quad \forall j \geqslant j_0 \ \forall x \in \overline{\Omega_j}. \tag{3.5}$$

We first observe that for every $\delta > 0$ there exists $R > 0$ such that, for every large j,

$$\int_{\Omega_j \cap \{|x| \geqslant R\}} (\widetilde{u}_j^2 + \widetilde{v}_j^2) \leqslant \delta. \tag{3.6}$$

This is proved in [16, Proposition 1.6] or [Proposition 3.4]; the key point in the proof consists in using the knowledge that $(\widetilde{u}_j, \widetilde{v}_j)$ converge in the H^1-norm to a *nonzero* function, followed by an argument envolving the information on the Morse index of the solutions (u_j, v_j).

Now, in view of a contradiction let us assume that there exist $z_j \in \Omega_j$, $|z_j| \to \infty$, such that $\widetilde{u}_j(z_j) + |\nabla \widetilde{u}_j(z_j)| \geqslant \rho$ for some $\rho > 0$. Then we let $\bar{x}_j := \varepsilon_j z_j + x_j \in \Omega$ and $\bar{u}_j(x) := u_j(\varepsilon_j x + \bar{x}_j)$, $\bar{v}_j(x) := v_j(\varepsilon_j x + \bar{x}_j)$, that is, $\bar{u}_j(x) = \widetilde{u}_j(x + z_j)$ and $\bar{v}_j(x) = \widetilde{v}_j(x + z_j)$. Since $\bar{u}_j(0) + |\nabla \widetilde{u}_j(0)| \geqslant \rho$, similarly as above we get that

$$\int_{\bar{\Omega}_j \cap \{|x| \geqslant \bar{R}\}} (\bar{u}_j^2 + \bar{v}_j^2) \leqslant \delta, \tag{3.7}$$

for some $\bar{R} > 0$, where $\bar{\Omega}_j = \Omega_j - z_j$. Since $|z_j| \to \infty$, we deduce from (3.6) and (3.7) that, for large j, $\int_{\Omega_j} (\widetilde{u}_j^2 + \widetilde{v}_j^2) \leqslant 2\delta$. Since δ is arbitrary, we conclude that $\int_{\mathbb{R}^N} (u^2 + v^2) = 0$, which is false. This contradiction shows that the rescaled solutions decay at infinity, i.e. for very $\delta > 0$ there exist $R > 0$ and $j_0 \in \mathbb{N}$ such that

$$|\widetilde{u}_j(x)| + |\widetilde{v}_j(x)| + |\nabla \widetilde{u}_j(x)| + |\nabla \widetilde{v}_j(x)| \leqslant \delta, \quad \forall |x| \geqslant R, j \geqslant j_0. \tag{3.8}$$

2) Now, given $0 < a < 1$, let $\Psi(x) = e^{a|x|}$. Then $|\nabla\Psi(x)| = a\Psi(x)$ and $|\Delta\Psi(x)| \leqslant C\Psi(x)$ for every $|x| \geqslant 1$. Since $f(s)/s \to 0$ and $g(s)/s \to 0$ as $s \to 0$, it follows from (3.8) that, for some large $R = R(a)$, both $(\Psi\widetilde{u}_j)$ and $(\Psi\widetilde{v}_j)$ are bounded in $H^1(\Omega_j \backslash B_R(0))$; to prove this we simply multiply the equations in (3.3) by $\widetilde{u}_j\Psi^2\varphi^2$ and $\widetilde{v}_j\Psi^2\varphi^2$ respectively (φ is a fixed cut-off function such that $\varphi = 0$ in $B_R(0)$ and $\varphi = 1$ in $\mathbb{R}^N \backslash B_{2R}(0)$), we integrate by parts and use straightforward computations. Since $R = R(a)$ is fixed, we conclude that

$$(\Psi\widetilde{u}_j) \text{ and } (\Psi\widetilde{v}_j) \text{ are bounded in } H^1(\Omega_j). \tag{3.9}$$

3) Using (3.9) and a standard bootstrap argument, the interior elliptic estimates imply that $(\Psi\widetilde{u}_j)$ and $(\Psi\widetilde{v}_j)$ are bounded in C^1, provided we stay away from the boundary of Ω_j, i.e.

$$\begin{aligned}\widetilde{u}_j(x) + \widetilde{v}_j(x) + |\nabla\widetilde{u}_j(x)| + |\nabla\widetilde{v}_j(x)|&\\ \leqslant Ce^{-a|x|}, \quad \forall x \in \Omega_j : \operatorname{dist}(x, \partial\Omega_j) > 1.&\end{aligned}$$

4) Since $\Psi\widetilde{u}_j$ and $\Psi\widetilde{v}_j$ vanish on the boundary of Ω_j, the local estimates in step 3 can be performed over a ball of fixed radius centered at any point lying in $\partial\Omega_j$. This readily leads to

$$\widetilde{u}_j(x) + \widetilde{v}_j(x) + |\nabla\widetilde{u}_j(x)| + |\nabla\widetilde{v}_j(x)| \leqslant \frac{C}{\varepsilon_j^q}\, e^{-a|x|}, \quad \forall x \in \Omega_j$$

for some $q = q(N)$, in connexion with the bootstrap argument.

5) The above estimate can be improved as far as $\widetilde{u}_j$ and $\widetilde{v}_j$ are concerned. Indeed, we can apply the maximum principle to the equation

$$-\Delta\widetilde{w}_j + \widetilde{w}_j = \frac{f(\widetilde{u}_j) + g(\widetilde{v}_j)}{\widetilde{u}_j + \widetilde{v}_j}\, \widetilde{w}_j, \quad \widetilde{w}_j := \widetilde{u}_j + \widetilde{v}_j$$

to conclude that

$$\widetilde{w}_j \leqslant Ce^{-a|x|} \qquad \forall x \in \Omega_j. \tag{3.10}$$

This completes the proof of the lemma. ☐

Remark. Lemma 3.3 can be improved *a posteriori*, by deleting ε^q from the estimate, since we will prove in Theorem 1.1 that $\liminf \operatorname{dist}(x_\varepsilon, \partial\Omega) > 0$ as $\varepsilon \to 0$.

Our next lemma concerns the reversed inequality in Lemma 3.3.

Lemma 3.4 *We have that*

$$u_j(x) + v_j(x) \geqslant e^{-(1+o(1))\rho_j} \quad \forall x \in \Omega : |x - x_j| \leqslant |x_j - y_j| - \varepsilon_j.$$

Proof. Let us concentrate on u_j. By considering again the rescaled solutions $\widetilde{u}_j$, we must prove that given $a > 1$ there exist $C > 0$ and $j_0 \in \mathbb{N}$ such that

$$\widetilde{u}_j(x) \geqslant C\, e^{-a\rho_j} \qquad \forall j \geqslant j_0 \;\; \forall x : |x| \leqslant \rho_j - 1. \tag{3.11}$$

To that purpose we let $\Psi(x) = \Psi(|x|) = e^{-\sqrt{1+a|x|^2}}$ and

$$w_j(x) = \alpha(\Psi(x) - \Psi(\rho_j)) - \widetilde{u}_j(x), \qquad |x| \leqslant \rho_j,$$

where α is a positive constant to be specified. Fix any $1 < a' < a$. A straightforward computation shows that $-\Delta\Psi + a'\Psi \leqslant 0$ in $\mathbb{R}^N \setminus B_R(0)$ for some large $R = R(a, a')$. Since $-\Delta\widetilde{u}_j + a'\widetilde{u}_j = g(\widetilde{v}_j) + (a' - 1)\widetilde{u}_j \geqslant 0$, we also have that

$$-\Delta w_j + a' w_j \leqslant 0 \qquad \forall x \in \Omega_j : \; |x| \geqslant R.$$

We can choose α in such a way that, for large values of j, $\alpha\Psi \leqslant \widetilde{u}_j$ in $B_R(0)$, since $\widetilde{u}_j$ converges uniformly in $B_R(0)$ to a positive function. In conclusion, we have that

$$-\Delta w_j + a' w_j \leqslant 0 \;\; \forall x : R \leqslant |x| \leqslant \rho_j \;\; \text{and} \;\; w_j \leqslant 0 \;\; \text{if} \;\; |x| \leqslant R \;\; \text{or if} \;\; |x| = \rho_j.$$

By the maximum principle we conclude that $w_j(x) \leqslant 0$ for any $|x| \leqslant \rho_j$. In particular, if $|x| \leqslant \rho_j - 1$,

$$\widetilde{u}_j(x) \geqslant \alpha\, (\Psi(\rho_j - 1) - \Psi(\rho_j)) \geqslant C e^{-\rho_j \sqrt{a}}$$

for some positive constant C. Since $a > 1$ is arbitrary, (3.11) follows. □

Remark. Of course, the same conclusion holds if we replace the assumption $|x - x_j| \leqslant |x_j - y_j| - \varepsilon_j$ by $|x - x_j| \leqslant |x_j - y_j| - c\,\varepsilon_j$ for some constant $c > 0$.

The following is a direct consequence of Lemma 3.4.

Lemma 3.5 *We have that*

$$\int_{\partial\Omega} \left(\left|\frac{\partial u_j}{\partial n}\right| + \left|\frac{\partial v_j}{\partial n}\right|\right) \geqslant \varepsilon_j^{N-2}\; e^{-(1+o(1))\rho_j}.$$

Proof. 1) We restrict ourselves to a fixed small neighborhood of y_0 and estimate $-\frac{\partial u_j}{\partial n}(y_j) = |\frac{\partial u_j}{\partial n}(y_j)|$ (cf. (3.1)). Let

$$z_j := y_j - \varepsilon_j n(y_j) \in \Omega$$

with ε_j so small (i.e. j so large) that $\overline{B}_{\varepsilon_j}(z_j) \cap \partial\Omega = \{y_j\}$. Since $-\varepsilon_j^2 \Delta u_j + u_j \geqslant 0$ it follows from Hopf lemma that

$$\left|\frac{\partial u_j}{\partial n}(y_j)\right| \geqslant \frac{C}{\varepsilon_j}\; \inf\{u_j(x) : x \in \Sigma_j\},$$

where

$$\Sigma_j = \{x \in \mathbb{R}^N : |x - z_j| \leqslant \varepsilon_j \text{ and } |x - y_j| = \varepsilon_j/2\}.$$

Now, since $x_j - y_j = \xi_j(z_j - y_j)$ with $\xi_j \to +\infty$ we see that there exists some small $c > 0$ such that

$$x \in \Sigma_j \Rightarrow |x - x_j| \leqslant |x_j - y_j| - c\varepsilon_j. \tag{3.12}$$

2) We can proceed similarly for any given point $\overline{y}_j \in \partial\Omega$ such that $|y_j - \overline{y}_j| \leqslant \nu\varepsilon_j$; now we replace Σ_j by a corresponding set $\overline{\Sigma}_j$ with $\overline{y}_j$ in the place of y_j and $\overline{z}_j := \overline{y}_j - \varepsilon_j n(\overline{y}_j)$ in the place of z_j. Since the normal derivative is assumed to be locally Lipschitz continuous, if ν is sufficiently small we have that $|x - z_j| \leqslant c_1\, \varepsilon_j$ and $|x - y_j| \geqslant c_2\, \varepsilon_j$ for some $c_1 > c_2 > 0$ and every $x \in \overline{\Sigma}_j$, and so (3.12) still holds for every such x (by taking a smaller constant c).

3) In conclusion, there exist positive constants ν, c, C such that, for every $y \in \partial\Omega \cap B_{\nu\varepsilon_j}(y_j)$,

$$|\frac{\partial u_j}{\partial n}(y)| \geqslant \frac{C}{\varepsilon_j} \inf\{u_j(x) : |x - x_j| \leqslant |x_j - y_j| - c\varepsilon_j\} \tag{3.13}$$

and similarly for $|\frac{\partial v_j}{\partial n}|$. By taking Lemma 3.4 into account, this implies that

$$|\frac{\partial u_j}{\partial n}(y)| + |\frac{\partial v_j}{\partial n}(y)| \geqslant \frac{C}{\varepsilon_j}\, e^{-(1+o(1))\rho_j}, \quad \forall y \in \partial\Omega \cap B_{\nu\varepsilon_j}(y_j). \tag{3.14}$$

The conclusion follows immediately. □

Our next goal is to extend the arguments in Proposition 2.2 to the present situation. For each j we let

$$\lambda_j(y) = \frac{\partial v_j/\partial n}{\partial u_j/\partial n}(y) \quad \text{ if } y \in \partial\Omega \text{ and } |y - y_j| \leqslant \nu\varepsilon_j \tag{3.15}$$

(cf. (3.13)) and we take, say, $\lambda_j(y) = 1$ if $|y - y_j| \geqslant 2\nu\varepsilon_j$; then $\lambda_j(y)$ can be extended to the whole boundary of Ω in a smooth way.

Lemma 3.6 *There exists $q \in \mathbb{N}$ such that, for any $\delta > 0$, we can find $j_0 \in \mathbb{N}$ such that*

$$\varepsilon_j^q e^{-\delta\rho_j} \leqslant \lambda_j(x) \leqslant \frac{e^{\delta\rho_j}}{\varepsilon_j^q} \quad \textit{and} \quad |D\lambda_j(x)| \leqslant \frac{e^{\delta\rho_j}}{\varepsilon_j^q} \quad \forall j \geqslant j_0, x \in \partial\Omega.$$

Proof. It follows from (3.14) and Lemma 3.3 that

$$u_j(x) + v_j(x) + \varepsilon^q|\nabla u_j(x)| + \varepsilon^q|\nabla v_j(x)| \leqslant e^{-(1-\delta)\rho_j} \quad \forall x \in B_j := B_{\nu\varepsilon_j}(y_j) \cap \overline{\Omega},$$

and that $\varepsilon_j|\frac{\partial u_j}{\partial n}(x)|+\varepsilon_j|\frac{\partial v_j}{\partial n}(x)|\geqslant C\,e^{-(1+\delta)\rho_j}$ for every $x\in B_j\cap\partial\Omega$. This implies that both $\lambda_j(x)$ and $1/\lambda_j(x)$ are bounded by $e^{2\delta\rho_j}/\varepsilon_j^q$ and that, for every $x\in B_j\cap\partial\Omega$,

$$|D\lambda_j(x)|\leqslant(|D^2u_j(x)|+|D^2v_j(x)|)\,\frac{e^{(1+3\delta)\rho_j}}{\varepsilon_j^{q-1}}.$$

On the other hand, since $-\Delta u_j=\frac{1}{\varepsilon_j^2}(g(v_j)-u_j)$, it follows from Schauder's estimates (see e.g. [8, Lemma 6.5]) that, for a fixed $0<\alpha<1$,

$$\varepsilon^2|D^2u_j(x)|\leqslant C(||u_j||_{L^\infty(B_j)}+\frac{1}{\varepsilon_j^2}(||u_j||_{C^{0,\alpha}(B_j)}+||g(v_j)||_{C^{0,\alpha}(B_j)})).$$

This implies that $\varepsilon_j^{4+q}|D^2u_j(x)|\leqslant Ce^{-(1-\delta)\rho_j}$, and similarly for v_j. In conclusion,

$$|D\lambda_j(x)|\leqslant\frac{C}{\varepsilon_j^{2q+3}}\,e^{4\delta\rho_j}.$$

By changing notations, this completes the proof. □

We are now in position to define a superharmonic extension of λ_j to a suitable neighborhood of Ω.

Proposition 3.7 *Let δ be any small positive constant. Then, for any large j and for some fixed constants $q\in\mathbb{N}$, $c>0$, we can find a smooth bounded open set $\Omega'\subset\mathbb{R}^N$ and a function $\lambda_j\in C(\Omega')\cap W^{1,\infty}(\Omega')$ with the following properties:*

(a) $\Omega\cup B_{\ell_j^2}(y_j)\subset\Omega'$, $\quad\ell_j:=c\varepsilon_j^q e^{-\delta\rho_j}$;

(b) $\lambda_j(x)=\dfrac{\partial v_j/\partial n}{\partial u_j/\partial n}(x)$ *for every* $x\in\partial\Omega\setminus\partial\Omega'$;

(c) $-\Delta\lambda_j\geqslant 0$ *in* Ω';

(d) $\ell_j\leqslant\lambda_j(x)\leqslant\ell_j^{-1}$ *and* $|\nabla\lambda_j(x)|\leqslant\ell_j^{-1}$ *for every* $x\in\Omega'$.

Proof. Let z_1 and z_2 be as in (2.3) and (2.4), except that we replace the value $\frac{\partial v/\partial n}{\partial u/\partial n}$ at the boundary of Ω by the above function λ_j. Then $z_1\in C^2(\Omega)\cap C^1(\overline{\Omega})$ and $z_2\in C^2(\Omega'\setminus\overline{\Omega})\cap C^1(\overline{\Omega'}\setminus\Omega)$. Moreover, by classical estimates for harmonic functions (cf. e.g. [10, Proposition 2.18]) together with Lemma 3.6 yield that $|Dz_1(x)|$ and $|Dz_2(x)|$ are bounded by $e^{\delta\rho_j}/\varepsilon_j^q$.

Next, let $\lambda_j(x)$ be defined similarly to (2.5), that is,

$$\lambda_j(x)=z_1(x)\quad\text{if } x\in\Omega,\qquad \lambda_j(x)=z_2(x)+Mh(x)\quad\text{if } x\in\Omega'\setminus\Omega,\qquad(3.16)$$

where $M>0$ is large and $h(x)$ is a fixed suitable superharmonic function in Ω'. As observed in the proof of Proposition 2.2, if $M=ce^{\delta\rho_j}\varepsilon_j^q$ then $\lambda_j(x)$ as given by (3.16) is superharmonic in Ω'. Moreover, by taking Lemma 3.6 into account, we have that

$$\lambda_j(x) + |\nabla\lambda_j(x)| \leqslant e^{\delta\rho_j}\varepsilon_j^{-q} \qquad \forall x \in \Omega'.$$

As a final step, we restrict further the neighborhood Ω'; namely, we use a smooth change of coordinates around y_j in order to construct a smooth bounded open set (still denoted by Ω') in such a way that

$$\Omega \cup B_{\nu\varepsilon_j^{2q}e^{-2\delta\rho_j^2}}(y_j) \subset \Omega' \subset \Omega \cup B_{2\nu\varepsilon_j^{2q}e^{-2\delta\rho_j^2}}(y_j)$$

with a small $\nu > 0$. In this way, property (b) follows from (3.15), while $|h(x)| = |h(x) - h(y_j)| \leqslant C\nu\varepsilon_j^{2q}e^{-2\delta\rho_j^2}$ over $\Omega' \setminus \Omega$, yielding that

$$z_2(x) \geqslant (1 - \nu cC)\,\varepsilon_j^q e^{-\delta\rho_j} \geqslant \frac{1}{2}\,\varepsilon_j^q e^{-\delta\rho_j} \quad \forall x \in \Omega' \setminus \Omega.$$

4 Proof of the main result

In this section we prove Theorem 1.1. As in the preceding section, we assume that (H) holds (with $q = p \in]2, 2N/N-2[$). We will work with the rescaled solutions (3.2). For any small $\delta > 0$ we apply Proposition 3.7 except that now we work with the rescaled functions and sets:

$$\widetilde{\lambda}_j(x) = \lambda_j(\varepsilon_j x + x_j), \qquad x \in \Omega'_j := \frac{1}{\varepsilon_j}(\Omega' - x_j).$$

Of course, for $\widetilde{\lambda}_j$ we have similar properties (a)–(d) as in Proposition 3.7. In fact, we should replace q by $q+1$ but in order to avoid cumbersome notations we introduce the notation $\varepsilon_j^{\pm \mathrm{O}(1)}$ to denote $\varepsilon_j^{\pm q}$ where $q \in \mathbb{N}$ may change from place to place but lies inside a fixed range $1 \leqslant q \leqslant q_0$ for some $q_0 \in \mathbb{N}$. Similarly, in this secton the notation $e^{\pm \mathrm{o}(1)\rho_j}$ stands for $e^{\pm\delta\rho_j}$ where δ may change from place to place but lies inside a fixed small interval $]0, \delta_0[$.

Keeping these notations in mind and by taking Proposition 3.7 (a) into account, let us fix a smooth cut-off function χ_j, $0 \leqslant \chi_j \leqslant 1$, such that

$$\chi_j = 1 \text{ in } B_{\frac{1}{2}e^{-\mathrm{o}(1)\rho_j}\varepsilon_j^{\mathrm{O}(1)}}\Big(\frac{y_j - x_j}{\varepsilon_j}\Big), \quad \chi_j = 0 \text{ in } \mathbb{R}^N \setminus B_j, \quad B_j := B_{e^{-\mathrm{o}(1)\rho_j}\varepsilon_j^{\mathrm{O}(1)}},$$

so that $|\nabla\chi_j(x)| \leqslant c e^{\mathrm{o}(1)\rho_j}\varepsilon_j^{-\mathrm{O}(1)}$. We also let

$$w_j(x) = \mu_j\,\chi_j(x), \tag{4.1}$$

where μ_j is a small constant to be defined later (see (4.5)). We anticipate that

$$0 \leqslant \widetilde{\lambda}_j(x) w_j(x) \leqslant 1 \quad \forall x \in B_j \qquad \text{and} \qquad ||w_j|| \leqslant 1. \tag{4.2}$$

Here $||u||^2 := \int_{\Omega'_j}(|\nabla u|^2 + u^2)$ for any $u \in H^1_0(\Omega'_j)$. We define

$$\overline{u}_j(x) = \widetilde{u}_j(x) + w_j(x), \qquad \overline{v}_j(x) = \widetilde{v}_j(x) + \widetilde{\lambda}_j(x) w_j(x), \qquad x \in \Omega'_j.$$

We denote by I_j the energy functional associated to the system (3.3) over the space $H^1_0(\Omega'_j) \times H^1_0(\Omega'_j)$, that is,

$$I_j(u,v) = \langle u, v\rangle - \int_{\Omega'_j} (F(u) + G(v)),$$

where $\langle u, v\rangle := \int_{\Omega'_j}(\langle \nabla u, \nabla v\rangle + uv)$. Our next lemma summarizes the content of Section 3.

Lemma 4.1 *The positive constant μ_j in* (4.1) *can be chosen in such a way that*

$$I_j(\overline{u}_j, \overline{v}_j) \leqslant I_j(\widetilde{u}_j, \widetilde{v}_j) - e^{-2(1+\mathrm{o}(1))\rho_j} \varepsilon_j^{\mathrm{O}(1)}.$$

Proof. Since $\overline{u}_j(x) = \widetilde{u}_j(x)$ and $\overline{v}_j(x) = \widetilde{v}_j(x)$ if $|x| \leqslant \rho_j - 1$, we have that

$$\begin{aligned} I_j(\overline{u}_j, \overline{v}_j) - I_j(\widetilde{u}_j, \widetilde{v}_j) = {} & langle e_j, \widetilde{\lambda}_j w_j\rangle + \int_{\partial\Omega \cap B_j} \left(\frac{\partial \widetilde{u}_j}{\partial n}\widetilde{\lambda}_j + \frac{\partial \widetilde{v}_j}{\partial n}\right) w_j \\ & + \int_{\{|x| \geqslant \rho_j - 1\}} (F(\widetilde{u}_j) - F(\widetilde{u}_j + w_j)) \\ & + \int_{\{|x| \geqslant \rho_j - 1\}} (G(\widetilde{v}_j) - G(\widetilde{v}_j + \widetilde{\lambda}_j w_j)) \\ & + \int_{\Omega'_j} f(\widetilde{u}_j) w_j + \int_{\Omega'_j} g(\widetilde{v}_j)\widetilde{\lambda}_j w_j. \end{aligned}$$

According to Lemma 3.5 (cf. (3.14)) and to Proposition 3.7 (b), at the boundary $\partial\Omega \cap B_j$ we have that

$$\frac{\partial \widetilde{u}_j}{\partial n}\widetilde{\lambda}_j + \frac{\partial \widetilde{v}_j}{\partial n} = -2\,|\frac{\partial \widetilde{v}_j}{\partial n}| \leqslant -2e^{(1+\mathrm{o}(1))\rho_j}. \tag{4.3}$$

The other terms in the expression of $I_j(\overline{u}_j, \overline{v}_j) - I_j(\widetilde{u}_j, \widetilde{v}_j)$ are of higher order with respect to (4.3). Indeed, we know that $||\widetilde{u}_j||_\infty \leqslant C$ and moreover (cf. (3.10))

$$|\widetilde{u}_j(x)| = \widetilde{u}_j(x) \leqslant e^{-(1-\mathrm{o}(1))\rho_j} \qquad \forall x \in B_j.$$

Since f is superlinear near zero (cf. (H), namely (1.3)) we see that, for some $\nu > 1$,

$$|f(\widetilde{u}_j) w_j| = f(\widetilde{u}_j) w_j \leqslant C e^{-(1-\mathrm{o}(1))\nu\rho_j} w_j.$$

Also $\widetilde{\lambda}_j(x) w_j(x)$ is uniformly bounded (cf. (4.2)) and so the other integral terms can be handled similarly, while $\langle w_j, \widetilde{\lambda}_j w_j\rangle \leqslant ||w_j||^2 e^{\mathrm{o}(1)\rho_j} \varepsilon_j^{-\mathrm{O}(1)}$.

Summing up, we conclude that

$$I_j(\overline{u}_j,\overline{v}_j)-I_j(\widetilde{u}_j,\widetilde{v}_j)\leqslant C\frac{e^{\mathrm{o}(1)\rho_j}}{\varepsilon_j^{\mathrm{O}(1)}}\,||w_j||^2-2\mu_j e^{-(1+\mathrm{o}(1))\rho_j}\int_{\partial\Omega\cap B_j}\chi_j$$
$$+\,C\mu_j e^{-(1-\mathrm{o}(1))\nu\rho_j}\frac{e^{\mathrm{o}(1)\rho_j}}{\varepsilon_j^{\mathrm{O}(1)}}\int_{B_j}\chi_j.$$

Since $\nu>1$, the quantity $e^{-(1-\mathrm{o}(1))\nu\rho_j}$ is dominated by $e^{-(1+\mathrm{o}(1))\rho_j}$. On the other hand, by construction

$$\varepsilon_j^{-\mathrm{O}(1)}\,|B_j|_N\leqslant|\partial\Omega\cap B_j|_{N-1}$$

(here $|\cdot|$ denotes N-dimensional Lebesgue measure), and so

$$I_j(\overline{u}_j,\overline{v}_j)-I_j(\widetilde{u}_j,\widetilde{v}_j)\leqslant c_1\mu_j^2-c_2\mu_j, \tag{4.4}$$

where

$$c_1=e^{\mathrm{o}(1)\rho_j}\,\varepsilon_j^{-\mathrm{O}(1)}\quad\text{ and }\quad c_2=e^{-(1+\mathrm{o}(1))\rho_j}\,\varepsilon_j^{\mathrm{O}(1)}.$$

Then we let

$$\mu_j=\frac{c_2}{2c_1}=\frac{1}{2}e^{-(1+\mathrm{o}(1))\rho_j}\varepsilon_j^{\mathrm{O}(1)}, \tag{4.5}$$

so that

$$I_j(\overline{u}_j,\overline{v}_j)-I_j(\widetilde{u}_j,\widetilde{v}_j)\leqslant-\frac{c_2^2}{4c_1}=-\frac{1}{4}e^{-2(1+\mathrm{o}(1))\rho_j}\varepsilon_j^{\mathrm{O}(1)}, \tag{4.6}$$

and this proves the lemma. □

A further estimate is required. As explained in Section 2, according to Benci-Rabinowitz's linking theorem, we know that the ground-state critical level of the energy over $H_0^1(\Omega_j')\times H_0^1(\Omega_j')$, call it $c_j(\Omega_j')$, is bounded above by

$$c_j(\Omega_j')\leqslant\sup\{I_j(s(\overline{u}_j,\overline{v}_j)+(\phi,-\widetilde{\lambda}_j\phi)):s\geqslant0,\ \phi\in H_0^1(\Omega_j')\}. \tag{4.7}$$

Let the right-hand member of (4.7) be assumed at

$$I_j(s_j(\overline{u}_j,\overline{v}_j)+t_j(\phi_j,-\widetilde{\lambda}_j\phi_j)),\ s_j\geqslant0,\ t_j\geqslant0,\int_{\Omega_j'}\widetilde{\lambda}_j^2(\phi_j^2+|\nabla\phi_j|^2)=1.$$

Lemma 4.2 *We have that*

$$I_j(s_j(\overline{u}_j,\overline{v}_j)+t_j(\phi_j,-\widetilde{\lambda}_j\phi_j))\leqslant I_j(\overline{u}_j,\overline{v}_j)+e^{\mathrm{o}(1)\rho_j}\varepsilon_j^{-\mathrm{O}(1)}||w_j||^2. \tag{4.8}$$

The proof of Lemma 4.2 is somewhat technical and therefore we give it in the Appendix.

Proof of Theorem 1.1 completed. The point in the inequality (4.8) is that the error term is of order $||w_j||^2$. Thus we can argue as in (4.4)-(4.6) in order to conclude that

$$I_j(s_j(\overline{u}_j,\overline{v}_j)+t_j(\phi_j,-\widetilde{\lambda}_j\phi_j))\leqslant I_j(\widetilde{u}_j,\widetilde{v}_j)-e^{-2(1+\mathrm{o}(1))\rho_j}\varepsilon_j^{\mathrm{O}(1)}. \tag{4.9}$$

Combining Corollary 2.3 with (4.7) and (4.9) yields that

$$c(\mathbb{R}^N)\leqslant I_j(\widetilde{u}_j,\widetilde{v}_j)-e^{-2(1+\mathrm{o}(1))\rho_j}\varepsilon_j^{\mathrm{O}(1)}. \tag{4.10}$$

Next, concerning (4.10), we observe that the reversed inequality is much simpler to prove. Namely, fix any point $P\in\Omega$ and let u,v be as in (3.4) except that now we translate them in such a way that $u(P)=\max_{\mathbb{R}^N}u$ and $v(P)=\max_{\mathbb{R}^N}v$. Fix any cut-off function χ such that $\chi=1$ in a neighborhood of P. Then we let

$$u_j^*(x)=u(\varepsilon_jx)\chi(\varepsilon_jx),\quad v_j^*(x)=v(\varepsilon_jx)\chi(\varepsilon_jx),\quad x\in\Omega_j^*:=\frac{1}{\varepsilon_j}(\Omega-P).$$

We conclude as before that the ground-state critical level $c_j(\Omega_j^*)$ of the energy over $H_0^1(\Omega_j^*)\times H_0^1(\Omega_j^*)$ is bounded above by

$$\begin{aligned}c_j(\Omega_j^*)&\leqslant\sup\{I_j(s(u_j^*,v_j^*)+(\phi,-\phi)):s\geqslant0,\ \phi\in H_0^1(\Omega_j^*)\}\\&\leqslant I_j(u_j^*,v_j^*)+e^{-2(1+(1-\mathrm{o}(1))\mathrm{dist}(P,\partial\Omega)/\varepsilon_j}.\end{aligned}$$

Using a change of variables we see that $c_j(\Omega_j^*)=c_j(\Omega_j)=I_j(\widetilde{u}_j,\widetilde{v}_j)$, and also that

$$I_j(u_j^*,v_j^*)\leqslant I_\infty(u,v)+e^{-2(1+(1-\mathrm{o}(1))\mathrm{dist}(P,\partial\Omega)/\varepsilon_j}.$$

(Here, I_∞ stands for the energy functioal defined over $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.) Thus we conclude that

$$I_j(\widetilde{u}_j,\widetilde{v}_j)\leqslant I_\infty(u,v)+e^{-2(1+(1-\mathrm{o}(1))\mathrm{dist}(P,\partial\Omega)/\varepsilon_j}. \tag{4.11}$$

Actually, we can replace u,v by any solutions of the system (3.4) lying in $H^1(\mathbb{R}^N)$, and so (4.11) shows in particular that $I_\infty(u,v)=c(\mathbb{R}^N)$. Combining (4.10) and (4.11) yields that

$$\varepsilon_j^{\mathrm{O}(1)}e^{-2(1+\mathrm{o}(1))\rho_j}\leqslant e^{-2(1-\mathrm{o}(1))\,\mathrm{dist}(P,\partial\Omega)/\varepsilon_j}\qquad\forall P\in\Omega.$$

This implies that $\liminf\mathrm{dist}(x_j,\partial\Omega)\geqslant\mathrm{dist}(P,\partial\Omega)$ for any $P\in\Omega$. Taking also Proposition 3.2 into account, this concludes the proof of Theorem 1.1. □

Appendix

Here we prove Lemma 4.2. For the sake of clarity we divide the proof into five steps.

Step 1. We show that

$$\nu_j := |I'_j(\overline{u}_j, \overline{v}_j)(\overline{u}_j, \overline{v}_j)| + |I'_j(\overline{u}_j, \overline{v}_j)(\phi_j, -\widetilde{\lambda}_j \phi_j)| \leqslant e^{\mathrm{o}(1)\rho_j} \varepsilon_j^{-\mathrm{O}(1)} ||w_j||. \tag{4.12}$$

Indeed, since $I'_j(\widetilde{u}_j, \widetilde{v}_j)(\widetilde{u}_j, \widetilde{v}_j) = 0$, we have that

$$\begin{aligned} I'_j(\overline{u}_j, \overline{v}_j)(\overline{u}_j, \overline{v}_j) &= 2\,\langle w_j, \widetilde{\lambda}_j w_j \rangle \\ &+ 2 \int_{\{|x| \geqslant \rho_j - 1\}} (\langle \nabla \widetilde{u}_j, \nabla(\widetilde{\lambda}_j w_j)\rangle + \langle \nabla \widetilde{v}_j, \nabla w_j \rangle + \widetilde{\lambda}_j \widetilde{u}_j w_j + \widetilde{v}_j w_j) \\ &+ \int_{\{|x| \geqslant \rho_j - 1\}} (f(\widetilde{u}_j)\widetilde{u}_j - f(\widetilde{u}_j + w_j)(\widetilde{u}_j + w_j)) \\ &+ \int_{\{|x| \geqslant \rho_j - 1\}} (g(\widetilde{v}_j)\widetilde{v}_j - g(\widetilde{v}_j + \widetilde{\lambda}_j w_j)(\widetilde{v}_j + \widetilde{\lambda}_j w_j)) \end{aligned}$$

Recalling that, for every j, x, $|\widetilde{v}_j(x)| + |\widetilde{\lambda}_j(x) w_j(x)| \leqslant C$, that $||w_j|| \leqslant 1$ (cf. (4.2)) and that $|g(s)| \leqslant C|s|$ for bounded $s \in \mathbb{R}$, we see that the last integral term is bounded by $C||\widetilde{\lambda}_j||_\infty ||w_j||$. Arguing similarly with the remaining terms and by taking Proposition 3.7 (d) into account yields the first estimate in (4.12). The second estimate in (4.12) is handled similarly. The main point now is to observe that, according to Proposition 3.7 (b),

$$\int_{\partial\Omega_j} (\frac{\partial \widetilde{v}_j}{\partial n} - \widetilde{\lambda}_j \frac{\partial \widetilde{u}_j}{\partial n})\, \phi_j = \int_{\partial\Omega_j \setminus \partial\Omega'_j} (\frac{\partial \widetilde{v}_j}{\partial n} - \widetilde{\lambda}_j \frac{\partial \widetilde{u}_j}{\partial n})\, \phi_j = 0;$$

this shows that

$$\begin{aligned} I'_j(\overline{u}_j, \overline{v}_j)(\phi_j, -\widetilde{\lambda}_j \phi_j) &= \int_{\Omega'_j} (w_j \langle \nabla \widetilde{\lambda}_j, \nabla \phi_j \rangle - \phi_j \langle \nabla w_j, \nabla \widetilde{\lambda}_j \rangle) \\ &+ \int_{\{|x| \geqslant \rho_j - 1\}} (f(\widetilde{u}_j) - f(\widetilde{u}_j + w_j))\phi_j \\ &+ \int_{\{|x| \geqslant \rho_j - 1\}} (g(\widetilde{v}_j + \widetilde{\lambda}_j w_j) - g(\widetilde{v}_j))\widetilde{\lambda}_j \phi_j \end{aligned}$$

with $||\phi_j|| \leqslant (\min \widetilde{\lambda}_j)^{-1/2}$, and this proves (4.12). We also observe that (4.1), (4.5) and (4.12) imply that

$$\nu_j \leqslant e^{-(1-\mathrm{o}(1))\rho_j}\, \varepsilon_j^{\mathrm{O}(1)}. \tag{4.13}$$

Step 2. Since

$$I_j(s_j(\overline{u}_j,\overline{v}_j)+t_j(\phi_j,-\widetilde{\lambda}_j\phi_j)) = I(\overline{u}_j,\overline{v}_j)+(s_j-1)I_j'(\overline{u}_j,\overline{v}_j)(\overline{u}_j,\overline{v}_j) \\ + t_jI_j'(\overline{u}_j,\overline{v}_j)(\phi_j,-\widetilde{\lambda}_j\phi_j)+\mathrm{O}((s_j-1)^2+t_j^2),$$

using (4.12) we see that the proof of Lemma 4.2 will be complete once we show that $s_j \to 1$, $t_j \to 0$ and

$$t_j + |s_j-1| \leqslant \nu_j\, e^{\mathrm{o}(1)\rho_j}\, \varepsilon_j^{-\mathrm{O}(1)}. \tag{4.14}$$

Step 3. Let us prove first that $s_j \to 1$ whenever, for some subsequence $j \to \infty$,

$$t_j \leqslant |s_j-1|\, e^{\mathrm{o}(1)\rho_j}\, \varepsilon_j^{-\mathrm{O}(1)}. \tag{4.15}$$

In this case, we have that

$$||\psi_j|| \leqslant e^{\mathrm{o}(1)\rho_j}\, \varepsilon_j^{-\mathrm{O}(1)}, \quad \text{with } \psi_j := \frac{t_j}{s_j-1}\,\phi_j,$$

and so we introduce the real function of one variable

$$\alpha_j(s) := I_j(s(\overline{u}_j,\overline{v}_j)+(1-s)(\psi_j,-\,\widetilde{lambda}_j\psi_j)).$$

A straightforward computation shows that

$$\alpha_j''(s) = I_j'(\overline{u}_j,\overline{v}_j)(\overline{u}_j,\overline{v}_j) - 2I_j'(\overline{u}_j,\overline{v}_j)(\psi_j,-\widetilde{\lambda}_j\psi_j) - 2\langle\psi_j,\widetilde{\lambda}_j\psi_j\rangle \\ + \int_{\Omega_j'} f(\overline{u}_j)\overline{u}_j + 2f(\overline{u}_j)\psi_j - f'(s\overline{u}_j+(1-s)\psi_j)(\overline{u}_j-\psi_j)^2 \\ + \int_{\Omega_j'} g(\overline{v}_j)\overline{v}_j - 2g(\overline{v}_j)\widetilde{\lambda}_j\psi_j - g'(s\overline{v}_j-(1-s)\widetilde{\lambda}_j\psi_j)(\overline{v}_j+\widetilde{\lambda}_j\psi_j)^2.$$

We have from Proposition 3.7 (c) that $\langle\psi_j,\widetilde{\lambda}_j\psi_j\rangle \geqslant 0$. Then, similarly to [16, Eqn. (3.3)], our assumption (H) (namely (1.3) and (1.4)) implies that

$$\alpha_j''(1) \leqslant A\nu_j e^{\mathrm{o}(1)\rho_j}\varepsilon_j^{-\mathrm{O}(1)} - B\int_{\Omega_j'}(f(\overline{u}_j)\overline{u}_j+g(\overline{v}_j)\overline{v}_j)$$

for some $A, B > 0$. Thus, according to (4.13),

$$\alpha_j''(1) \leqslant e^{-(1-\mathrm{o}(1))\rho_j}\varepsilon_j^{\mathrm{O}(1)} - B\int_{\Omega_j'}(f(\overline{u}_j)\overline{u}_j+g(\overline{v}_j)\overline{v}_j).$$

Since $\int_{\Omega_j'}(f(\overline{u}_j)\overline{u}_j+g(\overline{v}_j)\overline{v}_j) \to \int_{\mathbb{R}^N}(f(u)u+g(v)v) > 0$ we conclude that

$$\alpha_j''(1) < 0 \quad \text{for large } j.$$

More generally, we can assert that, for some $c > 0$,

$$\alpha_j''(s) \leqslant -c < 0 \qquad \forall s : |s-1| \leqslant c_j := e^{-o(1)\rho_j} \varepsilon_j^{O(1)}. \tag{4.16}$$

The main point here is to observe that, say,

$$|f(s\overline{u}_j + (1-s)\psi_j)(s\overline{u}_j + (1-s)\psi_j) - f(\overline{u}_j)\overline{u}_j| \leqslant C|s-1|(|\overline{u}_j|^2 + |\phi_j|^2 + |\overline{u}_j|^p + |\phi_j|^p),$$

since $|f'(s)| \leqslant C(1+|s|^{p-2})$ for every $s \in \mathbb{R}$ (and some $p > 2$).

Next we observe that $|\alpha_j'(1)| \leqslant \nu_j e^{o(1)\rho_j} \varepsilon_j^{-O(1)}$ and so, thanks to (4.13),

$$|\alpha_j'(1)| \leqslant \frac{c}{4} c_j \quad \text{for large } j. \tag{4.17}$$

Then (4.16) and (4.17) imply that

$$\alpha_j(1 \pm c_j) \leqslant \alpha_j(1) - \frac{c}{4} c_j^2. \tag{4.18}$$

At this point we can apply the argument in [16], namely Theorems 3.1 and 3.5; there, a further careful analysis of the graph of the functions α_j and α_j' (by means of an auxiliary function introduced in Eqns. (3.4) and (3.13) of [16]) leads to the conclusion that (4.18) implies that $|s_j - 1| \leqslant c_j$.

Summarizing, we have shown that whenever $t_j + |s_j - 1| \leqslant \nu_j e^{\delta \rho_j} \varepsilon_j^{-q}$ for some $\delta > 0$, $q \in \mathbb{N}$ then

$$|s_j - 1| \leqslant e^{-\delta' \rho_j} \varepsilon_j^{q'} \to 0$$

for some $\delta' > 0$, $q' \in \mathbb{N}$. Moreover, since $\alpha_j''(s) \leqslant -c < 0$ for every s between s_j and 1 and since $\alpha_j'(s_j) = 0$ and $|\alpha_j'(1)| \leqslant \nu_j e^{o(1)\rho_j} \varepsilon_j^{-O(1)}$, then

$$|s_j - 1| \leqslant \nu_j e^{o(1)\rho_j} \varepsilon_j^{-O(1)}. \tag{4.19}$$

Step 4. Our aim now is to show that the sequence (s_j) is bounded. Our starting point is the observation that, by definition,

$$\begin{aligned}
0 &\leqslant I_j(s_j(\overline{u}_j, \overline{v}_j) + t_j(\phi_j, -\widetilde{\lambda}_j \phi_j)) \\
&\leqslant t_j^2 \langle \phi_j, -\widetilde{\lambda}_j \phi_j \rangle + s_j^2 \langle \overline{u}_j, \overline{v}_j \rangle + s_j t_j (\langle \phi_j, \overline{v}_j \rangle + \langle \widetilde{\lambda}_j \phi_j, \overline{u}_j \rangle) \\
&\leqslant t_j^2 (-1 + C(\frac{s_j}{t_j})^2 + \frac{s_j}{t_j} e^{o(1)\rho_j} \varepsilon_j^{-O(1)}).
\end{aligned} \tag{4.20}$$

So, suppose at first that

$$|s_j - 1| \leqslant \frac{1}{4} t_j e^{-o(1)\rho_j} \varepsilon_j^{O(1)}. \tag{4.21}$$

In case $t_j e^{-o(1)\rho_j} \varepsilon_j^{O(1)} \to \infty$ then also $t_j \to \infty$ and, by (4.21) and for large j, $|s_j| \leqslant \frac{1}{2} t_j e^{-o(1)\rho_j} \varepsilon_j^{O(1)}$, contradicting (4.20). We conclude that the right-hand member of (4.21) must be bounded and thus so is (s_j).

Now suppose that (4.21) holds with the reversed inequality. Then we are in the situation of step 3 and we have proved that, in that case, $s_j \to 1$. Summarizing, in any case (s_j) is a bounded sequence.

Step 5. Once we know that (s_j) is bounded, the rest of the proof follows much as in [15, Lemma 3.3, steps 2 and 3] and so we only sketch the argument. By denoting

$$\theta_j(t) = I_j(s_j(\overline{u}_j, \overline{v}_j) + t(\phi_j, -\widetilde{\lambda}_j \phi_j)),$$

we see that $\theta_j'(t_j) = 0$, $\theta_j''(t) \leqslant -2$ for every t and

$$|\theta_j'(t_j)| \leqslant s_j \nu_j + |s_j - 1| \, ||\phi_j|| \leqslant s_j \nu_j + |s_j - 1| e^{\mathrm{o}(1)\rho_j} \varepsilon_j^{-\mathrm{O}(1)}.$$

As a consequence,

$$t_j \leqslant s_j \nu_j + |s_j - 1| e^{\mathrm{o}(1)\rho_j} \varepsilon_j^{-\mathrm{O}(1)}. \tag{4.22}$$

From (4.22) our final conclusion (4.14) follows easily. Indeed, (4.14) follows immediately from (4.22) in case $|s_j - 1| \leqslant \nu_j e^{\mathrm{o}(1)\rho_j} \varepsilon_j^{-\mathrm{O}(1)}$. While in case $|s_j - 1| \geqslant \nu_j e^{\mathrm{o}(1)\rho_j} \varepsilon_j^{-\mathrm{O}(1)}$ then (4.22) implies that we are in the situation of step 3 (namely, that (4.15) holds); in this case (4.19) holds and again we obtain the estimate (4.14). As was observed in step 2, this concludes the proof of Lemma 4.2.

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