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Weil zeta functions of group representations over finite fields

Ged Corob Cook[1] · Steffen Kionke[2] · Matteo Vannacci[3]

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Abstract

In this article we define and study a zeta function ζ_G—similar to the Hasse-Weil zeta function—which enumerates absolutely irreducible representations over finite fields of a (profinite) group G. This Weil representation zeta function converges on a complex half-plane for all UBERG groups and admits an Euler product decomposition. Our motivation for this investigation is the observation that the reciprocal value $\zeta_G(k)^{-1}$ at a sufficiently large integer k coincides with the probability that k random elements generate the completed group ring of G. The explicit formulas obtained so far suggest that ζ_G is rather well-behaved. A central object of this article is the Weil abscissa, i.e., the abscissa of convergence $a(G)$ of ζ_G. We calculate the Weil abscissae for free abelian, free abelian pro-p, free pro-p, free pronilpotent and free prosoluble groups. More generally, we obtain bounds (and sometimes explicit values) for the Weil abscissae of free pro-$\mathfrak{C}$ groups, where $\mathfrak{C}$ is a class of finite groups with prescribed composition factors. We prove that every real number $a \geq 1$ is the Weil abscissa $a(G)$ of some profinite group G. In addition, we show that the Euler factors of ζ_G are rational functions in p^{-s} if G is virtually abelian. For finite groups G we calculate ζ_G using the rational representation theory of G.

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✉ Steffen Kionke
steffen.kionke@fernuni-hagen.de

Ged Corob Cook
gcorobcook@gmail.com

Matteo Vannacci
matteo.vannacci@ehu.eus

1 University of Lincoln, Brayford Pool, Lincoln LN6 7TS, UK

2 Fakultät für Mathematik und Informatik, FernUniversität in Hagen, 58084 Hagen, Germany

3 Matematika Saila, UPV-EHU, Barrio Sarriena s/n, 48910 Bilbao, Spain

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1 Introduction

The use of zeta functions to study asymptotic properties of algebraic objects is a well-established branch of algebra, see for instance [13, 18, 33, 47, 48] and references therein. In particular, there is considerable interest in zeta functions enumerating all [1, 32], some [29] or equivalence classes of [45] complex irreducible representations of infinite groups. To be well-defined, the counting problem needs to be adapted to the class of groups under investigation. In this article we take a different approach suitable for the large class of UBERG groups: we define and study a *Weil representation zeta function* enumerating absolutely irreducible representations over finite fields.

For a profinite group G and a field F, we write $r^*(G, F, n)$ to denote the number of absolutely irreducible representations of G of dimension n defined over F. We say that G has *UBERG* if there exists a positive constant $c > 0$ such that $r^*(G, F, n) \leq |F|^{cn}$ for every finite field F. UBERG stands for 'uniformly bounded exponential representation growth' (over finite fields) and, maybe surprisingly, it shows up naturally in the study of probabilistic generation properties of profinite groups. In fact, a finitely presented profinite group is *positively finitely related* (PFR) exactly if it has UBERG. Moreover, a profinite group has UBERG if and only if the completed group algebra $\hat{\mathbb{Z}}[\![G]\!]$ is *positively finitely generated* (PFG, see [30]) as a $\hat{\mathbb{Z}}[\![G]\!]$-module. The properties of UBERG groups were the central object of the authors' paper [9]. The class of UBERG groups is 'large', meaning that it contains all finitely generated profinite groups not involving some finite simple group as an upper section [30, Theorem 6.10]. In particular, this implies that finitely generated prosoluble groups, congruence completions of S-arithmetic groups and congruence completions of branch groups have UBERG.

Since we are dealing with an exponential growth condition, we define the following complex function for an UBERG profinite group G:

$$\zeta_G(s) := \exp\left(\sum_{p\in\mathcal{P}}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{r^*(G,\mathbb{F}_{p^j},n)}{j}p^{-snj}|\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|\right) \quad \text{for } s\in\mathbb{C}. \tag{1.1}$$

We will see that, for an UBERG group G, the above sum converges on some complex half-plane (cf. Corollary 2.3). It admits an obvious Euler product decomposition $\zeta_G(s) = \prod_{p\in\mathcal{P}} \zeta_{G,p}(s)$. The formula for ζ_G is reminiscent of the Hasse–Weil zeta function of an algebraic variety V where absolutely irreducible representations of G over $\mathbb{F}_q$ now take the role of $\mathbb{F}_q$-rational points of V. Due to this ζ_G will be called the *Weil representation zeta function* of G. Thinking about maximal ideals explains the factor $|\mathbb{P}^{n-1}(\mathbb{F}_q)|$: for every absolutely irreducible $\mathbb{F}_q$-representation V of dimension n, there are $|\mathbb{P}^{n-1}(\mathbb{F}_q)|$ maximal ideals M in $\mathbb{F}_q[\![G]\!]$, such that $\mathbb{F}_q[\![G]\!]/M \cong V$. Moreover, the factor is necessary to make ζ_G a *probabilistic zeta function* (see Theorem A). The main objective of this article is to start the investigation of the function $\zeta_G(s)$ and the group-theoretical properties of G detected by it.

If G is an abstract group, any finite-dimensional representation of G over a finite field has to factor through a finite quotient; hence, for a group G with UBERG profinite completion $\widehat{G}$, we define $\zeta_G(s) := \zeta_{\widehat{G}}(s)$. For instance, for $G = \mathbb{Z}$ one obtains (see Example 2.5)

$$\zeta_{\mathbb{Z}}(s) = \frac{\zeta(s-1)}{\zeta(s)},$$

where ζ denotes Riemann's zeta function. In general, ζ_G is the Hasse–Weil zeta function of $\mathrm{Spec}(\mathbb{Z}[G])$ whenever G is a finitely generated abelian group. More examples can be found in Appendix A; our list of examples conveys the impression that ζ_G is rather well-behaved. It is worth noting that if one replaces absolutely irreducible by irreducible representations in the formula for the Weil representation zeta function the resulting zeta function is less well-behaved; see Appendix B.

Our initial motivation to study ζ_G is a significant connection between the values of $\zeta_G(s)$ at the natural numbers and the probability $P_R(R,k)$ that k random elements generate the group ring $R = \widehat{\mathbb{Z}}[\![G]\!]$ of G over $\widehat{\mathbb{Z}}$ as a $\widehat{\mathbb{Z}}[\![G]\!]$-module. In this setting, we have the following theorem (see Sect. 3).

Theorem A *Let G be an UBERG profinite group and let $R = \widehat{\mathbb{Z}}[\![G]\!]$ be its group ring over $\widehat{\mathbb{Z}}$. For all sufficiently large integers ℓ the following equality holds:*

$$P_R(R, \ell)^{-1} = \zeta_G(\ell).$$

Theorem A is evocative of the *probabilistic zeta function* introduced by Mann [36, Sec. 5, Conjecture]. In fact, both zeta functions are related to a probability: the former to the probability of generating a profinite group G with ℓ random elements, the latter to the probability $P_{\widehat{\mathbb{Z}}[\![G]\!]}(\widehat{\mathbb{Z}}[\![G]\!], \ell)$. However, it is important to remark that Mann's zeta function is not known to exist in general for positively finitely generated profinite groups (see [35]), while the Weil representation zeta function is well-defined for any UBERG group.

Following the tradition of the study of zeta functions associated to algebraic objects, we study the *abscissa of convergence* of the function $\zeta_G(s)$, i.e. the real number

$$a(G) = \inf\{t \in \mathbb{R}_{\geq 1} \mid \zeta_G(t) \text{ converges}\}. \tag{1.2}$$

We will refer to $a(G)$ as the *Weil abscissa*. In Sect. 4 we prove the following properties of the function $a(G)$.

Theorem B *Let G be an UBERG profinite group.*

(i) *If H is an open subgroup of G, then*

$$\frac{a(H)}{|G:H|} \leq a(G) \leq a(H) + 1 - \frac{1}{|G:H|}.$$

Moreover, if $G = H \times \Omega$ for a finite group Ω, then $a(G) = a(H)$.

(ii) If Δ a closed normal subgroup of G, then $a(G/\Delta) \leq a(G)$. Moreover, if the quotient map $G \to G/\Delta$ splits, then

$$a(G) \leq a(\Delta) + a(G/\Delta) + 1.$$

As a corollary we obtain that $a(G) = 1$ for all finite groups G, however there are infinite groups with Weil abscissa 1 (see Theorems C and E). We show in Example 5.12 that the lower bound in (i) is sharp, but it appears to us that the upper bound in (i) is not sharp. We construct examples of groups G that are split extensions of a finite index normal subgroup H such that $a(G) > a(H)$ in Sect. 7.

We move on to calculate the Weil abscissa of various families of profinite groups in Sect. 5. We recall briefly the notation used in the theorem. As usual, F_r^{sol}, F_r^{nil} and F_r^p denote the free prosoluble, pronilpotent and pro-p groups on r generators, respectively, and $\mathbb{Z}_p$ denotes the p-adic integers. Moreover, $c_{nil} = \frac{5\log(2)}{2\log(3)}$ and $c_{sol} = \frac{2}{3} + \frac{5\log(2)}{2\log(3)}$ are the constants from [50] and [40], respectively. Finally, $K(p)$ and $K'(p)$ are number-theoretic constants defined in Sects. 5.2 and 5.4 respectively, with $1 \leq K(p) \leq 2.1115$,

$K'(p) \leq K(p)$, and

$$K'(p) \begin{cases} = \frac{2\log(3)}{5\log(2)} & \text{if } p = 2; \\ = \frac{(p-1)\log(p+1)}{p\log(p)} < 1 & \text{if } p \text{ is a Mersenne prime;} \\ \geq 1 & \text{otherwise.} \end{cases}$$

Theorem C *Let p be a prime number and $r \in \mathbb{N}$.*

(i) $a(\mathbb{Z}^r) = r + 1$.
(ii) $\dfrac{r}{K(p)} \leq a(\mathbb{Z}_p^r) \leq \dfrac{r-1}{K(p)} + 1$. *In particular,* $a(\mathbb{Z}_p) = 1$.
(iii) $a(F_r^p) = \dfrac{r-1}{K'(p)} + 1$ *if p is odd and* $a(F_r^2) = c_{nil}(r-1) + 1$.
(iv) $a(F_r^{nil}) = c_{nil}(r-1) + 1$ *if $r > 2$ and* $a(F_2^{nil}) = 3$.
(v) $a(F_r^{sol}) = c_{sol}(r-1) + 1$.
(vi) *If G is a pronilpotent group of finite rank r, then $a(G) \leq r + 1$.*

It is worth pointing out that the calculation of the Weil abscissa of $\mathbb{Z}_p^r$ touches on some deep number-theoretic conjectures and results concerning the distribution of primes in arithmetic progressions: see Sect. 5.2. Strong results are needed to show that $a(\mathbb{Z}_p^2) > 1$.

On the other hand, we can estimate (and sometimes calculate exactly) the abscissa of some free groups with restricted non-abelian composition factors of *large degree*, as we explain now. We recall that an *NE-formation* is a class of finite groups which is closed under quotients, finite subdirect products, taking normal subgroups and extensions (see [43, §2.1]). Let $\mathfrak{C}$ be an NE-formation containing the cyclic groups of prime order. Let $F_r^{\mathfrak{C}}$ be the free pro-$\mathfrak{C}$ group on r generators. If $\mathfrak{C}$ contains alternating groups of arbitrarily large degree, or classical groups with natural representation of arbitrarily large dimension, one can show that $F_r^{\mathfrak{C}}$ does not have UBERG (cf. Theorem 5.5). Assume it does not. Let c_0 be maximal such that $Alt(c_0) \in \mathfrak{C}$. In Theorem 5.22 we show that, for large enough c_0, the Weil abscissa of $F_r^{\mathfrak{C}}$ is dominated by representations of a very specific type. More concretely, in Sect. 5.9, we define an effectively computable absolute constant C_0.

Theorem D *Let $\mathfrak{C}$ be a NE-formation of finite groups containing the cyclic groups of prime order and let c_0 be maximal such that $Alt(c_0) \in \mathfrak{C}$. Suppose that $c_0 > C_0$. Then $\mathfrak{C}$ does not contain alternating groups of degree greater than c_0. Let $F_r^{\mathfrak{C}}$ be the free pro-$\mathfrak{C}$ group on r generators. Then:*

$$c_{\mathfrak{C},space}(r-1) + 1 \leq a(F_r^{\mathfrak{C}}) \leq \max(c_{\mathfrak{C},space}(r-1) + 1, c_{\mathfrak{C},time}(r-1) + 2).$$

The constants $c_{\mathfrak{C},space}$ and $c_{\mathfrak{C},time}$ are defined just before Theorem 5.34. Note that $c_{\mathfrak{C},time} \leq c_{\mathfrak{C},space}$, so these bounds differ by at most 1. The inspiration for these names is that the convergence of ζ_G is often dominated either by the representations in all dimensions over a specific field, or by the representations in a specific dimension over all fields; we think of the former behaviour as 'space-like' and the latter as 'time-like'.

This should not be taken as a claim of a detailed analogy to the concept from general relativity, but the intuition is as follows. In general relativity, an interval between two events is called time-like if, in some inertial frame, the interval in space-time is parallel to the time axis; the interval is called space-like if, in some inertial frame, the interval is parallel to the space axis. We imagine the prime powers on a number line as a time axis, and the positive integers on a number line as a (1-dimensional) space axis. We assign to each point (p^j, n) the value $\nu(p^j, n) = \log_{p^{nj}}(\frac{r^*(G,\mathbb{F}_{p^j},n)}{j}\frac{p^{nj}-1}{p^j-1})$, and let $\nu = \limsup_{p^{nj}\to\infty} \nu(p^j, n)$. Then we fix a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive real numbers converging to 0, and for each $k \in \mathbb{N}$ we pick a point $(p_k^{j_k}, n_k)$ in space-time such that $\nu(p_k^{j_k}, n_k) > \nu - \varepsilon_k$ and $p_k^{n_k j_k} \geq p_{k-1}^{n_{k-1} j_{k-1}}$. The sequence $(p_k^{n_k j_k})$ plots a path through space-time: if, for any sequence (ε_k) and any choice of $(p_k^{n_k j_k})$, the path is eventually parallel to the time axis, we say G is time-like; if, for any (ε_k) and any $(p_k^{n_k j_k})$, the path is eventually parallel to the space axis, we say G is space-like.

Not all groups fit into one of these classes, and the time-like groups could be broken down further into being dominated by representations in a specific characteristic, versus in a specific power of the characteristic; but at any rate, this categorisation is useful for the groups we have studied. For a surprising example, F_r^{nil} is space-like for $r > 2$ and time-like for $r = 2$, which is what causes the behaviour of the abscissa in Theorem C(iv).

Additionally, we can calculate the Weil abscissa of $F_r^{\mathfrak{C}}$ for two specific NE-formations $\mathfrak{C}$ where the zeta function is space-like: the NE-formation $\mathfrak{C}_{\mathrm{Alt}(c_0)}$ generated by all cyclic groups of prime order and the alternating groups of degree at most c_0; and the NE-formation $\mathfrak{C}_{\Sigma(c_0)}$ generated by all the simple groups in $\mathfrak{C}_{\mathrm{Alt}(c_0)}$, all the sporadic groups, all the exceptional groups of Lie type, and all the classical simple groups whose natural representation has dimension at most c_0. In these cases,

$$a(F_r^{\mathfrak{C}_{\mathrm{Alt}(c_0)}}) = \frac{c_0 \log_2(c_0!)}{(c_0 - \delta(c_0))(c_0 - 1)}(r-1) + 1$$

where $\delta(c_0) = 1$ or 2 for c_0 odd or even, respectively, and

$$a(F_r^{\mathfrak{C}_{\Sigma(c_0)}}) = (c_0 + \frac{(c_0 - 1)\log(\prod_{i=1}^{c_0}(1 - 2^{-i})) + \log(c_0!)}{c_0(c_0 - 1)\log(2)})(r-1) + 1.$$

In general determining the Weil abscissa for a profinite group seems to be a hard problem. One might wonder whether all real numbers $\alpha \geq 1$ are the Weil abscissa of some profinite group. We answer this positively in Sect. 6.

Theorem E *Let α be a non-negative real number and set*

$$G_\alpha = \prod_{p\in\mathcal{P}} \mathrm{SL}(2, p)^{\lfloor p^\alpha \rfloor}.$$

Then $\zeta_{G_\alpha}(s)$ has abscissa of convergence $\alpha/2 + 1$.

The analogous theorem for representations over the complex numbers was proved in [17, Chapter 6]. It is interesting to point out that certain classes of groups have better behaved abscissae; for instance, the abscissa of convergence of the complex representation zeta function of an arithmetic group is always a rational number [1]. We suspect that an analogous result for arithmetic groups might hold for the Weil representation zeta function. However, some natural profinite groups have irrational abscissa of convergence: by Theorem C the Weil abscissa of free pronilpotent and prosoluble groups is irrational.

Finally, we concentrate our attention on analytic properties of the function $\zeta_G(s)$ in special cases. In the case where G is a finite group, we can give in Sect. 8 an explicit formula for the Weil representation zeta function, depending on the rational representations of G, up to rational functions. First some notation: for a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ and a natural number n we define

$$f^{\# n}(s) := \prod_{j=0}^{n-1} f(ns - j).$$

We write $f \sim g$ if there exists $K \in \mathbb{N}$ such that f/g is of the form

$$\prod_{k=1}^{K} (1 - M_k^{-a_k s + b_k})^{\varepsilon_k} \tag{1.3}$$

with $M_k \in \mathbb{N}_{\geq 2}$, $a_k > b_k \in \mathbb{N}_0$ and $\varepsilon_k \in \{\pm 1\}$, for $k = 1, \ldots, K$.

Let G be a finite group and let V be an irreducible rational representation with character χ. We write K_χ for the centre of the endomorphism algebra of $\mathrm{End}_G(V)$ and $m(\chi)$ denotes the Schur index of χ. We note that K_χ is an algebraic number field and $\dim_{K_\chi} \mathrm{End}_G(V) = m(\chi)^2$.

Theorem F *Let G be a finite group. Then*

$$\zeta_G(s) \sim \prod_{\chi \in \mathrm{Irr}(G,\mathbb{Q})} \zeta_{K_\chi}^{\# n_\chi}(s).$$

where ζ_{K_χ} denotes the Dedekind zeta function of K_χ and $n_\chi = \frac{\chi(1)}{[K_\chi:\mathbb{Q}]m(\chi)}$. The Weil representation zeta function $\zeta_G(s)$ admits a meromorphic continuation to $\mathbb{C}$, it has a pole of order $|\mathrm{Irr}(G, \mathbb{Q})|$ at $s = 1$ and all other poles are located at rational numbers in the interval $[0, 1 - \sqrt{|G|^{-1}}]$.

By a result of Solomon [46], the formula in Theorem F agrees (up to rational factors) with the ideal zeta function of $\mathbb{Z}[G]$, i.e., the Dirichlet series enumerating ideals of a given index in $\mathbb{Z}[G]$. We thank an anonymous referee for pointing us to this remarkable equality. It can be traced back to the case of algebras of the form $M_{n,n}(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers in a p-adic field (cf. [46, Lemma 5]). It prompts a natural question: Is there relation between the ideal zeta function of $\widehat{\mathbb{Z}}[\![G]\!]$ and the Weil representation zeta function for general profinite groups?

Since the Weil representation zeta function bears a resemblance to the Hasse–Weil zeta function of an algebraic variety, it is a very natural question to ask if (or when) the local factors $\zeta_{G,p}$ satisfy parts of Weil's conjectures. In Sect. 9 we show the following.

Theorem G *Let p be a prime and let G be a finitely generated virtually abelian group. Then $\zeta_{G,p}(s)$ is a rational function in p^{-s}.*

The proof uses the *moduli varieties* M_n of n-dimensional absolutely irreducible representations and the classical solution of the Weil conjectures. We suspect that the Weil representation zeta functions of a larger class of UBERG groups have rational local factors. To support this, we show in Appendix A that the local factors of the Weil representation zeta function of the lamplighter groups $C_2 \wr \mathbb{Z}$ and $C_3 \wr \mathbb{Z}$ are rational.

To conclude, one natural direction of future work is the study of meromorphic continuation of Weil representation zeta functions. At the moment, we only have some partial calculations.

Notation As it is customary when working with profinite groups and rings, all subgroups will be assumed to be closed and all homomorphisms will be assumed to be continuous.

2 The zeta function à la Weil of PFG profinite rings

Let R be a profinite ring. We say that R is PFG if it is positively finitely generated as left module over itself. Let F be a finite field, and let $\overline{F}$ denote the algebraic closure. An $F \otimes_{\mathbb{Z}} R$ module M is *absolutely simple* if it is simple and $\overline{F} \otimes_F M$ is simple. We define $r^*(R, F, n)$ to be the number of isomorphism classes of absolutely simple $F \otimes_{\mathbb{Z}} R$ modules of dimension n over F. We observe that if $R = \widehat{\mathbb{Z}}[\![G]\!]$ is the completed group ring of a profinite group G, then $r^*(R, F, n) = r^*(G, F, n)$.

Lemma 2.1 *A profinite ring R is PFG if and only if there is a constant $c > 0$ such that for all finite fields F and all n the following inequality holds:*

$$r^*(R, F, n) \leq |F|^{cn}.$$

Proof The proofs of [30, Prop. 6.1] and [30, Lem. 6.8] go through without changes in our current situation. □

Suppose R is a PFG profinite ring. Let $c_0(R)$ be the infimum of the numbers c such that, for all but finitely many tuples (p, j, n), $r^*(R, \mathbb{F}_{p^j}, n) \leq p^{cnj}$. We may define $c_0(R) = \infty$ if R is not PFG.

Lemma 2.2 *The series*

$$\sum_{p \in \mathcal{P}} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{r^*(R, \mathbb{F}_{p^j}, n)}{j} p^{-snj} |\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|$$

converges absolutely for all complex numbers $s = \sigma + i\tau$ with $\sigma > c_0(R) + 1$, and diverges for $s = \sigma + i\tau$ with $\sigma < c_0(R)$.

Proof Since all the coefficients are real and positive, absolute convergence is the same as convergence, and we may assume s is real and rearrange the terms. Let $\varepsilon > 0$. Replacing p^j with q, we get the upper bound

$$\sum_{p\in\mathcal{P}}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{r^*(R,\mathbb{F}_{p^j},n)}{j}p^{-snj}|\mathbb{P}^{n-1}(\mathbb{F}_{p^j})| \le \kappa\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}q^{n(c_0(R)+\varepsilon)}q^{-\sigma n}q^{n-1}.$$

for a suitable constant $\kappa > 0$. When $\sigma > c_0(R) + 1 + 2\varepsilon$, we have

$$\sum_{n=1}^{\infty}q^{n(c_0(R)+\varepsilon)}q^{-\sigma n}q^{n-1} = \frac{q^{c_0(R)+\varepsilon-\sigma}}{1-q^{c_0(R)+\varepsilon+1-\sigma}} \ll q^{-1-\varepsilon},$$

so the whole sum converges.

On the other hand, when $\sigma < c_0(R)$, fix $\varepsilon > 0$ such that $\sigma + \varepsilon < c_0(R)$. For infinitely many tuples (p, j, n) we have $r^*(R, \mathbb{F}_{p^j}, n) > p^{(\sigma+\varepsilon)nj}$; call the set of such tuples S. So

$$\sum_{p\in\mathcal{P}}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{r^*(R,\mathbb{F}_{p^j},n)}{j}p^{-\sigma nj}|\mathbb{P}^{n-1}(\mathbb{F}_{p^j})| > \sum_{(p,j,n)\in S}p^{\varepsilon j}/j.$$

Since $p \ge 2$, a calculation shows $p^{\varepsilon j}/j \ge \varepsilon\log(2)2^{1/\log(2)}$, so the sum diverges. Hence, by standard results on Dirichlet series, the sum diverges for all s with $\sigma < c_0(R)$. □

Corollary 2.3 *If R is a PFG profinite ring, then*

$$\zeta_R(s) := \exp\left(\sum_{p\in\mathcal{P}}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{r^*(R,\mathbb{F}_{p^j},n)}{j}p^{-snj}|\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|\right)$$

defines a holomorphic function in some right half plane.

We will call $\zeta_R(s)$ the Weil zeta function of R. Now we compute some examples that will be useful in the proof of Theorem A.

Example 2.4 Let $R = \widehat{\mathbb{Z}} = \widehat{\mathbb{Z}}[\![1]\!]$. Let F be any finite field. Then $\widehat{\mathbb{Z}}$ has exactly one one-dimensional absolutely simple module over every finite field. Therefore (using the logarithmic series)

$$\zeta_{\widehat{\mathbb{Z}}}(s) = \exp\left(\sum_{p\in\mathcal{P}}\sum_{j=1}^{\infty}\frac{1}{j}p^{-sj}\right) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^s}\right)^{-1} = \zeta(s)$$

is the Riemann zeta function.

More generally, let $\mathcal{O}_K$ denote the ring of integers of an algebraic number field K. Then for $R = \widehat{\mathcal{O}}_K$ the Weil zeta function ζ_R agrees with the Dedekind zeta function of K. In the following we will mostly be studying $\widehat{\mathbb{Z}}[\![G]\!]$ for profinite groups G. It seems interesting to replace $\mathbb{Z}$ by $\mathcal{O}_K$ in this situation; we will not consider this more general setting here.

Example 2.5 Let $R = \widehat{\mathbb{Z}}[\![\widehat{\mathbb{Z}}]\!]$. Let F be any finite field. $\widehat{\mathbb{Z}}[\![\widehat{\mathbb{Z}}]\!]$ has $q-1$ one-dimensional absolutely simple modules over $\mathbb{F}_q$. Therefore

$$\zeta_R(s) = \exp\left(\sum_{p\in\mathcal{P}}\sum_{j=1}^{\infty}\frac{p^j-1}{j}p^{-sj}\right) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^{s-1}}\right)^{-1}\left(1-\frac{1}{p^s}\right) = \frac{\zeta(s-1)}{\zeta(s)}$$

Example 2.6 Let p be a fixed prime. Let $R = M_n(\mathbb{F}_{p^k})$ be the matrix algebra over the finite field $\mathbb{F}_{p^k}$. The ring has k absolutely simple modules of dimension n over F if and only if F contains $\mathbb{F}_{p^k}$, i.e.,

$$r^*(R, F, m) = \begin{cases} k & \text{if } m = n \text{ and } \mathbb{F}_{p^k} \subseteq F; \\ 0 & \text{otherwise.} \end{cases}$$

The zeta function agrees with the Weil zeta function of the projective space $\mathbb{P}^{n-1}$ defined over $\mathbb{F}_{p^k}$ evaluated at sn:

$$\zeta_R(s) = \exp\left(\sum_{j=1}^{\infty}\frac{k}{jk}p^{-sjkn}|\mathbb{P}^{n-1}(\mathbb{F}_{p^{kj}})|\right) = \prod_{i=0}^{n-1}\left(1-\frac{p^{ki}}{p^{skn}}\right)^{-1}.$$

Next we show that the Weil zeta function of a profinite ring is the pointwise limit of the Weil zeta functions of its finite quotients.

Proposition 2.7 (Continuity) *Let $R = \varprojlim_{i\in\mathbb{N}} R_i$ be a PFG profinite ring which is given a an inverse limit of finite rings R_i. If $\zeta_R(s)$ converges absolutely at $s \in \mathbb{C}$, then*

$$\zeta_R(s) = \lim_{i\to\infty}\zeta_{R_i}(s).$$

Proof Since every absolutely simple module factors over some R_i, we have

$$r^*(R, F, n) = \lim_{i\to\infty} r^*(R_i, F, n)$$

for all finite fields F and the sequence $r^*(R_i, F, n)$ is monotonically increasing. By assumption there is $c > 0$ such that

$$r^*(R_i, F, n) \leq r^*(R, F, n) \leq |F|^{nc}$$

for all finite fields F and every n. Let $s \in \mathbb{C}$ and assume that the series defining $\zeta_R(s)$ converges absolutely at s, then

$$\lim_{i\to\infty} \zeta_{R_i}(s) = \zeta_R(s)$$

by Lebesgue's theorem of dominated convergence (e.g., [5, 5.6 Thm.]). □

We end this preliminary section with a formula for the Weil zeta function of the direct product of rings and of groups.

Proposition 2.8 (Products of rings) *Let R, S be two PFG profinite rings. Then $R \times S$ is PFG and*

$$\zeta_{R\times S}(s) = \zeta_R(s) \cdot \zeta_S(s)$$

for all $s \in \mathbb{C}$ where $\zeta_R(s)$, $\zeta_S(s)$ are defined.

Recall that, given two functions $f, g : \mathbb{N} \to \mathbb{N}$ the *Dirichlet convolution* of f and g is the function

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

Recall that if G and H have UBERG, then the direct product $G \times H$ does too; see [30, Theorem 6.4].

Proposition 2.9 (Products of groups) *Let G, H be two profinite groups. Then $r^*(G \times H, F, n) = (r^*(G, F, -) * r^*(H, F, -))(n)$, where $*$ denotes Dirichlet convolution. If G and H have UBERG, we can write $\zeta_{G\times H}(s)$ as*

$$\exp\left(\sum_{p\in\mathcal{P}}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{(r^*(G,\mathbb{F}_{p^j},-)*r^*(H,\mathbb{F}_{p^j},-))(n)}{j}p^{-snj}|\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|\right).$$

Note that the above formula does not directly give a nice formula for the Weil representation zeta function of $G \times H$ and, even for finite groups, it might be hard to compute it.

3 Weil zeta functions and probability

In this section we will prove Theorem A. In fact, Theorem A holds for arbitrary PFG profinite rings and the following proof is written in this general setting.

Proof of Theorem A Let R be a PFG profinite ring and let $J(R)$ denote the Jacobson radical. A tuple of elements generates R if and only if it generates $R/J(R)$. Hence

$P_R(R, \ell) = P_{R/J(R)}(R/J(R), \ell)$ for all $\ell \in \mathbb{N}$. Similarly, the Jacobson radical annihilates all simple modules, hence

$$r^*(R, F, n) = r^*(R/J(R), F, n).$$

So we may replace R with $R/J(R)$, and assume that R is semisimple, i.e., a product of matrix algebras over finite fields. Since R is PFG, it has countably many simple quotients, so this product is countably based.

Thus we may write $R = \varprojlim_{i\in\mathbb{N}} R_i$ as an inverse limit of finite semisimple rings. Then $P_R(R, \ell) = \lim_{i\in\mathbb{N}} P_{R_i}(R_i, \ell)$ and so by Proposition 2.7 it is sufficient to prove the theorem for finite semisimple rings. Since probabilities and zeta functions behave well under finite direct products of rings (see Proposition 2.8), it is sufficient to prove the equality for a single matrix algebra. The probability that ℓ random elements generate $M_n(\mathbb{F}_{p^k})$ is the probability that a random $(n \times n\ell)$ matrix has rank n, i.e.,

$$P_{M_n(\mathbb{F}_{p^k})}(M_n(\mathbb{F}_{p^k}), \ell) = p^{-k\ell n^2} \prod_{i=0}^{n-1}(p^{k\ell n} - p^{ki}) = \prod_{i=0}^{n-1}\Big(1 - \frac{p^{ki}}{p^{\ell kn}}\Big),$$

and the assertion follows from Example 2.6. □

4 Abscissae of convergence

Let G be an UBERG profinite group. Write $a(G)$ for the abscissa of convergence of $\zeta_G(s)$, as defined in (1.2). Observing that all the coefficients are real and positive, we deduce that the abscissa of convergence of $\zeta_G(s)$ equals its abscissa of absolute convergence. Therefore, we can rearrange terms in $\zeta_G(s)$ and only study convergence for real s.

We start by comparing the abscissa of converge of a quotient with that of the group. The following lemma is clear.

Lemma 4.1 *For any profinite group G and $\Delta \trianglelefteq G$, we have $a(G/\Delta) \leq a(G)$.*

In particular, $a(G) \geq 1$ for any G, because $a(\{1\}) = 1$ by Example 2.4. The next lemma will be used repeatedly for calculations of abscissae.

Lemma 4.2 *Suppose $f : \mathcal{P} \times \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{R}_{\geq 1}$ is such that $f(p, n, j) = p^{o(nj)}$. For any function $g : \mathcal{P} \times \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{R}$, the series*

$$A(s) = \sum_{p\in\mathcal{P}} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{g(p, n, j)}{j} p^{-snj} |\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|$$

and

$$B(s) = \sum_{p\in\mathcal{P}} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} f(p, n, j) \frac{g(p, n, j)}{j} p^{-snj} |\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|$$

have the same abscissae of convergence.

Proof When $A(s)$ converges, $B(s) = O_\varepsilon(A(s+\varepsilon))$ converges too for all $\varepsilon > 0$, because $p^{\varepsilon nj} \geq f(p,n,j)$ for large p, n, j. The converse is clear. $\square$

Next we deal with the abscissa of open subgroups and we prove the first item in Theorem B.

Proposition 4.3 *Suppose G is a profinite group and $H \leq_o G$. Then:*

(i) $a(G) \leq a(H) + 1 - |G:H|^{-1}$;
(ii) $a(H) \leq |G:H| a(G)$.

If Ω is a finite group and $G = H \times \Omega$, then $a(G) = a(H)$.

Proof G has UBERG if and only if H does, by [30, Thm. 6.2]. If neither has UBERG, ζ_G and ζ_H have abscissa of convergence ∞. So we may assume they both have UBERG.

For each absolutely irreducible $\mathbb{F}_{p^j}[\![G]\!]$-module N, pick one irreducible $\mathbb{F}_{p^j}[\![H]\!]$-submodule M of $\mathrm{Res}^G_H(N)$. We note that M has the structure of an $\mathbb{F}_{p^{je}}[\![H]\!]$-module for e such that $\mathbb{F}_{p^{je}} \cong \mathrm{End}_H(M)$; by [22, (9.2)] this module is absolutely irreducible and henceforth we will abuse notation by writing M for this $\mathbb{F}_{p^{je}}[\![H]\!]$-module. We get a map $\phi : N \mapsto M$ from absolutely irreducible G-modules over finite fields to absolutely irreducible H-modules over finite fields.

Fix an absolutely irreducible $\mathbb{F}_{p^j}[\![H]\!]$-module M of dimension m: we want to consider $\phi^{-1}(M)$. Any absolutely irreducible G-module N in $\phi^{-1}(M)$ is defined over some field $\mathbb{F}_{p^\ell}$ with $\ell \mid j$: note that $|\mathrm{End}_H(M)| \leq |\mathrm{Hom}_H(M, \mathrm{Res}^G_H(N)| = |\mathrm{Hom}_G(\mathrm{Ind}^G_H(M), N)| \leq |\mathrm{End}_G(N)|^{|G:H|}$, so $p^j \leq p^{\ell|G:H|}$ and $j/\ell \leq |G:H|$. In particular, there are at most $|G:H|$ possibilities for ℓ. Denote by L the restriction of scalars from $\mathbb{F}_{p^\ell}$ to $\mathbb{F}_{p^j}$ of M. As L is irreducible and N appears as a quotient of $\mathrm{Ind}^G_H(L)$, for a fixed ℓ there are at most $|G:H|$ possibilities for $N \in \phi^{-1}(M)$. Hence there are at most $|G:H|^2$ possibilities for $N \in \phi^{-1}(M)$. Moreover, such an N can have dimension n at most $|G:H| \dim_{\mathbb{F}_{p^\ell}} L = |G:H| jm/\ell$ over $\mathbb{F}_{p^\ell}$, with $n\ell \geq mj$.

Write $\mathrm{Irr}^*(G, \mathbb{F}_{p^\ell}, n)$ for the set of absolutely irreducible $\mathbb{F}_{p^\ell}[\![G]\!]$-modules of dimension n, and define $\alpha(N) = \dfrac{p^{-sn\ell}}{\ell}\dfrac{p^{\ell n}-1}{p^\ell - 1}$ for $N \in \mathrm{Irr}^*(G, \mathbb{F}_{p^\ell}, n)$. Then we may write the log of the Weil representation zeta function of G as

$$\sum_{p \in \mathcal{P}} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\mathrm{Irr}^*(G, \mathbb{F}_{p^\ell}, n)} \alpha(N).$$

Grouping these terms by their image under ϕ, this is equal to

$$\sum_{p \in \mathcal{P}} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\mathrm{Irr}^*(H, \mathbb{F}_{p^j}, m)} \sum_{N \in \phi^{-1}(M)} \alpha(N).$$

Now, for N as above in $\phi^{-1}(M)$,

$$\begin{aligned}\alpha(N) &= \frac{p^{-sn\ell}}{\ell}\frac{p^{\ell n}-1}{p^{\ell}-1} = \frac{p^{-sn\ell}}{\ell}\frac{p^{j}-1}{p^{\ell}-1}\frac{p^{jm}-1}{p^{j}-1}\frac{p^{\ell n}-1}{p^{jm}-1}\\ &\leq \frac{p^{-sn\ell}}{\ell}\frac{jp^{j-\ell}}{\ell}\frac{p^{jm}-1}{p^{j}-1}\frac{\ell n p^{\ell n-jm}}{jm} = \frac{np^{-sn\ell+j-\ell+\ell n-jm}}{\ell m}\frac{p^{jm}-1}{p^{j}-1}\\ &= \frac{nj}{\ell m}p^{(\ell n-jm)(1-s)+j-\ell}\alpha(M). \end{aligned} \tag{4.1}$$

We have $j/\ell \leq |G:H|$ and $n/m \leq |G:H|j/\ell \leq |G:H|^2$, so for $s > 1$, $\alpha(N) \leq |G:H|^3 p^{(\ell n-jm)(1-s)+j-\ell}\alpha(M) \leq |G:H|^3 p^{(1-|G:H|^{-1})j}\alpha(M)$, and hence $\sum_{N\in\phi^{-1}(M)}\alpha(N) \leq |G:H|^5 p^{(1-|G:H|^{-1})j}\alpha(M)$. Finally, for $s > a(H)+1-|G:H|^{-1}$, we conclude that

$$\begin{aligned}\log(\zeta_G)(s) &\leq \sum_{\mathbb{F}_{p^j}}\sum_{m=1}^{\infty}\sum_{\mathrm{Irr}^*(H,\mathbb{F}_{p^j},m)} |G:H|^5 p^{(1-|G:H|^{-1})j}\frac{p^{-smj}}{j}\frac{p^{jm}-1}{p^{j}-1}\\ &\leq |G:H|^5\sum_{\mathbb{F}_{p^j}}\sum_{m=1}^{\infty}\sum_{\mathrm{Irr}^*(H,\mathbb{F}_{p^j},m)}\frac{p^{(1-|G:H|^{-1}-s)mj}}{j}\frac{p^{jm}-1}{p^{j}-1}\\ &= |G:H|^5\log(\zeta_H)(s-1+|G:H|^{-1}),\end{aligned}$$

which converges.

Conversely, by the same approach, we may define a map ψ from absolutely irreducible H-modules M to absolutely irreducible G-modules N: if M is defined over $\mathbb{F}_{p^j}$, we take N to be an irreducible quotient of $\mathrm{Ind}_H^G(M)$, with the structure of an $\mathbb{F}_{p^{je}}[\![G]\!]$-module, where $\mathbb{F}_{p^{je}} \cong \mathrm{End}_G(N)$.

Fix an absolutely irreducible $\mathbb{F}_{p^{\ell}}[\![G]\!]$-module N of dimension n. Suppose $M \in \psi^{-1}(N)$ is defined over $\mathbb{F}_{p^j}$ (with $j \mid \ell$) and has dimension m. As before, we have $|\mathrm{End}_G(N)| \leq |\mathrm{End}_H(M)|^{|G:H|}$, so $p^{\ell} \leq p^{j|G:H|}$ and $\ell/j \leq |G:H|$. Denote by L the restriction of scalars from $\mathbb{F}_{p^{\ell}}$ to $\mathbb{F}_{p^j}$ of N. As L is irreducible and M appears as a submodule of L, for a fixed j there are at most $|G:H|$ possibilities for $M \in \psi^{-1}(N)$ and so at most $|G:H|^2$ possibilities for M altogether; moreover, such an N must have dimension n at least $(\dim_{\mathbb{F}_{p^j}} L)/|G:H|$ over $\mathbb{F}_{p^j}$.

Grouping terms as before, we may write the log of the Weil representation zeta function of H as

$$\sum_{p\in\mathcal{P}}\sum_{\ell=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\mathrm{Irr}^*(G,\mathbb{F}_{p^{\ell}},n)}\sum_{M\in\psi^{-1}(N)}\alpha(M).$$

Now, for M as above in $\psi^{-1}(N)$,

$$\begin{aligned}\alpha(M) &= \frac{p^{-smj}}{j}\frac{p^{jm}-1}{p^j-1} = \frac{p^{-smj}}{j}\frac{p^\ell-1}{p^j-1}\frac{p^{\ell n}-1}{p^\ell-1}\frac{p^{jm}-1}{p^{\ell n}-1}\\ &\leq \frac{p^{-smj}}{j}\frac{\ell p^{\ell-j}}{j}\frac{p^{\ell n}-1}{p^\ell-1}\frac{p^{jm}}{p^{\ell n}(1-p^{-\ell n})}\\ &= \frac{2\ell^2 p^{sn\ell-smj+\ell-j+jm-\ell n}}{j^2}\alpha(N).\end{aligned}$$

We have $\ell/j \leq |G:H|$, $n\ell/mj \leq |G:H|$ and

$$(s-1)n\ell(1-|G:H|^{-1}) + \ell(1-|G:H|^{-1}) - sn\ell \leq -|G:H|^{-1}sn\ell,$$

and thus we conclude that $\log(\zeta_H)(s)$ is at most

$$\begin{aligned}&\sum_{p\in\mathcal{P}}\sum_{\ell=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\mathrm{Irr}^*(G,\mathbb{F}_{p^\ell},n)} 2|G:H|^4 p^{\ell(1-|G:H|^{-1})((s-1)n+1)}\frac{p^{-sn\ell}}{\ell}\frac{p^{\ell n}-1}{p^\ell-1}\\ &\leq 2|G:H|^4\sum_{p\in\mathcal{P}}\sum_{\ell=1}^{\infty}\sum_{n=1}^{\infty} r^*(G,\mathbb{F}_{p^\ell},n)\frac{p^{-|G:H|^{-1}sn\ell}}{\ell}\frac{p^{\ell n}-1}{p^\ell-1}\\ &= 2|G:H|^4\log(\zeta_G)(s/|G:H|),\end{aligned}$$

which converges for $s > |G:H|a(G)$.

For the final statement, if $G = H\times\Omega$, $a(H)\leq a(G)$ by Lemma 4.1. For the converse, we can argue exactly as in the proof of (i), except that, by [15, Theorem 2.7], we always have $j=\ell$. We then have $n/m \leq |G:H|$, so from (4.1) we see that, for $s>1$, $\alpha(N)\leq |G:H|\alpha(M)$, and hence $\sum_{N\in\phi^{-1}(M)}\alpha(N)\leq |G:H|^2\alpha(M)$. We conclude as in (i) that $\log(\zeta_G)(s)\leq |G:H|^2\log(\zeta_H)(s)$, which converges for $s>a(H)$. □

Corollary 4.4 (Finite groups) *Let G be a finite group, then $a(G)=1$.*

Proof Apply Proposition 4.3 to G and the trivial open subgroup. □

Remark 4.5 The bound in Proposition 4.3(i) can be improved under additional assumptions. For instance, suppose in addition that H is perfect and normal in G. Any irreducible representation N of G such that $N\in\phi^{-1}(M)$ with M 1-dimensional can then be seen as a representation of G/H. In particular,

$$\sum_{p,j}\sum_{\mathrm{Irr}^*(H,\mathbb{F}_{p^j},1)}\sum_{N\in\phi^{-1}(M)}\alpha(N)\leq\log(\zeta_{G/H})(s),$$

which converges for all $s > 1$ by Corollary 4.4. On the other hand,

$$\sum_{p,j}\sum_{m=2}^{\infty}\sum_{\mathrm{Irr}^*(H,\mathbb{F}_{p^j},m)} |G:H|^5 p^{(1-|G:H|^{-1})j}\frac{p^{-smj}}{j}\frac{p^{jm}-1}{p^j-1}$$
$$\leq |G:H|^5\sum_{p,j}\sum_{m=1}^{\infty}\sum_{\mathrm{Irr}^*(H,\mathbb{F}_{p^j},m)}\frac{p^{((1-|G:H|^{-1})/2-s)mj}}{j}\frac{p^{jm}-1}{p^j-1}$$
$$= |G:H|^5\log(\zeta_H)(s-(1-|G:H|^{-1})/2)$$

converges when $s > a(H) + (1-|G:H|^{-1})/2$. So we conclude $a(G) \leq a(H) + (1-|G:H|^{-1})/2$. Further variations of this argument are possible.

Now we will look at the Weil abscissae of direct products of profinite groups. We start by considering the case of profinite rings.

Lemma 4.6 (Products of rings) *For PFG profinite rings* R, S, $a(R \times S) = \max(a(R), a(S))$.

Proof This is clear from Proposition 2.8. □

Unfortunately, as we have seen in Sect. 2, the Weil representation zeta function of the product of two groups is not as well-behaved. Nonetheless, we can produce an upper bound.

Proposition 4.7 (Products) $a(G \times H) \leq a(G) + a(H)$.

Proof By [15, Theorem 2.7], absolutely irreducible $\mathbb{F}_{p^j}[\![G \times H]\!]$-modules have the form $M \otimes_{\mathbb{F}_{p^j}} N$, with M an absolutely irreducible $\mathbb{F}_{p^j}[\![G]\!]$-module and N an absolutely irreducible $\mathbb{F}_{p^j}[\![H]\!]$-module. Fix such an M, of dimension m, say. By the contribution of M to the sum $\log(\zeta_{G\times H})(s)$, we mean the sum, over all $\mathbb{F}_{p^j}[\![G \times H]\!]$-modules of the form $M \otimes_{\mathbb{F}_{p^j}} N$, of $\frac{1}{j}p^{-smkj}|\mathbb{P}^{mk-1}(\mathbb{F}_{p^j})|$, where k is the dimension of N.

When $s = a(H) + \varepsilon + t$, with $\varepsilon > 0$, the contribution of M is therefore

$$\sum_{k=1}^{\infty}\frac{r^*(H,\mathbb{F}_{p^j},k)}{j}p^{-(a(H)+\varepsilon)kj}\frac{p^{kj}-1}{p^j-1}p^{-(sm-a(H)-\varepsilon)kj}\frac{p^{mkj}-1}{p^{kj}-1}.$$

Now

$$p^{-(sm-a(H)-\varepsilon)kj}\frac{p^{mkj}-1}{p^{kj}-1} \leq \frac{p^{kj(m-1-sm+a(H)+\varepsilon)}}{1-p^{-kj}}$$
$$\leq \frac{p^{j(m-1-sm+a(H)+\varepsilon)}}{1-p^{-j}} \leq \frac{p^{jm-tjm}}{p^j-1}$$

where we use that

$$m-1-sm+a(H)+\varepsilon = -tm-(a(H)+\varepsilon-1)(m-1) < -tm < 0$$

in the second inequality. So the contribution of M is bounded above by

$$\log(\zeta_H)(a(H)+\varepsilon)\frac{p^{-tmj}p^{mj}}{p^j-1} \leq 2\log(\zeta_H)(a(H)+\varepsilon)p^{-tmj}|\mathbb{P}^{m-1}(\mathbb{F}_{p^j})|.$$

Hence, when $t = a(G)+\delta$, $\delta > 0$, the total sum over all M is

$$\leq 2\log(\zeta_H)(a(H)+\varepsilon)\sum_{p\in\mathcal{P}}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty} r^*(G, \mathbb{F}_{p^j}, m)p^{-(a(G)+\delta)mj}|\mathbb{P}^{m-1}(\mathbb{F}_{p^j})|,$$

which converges by Lemma 4.2. This holds for all $\varepsilon, \delta > 0$, so the abscissa of convergence is at most $a(G)+a(H)$. □

Recall from [9, Theorem 4.7] that split extensions of profinite groups with UBERG have UBERG. We cannot get the same bound on the Weil abscissa for such groups as we do for direct products, but we can get close.

Theorem 4.8 (Split extensions) *Suppose G is a profinite group, $\Delta \trianglelefteq G$ and the quotient map $G \to G/\Delta$ splits. Then $a(G) \leq a(\Delta)+a(G/\Delta)+1$.*

We will use the following elementary lemma.

Lemma 4.9 *For all $a \in \mathbb{N}$, $p \geq 2$ and $t > 0$, there is a constant c, depending only on t, such that*

$$\sum_{l\geq a} l^2 p^{-tl} \leq ca^2 \cdot p^{-at}.$$

Proof Divide by a^2p^{-at} and observe that

$$\sum_{l=a}^{\infty}\left(\frac{l}{a}\right)^2 p^{-t(l-a)} \leq \sum_{l=0}^{\infty}(l+1)^2p^{-tl} \leq \sum_{l=0}^{\infty}(l+1)^2 2^{-tl}$$

converges to a value which does not depend on p and a. □

To simplify the notation, sums in this proof will mean sums from 1 to ∞ unless stated otherwise.

Proof of Theorem 4.8 For each absolutely irreducible $\mathbb{F}_{p^j}[\![G]\!]$-module N of dimension n, fix an irreducible summand M of $\mathrm{Res}^G_\Delta N$. Then M is absolutely irreducible over $\mathbb{F}_{p^k}$, some multiple k of j, of dimension m such that $mk \leq nj$. This gives a map ϕ from the set of absolutely irreducible G-modules to the set of absolutely irreducible Δ-modules, and we write $r^*(G, \mathbb{F}_{p^j}, n, M)$ for the number of absolutely irreducible $\mathbb{F}_{p^j}[\![G]\!]$-modules N of dimension n such that $\phi(N) = M$. Similarly, if N is an irreducible $\mathbb{F}_{p^j}[\![G]\!]$-module, we may think of N as an absolutely irreducible $\mathrm{End}_G(N)[\![G]\!]$-module N', and define a map $N \mapsto \phi(N')$ from irreducible $\mathbb{F}_{p^j}[\![G]\!]$-modules to absolutely irreducible Δ-modules. Then we write $R(G, \mathbb{F}_{p^j}, n, M)$ for the number of irreducible $\mathbb{F}_{p^j}[\![G]\!]$-modules of dimension at most n in the preimage of M.

We call the contribution of M to $\log(\zeta_G)(s)$ the sum $\sum_{\phi^{-1}(M)} \alpha(N)$, where $\alpha(N) = \frac{1}{j} p^{-snj} \frac{p^{nj}-1}{p^j-1}$. Thus, the contribution of M is

$$\sum_{n,j} \frac{r^*(G, \mathbb{F}_{p^j}, n, M)}{j} p^{-snj} \frac{p^{jn}-1}{p^j-1}.$$

Now $r^*(G, \mathbb{F}_{p^j}, n, M) \leq R(G, \mathbb{F}_{p^j}, n, M)$, and the proof of [9, Theorem 4.7] shows that $R(G, \mathbb{F}_{p^j}, n, M) \leq nR(G/, \mathbb{F}_{p^j}, n)$; finally,

$$R(G/\Delta, \mathbb{F}_{p^j}, n) \leq \sum_{l=1}^{n} \sum_{i|l} r^*(G/\Delta, \mathbb{F}_{p^{ji}}, l/i).$$

Let $s' = a(\Delta) + \varepsilon + s$, for some small $\varepsilon > 0$. Then the contribution of M to $\log(\zeta_G)(s')$ is

$$\begin{aligned}
&\sum_{n,j} \frac{r^*(G, \mathbb{F}_{p^j}, n, M)}{j} p^{-s'nj} \frac{p^{jn}-1}{p^j-1} \\
&\quad \leq p^{-(a(\Delta)+\varepsilon)mk} \sum_{n,j} \sum_{u,v: v \geq j, uv \leq nj} \frac{n}{j} r^*(G/\Delta, \mathbb{F}_{p^v}, u) p^{-snj} \frac{p^{jn}-1}{p^j-1} \\
&\quad = p^{-(a(\Delta)+\varepsilon)mk} \sum_{u,v} r^*(G/\Delta, \mathbb{F}_{p^v}, u) \sum_{n,j: j \leq v, nj \geq uv} \frac{n}{j} p^{-snj} \frac{p^{jn}-1}{p^j-1}.
\end{aligned}$$

Now grouping the terms with the same value of nj together, we get

$$\sum_{n,j:nj \geq uv} \frac{n}{j} p^{-snj} \frac{p^{jn}-1}{p^j-1} \leq \sum_{n,j:nj \geq uv} njp^{(1-s)nj} \leq \sum_{l \geq uv} l^2 p^{(1-s)l} \leq (c_0 u^2 v^2) p^{vu(1-s)}$$

for some constant c depending only on s, by Lemma 4.9 for $a = uv$ and $t = 1 - s$.

So the contribution of M is

$$\begin{aligned}
&\leq p^{-(a(\Delta)+\varepsilon)mk} \sum_{u,v} cu^2 v^2 r^*(G/\Delta, \mathbb{F}_{p^v}, u) p^{vu(1-s)} \\
&\leq p^{-(a(\Delta)+\varepsilon)mk} \sum_{u,v} cu^2 v^3 \frac{r^*(G/\Delta, \mathbb{F}_{p^v}, u)}{v} p^{vu(1-s)} |\mathbb{P}^{u-1}(\mathbb{F}_{p^v})|
\end{aligned}$$

which converges, by Lemma 4.2, for all $s > a(G/\Delta) + 1$, to $p^{-(a(\Delta)+\varepsilon)mk} f(s) \ll \frac{1}{k} p^{-(a(\Delta)+\varepsilon/2)mk} |\mathbb{P}^{m-1}(\mathbb{F}_{p^k})| f(s)$ for some f which is independent of M. Summing over all absolutely irreducible Δ-modules M, we see that $\log(\zeta_G)(s')$ converges for all $s' > a(\Delta) + a(G/\Delta) + 1$. $\square$

5 Examples of abscissae

In this section we will prove Theorem C.

5.1 Free abelian groups

Finitely generated abelian groups have UBERG by [30], and all their absolutely irreducible representations have dimension 1. For the free abelian group $\mathbb{Z}^r$ of rank r and its profinite completion $\hat{\mathbb{Z}}^r$, $r^*(\mathbb{Z}^r, \mathbb{F}_{p^j}, 1) = r^*(\hat{\mathbb{Z}}^r, \mathbb{F}_{p^j}, 1)$ is the number of homomorphisms from $\hat{\mathbb{Z}}^r$ to $\mathbb{F}_{p^j}^{\times}$, that is, $(p^j - 1)^r$. So

$$\log(\zeta_{\hat{\mathbb{Z}}^r}(s)) = \sum_{p \in \mathcal{P}} \sum_{j=1}^{\infty} \frac{(p^j - 1)^r}{j} p^{-sj} \leq \sum_{n} n^{r-s}$$

which converges for $s > r + 1$. When $s = r + 1$, we get

$$\log(\zeta_{\hat{\mathbb{Z}}^r}(s)) > \sum_{p \in \mathcal{P}} \frac{(p-1)^r}{p^{r+1}} = \sum_{p \in \mathcal{P}} \sum_{i=0}^{r} \frac{(-1)^i}{p^{i+1}} \binom{r}{i}.$$

For each $i > 0$, the sum over the primes p converges absolutely, whereas for $i = 0$, it diverges [37]. Therefore the whole sum, over i and p, diverges. So $a(\hat{\mathbb{Z}}^r) = r + 1$.

Note that, by expanding $(p^j - 1)^r = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} p^{ij}$, we get

$$\log(\zeta_{\hat{\mathbb{Z}}^r}(s)) = \sum_{i=0}^{r} \sum_{p \in \mathcal{P}} \sum_{j=1}^{\infty} (-1)^{r-i} \binom{r}{i} \frac{p^{(i-s)j}}{j} = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \log(\zeta(s-i)),$$

so $\zeta_{\hat{\mathbb{Z}}^r}(s) = \prod_{i=0}^{r} \zeta(s-i)^{(-1)^{r-i}\binom{r}{i}}$, where $\zeta(s)$ is the Riemann zeta function. In particular, $\zeta_{\hat{\mathbb{Z}}^r}(s)$ admits a meromorphic extension to $\mathbb{C}$ and has a simple pole at $r+1$. It is interesting to remark that the Weil representation zeta function in this case is very similar to the subgroup growth zeta function as defined in [18]: in that case $\zeta^{\leq}_{\hat{\mathbb{Z}}^r}(s) = \prod_{i=1}^{r-1} \zeta(s-i)$ by [18, Proposition 1.1].

5.2 Free abelian pro-p groups

Fix a prime number p. Here we study the Weil abscissa of free abelian pro-p groups $\mathbb{Z}_p^r$. Let $w_p(m) = |m|_p^{-1}$ denote the highest p-power dividing m. If q is a prime power, then

$$r^*(\mathbb{Z}_p^r, \mathbb{F}_q, 1) = w_p(q-1)^r,$$

i.e., there are many absolutely irreducible representations, if q is congruent 1 modulo a high power of p. We will see that this observation links the Weil abscissa closely to small prime powers in the arithmetic progressions $np^k + 1$.

We begin with a short summary of results on small *primes* in arithmetic progressions. By Linnik's theorem there are constants c, L such that for every $d \geq 2$ the least prime $p_{\min}(d)$ congruent 1 modulo d satisfies

$$p_{\min}(d) \leq cd^L.$$

Currently, the best known value for the exponent is $L = 5$; see [51]. Assuming the extended Riemann hypothesis or the generalized Riemann hypothesis, we have $L = 2+\varepsilon$ for every $\varepsilon > 0$; see [3] or [19]. A folklore conjecture (sometimes attributed to Chowla) states that $L = 1 + \varepsilon$ for every $\varepsilon > 0$.

For our purposes the only relevant case is $d = p^j$ a power of the fixed prime number p. In this case better results are known. Let $L(p)$ be defined as

$$L(p) = \limsup_{j\to\infty} \frac{\log(p_{\min}(p^j))}{j\log(p)}.$$

In other words $L(p)$ is the infimum over all real numbers $L > 0$ such that $p_{\min}(p^j) \leq c_L p^{jL}$ for some $c_L > 0$ and all $j \geq 1$. Barban, Linnik and Tshudakov proved $L(p) \leq \frac{8}{3}$. Gallagher [16] (see also Iwaniec [23]) established $L(p) < 2.5$ and Huxley [21] improved this to $L(p) \leq 2.4$. Currently the best bound appears in a paper of Banks-Shparlinski [4], who show that $L(p) < 2.1115$.

We will now see that the Weil abscissa for $\mathbb{Z}_p^r$ is related to a very similar, but less studied constant. For our purposes, we can replace lim sup by lim inf and, in addition, we are interested in *prime powers* in arithmetic progressions (which should not make a big difference asymptotically).

Definition 5.1 Let $pp_{\min}(d)$ denote the least prime power congruent 1 modulo d. We define

$$K(p) = \liminf_{j\to\infty} \frac{\log(pp_{\min}(p^j))}{j\log(p)}.$$

Proposition 5.2 *Let p be a prime number. Then*

$$\frac{r}{K(p)} \leq a(\mathbb{Z}_p^r) \leq \frac{r-1}{K(p)} + 1.$$

In particular, $a(\mathbb{Z}_p) = 1$.

We note that $a(\mathbb{Z}_p) \geq 1$ for all groups, so the second assertion follows immediately from the upper bound.

It is clear that $1 \leq K(p) \leq L(p)$. As mentioned before, it is conjectured that $L(p) = 1$ and so one might conjecture $K(p) = 1$. In this case the upper and lower bounds agree and $a(\mathbb{Z}_p^r) = r$.

Proof of the lower bound in 5.2 Let $q_j^{k_j} = pp_{\min}(p^j)$ be the least prime power congruent 1 modulo p^j (where q_j is a prime).

Let $\varepsilon > 0$. By assumption, $q_j^{k_j} > p^{j(K(p)-\varepsilon)}$ for all sufficiently large j, say $j \geq j_0$. Since $q_j^{k_j} \leq cp^{5j}$ by [51], we have $k_j \log(q_j) \leq \log(c) + 5j\log(p)$. Then, for all real $s > 1$, $\log \zeta_{\mathbb{Z}_p^r}(s)$ is at least

$$\sum_{j \geq j_0} \frac{p^{jr}}{k_j q_j^{k_j s}} \geq \sum_{j \geq j_0} \frac{p^{jr}}{k_j p^{js(K(p)-\varepsilon)}} \geq \sum_{j \geq j_0} \frac{p^{jr}\log(q_j)}{(\log(c)+5j\log(p))p^{js(K(p)-\varepsilon)}}$$

and this series diverges for $s < \frac{r}{K(p)-\varepsilon}$. We deduce that $a(\mathbb{Z}_p^r) \geq \frac{r}{K(p)}$. □

Let $\zeta_{\mathbb{Z}_p^r}$ be the Weil representation zeta function of $\mathbb{Z}_p^r$. Then

$$\log \zeta_{\mathbb{Z}_p^r}(s) = \sum_{q \text{ prime}} \sum_{k=1}^{\infty} \frac{w_p(q^k-1)^r}{k} q^{-sk};$$

recall that $w_p(m)$ denotes the highest p-power dividing m with the convention that $w_p(0) = 0$. It is well-known that the Riemann zeta function satisfies

$$\log \zeta(s) = \sum_n \frac{\Lambda(n)}{\log(n)} n^{-s}$$

where Λ denotes the von Mangoldt function. In the same way we can rewrite $\zeta_{\mathbb{Z}_p^r}(s)$ and obtain

$$\log \zeta_{\mathbb{Z}_p^r}(s) = \sum_n \frac{\Lambda(n)w_p(n-1)^r}{\log(n)} n^{-s}.$$

Proof of the upper bound in Proposition 5.2 Let $K := K(p)$. Let $\varepsilon > 0$ be given. We may assume that $K - \varepsilon \geq 1$, since for $K = 1$, we can even take $\varepsilon = 0$.

By assumption, there are only finitely many pairs (j, n) where n is a prime power with $n \equiv 1 \bmod p^j$ and $n^{1/(K-\varepsilon)} \leq p^j$. We define $w_p^\varepsilon(n-1)$ to be the highest power of p which divides $n-1$ and is at most $n^{1/(K-\varepsilon)}$. Using $\Lambda(n)/\log(n) \leq 1$ we obtain for all real $s > 1$:

$$\begin{aligned}
\log \zeta_{\mathbb{Z}_p^r}(s) &= \sum_n \frac{\Lambda(n)w_p(n-1)^r}{\log(n)} n^{-s} \\
&= \sum_n \frac{\Lambda(n)w_p^\varepsilon(n-1)^r}{\log(n)} n^{-s} + O(1) \\
&\leq \sum_n w_p^\varepsilon(n-1)^r n^{-s} + O(1).
\end{aligned}$$

Let $W^{\varepsilon}_{p,r}(x) = \sum_{n \le x} w^{\varepsilon}_p(n-1)^r$. Using a standard trick we get

$$\begin{aligned}
W^{\varepsilon}_{p,r}(x) = \sum_{n \le x} w^{\varepsilon}_p(n-1)^r &\le \lfloor x-1 \rfloor + \sum_{k=1}^{\lfloor \log_p(x)/(K-\varepsilon) \rfloor} \left\lfloor \frac{x-1}{p^k} \right\rfloor (p^{kr} - p^{(k-1)r}) \\
&\le (x-1) + \sum_{k=1}^{\lfloor \log_p(x)/(K-\varepsilon) \rfloor} \frac{x-1}{p^k}(p^{rk} - p^{r(k-1)}) \\
&= (x-1) + (1-p^{-r})(x-1) \sum_{k=1}^{\lfloor \log_p(x)/(K-\varepsilon) \rfloor} p^{(r-1)k} \\
&\le (x-1)\left(1 + \frac{1-p^{-r}}{1-p^{-(r-1)}} x^{(r-1)/(K-\varepsilon)}\right) \\
&\ll x^{(r-1)/(K-\varepsilon)+1}.
\end{aligned}$$

We use Abel's summation formula to obtain for all $x > p$

$$\begin{aligned}
\sum_{p<n \le x} w^{\varepsilon}_p(n-1)^r n^{-s} &= W^{\varepsilon}_{p,r}(x)x^{-s} - W^{\varepsilon}_{p,r}(p)p^{-s} - \int_p^x W^{\varepsilon}_{p,r}(u)(-su^{-s-1})du \\
&\ll x^{\frac{r-1}{K-\varepsilon}+1-s} + s\int_p^x u^{(r-1)/(K-\varepsilon)-s} du \\
&\ll x^{\frac{r-1}{K-\varepsilon}+1-s} + \frac{s}{\frac{r-1}{K-\varepsilon}+1-s} x^{\frac{r-1}{K-\varepsilon}+1-s}
\end{aligned}$$

the latter expression is bounded for all $s > \frac{r-1}{K-\varepsilon} + 1$ as $x \to \infty$. We conclude that $\log \zeta_{\mathbb{Z}_p^r}(s)$ converges absolutely for $\mathrm{Re}(s) > \frac{r-1}{K-\varepsilon} + 1$. □

Remark 5.3 Assuming $K(p) > 1$, it seems that the upper bound in Proposition 5.2 is of the right order of magnitude. In fact, from the prime number theorem in arithmetic progressions one would expect that there are roughly

$$\frac{p^{j(K(p)+\varepsilon)}}{j\varphi(p^j)} = \frac{p^{j(K(p)+\varepsilon-1)}}{j(1-p^{-1})}$$

primes congruent 1 mod p^j below $p^{j(K(p)+\varepsilon)}$ (at least for large j). An estimate of this form, i.e. a lower bound on $\psi(x; p^m, 1) = \sum_{\substack{n \le x \\ n \equiv 1 \bmod p^m}} \Lambda(n)$ for small x and infinitely many m, gives rise to a lower bound which matches the upper bound in Proposition 5.2. Some quantitative results concerning the amount of small primes in arithmetic progressions are available. For instance, it is a result of Banks-Shparlinski [4, Theorem 3.6] that $\psi(x; p^m, 1) \gg \frac{x}{\varphi(p^m)}$ for m large and $x \ge p^{mL}$ with $L \approx 2.1115$ which implies

$$a(\mathbb{Z}_p^r) \geq \frac{r-1}{L} + 1.$$

This improves our best numerical lower bound of $a(\mathbb{Z}_p^r) \geq \frac{r}{L}$ which can be deduced from Proposition 5.2, though whether we have $\frac{r-1}{L} + 1 \geq \frac{r}{K(p)}$ or the converse will depend on the value of $K(p)$. For $r = 2$, this gives $a(\mathbb{Z}_p^2) \geq 1 + L^{-1} \approx 1.4736 > 1$.

Remark 5.4 The proof of the upper bound shows similarly that any finite product $\prod \mathbb{Z}_p$ over distinct primes p has Weil abscissa 1 (though, as we have already seen, the product over all primes has abscissa of convergence 2). But in fact, we can show that in some cases, $\prod_k \mathbb{Z}_{p_k}$ has Weil abscissa 1 for an infinite sequence of primes (p_k).

Indeed, suppose we choose $p_k > 2^{(2^k)}$, and let $G = \prod_k \mathbb{Z}_{p_k}$. We may define $c_{(p_k)}(n)$ to be the largest divisor of n all of whose prime factors are in (p_k), so that $\log(\zeta_G)(s) \leq \sum_n c_{(p_k)}(n-1)n^{-s}$. Write $f(x)$ for the number of positive integers n less than x such that $c_{(p_k)}(n) = n$. As in Proposition 5.2, Tschebyscheff's trick shows that $\sum_{n \leq x} c_{(p_k)}(n-1)$ is at most $(x-1)f(x)$.

Let $x = 2^{(2^k)}$; then $f(x)$ is at most the number of partitions of 2^k into powers of 2. It is shown in [7, (1.3)] that the number of such partitions is $2^{O(k^2)}$. Therefore, $\sum_{n \leq 2^{(2^k)}} c_{(p_k)}(n-1) \leq 2^{2^k + O(k^2)}$. Finally, we can use Abel's summation formula as before to show that $\sum_{n \leq 2^{(2^k)}} c_{(p_k)}(n-1)n^{-s} = O(2^{2^k + O(k^2) - 2^k s})$ converges when $s > 1$.

5.3 Free pro-C groups, I

Let $\mathfrak{C}$ be a class of finite groups which is closed under quotients, finite subdirect products, taking normal subgroups and extensions (i.e., $\mathfrak{C}$ an NE-formation in the sense of [43, §2.1]). Let $F_r^{\mathfrak{C}}$ be the free pro-$\mathfrak{C}$ on r generators; these exist by [43, Section 3.3] and satisfy the usual universal property. An open normal subgroup of index m is again a free pro-$\mathfrak{C}$ group of rank $m(r-1)+1$ by [43, Theorem 3.6.2]. Here we prove a general lower bound result for the Weil abscissa of $F_r^{\mathfrak{C}}$.

Theorem 5.5 *Let q be a prime power. Assume that $\mathfrak{C}$ contains a non-trivial abelian group of order coprime to q and an absolutely irreducible subgroup $S \subseteq \mathrm{GL}(n_0, q)$ for some $n_0 \geq 1$. Then*

$$a(F_r^{\mathfrak{C}}) \geq \frac{\log |S|}{n_0 \log q}(r-1) + 1.$$

Lemma 5.6 *Let p be a prime and let L/F be an extension of finite fields such that p divides $(|L|-1)/(|F|-1)$. Let G be a profinite group and let $N \leq G$ be a normal subgroup of index p. Every G-invariant absolutely irreducible representation of N over F extends to an absolutely irreducible representation of G over L.*

Proof Let θ be an absolutely irreducible G-invariant representation of N defined over F. Let $\omega \in H^2(G/N, F^\times)$ be the obstruction cocycle (see [26, Section 9]). Since G/N is cyclic of order p, we have $H^2(G/N, A) = A/pA$ for all modules A with trivial G/N-action (see [26, Lemma 5.5]). Since $F^\times$ and $L^\times$ are cyclic the map $H^2(G/N, F^\times) \to H^2(G/N, L^\times)$ is trivial, i.e., by [26, Thm. 9.6] θ extends to G over L. $\square$

Proof of Theorem 5.5 We assume $r \geq 2$; in fact, for $r = 1$ there is nothing to do. We may also assume that S is non-trivial. Since $\mathfrak{C}$ contains a non-trivial abelian group, there is a prime number p (not dividing q) such that $\mathfrak{C}$ contains all p-groups. We only consider representations over the field $\mathbb{F}_q$. Let $m = |S|$. The number of surjective homomorphisms from $F_r^{\mathfrak{C}}$ onto S is $m^r - O((m/2)^r)$, where the error term depends on the maximal subgroups of S. A conjugacy class of homomorphisms contains at most $\frac{|\operatorname{GL}(n_0,q)|}{q-1}$ elements. We deduce that the number of equivalence classes of representations $r^S(F_r^{\mathfrak{C}})$ with image $S \subseteq \operatorname{GL}(n_0, q)$ satisfies

$$r^S(F_r^{\mathfrak{C}}) \geq \lambda m^{r-1}\big(1 - O(2^{-r+1})\big)$$

as r tends to infinity with $\lambda = \frac{m(q-1)}{|\operatorname{GL}(n_0,q)|}$.

Let $k \geq 1$ and let $\phi\colon F_r^{\mathfrak{C}} \to \mathbb{Z}/p^k\mathbb{Z}$ be a surjective homomorphism. Let $N = \ker(\phi)$ and let N^+ be the unique normal subgroup containing N with index p. We note that N is free of rank $p^k(r-1)+1$ and N^+ is free of rank $d = p^{k-1}(r-1)+1$. We claim that most irreducible representations of N with image $S \subseteq \operatorname{GL}(n_0, q)$ are not N^+-invariant. From Lemma 5.6 we know that an N^+-invariant representation of N over $\mathbb{F}_q$ extends to N^+ over a suitable field extension $L/\mathbb{F}_q$ (whose degree depends only on p and q). Let $x_1, \ldots, x_d$ be a free generating set of N^+ with $x_1, \ldots, x_{d-1} \in N$ (To find such a generating set, one can start with an arbitrary free generating set of N^+ such that $\phi(x_d)$ has order p and replace x_i by $x_i x_d^{k_i}$ for suitable $k_i \in \mathbb{Z}$). The number of homomorphisms of N^+ into $\operatorname{GL}_{n_0}(L)$ that map $x_1, \ldots, x_{d-1}$ into $S \subseteq \operatorname{GL}(n_0, q)$ is $m^{d-1} \cdot |\operatorname{GL}_{n_0}(L)|$ and we conclude that the number of N^+-invariant representations satisfies

$$r^S(N)^{N^+} = O(m^{p^{k-1}(r-1)}).$$

and so

$$r^S(N) - r^S(N)^{N^+} \geq \lambda m^{p^k(r-1)}\left(1 - O\big(2^{-p^{k-1}(r-1)}\big)\right).$$

If θ is an absolutely irreducible representation of N which is not N^+-invariant, then θ has trivial inertia subgroup in $F_r^{\mathfrak{C}}$ since N^+ is the unique minimal normal subgroup of in G/N. Therefore the induced representation $\operatorname{Ind}_N^{F_r^{\mathfrak{C}}}(\theta)$ is absolutely irreducible. Moreover, only $|G/N| = p^k$ distinct conjugates of θ give rise to the same induced representation. We obtain that

$$r^*(F_r^{\mathfrak{C}}, \mathbb{F}_q, p^k n_0) \geq \kappa p^{-k} m^{p^k(r-1)}$$

for all sufficiently large k and a suitable constant κ. This lower bound implies that the series

$$\log(\zeta_{F_r^{\mathfrak{C}}})(s) = \sum_{n,\ell,j} \frac{r^*(F_r^{\mathfrak{C}}, \mathbb{F}_{\ell^j}, n)}{j} \ell^{-jns} \frac{\ell^{jn}-1}{\ell^j - 1} \geq \kappa \sum_k m^{p^k(r-1)} p^{-k} q^{(1-s)p^k n_0}$$

diverges for real numbers $s < \frac{\log(m)}{n_0 \log(q)}(r-1)+1$. □

Corollary 5.7 *In Theorem 5.5 assume in addition that $\mathfrak{C}$ contains a transitive permutation group T of degree $d > 1$. Then*

$$a(F_r^{\mathfrak{C}}) \geq \frac{\log\big(|S|^{(d-1)}|T|\big)}{(d-1)n_0 \log q}(r-1)+1.$$

Proof Let $k \geq 1$. We consider the wreath product

$$W_k = S \wr \underbrace{T \wr T \wr \cdots \wr T}_{k \text{ times}}$$

constructed from the permutation representation of T on d elements. Since W_k is an extension of direct products of groups in $\mathfrak{C}$, it belongs to $\mathfrak{C}$. We will show that W_k has a faithful absolutely irreducible representation of degree $d^k n_0$ over $\mathbb{F}_q$. Then Theorem 5.5 gives

$$\begin{aligned} a(F_r^{\mathfrak{C}}) - 1 &\geq \frac{\log|W_k|}{d^k n_0 \log(q)}(r-1) = \frac{\log(|S|^{d^k}|T|^{(d^k-1)/(d-1)})}{d^k n_0 \log(q)}(r-1) \\ &= \frac{\log(|S|^{d-1}) + (1-\frac{1}{d^k})\log(|T|))}{(d-1)n_0\log(q)}(r-1) \end{aligned}$$

and the result follows by letting k tend to infinity.

Let $N = \prod_{i=1}^{d^k} S_i$ be the normal base group of W_k. To see that W_k admits an absolutely irreducible faithful representation of degree $d^k n_0$ we fix a copy $S_1 \subseteq N$ of S in the base group. Then S_1 has a faithful representation θ of degree n_0 over $\mathbb{F}_q$. We extend this representation trivially to a representation θ' of the normalizer $N_{W_k}(S_1)$. This is possible since the normalizer is a direct product $S_1 \times W'$; indeed it is generated by N and a point stabilizer in $T \wr T \wr \cdots \wr T$. In particular, the normalizer has index d^k in W_k. The induced representation $\mathrm{Ind}_{N_{W_k}(S_1)}^{W_k}(\theta')$ has degree $d^k n_0$. It is absolutely irreducible because the restriction to N consists of a single W_k-orbit of irreducible representations. Moreover, it is faithful, because both θ and the permutation representation of $T \wr T \wr \cdots \wr T$ on d^k elements are faithful. □

We also include here a result which will be useful in proving upper bounds.

Proposition 5.8 *Let $G \leq GL(n, p^j)$. Then the number of absolutely irreducible representations of dimension n over $\mathbb{F}_{p^j}$ of an r-generated group F with image contained* *in G is at most $|G|^{r-1}|G \cap \mathbb{F}_{p^j}^\times| \le |G|^{r-1} p^j$, where $\mathbb{F}_{p^j}^\times \le GL(n, p^j)$ is the group of diagonal matrices.*

Proof An absolutely irreducible representation of F is just a homomorphism $F \twoheadrightarrow J \le GL(n, p^j)$, up to conjugacy in $GL(n, p^j)$, with J absolutely irreducible. The number of such homomorphisms with $J \le G$ is at most $|G|^r$. The size of a conjugacy class is at least $|N_{GL(n,p^j)}(G)|/|C_{GL(n,p^j)}(J)| \ge |G|/|C_{GL(n,p^j)}(J) \cap G| \ge |G|/p^j$, because J is absolutely irreducible. Therefore the number of conjugacy classes is at most $|G|^r \frac{p^j}{|G|}$, as required. □

5.4 Free pro-p groups (odd p)

Let p be an odd prime number. We define

$$K'(p) = \inf_{k \ge 1} \frac{\log(p p_{\min}(p^k))}{(k + \frac{1}{p-1}) \log(p)}.$$

Theorem 5.9 *Let F_r^p be the free pro-p group on r generators with $r \ge 2$. The abscissa of convergence is*

$$a(F_r^p) = \frac{r-1}{K'(p)} + 1.$$

Before we give the proof, some comments on the constant $K'(p)$ are in order. We note that

$$K'(p) \le \inf_{k \ge 1} \frac{\log(p p_{\min}(p^k))}{k \log(p)} \le K(p),$$

where $K(p)$ is the constant from Sect. 5.2. The discussion there gives $K'(p) \le 2.1115$ and conjecturally $K'(p) \le 1$. One can determine the precise value of $K'(p)$ if p is a Mersenne prime.

Proposition 5.10 *Let p be an odd prime.*

(i) *If p is not a Mersenne prime, $K'(p) \ge 1$.*
(ii) *If p is a Mersenne prime, $K'(p) = \frac{(p-1)\log(p+1)}{p \log(p)} < 1$.*

Proof (i) Since the size $S(n, q^j)$ of a Sylow p-subgroup of $GL(n, q^j)$ is at most $w_p(q^j - 1)^n p^{\frac{n-1}{p-1}}$, with equality when n is a power of p, we see that $K'(p)$ is maximal such that $|S(n, q^j)|^{K'(p)} \le q^{nj}$. So when p is not a Mersenne prime, $K'(p) \ge 1$ by [50, Theorem 1.6(ii)]; the result follows.

(ii) Suppose $p = 2^m - 1$ is a Mersenne prime. Let $f_p(k) = \frac{\log(p^k+1)}{(k+1/(p-1))\log(p)} \le \frac{\log(p p_{\min}(p^k))}{(k+1/(p-1))\log(p)}$. When $p > 3$, a calculation shows $f_p(k)$ has a minimum when $k = 1$, and so $K'(p) = f_p(1)$. When $p = 3$, $f_3(k) > f_3(1)$ for all $k \ge 3$. Since

19 is the smallest prime power congruent 1 modulo 9, we conclude that for $k = 2$ we have $\frac{\log(19)}{(2+1/(p-1))\log(p)} > f_3(1)$, so we conclude that $K'(3) = f_3(1)$. □

Example 5.11 For the first four Mersenne primes one obtains

$$K'(3) \approx 0.8412, \quad K'(7) \approx 0.9160, \quad K'(31) \approx 0.9767, \quad K'(127) \approx 0.9937.$$

Even if p is not a Mersenne prime the formula for $K'(p)$ yields upper bounds close to 1 already for small values of k. For instance,

$$K'(5) \leq \frac{\log(1251)}{(4+\frac{1}{4})\log(5)} \approx 1.0426.$$

Example 5.12 Let $G = F_r^p$. Then $a(G) = \frac{r-1}{K'(p)} + 1$. If H is an open subgroup of index i, by the Nielsen-Schreier theorem for free pro-p groups [43, Theorem 3.6.2], H is free pro-p on $i(r-1)+1$ generators; hence $a(H) = \frac{i(r-1)}{K'(p)} + 1$. This gives $a(H)/a(G) = i\frac{1+\frac{K'(p)}{i(r-1)}}{1+\frac{K'(p)}{(r-1)}}$. Since the right hand side approaches i as r tends to infinity, this shows that the upper bound $a(H)/a(G) \leq i$ given in Proposition 4.3(ii) is sharp.

As in the previous section $w_p(n) = |n|_p^{-1}$ denotes the highest p-power dividing n. We begin with an upper bound result.

Lemma 5.13 *Let q be a prime power and let $k \geq 0$. If p divides $q-1$, then*

$$r^*(F_r^p, \mathbb{F}_q, p^k) \leq w_p(q-1)^{p^k(r-1)+1} p^{\frac{(p^k-1)(r-1)}{p-1}}.$$

If n is not a p-power or $n > 1$ and p doesn't divide $q-1$, then $r^*(F_r^p, \mathbb{F}_q, n) = 0$.

Proof It is well-known that the degrees of absolutely irreducible representations of finite p-groups are p-powers. Since the centre of the image of a non-trivial absolutely irreducible representation acts by a non-trivial character, such representations require the existence of pth roots of units, i.e. $p \mid q-1$.

Write $n = p^k$ and assume $p \mid q-1$. Fix a Sylow p-subgroup S_p of $\mathrm{GL}(n,q)$, which is isomorphic to

$$C_{w_p(q-1)} \underbrace{\wr C_p \wr \cdots \wr C_p}_{k \text{ times}},$$

and in general $|S_p| \leq w_p(q-1)^n p^{\frac{n-1}{p-1}}$: see [49, Section 2]. The claimed upper bound follows from Proposition 5.8. □

Proof of Theorem 5.9 The upper bound $a(F_r^p) \leq \frac{r-1}{K'(p)} + 1$ for the Weil abscissa follows from Proposition 5.2 in combination with $r^*(F_r^p, \mathbb{F}_q, n) \leq q^{n(r-1)/K'(p)} w_p(q-1)$ (which can be deduced from Lemma 5.13). In fact, for real s we have

$$\begin{aligned}\log \zeta_{F_r^p}(s) &\leq \sum_{q,n} q^{n(r-1)/K'(p)} w_p(q-1) q^{-sn} \frac{q^n - 1}{q-1} \\ &\ll \sum_q w_p(q-1) q^{-1} \sum_{n=1}^{\infty} q^{n\left((r-1)/K'(p)+(1-s)\right)} \\ &\ll \sum_q w_p(q-1) q^{(r-1)/K'(p)+s} = \log \zeta_{\mathbb{Z}_p}\left(s - \frac{r-1}{K'(p)}\right)\end{aligned}$$

Let $k \geq 1$. The cyclic group of order p^k admits a 1-dimensional faithful absolutely irreducible representation over $\mathbb{F}_q$ with $q = pp_{\min}(p^k)$ and the cyclic group of order p admits a faithful permutation representation on p elements. Corollary 5.7 implies that

$$a(F_r^p) \geq \frac{\log(p^{k(p-1)+1})}{(p-1)\log(q)}(r-1) + 1 = \frac{(k + \frac{1}{p-1})\log(p)}{\log(pp_{\min}(p^k))}(r-1) + 1.$$

Taking the supremum gives the lower bound. □

5.5 Free pro-2 groups

We consider separately the case $p = 2$. Let F_r^2 be the free pro-2 group on r generators, with $r \geq 2$.

Theorem 5.14 *The abscissa of convergence for the free pro-2 group F_r^2 of rank $r \geq 2$ is $a(F_r^2) = \frac{5\log(2)}{2\log(3)}(r-1) + 1$.*

We take an approach similar to that used for odd primes, however, the proof requires a number of modifications. We first discuss the upper bound for $a(F_r^2)$. As usual we have $r^*(F_r^2, \mathbb{F}_{2^j}, n) = 1$ for $n = 1$ and 0 otherwise.

Lemma 5.15 *Let q be an odd prime power. If $q \equiv 1 \bmod 4$, then*

$$r^*(F_r^2, \mathbb{F}_q, 2^k) \leq 2^{(2^k-1)(r-1)} w_2(q-1)^{2^k(r-1)+1}.$$

If $q \equiv 3 \bmod 4$, then

$$r^*(F_r^2, \mathbb{F}_q, 2^k) \leq 2^{(2^k+2^{k-1}-1)(r-1)+1} w_2(q+1)^{2^{k-1}(r-1)}.$$

Proof Let S be a Sylow 2-subgroup of $\mathrm{GL}(2^k, q)$. By Proposition 5.8, we have

$$r^*(F_r^2, \mathbb{F}_q, 2^k) \leq |S|^{r-1} |Z \cap S|$$

where $Z \subseteq \mathrm{GL}_{2^k}(\mathbb{F}_q)$ denotes the group of scalar matrices. The order of the Sylow 2-subgroups of $\mathrm{GL}(2^k, q)$ can be found in [8, Section 1]. When $q \equiv 1 \bmod 4$ we have $|S| = 2^{2^k-1} w_2(q-1)^{2^k}$ and $|S \cap Z| = w_2(q-1)$. When $q \equiv 3 \bmod 4$ we have $|S| = 2^{2^k+2^{k-1}-1} w_2(q+1)^{2^{k-1}}$ and $|S \cap Z| = 2$. □

This gives an upper bound for the zeta function:

$$\log(\zeta_{F_r^2})(s) \leq \sum_{q,n} \frac{\Lambda(q)}{\log(q)} 2^{(n-1)(r-1)} w_2(q-1)^{n(r-1)+1} \frac{q^n-1}{q-1} q^{-ns} + 2 \sum_{\substack{q \equiv 3 \bmod 4 \\ n \text{ even}}} \frac{\Lambda(q)}{\log(q)} 2^{(n+\frac{n}{2}-1)(r-1)} w_2(q+1)^{\frac{n}{2}(r-1)} \frac{q^n-1}{q-1} q^{-ns}.$$

We show exactly as for F_r^p with p odd that the first sum converges when $s > (r-1)/K'(2) + 1$ with

$$K'(2) = \inf_{k \geq 1} \frac{\log(pp_{\min}(2^k))}{(k+1)\log(2)}.$$

Recall that $pp_{\min}(2^k)$ denotes the smallest prime power congruent 1 modulo 2^k. Similarly, let $pp_{\min}^{-}(2^k)$ denote the smallest prime power congruent -1 modulo 2^k. Define

$$K^{-} = \inf_{k \geq 1} \frac{2\log(pp_{\min}^{-}(2^k))}{(k+3)\log(2)}.$$

Then, for $q \equiv 3 \bmod 4$ and n even, $r^*(F_r^2, \mathbb{F}_q, n) \leq q^{n(r-1)/K^{-}}$, so the second sum converges when $s > (r-1)/K^{-} + 1$. As in Proposition 5.10, [50, Theorem 1.6(i), Proposition 1.7] shows that $K'(2) \geq K^{-} = \frac{2\log(3)}{5\log(2)} \approx 0.633985$. Overall, we conclude that $a(F_r^2) \leq \frac{5\log(2)}{2\log(3)}(r-1) + 1$.

We now turn to the lower bound. The proof for the upper bound suggests that the crucial prime power is $q = 3$ and it turns out that it is sufficient to consider this case.

Here the 2-dimensional representations will serve as our base case. A Sylow 2-subgroup S of $\mathrm{GL}(2,3)$ is a semidihedral group of order 16. We observe that S is absolutely irreducible, because it is non-abelian. Indeed, an abelian group cannot be absolutely irreducible in dimension 2. Conversely, since S is a non-abelian 2-group, the representation on $\overline{\mathbb{F}_3}$ is semisimple and cannot decompose into 1-dimensional pieces. Using the action of the 2-element group on two points, we deduce from Corollary 5.7 that

$$a(F_r^2) \geq \frac{\log(16 \cdot 2)}{2\log(3)}(r-1) + 1 = \frac{5\log(2)}{2\log(3)}(r-1) + 1.$$

5.6 Free prosoluble groups

In this section we discuss an upper bound for the Weil abscissa of free prosoluble groups. Let $c_{sol} = \frac{2}{3} + \frac{5\log(2)}{2\log(3)} \approx 2.24399$.

Theorem 5.16 *Let F_r^{sol} be the free prosoluble group on r generators, $r \geq 2$. Then $a(F_r^{sol}) = c_{sol}(r-1) + 1$.*

We collect some results needed in the proof. By [50, Theorem 3.1] and [44, Corollary to Theorem A] we have:

Lemma 5.17 *Let q be a prime power. The order of a soluble irreducible subgroup of $\mathrm{GL}(n, q)$ is $\leq q^{c_{sol}n}$. For $q \geq 11$, the order of a soluble irreducible subgroup of $\mathrm{GL}(n, q)$ is $\leq q^{(1+\log_q(24)/3)n}$.*

We also use [6, Theorem 14.1]:

Lemma 5.18 *The number of conjugacy classes of maximal irreducible soluble subgroups of $\mathrm{GL}(n, q)$ is at most $n^{4\log^3(n)+4\log^2(n)+\log(n)+3}$.*

Fix a prime power q and fix a maximal soluble subgroup M in each conjugacy class. By Proposition 5.8, the number of $\mathrm{GL}(n, q)$-conjugacy classes of homomorphisms to M with absolutely irreducible image is at most $|M|^{r-1}(q-1)$. We conclude that

$$r^*(F_r^{sol}, \mathbb{F}_q, n) \leq n^{4\log^3(n)+4\log^2(n)+\log(n)+3}(q-1)q^{c_{sol}n(r-1)}$$

for all q, and

$$r^*(F_r^{sol}, \mathbb{F}_q, n) \leq n^{4\log^3(n)+4\log^2(n)+\log(n)+3}(q-1)q^{(1+\log_q(24)/3)n(r-1)}$$

for $q \geq 11$.

For any $\varepsilon > 0$, pick Q such that $\log_Q(24)/3 < \varepsilon$. We split the series defining $\log(\zeta_{F_r^{sol}})(s)$ into terms with $q > Q$ and terms with $q \leq Q$. By Lemmas 2.2 and 4.2, the first sum converges when $s > (1+\varepsilon)(r-1)+2$, while, for each $q \leq Q$, we have the series

$$\sum_n \frac{n^{O(\log^3(n))}}{j} q^{(c_{sol}(r-1)+1-s)nj},$$

which converges when $s > c_{sol}(r-1)+1$. Since the second of these bounds is larger for $r \geq 2$, we conclude $a(F^{sol}) \leq c_{sol}(r-1)+1$.

The lower bound follows from Corollary 5.7. We note that $\mathrm{GL}(2, 3)$ is an absolutely irreducible soluble group of order 48 and that S_4 is a soluble group of order 24. Corollary 5.7 allows us to deduce

$$a(F_r^{sol}) - 1 \geq \frac{\log(48^3 \cdot 24)}{2 \cdot 3 \cdot \log(3)}(r-1) = c_{sol}(r-1)$$

5.7 Free pronilpotent groups

The results for free pro-p groups may be used to determine the Weil abscissa for free pronilpotent groups. We note that the class of nilpotent groups is not closed under extensions, so that the results from Sect. 5.3 cannot be applied directly. Let $F_r^{nil} = \prod_p F_r^p$ be the free pronilpotent group on r generators, with $r \geq 2$.

Theorem 5.19 *Let $r \geq 2$. The Weil abscissa of $\zeta_{F_r^{nil}}$ is*

$$a(F_r^{nil}) = \begin{cases} 3 & \text{if } r = 2 \\ \frac{5\log(2)}{2\log(3)}(r-1)+1 & \text{if } r > 2. \end{cases}$$

In other words, for $r \geq 3$ the free pro-2 factor of F_r^{nil} is responsible for most of the representations of F_r^{nil}. In particular, the lower bound follows immediately from 5.14. For $r = 2$ the lower bound follows from $a(\widehat{\mathbb{Z}}^2) = 3$ obtained in Sect. 5.1. We will study the corresponding upper bound now.

Lemma 5.20 *Let $n \geq 1$ and let q be a prime power. Then*

$$r^*(F_r^{nil}, \mathbb{F}_q, n) \leq (q-1)^r q^{\frac{5\log(2)}{2\log(3)}(r-1)n}.$$

Proof Let $n = p_1^{k_1} \cdots p_l^{k_l}$ be the prime decomposition. Every absolutely irreducible representation of dimension n over $\mathbb{F}_q$ is, by [15], of the form

$$\theta_1 \otimes \cdots \otimes \theta_l \otimes \chi$$

where θ_i is an absolutely irreducible representation of dimension $p_i^{k_i}$ of $F_r^{p_i}$ over $\mathbb{F}_q$ and $\chi : \prod_{p \nmid n} F_r^p \to \mathbb{F}_q^\times$ is a homomorphism. Now Proposition 5.8 gives

$$r^*(F_r^{nil}, \mathbb{F}_q, n) \leq (q-1)^r \prod_{i=1}^{l} S(p_i, q)^{r-1}$$

where $S(p_i, q)$ is the order of a Sylow p_i-subgroup of $\mathrm{GL}(p_i^{k_i}, q)$.

We note that q and n are coprime. So we know by [50, Theorem 1.6] that $S(p_i, q) \leq q^{\frac{5\log(2)}{2\log(3)} p_i^{k_i}}$. Therefore we have

$$r^*(F_r^{nil}, \mathbb{F}_{q^j}, n) \leq (q-1)^r q^{\frac{5\log(2)}{2\log(3)}(r-1)\sum_{i=1}^{l} p_i^{k_i}} \leq (q-1)^r q^{\frac{5\log(2)}{2\log(3)}(r-1)n}.$$

$\square$

Proof of the upper bounds in Theorem 5.19 We adapt the argument used for the free prosoluble group. For any $\varepsilon > 0$, pick Q such that $\log_Q(24)/3 < \varepsilon$. We split the sum $\log(\zeta_{F_r^{nil}})(s)$ into terms with $q > Q$ and terms with $q \leq Q$. As for prosoluble groups we infer $r^*(F_r^{nil}, \mathbb{F}_q, n) \le n^{O(\log^3(n))}(q-1)q^{(1+\log_q(24)/3)n(r-1)}$ for $q \ge 11$ from Lemmas 5.17 and 5.18. Thus we see by Lemmas 2.2 and 4.2 that the first sum converges when $s > (1+\varepsilon)(r-1)+2$, while, for each $q \le Q$, we have the upper bound

$$\sum_n (q-1)^r q^{(\frac{5\log(2)}{2\log(3)}(r-1)+1-s)n} \le Q^r \sum_n q^{(\frac{5\log(2)}{2\log(3)}(r-1)+1-s)n},$$

which converges when $s > \frac{5\log(2)}{2\log(3)}(r-1)+1$.

We note that $\frac{5\log(2)}{2\log(3)} > \frac{3}{2}$. Therefore, when $r > 2$, the second bound is larger, and we conclude $a(F_r^{nil}) \le \frac{5\log(2)}{2\log(3)}(r-1)+1$. When $r = 2$, the first bound is larger, and we conclude $a(F_2^{nil}) \le 3$. □

5.8 Pronilpotent groups of finite rank

Let G be a profinite group. Recall that the rank of G is the supremum of the minimal number of generators of all open subgroups; see [12, 3.11].

Proposition 5.21 *Let G be a pronilpotent group of finite rank r. Then $a(G) \le r+1$.*

The bound is sharp for the free abelian profinite group $\hat{\mathbb{Z}}^r$.

Proof We consider the one-dimensional and higher dimensional representations separately. The one-dimensional representations factor through the abelianisation, which itself is a factor of $\hat{\mathbb{Z}}^r$. Hence the sum of the contributions for all one-dimensional representations converges for $\mathrm{Re}(s) > r+1$; see 5.1.

Let $n \ge 1$ and let q be a prime power. When $q \equiv 1 \bmod 4$, or when n is odd, every absolutely irreducible representation of G over $\mathbb{F}_q$ is monomial by [11, Theorem 5.3]. When $q \equiv 3 \bmod 4$ and n is even, every absolutely irreducible representation of G over $\mathbb{F}_{q^j}$ is induced from one of dimension ≤ 2, by [11, Theorem 5.1]. We may count the absolutely irreducible primitive representations of dimension 2 using the structure theory of [10]: by [10, Proposition 4.2], an absolutely irreducible primitive nilpotent subgroup G of $\mathrm{GL}(2, q)$ is a direct product $G_2 \times C$ where G_2 is a primitive 2-group and C is a group of scalars of odd order. If $q \equiv 3 \bmod 4$, then $4w_2(q+1)$ is the order of a Sylow 2-subgroup of $\mathrm{GL}(2, q)$. Let $a_n(G)$ denote the number of subgroups of index n in G. Taking into account the monomial representations and, if needed, those induced from 2-dimensional representations, we deduce

$$r^*(G, \mathbb{F}_q, n) \le a_n(G)(q-1)^r + \delta_{n,q} a_{n/2}(G)(4c_2(q+1))^r \left(\frac{q-1}{2}\right)^r$$

where $\delta_{n,q} \in \{0, 1\}$ depending on whether $q \equiv 3 \bmod 4$ and n is even. Since G has finite rank, it has polynomial subgroup growth (see [34, 10.1]); hence, we get an upper bound of the form

$$r^*(G, \mathbb{F}_q, n) \le b_n q^{2r}$$

where b_n is a function which grows at most polynomially in n.

We use this to estimate the contribution to ζ_G from representations of dimension at least 2 and obtain

$$\sum_{q,j}\sum_{n\geq 2}\frac{r^*(G,\mathbb{F}_{q^j},n)}{j}q^{-jns}\frac{q^{jn}-1}{q^j-1} \leq \sum_{q,j}\frac{1}{j}\sum_{n\geq 2}b_n q^{2rj}q^{-jns}q^{j(n-1)}$$
$$\leq \sum_{q,j}\frac{q^{(2r-1)j}}{j}\sum_{n\geq 2}b_n q^{j(1-s)n}$$

The inner sum converges for $\mathrm{Re}(s) > 1$ and the result is bounded by $q^{2(1-s)j+\varepsilon}$ for any $\varepsilon > 0$ up to a constant. We deduce that the series converges for $\mathrm{Re}(s) > r + 1 + \varepsilon/2$. Since ε was arbitrary, the result follows. □

5.9 Free pro-C groups, II

Let $\mathfrak{C}$ be a NE-formation of finite groups containing the cyclic groups of prime order. Let $F_r^{\mathfrak{C}}$ be the free pro-$\mathfrak{C}$ group on r generators. If $\mathfrak{C}$ contains alternating groups of arbitrarily large degree, or classical groups with natural representation of arbitrarily large dimension, an application of Theorem 5.5 shows that $F_r^{\mathfrak{C}}$ does not have UBERG. Assume it does not. Let c_0 be maximal such that $Alt(c_0) \in \mathfrak{C}$.

To make the results in this section work, we will need to assume that c_0 is sufficiently large. Specifically, let C_0 be an absolute constant such that $(C_0!)^{1/(C_0-1)} \geq \max((2c_t)^4, 2^{2c_4+4c_i+2})$, where c_t, c_4, c_i are the (effectively computable) constants defined in [24]; we will refer to these constants without further comment in the rest of this section. (By Stirling's approximation, $(C_0!)^{1/(C_0-1)} \to \infty$ as $C_0 \to \infty$, so the required C_0 does exist.) We will assume for the rest of Sect. 5.9 that $c_0 > C_0$.

To make the calculation of the Weil abscissa of $F_r^{\mathfrak{C}}$ easier, we start by showing that only certain types of absolutely irreducible representations need to be considered. Just as, for soluble groups, the Weil abscissa is dominated by representations whose image is in a wreath product $GL(2,\mathbb{F}_3)\wr Sym(4)\wr\cdots\wr Sym(4)$, here the abscissa is dominated by representations whose image is in a wreath product $N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta,\mathbb{F}_{p^k})}(E)\wr Sym(c_0)\wr\cdots\wr Sym(c_0)$, for E a classical or alternating group with natural representation over $\mathbb{F}_{p^k}$ of dimension β. (Here, for an alternating group, by the natural representation we mean the fully deleted permutation module defined in [28, Section 5.3, Alternating groups].) Note that, when $E \in \mathfrak{C}$ is classical or alternating, $\mathrm{Out}(E)$ and $C_{\Gamma L_{\mathbb{F}_{p^j}}(\beta,\mathbb{F}_{p^k})}(E)$ are soluble, so $N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta,\mathbb{F}_{p^k})}(E) \in \mathfrak{C}$ and hence $N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta,\mathbb{F}_{p^k})}(E)\wr Sym(c_0)\wr\cdots\wr Sym(c_0)$ is too.

For p a prime and $j, n \geq 1$ integers, consider the set of all classical or alternating groups $E \in \mathfrak{C}$ with natural representation over $\mathbb{F}_{p^k}$ of dimension β, such that $j|k$ and $n = \beta\frac{k}{j}c_0^l$, for some l. We define $r_D^*(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ to be the sum over all such E of

$$|N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta,\mathbb{F}_{p^k})}(E)\wr\underbrace{Sym(c_0)\wr\cdots\wr Sym(c_0)}_{l\text{ times}}|^{r-1}p^j.$$

Theorem 5.22 *The abscissa of convergence of* $\zeta_{F_r^{\mathfrak{C}}}(s)$ *is at most that of* $\Sigma_D = \sum_{p,j,n} \frac{r_D^*(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)}{j} p^{-snj} |\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|$.

We will prove this in a series of smaller results.

5.9.1 Permutation group results

We recall following the well known result of

Proposition 5.23 *Let* P *be a primitive permutation group of degree* n *and suppose that* P *does not contain* $Alt(n)$. *Then the order of* P *is at most* 4^n.

Proposition 5.24 *If* $G \leq Sym(n)$ *is a transitive permutation group in* $\mathfrak{C}$ *then* $\log(|G|) \leq \frac{n-1}{c_0-1} \log(c_0!)$, *and the orders of the iterated wreath products*

$$S_k = \underbrace{Sym(c_0) \wr \cdots \wr Sym(c_0)}_{k \text{ times}} \leq Sym(c_0^k)$$

attain this bound. Thus, $|G| \leq (c_0!)^{(n-1)/(c_0-1)}$, *and the supremum of* $\log(|G|)/n$ *over all* n *and* $G \in \mathfrak{C}$ *is* $\frac{\log(c_0!)}{c_0-1}$.

Proof We have $G \leq P_1 \wr \cdots \wr P_l$ for some primitive $P_1, \ldots, P_l$ of degrees $s_1, \ldots, s_l$ respectively, with P_i induced by the stabiliser in G of a minimal block in the quotient action of G on a set of $n/(s_1 \cdots s_{i-1})$ elements. If $s_1, \ldots, s_l \leq c_0$, the proof is a straightforward induction which we leave to the reader. Suppose instead that (without loss of generality) $s_1, \ldots, s_{i-1} \leq c_0$, and $s_i > c_0$. Write G_i (respectively, G_{i+1}) for the image of G in the quotient $P_i \wr \cdots \wr P_l$ (respectively, $P_{i+1} \wr \cdots \wr P_l$). By the inductive hypothesis, $\log(|G_{i+1}|) \leq \frac{s_{i+1} \cdots s_l - 1}{c_0-1} \log(c_0!)$. If P_i does not contain $Alt(s_i)$, $|P_i| \leq 4^{s_i}$ by Proposition 5.23, and we calculate $\log(|G_i|) \leq s_i \cdots s_l \log(4) + \log(|G_{i+1}|) \leq s_i \cdots s_l \log(4) + \frac{s_{i+1} \cdots s_l - 1}{c_0-1} \log(c_0!) \leq \frac{s_i \cdots s_l - 1}{c_0-1} \log(c_0!)$. Indeed, because $s_i > c_0 > C_0 > 9$, this implies that $\frac{\log(c_0!)}{c_0-1} > \frac{c_0+1}{c_0} \log(4) \geq \frac{s_i}{s_i-1} \log(4)$, and hence $s_i \cdots s_l \log(4) \leq (s_i - 1)s_{i+1} \cdots s_l \frac{\log(c_0!)}{c_0-1}$ and the result follows. If P_i contains $Alt(s_i)$, consider $H = G_i \cap P_i^{s_{i+1} \cdots s_l}$: this is normal in G_i, so in $\mathfrak{C}$. Since G permutes the factors transitively, if H is non-trivial, the image of H in each copy of P_i is a non-trivial normal subgroup of P_i, so it contains $Alt(s_i)$, giving a contradiction. Therefore H is trivial, and $\log(|G_i|) \leq \frac{s_{i+1} \cdots s_l - 1}{c_0-1} \log(c_0!) \leq \frac{s_i \cdots s_l - 1}{c_0-1} \log(c_0!)$.

Finally, $\log(|G|) \leq s_i \cdots s_l \log(|P_1 \wr \cdots P_{i-1}|) + \log(|G_i|)$, so by the inductive hypothesis, $\log(|G|) \leq \frac{\log(c_0!)}{c_0-1}((s_1 \cdots s_{i-1} - 1)s_i \cdots s_l + s_i \cdots s_l - 1) = \frac{n-1}{c_0-1} \log(c_0!)$, as required.

For the second claim, we have $|S_k| = |Sym(c_0)|^{c_0^{k-1} + c_0^{k-2} + \cdots + c_0^1}$, so $\log(|S_k|) = \frac{c_0^k - 1}{c_0-1} \log(c_0!)$. □

In fact the proof shows more:

Corollary 5.25 *Suppose $G \leq P_1 \wr \cdots \wr P_l$ is in $\mathfrak{C}$, with $P_1, \ldots, P_l$ primitive permutation groups of degree $s_1, \ldots, s_l$. If $s_1, \ldots, s_{i-1} \leq c_0$ and $s_i > c_0$, $|G| \leq 4^{s_i \cdots s_l} (c_0!)^{\frac{n - s_i \cdots s_l + s_{i+1} \cdots s_l - 1}{c_0 - 1}}$.*

Proof The proof of the proposition shows $\log(|G_i|) \leq s_i \log(4) + \frac{s_{i+1} \cdots s_l - 1}{(c_0 - 1)} \log(c_0!)$. So $\log(|G|) \leq s_i \cdots s_l \log(|P_1 \wr \cdots \wr P_{i-1}|) + s_i \cdots s_l \log(4) + \frac{s_{i+1} \cdots s_l - 1}{(c_0 - 1)} \log(c_0!) \leq \frac{n - s_i \cdots s_l + s_{i+1} \cdots s_l - 1}{c_0 - 1} \log(c_0!) + s_i \cdots s_l \log(4)$. □

Lemma 5.26 *Let $s_1, \ldots, s_l$ be fixed. Suppose $G \in \mathfrak{C}$ is an r-generated transitive permutation group such that $G \leq P_1 \wr \cdots \wr P_l$ for some primitive $P_1, \ldots, P_l$ of degrees $s_1, \ldots, s_l$. Suppose $Alt(s_1), \ldots, Alt(s_{i-1}) \in \mathfrak{C}$ and $Alt(s_i) \notin \mathfrak{C}$. Then up to conjugacy in $Sym(n)$, for the constant c_t defined in* [24, Theorem 3.1], *G is contained in one of at most $c_t^{r s_i \cdots s_l}$ subgroups of $Sym(s_1) \wr \cdots \wr Sym(s_{i-1}) \wr P_i \wr \cdots \wr P_l$ which are in $\mathfrak{C}$.*

Proof By [24, Theorem 3.1], there are at most $c_t^{r s_i \cdots s_l}$ possibilities for the image G' of G in $P_i \wr \cdots \wr P_l$ up to conjugacy. The result follows immediately. □

5.9.2 Linear group results

Suppose $G \leq GL(n, p^j)$ is in $\mathfrak{C}$ and irreducible. We can write $G \leq P \wr T$, where $P \leq GL(b, p^j)$ is primitive, and $T \leq Sym(t)$, $t = n/b$, is transitive and in $\mathfrak{C}$ – but note that in general P need not be in $\mathfrak{C}$. We can also write $T \leq P_1 \wr \cdots \wr P_l$ as in the last section, with $P_1, \ldots, P_l$ primitive permutation groups of degrees $s_1, \ldots, s_l$ such that $s_1 \cdots s_l = t$.

In the case $|P| > p^{c_4 bj}$, we will need to fix some additional notation, following [24, Proposition 5.7]. Suppose $\mathbb{F}_{p^{k'}} = Z(\mathrm{End}_{F^*(P)}(\mathbb{F}_{p^j}^b))$. In this case, there exist $A \leq \Gamma L_{\mathbb{F}_{p^j}}(\alpha, \mathbb{F}_{p^{k'}})$, $B \leq \Gamma L_{\mathbb{F}_{p^j}}(\beta, \mathbb{F}_{p^{k'}})$ such that $P \leq A \odot B$ (so that $\alpha\beta k' = bj$ and $t = (nj)/(\alpha\beta k')$). These have the property that $\mathbb{F}_{p^{k'}}^{\times} \leq A$, $\mathbb{F}_{p^{k'}}^{\times} \leq B$, and $\beta \geq \alpha$, so that $|A| \leq p^{(\alpha^2+1)k'} \leq p^{bj+k'}$. Also, $E(B)$ is either a classical group over a subfield $\mathbb{F}_{p^k}$ of $\mathbb{F}_{p^{k'}}$ with natural representation in dimension β with scalars extended to $\mathbb{F}_{p^{k'}}$, or an alternating group with natural representation over $\mathbb{F}_{p^{k'}}$ in dimension β.

For each classical group $E \in \mathfrak{C}$ over $\mathbb{F}_{p^k}$ with natural representation in dimension β, define c_E by $|N_{\Gamma L_{\mathbb{F}_p}(\beta, \mathbb{F}_{p^k})}(E)| = p^{c_E \beta k}$ (so that $|N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta, \mathbb{F}_{p^k})}(E)| = \frac{1}{j} p^{c_E \beta k}$). For each alternating group $E \in \mathfrak{C}$ with natural representation over $\mathbb{F}_{p^k}$ in dimension β, define c_{E,p^k} by $|N_{\Gamma L_{\mathbb{F}_p}(\beta, \mathbb{F}_{p^k})}(E)| = p^{c_{E,p^k} \beta k}$. We have $|N_{\Gamma L_{\mathbb{F}_p}(\beta, \mathbb{F}_{p^k})}(Alt(d))| = |\mathrm{Out}(Alt(d))| k(p^k - 1)$ (with $\mathrm{Out}(Alt(d)) = Sym(d)$ for $d \neq 6$), and $\beta = d - 1$ or $d - 2$, so an easy calculation shows that $c_{Alt(d), p^k} < c_{Alt(d), 2}$ for any $p^k > 2$ and $d \geq 5$. When the choice of field is clear, we may suppress the dependence on p^k in the notation.

We want to count representations of $F_r^{\mathfrak{C}}$ with image G. We consider several classes of possibilities for G. The strategy for each class is to show that G is contained in some larger subgroup of $GL(n, p^j)$ in $\mathfrak{C}$ which has an easy-to-describe form, such that we can bound both the order of these larger subgroups and the number of such subgroups up to conjugacy.

1. $|P| \leq p^{c_4 bj}$.

Proposition 5.27 *The number $r_1^*(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ of absolutely irreducible representations of $F_r^{\mathfrak{C}}$ with image in class 1 is at most*

$$p^{c_4(r-1)nj+j+c_i rnj}(c_0!)^{\frac{(r-1)(n/2-1)}{c_0-1}}. \tag{5.1}$$

Proof Suppose G is in class 1. By Proposition 5.24, $|G| \leq p^{c_4 nj}(c_0!)^{\frac{t-1}{c_0-1}}$, where $t \leq n/2$. By [24, Proposition 6.1], up to conjugacy in $GL(n, p^j)$, there are at most $p^{c_i rnj}$ possibilities for the image G of $F_r^{\mathfrak{C}}$. The result follows by Proposition 5.8. □

2. G is not in class 1, and $E(B)$ is not in $\mathfrak{C}$.

Proposition 5.28 *The number $r_2^*(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ of absolutely irreducible representations of $F_r^{\mathfrak{C}}$ with image in class 2 is at most $p^{2(r-1)nj+j+c_i rnj}(c_0!)^{\frac{(r-1)(n/2-1)}{c_0-1}}$.*

Proof Suppose G is in class 2. Consider $H = G \cap E(B)^t$, which is normal in G, so in $\mathfrak{C}$. If $H \not\leq Z(E(B))^t$, since the factors are permuted transitively, the image of H in each copy of $E(B)$ is a non-trivial normal subgroup of $E(B)$, so it contains $E(B)$, giving a contradiction. Therefore $H \leq Z(E(B))^t$. Also, $|N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta, \mathbb{F}_{p^{k'}})}(E(B))/E(B)|$ is at most

$$|C_{\Gamma L_{\mathbb{F}_{p^j}}(\beta, \mathbb{F}_{p^{k'}})}(E(B))||\operatorname{Out}(E(B)/Z(E(B)))| \leq \frac{k'}{j} p^{2bj} \leq \frac{1}{j} p^{3bj}$$

by [24, Lemma 2.6, Proposition 5.7]. So, by Proposition 5.24, $|G| \leq \frac{1}{j^t} p^{3nj}(c_0!)^{\frac{t-1}{c_0-1}}$, with $t \leq n/2$.

As for class 1, there are at most $p^{c_i rnj}$ possibilities for G. The result follows by Proposition 5.8. □

3. G is not in classes 1 or 2.

In this case, we have

$$|B| \leq |N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta, \mathbb{F}_{p^{k'}})}(E(B))| \leq \frac{k'}{kj} p^{c_{E(B)}\beta k + k'} \leq \frac{1}{j} p^{c_{E(B)}\beta k + 2k'}.$$

Moreover, $B \in \mathfrak{C}$, because $C_{GL(\beta, \mathbb{F}_{p^{k'}})}(E(B))$ is cyclic and $\operatorname{Out}(E(B))$ is soluble.

We divide class 3 into classes $3.E$ of groups G in class 3 such that $E(B) = E$, over all classical and alternating groups E in $\mathfrak{C}$. We subdivide these classes further.

$3.E.1$. $\alpha > 1$.

Proposition 5.29 *The number $r^*_{E.1}(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ of absolutely irreducible representations of $F_r^{\mathfrak{C}}$ with image in class 3.E.1 is at most*

$$\frac{nj}{\beta k} c_t^{rnj/(2\beta k)} \left(\frac{1}{j}\right)^{(r-1)(nj)/(\beta k)} p^{(c_E/2+2)(r-1)nj+j} (c_0!)^{\frac{(r-1)((nj)/(2\beta k)-1)}{c_0-1}}.$$

Proof Since $\beta k' \leq bj/2$, we have $|B| \leq \frac{1}{j} p^{c_{E(B)} bj/2+bj}$. Therefore $|P| \leq |A \odot B| = |A||B|/p^{k'} \leq \frac{1}{j} p^{c_{E(B)} bj/2+2bj}$, and hence, by Proposition 5.24,

$$|G| \leq \left(\frac{1}{j}\right)^{(nj)/(\beta k')} p^{(c_{E(B)}/2+2)nj} (c_0!)^{\frac{(nj)/(2\beta k')-1}{c_0-1}}.$$

In this class, we have $G \leq (\Gamma L_{\mathbb{F}_{p^j}}(\alpha, \mathbb{F}_{p^{k'}}) \odot N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta, \mathbb{F}_{p^{k'}})}(E(B))) \wr T$. There are at most $\frac{nj}{\beta k}$ choices for α and then, by [24, Theorem 3.1], at most $c_t^{rnj/(2\beta k)}$ choices for T up to conjugacy, giving $\frac{nj}{\beta k} c_t^{rnj/(2\beta k)}$ possibilities altogether. The result follows by Proposition 5.8. □

3.E.2. G is not in class 3.E.1, and $k' > k$.

Proposition 5.30 *The number $r^*_{E.2}(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ of absolutely irreducible representations of $F_r^{\mathfrak{C}}$ with image in class 3.E.2 is at most*

$$c_t^{rnj/(\beta k)} \left(\frac{1}{j}\right)^{(r-1)(nj)/(\beta k)} p^{(c_E/2+3)(r-1)nj+j} (c_0!)^{\frac{(r-1)((nj)/(\beta k)-1)}{c_0-1}}.$$

Proof Here we may choose B to be P, so

$$|P| = |B| \leq \frac{1}{j} p^{c_{E(B)} \beta k + 2k'} \leq \frac{1}{j} p^{(c_{E(B)}/2+2)\beta k'} = \frac{1}{j} p^{(c_{E(B)}/2+2)bj},$$

and hence $|G| \leq \left(\frac{1}{j}\right)^{(nj)/(\beta k')} p^{(c_{E(B)}/2+2)nj} (c_0!)^{\frac{(nj)/(\beta k')-1}{c_0-1}}$ by Proposition 5.24.

In this class, we have $G \leq N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta, \mathbb{F}_{p^{k'}})}(E(B))) \wr T$. There are at most $c_t^{rnj/(\beta k)}$ choices for T up to conjugacy, by [24, Theorem 3.1], and hence at most $c_t^{rnj/(\beta k)}$ possibilities for $N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta, \mathbb{F}_{p^{k'}})}(E(B))) \wr T$. The result follows by Proposition 5.8. □

3.E.3. G is not in classes 3.E.1 or 3.E.2, and some s_i is greater than c_0.

Proposition 5.31 *The number $r^*_{E.3}(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ of absolutely irreducible representations of $F_r^{\mathfrak{C}}$ with image in class 3.E.3 is at most*

$$2^{\log_2((nj)/(\beta k))^2} c_t^{rs_i\cdots s_l} \left(\frac{1}{j}\right)^{(r-1)(nj)/(\beta k)} p^{c_E(r-1)nj+j} 4^{(r-1)s_i\cdots s_l} \cdot \cdot (c_0!)^{\frac{(r-1)((nj)/(\beta k)-s_i\cdots s_l+s_{i+1}\cdots s_l-1)}{c_0-1}}.$$

Proof Note that $l \le \log_2((nj)/(\beta k))$, and there are at most $(nj)/(\beta k)$ choices for each s_i, so there are at most $((nj)/(\beta k))^{\log_2((nj)/(\beta k))} = 2^{\log_2((nj)/(\beta k))^2}$ possibilities for $s_1, s_2, \ldots, s_l$. Fix one possibility.

Without loss of generality, we assume that $s_1, \ldots, s_{i-1} \le c_0$. Because $k = k'$, we have $|P| \le \frac{1}{j} p^{c_{E(B)}bj}$. By Corollary 5.25, we have

$$|G| \le \left(\frac{1}{j}\right)^{(nj)/(\beta k)} p^{c_{E(B)}nj} 4^{s_i\cdots s_l} (c_0!)^{\frac{(nj)/(\beta k)-s_i\cdots s_l+s_{i+1}\cdots s_l-1}{c_0-1}}.$$

By Lemma 5.26, for fixed $s_1, \ldots, s_l$, G is contained in one of at most $c_t^{rs_i\cdots s_l}$ possibilities in $\mathfrak{C}$ in this class, up to conjugacy. The result follows by Proposition 5.8. □

3.E.4. G is not in classes 3.E.1 or 3.E.2 or 3.E.3.

Proposition 5.32 *The number $r^*_{E.4}(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ of absolutely irreducible representations of $F_r^{\mathfrak{C}}$ with image in class 3.E.4 is at most*

$$2^{\log_2((nj)/(\beta k))^2} \left(\frac{1}{j}\right)^{(r-1)(nj)/(\beta k)} p^{c_E(r-1)nj+j} (c_0!)^{\frac{(r-1)((nj)/(\beta k)-1)}{c_0-1}}.$$

Proof As before, there are at most $2^{\log_2((nj)/(\beta k))^2}$ possibilities for $s_1, \ldots, s_l$. Fix one. Then $G \le N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta, \mathbb{F}_{p^{k'}})}(E(B))) \wr Sym(s_1) \wr \cdots \wr Sym(s_l)$, so

$$|G| \le \left(\frac{1}{j}\right)^{(nj)/(\beta k)} p^{c_{E(B)}nj} (c_0!)^{\frac{(nj)/(\beta k)-1}{c_0-1}}$$

by Proposition 5.24. The result follows by Proposition 5.8. □

Proof of Theorem 5.22 Recall, for G in class 3.E, that $c_{E(B)} > c_4 > 4$, and that $Alt(s_i) \in \mathfrak{C}$ for $s_i < 5$. Since $(c_0!)^{1/(c_0-1)} \ge (2c_t)^4$, a calculation shows that the bounds for $r^*_{E.1}(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$, $r^*_{E.2}(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ and $r^*_{E.3}(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ are at most

$$2^{\log_2((nj)/(\beta k))^2} \left(\frac{1}{j}\right)^{(r-1)(nj)/(\beta k)} p^{c_E(r-1)nj+j} (c_0!)^{\frac{(r-1)((nj)/(\beta k)-1)}{c_0-1}}. \tag{5.2}$$

Also, it is clear that the upper bound for $r^*_2(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ is less than the upper bound for $r^*_1(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$.

We have shown that $r^*(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ is at most

$$2p^{c_4(r-1)nj+j+c_i rnj}(c_0!)^{\frac{(r-1)(n/2-1)}{c_0-1}} \tag{5.3}$$

$$+4\sum_{E\in\mathfrak{C}} 2^{\log_2((nj)/(\beta k))^2}\left(\frac{1}{j}\right)^{(r-1)(nj)/(\beta k)} p^{c_E(r-1)nj+j}(c_0!)^{\frac{(r-1)((nj)/(\beta k)-1)}{c_0-1}}, \tag{5.4}$$

so $\log(\zeta_G) \le 2\Sigma_1 + 4\Sigma_2$ where

$$\Sigma_1 = \sum_{p\in\mathcal{P}}\sum_{j,n=1}^{\infty} \frac{f_1(p,n,j)}{j} p^{-snj}|\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|,$$

$$\Sigma_2 = \sum_{E\in\mathfrak{C}}\sum_{p\in\mathcal{P}}\sum_{j,n=1}^{\infty} \frac{f_2(p,n,j)}{j} p^{-snj}|\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|,$$

$f_1(p,n,j)$ and $f_2(p,n,j)$ are the functions appearing in (5.3) and (5.4). Therefore, $a(G)$ is at most the maximum of the abscissae of Σ_1 and Σ_2. We have $Sym(c_0) \in \mathfrak{C}$, with (absolutely irreducible) natural representation over $\mathbb{F}_2$ of dimension $c_0 - \delta(c_0)$, where $\delta(c_0) = 1$ or 2 for odd or even c_0, respectively. By Theorem 5.5, we have $a(F_r^{\mathfrak{C}}) \ge \frac{\log(c_0!)}{(c_0-\delta(c_0))\log(2)}(r-1)+1$. On the other hand, the abscissa of Σ_1 is at most $c_4(r-1)+c_i r+\frac{\log(c_0!)(r-1)}{2\log(2)(c_0-1)}+2$ by Lemma 2.2. So when $(c_0!)^{1/(c_0-1)} > 2^{2c_4+4c_i+2}$, a calculation shows that we have $a(G)$ larger than the abscissa of Σ_1, and hence at most that of Σ_2.

For each $E \in \mathfrak{C}$, and each $n = \beta\frac{k}{j}c_0^l$, some l, we have

$$|N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta,\mathbb{F}_{p^k})}(E) \wr Sym(c_0) \wr \cdots \wr Sym(c_0)| = \left(\frac{1}{j}\right)^{(nj)/(\beta k)} p^{c_E nj}(c_0!)^{\frac{(nj)/(\beta k)-1}{c_0-1}},$$

by Proposition 5.24. So the sum

$$\sum_{(nj)/(\beta k)=1}^{c_0^l} 2^{\log_2((nj)/(\beta k))^2}\left(\frac{1}{j}\right)^{(r-1)(nj)/(\beta k)} p^{c_E(r-1)nj+j}(c_0!)^{\frac{(r-1)((nj)/(\beta k)-1)}{c_0-1}}$$

is at most $c_0^l 2^{\log_2(c_0^l)^2}|N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta,\mathbb{F}_{p^k})}(E) \wr Sym(c_0) \wr \cdots \wr Sym(c_0)|^{r-1}p^j$. It follows, using the usual techniques, that the abscissa of Σ_2 is at most that of Σ_D, by summing both over n for a fixed E and j, and then by summing over E and j. □

In fact, we can do slightly better: for p a prime and $n \ge 1$ an integer, consider the set $S_{p,n}$ of all classical or alternating groups $E \in \mathfrak{C}$ with natural representation over $\mathbb{F}_{p^k}$ of dimension β, such that $n = \beta k c_0^l$, for some l. We define $r_{D'}^*(F_r^{\mathfrak{C}}, \mathbb{F}_p, n)$ to be the sum over $S_{p,n}$ of

$$|N_{\Gamma L(\beta,\mathbb{F}_{p^k})}(E) \wr \underbrace{Sym(c_0) \wr \cdots \wr Sym(c_0)}_{l \text{ times}}|^{r-1} p.$$

Corollary 5.33 *The abscissa of convergence of $\zeta_{F_r^{\mathfrak{C}}}(s)$ is at most that of $\Sigma_{D'} = \sum_{p,n} r^*_{D'}(F_r^{\mathfrak{C}}, \mathbb{F}_p, n) p^{-ns} |\mathbb{P}^{n-1}(\mathbb{F}_p)|$.*

Proof It is enough to show that $\Sigma_{D'}$ has the same abscissa as Σ_D. Clearly $\Sigma_{D'} \leq \Sigma_D$, so one direction is trivial. We prove the converse.

For a fixed classical or alternating group E with natural representation over $\mathbb{F}_{p^k}$ of dimension β, and fixed l (so that $nj = \beta k c_0^l$), the contribution to Σ_D is

$$\begin{aligned}
&\sum_{j:j|k} \frac{|N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta,\mathbb{F}_{p^k})}(E) \wr Sym(c_0) \wr \cdots \wr Sym(c_0)|^{r-1} p^j}{j} p^{-snj} |\mathbb{P}^{n-1}(\mathbb{F}_{p^j})| \\
&\leq 2 \sum_{j:j|k} |N_{\Gamma L_{\mathbb{F}_{p^j}}(\beta,\mathbb{F}_{p^k})}(E) \wr Sym(c_0) \wr \cdots \wr Sym(c_0)|^{r-1} p^{-s\beta k c_0^l} (p^{\beta k c_0^l - 1}) \\
&\leq 2k |N_{\Gamma L(\beta,\mathbb{F}_{p^k})}(E) \wr Sym(c_0) \wr \cdots \wr Sym(c_0)|^{r-1} p^{-s\beta k c_0^l} (p^{\beta k c_0^l - 1}),
\end{aligned}$$

which is at most $2k$ times the contribution of this fixed E and l to $\Sigma_{D'}$. Summing over E and l, Lemma 4.2 now gives the result. $\square$

Let $c_{\mathfrak{C},\text{space}}$ be the supremum of $c_E + \frac{\log_p(c_0!)}{\beta k(c_0-1)}$ over all classical and alternating groups $E \in \mathfrak{C}$. For a constant K, let $c_{\mathfrak{C},\text{time},K}$ be the supremum of c_E over all classical and alternating groups E such that $p^k \geq K$, and let $c_{\mathfrak{C},\text{time}} = \inf_K(c_{\mathfrak{C},\text{time},K})$; clearly $c_{\mathfrak{C},\text{time}} \leq c_{\mathfrak{C},\text{time},K} \leq c_{\mathfrak{C},\text{space}}$ for any K.

Theorem 5.34 *$c_{\mathfrak{C},space}(r-1)+1 \leq a(F_r^{\mathfrak{C}}) \leq \max(c_{\mathfrak{C},space}(r-1)+1, c_{\mathfrak{C},time}(r-1)+2)$.*

Proof The lower bound follows from Corollary 5.7 by taking, for all $\varepsilon > 0$, $S = N_{\Gamma L_{\mathbb{F}_p}(n,\mathbb{F}_{p^k})}(E)$ for some $E \in \mathfrak{C}$ with $c_E + \frac{\log_p(c_0!)}{\beta k(c_0-1)} > c_{\mathfrak{C},\text{space}} - \varepsilon$, and $T = Sym(c_0)$.

The number of groups E contributing to $r^*_D(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)$ is at most $7n^2$: there are at most n choices for each of β and k, and then at most 7 choices of classical or alternating group. So $r^*_D(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n) \leq 7n^2 p^{c_{\mathfrak{C},\text{space}}(r-1)nj+j}$.

To prove the upper bound, we may split our sum into the sum

$$\sum_{p,j,n:p^j \geq K} \frac{r^*_D(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)}{j} p^{-njs} \frac{p^{nj}-1}{p^j-1}$$

and finitely many sums of the form

$$\sum_n \frac{r^*_D(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n)}{j} p^{-njs} \frac{p^{nj}-1}{p^j-1}$$

with p^j fixed. Each of the sums with p^j fixed converges for $s > c_{\mathfrak{C},\text{space}}(r-1)+1$. On the other hand, $\frac{\log_p(c_0!)}{\beta k(c_0-1)} \to 0$ as $p^k \to \infty$. So for all $\varepsilon > 0$, we can pick K large enough that for $p^k \geq K$, we have $r_D^*(F_r^{\mathfrak{C}}, \mathbb{F}_{p^j}, n) \leq 7n^2 p^{(c_{\mathfrak{C},\text{time}}+\varepsilon)(r-1)nj+j}$. Since ε is arbitrary, we get the result by Lemmas 2.2 and 4.2. □

As examples, we will now calculate the abscissa in more detail for two specific NE-formations. Let $\mathfrak{C}_{Alt(c_0)}$ be the NE-formation generated by the cyclic groups of prime order and the alternating groups of degree $\leq c_0$; let $\mathfrak{C}_{\Sigma(c_0)}$ be the NE-formation generated by the class of groups defined in [2]: that is, all the simple groups in $\mathfrak{C}_{Alt(c_0)}$, all exceptional groups of Lie type, all non-abelian simple groups of order at most c_0, and all classical simple groups whose natural representation has dimension at most c_0. None of the calculations below would be affected if we followed [34, Window: Permutation groups, Section 2] in also including all the sporadic simple groups (this is not done in [2] only because the classification of finite simple groups was incomplete when the paper was written), so our results also hold in this case.

5.9.3 $\mathfrak{C}_{Alt(c_0)}$

Let $\delta(c_0) = 1$ or 2 for odd or even c_0, respectively.

Theorem 5.35 *Suppose that $c_0 > C_0$. For any $r \geq 2$,*

$$a(F_r^{\mathfrak{C}_{Alt(c_0)}}) = \frac{c_0 \log_2(c_0!)}{(c_0 - \delta(c_0))(c_0-1)}(r-1)+1.$$

Proof Recall that, for E alternating, of degree ≥ 5, that c_{E,p^k}, over all fields $\mathbb{F}_{p^k}$, is maximised when $p^k = 2$. In this case, we have $|N_{\Gamma L_{\mathbb{F}_p}(\beta,\mathbb{F}_{p^k})}(E)| = c_0!$, and $\beta = c_0 - \delta(c_0)$. So $c_0! = 2^{c_{Alt(c_0)}(c_0-\delta(c_0))}$ and $c_{Alt(c_0)} = \frac{\log_2(c_0!)}{c_0-\delta(c_0)}$.

In fact, $\frac{\log_p(c_0!)}{\beta k(c_0-1)}$ is also maximised in this case, giving $\frac{\log_2(c_0!)}{\beta(c_0-1)}$. So $c_{\mathfrak{C}_{Alt(c_0)},\text{space}} = \sup_{a\leq c_0}(\frac{\log_2(a!)}{a-\delta(a)} + \frac{\log_2(c_0!)}{(a-\delta(a))(c_0-1)}) = \frac{\log_2(c_0!)}{c_0-\delta(c_0)} + \frac{\log_2(c_0!)}{(c_0-\delta(c_0))(c_0-1)} = \frac{c_0\log_2(c_0!)}{(c_0-\delta(c_0))(c_0-1)}$.

On the other hand, the same proof shows that for $p^k \geq K$ large, $c_{E,p^k} \leq \frac{\log_K(c_0!)+1}{c_0-2}$. So $c_{\mathfrak{C}_{Alt(c_0)},\text{time}} \leq \frac{1}{c_0-2}$, and hence, for any $r \geq 2$,

$$a(F_r^{\mathfrak{C}_{Alt(c_0)}}) \leq \frac{c_0 \log_2(c_0!)}{(c_0 - \delta(c_0))(c_0-1)}(r-1)+1.$$

This is also the lower bound for $a(F_r^{\mathfrak{C}_{Alt(c_0)}})$ given by Theorem 5.34. □

Stirling's approximation shows that $\frac{c_0\log_2(c_0!)}{(c_0-\delta(c_0))(c_0-1)} \sim \log_2(c_0)$ as $c_0 \to \infty$.

Remark 5.36 In fact, this approach can be used to show that we could modify the definition of $\Sigma_{D'}$ to consider, in the contributions of alternating groups to the sum, only their natural representations over $\mathbb{F}_2$, and the statement of Corollary 5.33 would still hold.

5.9.4 $\mathfrak{C}_{\Sigma(c_0)}$

For a prime power p^k, define $\rho(p^k) = \frac{(c_0-1)\log(k\prod_{i=1}^{c_0}(1-p^{-ik}))+\log(c_0!)}{c_0(c_0-1)\log(p^k)}$.

Theorem 5.37 *Suppose that $c_0 > C_0$. For any $r \geq 2$, $a(F_r^{\mathfrak{C}_{\Sigma(c_0)}}) = (c_0 + \rho(2))(r - 1) + 1$. Moreover, $\frac{\log(c_0)}{(c_0-1)\log(2)} - \frac{3}{c_0\log(2)} < \rho(2) < \frac{\log(c_0)}{(c_0-1)\log(2)}$.*

Proof By hypothesis, for any prime power p^k and any classical or alternating $E \in \mathfrak{C}_{\Sigma(c_0)}$ defined over $\mathbb{F}_{p^k}$, E has natural representation of dimension β, some $\beta \leq c_0$. Since $GL(c_0, \mathbb{F}_{p^k}) \in \mathfrak{C}_{\Sigma(c_0)}$, it is straightforward to show that, for any prime power p^k, $|N_{\Gamma L_{\mathbb{F}_p}(\beta,\mathbb{F}_{p^k})}(E)| \leq p^{c_{SL(c_0,\mathbb{F}_{p^k})}\beta k}$ (see [28, Table 2.1.C]). We calculate $|\Gamma L_{\mathbb{F}_p}(c_0, \mathbb{F}_{p^k})| = kp^{c_0^2 k}\prod_{i=1}^{c_0}(1 - p^{-ik}) = p^{c_{SL(c_0,\mathbb{F}_{p^k})}c_0 k}$. As $p^k \to \infty$, $c_{p^k} \to c_0$, so $c_{\mathfrak{C}_{\Sigma(c_0)},\text{time}} = c_0$. Similarly, we see that $\frac{\log_p(|N_{\Gamma L_{\mathbb{F}_p}(\beta,\mathbb{F}_{p^k})}(E)|)}{\beta k} + \frac{\log_p(c_0!)}{\beta k(c_0-1)}$ is maximised, for each p^k, by $\beta = c_0$ and $E = SL(c_0, \mathbb{F}_{p^k})$, so $c_{\mathfrak{C}_{\Sigma(c_0)},\text{space}}$ is given by $\sup_{p^k}(c_{SL(c_0,\mathbb{F}_{p^k})} + \frac{\log_p(c_0!)}{kc_0(c_0-1)}) = c_0 + \sup_{p^k}\rho(p^k)$.

We split the sum $\sum_{p,j,n}\frac{r_D^*(F_r^{\mathfrak{C}},\mathbb{F}_{p^j},n)}{j}p^{-njs}\frac{p^{nj}-1}{p^j-1}$ as follows. For any $\varepsilon > 0$, set K large enough that $\rho(p^k) < \varepsilon$ for $p^j \geq K$. For each $p^j < K$, $\Sigma_1(p^j)$ is the sum over n of terms with p^j fixed. Σ_2 is the sum over p, j, n of terms with $p^j \geq K$ and $n \geq c_0$. For each $n < c_0$, $\Sigma_3(n)$ is the sum over p, j such that $p^j \geq K$, with n fixed. This partitions the terms of $\sum_{p,j,n}\frac{r_D^*(F_r^{\mathfrak{C}},\mathbb{F}_{p^j},n)}{j}p^{-njs}\frac{p^{nj}-1}{p^j-1}$ into finitely many parts.

Each $\Sigma_1(p^j)$ converges for $s > c_{\mathfrak{C}_{\Sigma(c_0)},\text{space}}(r-1)+1 = (c_0 + \sup_{p^k}\rho(p^k))(r - 1) + 1$.

Next, $\Sigma_2 \leq \sum_{p,j,n:p^j\geq K,n\geq c_0} 7n^2 2^{\log_2(nj)^2}p^{((c_0+\varepsilon)(r-1)+1-s)nj}$. This has the same abscissa of convergence as $\sum_{p,j,n:p^j\geq K,n\geq c_0} p^{-tnj}$, by Lemma 4.2, for $s = (c_0 + \varepsilon)(r - 1) + 1 + t$. Summing over n first, we get $\sum_{p,j:p^j\geq K}\frac{p^{-tc_0 j}}{1-p^{-tj}} \ll \sum_{p,j:p^j\geq K}p^{-tc_0 j}$, where the implicit constant depends only on t; the latter sum converges when $t > 1/c_0$.

Finally, for each $n < c_0$, $|\Gamma L_{\mathbb{F}_p}(n, p^j)| \leq p^{(n+\varepsilon)nj}$. As before, we get that $\Sigma_3(n)$ has the same abscissa of convergence as $\sum_{p,j:p^j\geq K}p^{((n+\varepsilon)(r-1)+1-s)nj}$, which converges when $s > (n+\varepsilon)(r-1)+1+1/n$, and this bound is $< (c_0+\varepsilon)(r-1)+1+1/c_0$.

Since ε was arbitrary, to show $a(F_r^{\mathfrak{C}_{\Sigma(c_0)}}) \leq (c_0 + \sup_{p^k}\rho(p^k))(r - 1) + 1$, it remains to show that $(c_0 + \sup_{p^k}\rho(p^k))(r - 1) + 1 \geq c_0(r - 1) + 1 + 1/c_0$, or equivalently that $\sup_{p^k}\rho(p^k)(r-1) \geq 1/c_0$. We first find a bound for $\log(\prod_{i=1}^{c_0}(1 - 2^{-i})) \geq \sum_{i=1}^{\infty}\log(1 - 2^{-i})$. We can write $\log(1 - 2^{-i})$ as a Taylor series to get $\sum_{i=1}^{\infty}\log(1 - 2^{-i}) = -\sum_{i=1}^{\infty}\sum_{\iota=1}^{\infty}\frac{2^{-\iota i}}{\iota}$. Changing the order of summation, we get $-\sum_{\iota=1}^{\infty}\sum_{i=1}^{\infty}\frac{2^{-\iota i}}{\iota} = -\sum_{\iota=1}^{\infty}\frac{2^{-\iota}}{\iota(1-2^{-\iota})} = -\sum_{\iota=1}^{\infty}\frac{1}{\iota(2^{\iota}-1)} > -\sum_{\iota=1}^{\infty}\frac{1}{2^{\iota-1}} = -2$. So the whole sum is absolutely convergent, and we conclude $\log(\prod_{i=1}^{c_0}(1 - 2^{-i})) > -2$. Moreover, $\log(c_0!) > c_0(\log(c_0) - 1)$ by Stirling's approximation, so $\rho(2) >$

$\frac{\log(c_0)-1}{(c_0-1)\log(2)} - \frac{2}{c_0\log(2)} > \frac{\log(c_0)}{(c_0-1)\log(2)} - \frac{3}{c_0\log(2)}$. Therefore, because $c_0 > C_0 > 2e^3$, the result holds.

We wish to show that $\sup_{p^k} \rho(p^k) = \rho(2) \le \frac{\log(c_0)}{(c_0-1)\log(2)}$; we have already shown the lower bound for $\rho(2)$. For each p^k, we have the upper bound $\rho(p^k) < \frac{\log(k)+\log(c_0!)}{c_0(c_0-1)\log(p^k)} < \frac{\log(k)+\log(c_0)}{(c_0-1)\log(p^k)}$. Since $c_0 > C_0 > 8$, a calculation shows, for $p^k \ge 3$, $\rho(p^k) < \frac{\log(k)+\log(c_0)}{(c_0-1)\log(p^k)} < \frac{\log(c_0)-3}{c_0\log(2)} < \rho(2) < \frac{\log(c_0)}{(c_0-1)\log(2)}$.

Finally, for the lower bound on $a(F_r^{\mathfrak{C}_{Alt(c_0)}})$, $(c_0 + \rho(2))(r-1)+1$ is given by Theorem 5.34. □

The reader may generate their own examples, but we mention the following generalisation of the previous two theorems, which follows from essentially the same proof without any additional work, as the reader may verify. Suppose, for each of the types $A_d, B_d, C_d, D_d, ^2A_d, ^2D_d$ of classical groups (where d denotes the Lie rank), we pick some non-negative integers $d_A, d_B, d_B, d_D, d_{(^2A)}, d_{(^2D)}$. Let $\mathfrak{C}$ be the NE-formation generated by some subset of the cyclic groups of prime order including C_2 and C_3, some subset of the sporadic groups, some subset of the exceptional groups of Lie type, some subset of the alternating groups of degree $\le c_0$ including $Alt(c_0)$, some subset of the classical groups of type $A_d(q)$ (if $d_A > 0$) over all $d \le d_A$ and all prime powers q including $A_{d_A}(2)$, and similarly for the other classical groups (for the Steinberg groups $^2A_{d_{(^2A)}}(q^2)$ and $^2D_{d_{(^2D)}}(q^2)$, we assume $^2A_{d_{(^2A)}}(4)$ and $^2D_{d_{(^2D)}}(4)$ are in $\mathfrak{C}$). Including C_3 ensures that the normalisers of the Steinberg groups in $\mathfrak{C}$ in the corresponding general linear groups over $\mathbb{F}_4$ are also in $\mathfrak{C}$; including C_2 ensures $Sym(c_0) \in \mathfrak{C}$. C_2 and C_3 also ensure that Theorem 5.5 applies.

Theorem 5.38 *Suppose $c_0 > C_0$. For any $r \ge 2$, $a(F_r^{\mathfrak{C}}) = c_{\mathfrak{C},space}(r-1)+1$, where*

$$\begin{aligned}
c_{\mathfrak{C},space} = \max\Bigg\{& c_{Alt(c_0)} + \frac{\log_2(c_0!)}{(c_0-\delta(c_0))(c_0-1)}, c_{A_{d_A}(2)} + \frac{\log_2(c_0!)}{(d_A+1)(c_0-1)},\\
& c_{B_{d_B}(2)} + \frac{\log_2(c_0!)}{(2d_B+1)(c_0-1)}, c_{C_{d_C}(2)} + \frac{\log_2(c_0!)}{2d_C(c_0-1)}, c_{D_{d_D}(2)} + \frac{\log_2(c_0!)}{2d_D(c_0-1)},\\
& c_{(^2A_{d_{(^2A)}})(4)} + \frac{\log_2(c_0!)}{2(d_{(^2A)}+1)(c_0-1)}, c_{(^2D_{d_{(^2D)}})(4)} + \frac{\log_2(c_0!)}{4d_{(^2D)}(c_0-1)}\Bigg\}.
\end{aligned}$$

Note that analogous results can still be proved without the assumption that $A_{d_A}(2)$, etc., are in $\mathfrak{C}$, but in this case the constant C_0 may have to be increased, as a function of the smallest p^k such that $A_{d_A}(p^k)$, etc., are in $\mathfrak{C}$. By contrast, C_0 does not depend on d_A, etc.

Up to error terms which tend to 0 as $q \to \infty$, the values given in [28, Table 5.1.A, Table 5.4.C] show that $c_{A_{d_A}(q)} = d_A + 1$; $c_{B_{d_B}(q)} = d_B + 1/(2d_B+1)$; $c_{C_{d_C}(q)} = d_C + 1/2 + 1/(2d_C)$; $c_{D_{d_D}(q)} = d_D - 1/2 + 1/(2n)$; $c_{(^2A_{d_{(^2A)}})(q^2)} = d_{(^2A)}/2 + 1/2$; $c_{(^2D_{d_{(^2D)}})(q^2)} = d_{(^2D)}/2 - 1/4 + 1/(2n)$.

6 Groups with arbitrary abscissae

In this section we will prove Theorem E. In passing we mention that similar results for other types of zeta functions have been obtained by Kassabov [27] and Klopsch-Piccolo (unpublished).

For n a positive integer, let $\pi(n) \leq \log_2(n)$ denote the number of prime factors of n with repetitions (for example, $\pi(2^k) = k$). In particular, we set $\pi(1) = 0$.

Theorem 6.1 *Let $G_\alpha = \prod_{p>3} \mathrm{SL}(2, p)^{\lfloor p^\alpha \rfloor}$, for any real $\alpha \geq 0$. Then $\zeta_{G_\alpha}(s)$ has abscissa of convergence $\alpha/2 + 1$.*

Proof of Theorem E Let $p > 3$ be a prime. We first list a few general facts about the representations of $\mathrm{SL}(2, p)$ that we will use. Since $p > 3$, the group $\mathrm{PSL}(2, p)$ is simple. It is known that (see [20, Section 8]) $\mathrm{SL}(2, p)$ has exactly one absolutely irreducible representation in each dimension $d \in \{1, \dots, p\}$, over the field $\mathbb{F}_{p^j}$ for all j. Let $q \neq p$ be a prime. The group $\mathrm{SL}(2, p)$ has non-trivial absolutely irreducible representations over $\mathbb{F}_{q^j}$ only in dimensions $\geq (p-1)/2$ by [28, Theorem 5.3.9]. By the Wedderburn-Artin theorem, the number of non-trivial absolutely irreducible representations of $\mathrm{SL}(2, p)$ over $\mathbb{F}_{q^j}$ of dimension $n \geq (p-1)/2$ is at most

$$|\mathrm{SL}(2, p)|/n^2 \leq 4(p^3 - p)/(p-1)^2 = 4p + 8 + 8/(p-1) \leq 4p + 10. \quad (6.1)$$

Now fix $\alpha \geq 0$ and denote $G = G_\alpha$. We will split the proof into two parts: showing that $a(G) \leq \alpha/2 + 1$ and showing that $a(G) \geq \alpha/2 + 1$.

We start by fixing a finite field $\mathbb{F}_{q^j}$, q prime, and count absolutely irreducible representations in dimension n. All copies of $\mathrm{SL}(2, p)$ with $p \neq q$ and $p > 2n + 1$ must act trivially.

By [15] (see also [28, Lemma 5.5.5]), we may write an n-dimensional absolutely irreducible representation of G as a tensor product of one absolutely irreducible m-dimensional representation of $\mathrm{SL}(2, q)^{\lfloor q^\alpha \rfloor}$ and one absolutely irreducible n/m-dimensional representation of $H = \prod \mathrm{SL}(2, p)^{\lfloor p^\alpha \rfloor}$, where the product of H ranges over the primes p such that $p \leq 2n/m + 1$ and $p \neq 2, 3, q$. For the case $q \leq 3$, we simply assume $m = 1$. As an upper bound for $r^*(H, \mathbb{F}_{q^j}, n/m)$, note that any such representation must be a tensor product of at most $\log_2(n/m)$ non-trivial absolutely irreducible representations of one of the special linear groups in H. Additionally, the number of SL-factors appearing in H is at most

$$\sum_{p \leq 2n/m+1} \lfloor p^\alpha \rfloor \leq (2n/m + 1)^{\alpha+1},$$

and by (6.1) each factor has at most $4(2n/m + 1) + 10 \leq 8n/m + 14$ non-trivial absolutely irreducible representations. This gives the upper bound

$$r^*(H, \mathbb{F}_{q^j}, n/m) \leq ((2n/m + 1)^{\alpha+1}(8n/m + 14))^{\log_2(n/m)} = 2^{O_\alpha(\log_2(n/m)^2)},$$

where the implied constant depends only on α.

Now we estimate $r^*(\mathrm{SL}(2,q)^{\lfloor q^\alpha \rfloor}, \mathbb{F}_{q^j}, m)$ similarly: any such representation must be a tensor product of non-trivial absolutely irreducible representations of the $\mathrm{SL}(2,q)$ factors, of which there are $\lfloor q^\alpha \rfloor$, each with at most one absolutely irreducible representation in each dimension. Thus $r^*(\mathrm{SL}(2,q)^{\lfloor q^\alpha \rfloor}, \mathbb{F}_{q^j}, m)$ is bounded by the number of possible distributions of the $\pi(m)$ prime factors of m to the $\lfloor q^\alpha \rfloor$ copies of $\mathrm{SL}(2,q)$. This gives the upper bound $r^*(\mathrm{SL}(2,q)^{\lfloor q^\alpha \rfloor}, \mathbb{F}_{q^j}, m) \leq q^{\alpha\pi(m)}$.

In total, summing over m, we have $r^*(G, \mathbb{F}_{q^j}, n) = q^{\alpha\pi(n)} 2^{O_\alpha(\log_2(n)^2)}$.

Putting all this together, we get the upper bound

$$\log(\zeta_G)(s) \leq \sum_{q\in\mathcal{P}} \sum_{n,j\in\mathbb{N}} \frac{q^{\alpha\pi(n)} 2^{O_\alpha(\log_2(n)^2)}}{j} q^{-snj} \frac{q^{nj}-1}{q^j-1}.$$

For a fixed $k = nj$, because $q \geq 2$, we have $\frac{q^{nj}-1}{q^j-1} \leq \frac{q^{nj}-1}{q-1} \leq q^{nj-1}\frac{q}{q-1} \leq 2q^{k-1}$. So replacing nj with k, we find the upper bound

$$2\sum_{q\in\mathcal{P}} \sum_{k} 2^{O_\alpha(\log_2(k)^2)} q^{\alpha\pi(k)+(1-s)k-1},$$

which converges when $\alpha\pi(k) + (1-s)k < -\delta k$ for some $\delta > 0$, or equivalently when $\alpha\pi(k) + (1-s)k < 0$ for all integers $k \geq 1$. Indeed, summing over k, $\sum 2^{O_\alpha(\log_2(k)^2)} q^{-1-\delta k} = O(q^{-1-\delta'}/(1-q^{-\delta'}))$ for any $0 < \delta' < \delta$, for an implied constant independent of q, so the sum converges by the integral test.

In particular, it converges when $s > \alpha \max\{\pi(k)/k \mid k \in \mathbb{N}\} + 1$. Using $\pi(k) \leq \log_2(k)$, it is easy to check that $\alpha(\log_2(k)/k) + 1$ attains its upper bound $\alpha/2 + 1$ for $k = 2$ and $k = 4$, and is otherwise smaller. We conclude that $a(G) \leq \alpha/2 + 1$.

To show that $a(G) \geq \alpha/2 + 1$, we count representations of dimension 2 over fields of prime order. Here we have $\log(\zeta_G)(s) \geq \sum_{q\geq 5} \lfloor q^\alpha \rfloor q^{-2s}(q+1)$. For large q, $\lfloor q^\alpha \rfloor \geq (1-\varepsilon)q^\alpha$ for any $\varepsilon > 0$, so this sum is at least $(1-\varepsilon)\sum_{q\geq Q} q^{\alpha+1-2s} + q^{\alpha-2s}$, which diverges when $\alpha + 1 - 2s > -1$, or equivalently when $s < \alpha/2 + 1$. So $a(G) \geq \alpha/2 + 1$. □

7 Finite extensions with large Weil abscissae

Here we will construct examples of groups G that are *split extensions* of a finite index normal subgroup H such that $a(G) > a(H)$. We will use a product of groups of Lie type, acted on by the cyclic groups C_f of order f via Frobenius automorphisms.

Theorem 7.1 *Let $f \geq 5$ be a prime and let $G = \prod_{p\in\mathcal{P}, p>3} \mathrm{SL}(2, p^f)^{p^f}$. Let the cyclic group C_f act diagonally on G by Frobenius automorphisms on the factors. Then*

$$a(G) = \frac{3}{2} - \frac{f-1}{4f} < \frac{3}{2} \leq a(G \rtimes C_f).$$

Note that $a(G \rtimes C_f) - a(G) \geq (1 - f^{-1})/4$, that is, one quarter of the maximum increase predicted by Proposition 4.3 (see also Remark 4.5) as f tends to infinity.

Proof We copy the proof of Theorem E to show convergence. In characteristic $q \neq p$, the group $\mathrm{SL}(2, p^f)$ has non-trivial absolutely irreducible representations only in dimensions $\geq (p^f - 1)/2$ when $p > 3$ by [28, Theorem 5.3.9], and the number of such representations over $\mathbb{F}_{q^j}$ of dimension $n \geq (p^f - 1)/2$ is at most $|\mathrm{SL}(2, p^f)|/n^2 \leq 4(p^{3f} - p^f)/(p^f - 1)^2 = 4p^f + 8 + 8/(p^f - 1) \leq 4p^f + 9$.

Now we consider characteristic p. The splitting field for $\mathrm{SL}(2, p^f)$ is $\mathbb{F}_{p^f}$ by [28, Proposition 5.4.4], and we consider first the irreducible representations over a field $\mathbb{F}_{p^{fj'}}$. We can use [28, Theorem 5.4.5] to see that $r^*(\mathrm{SL}(2, p^f), \mathbb{F}_{p^{fj'}}, n)$ is the number of ways of writing n as an ordered product of f numbers between 1 and p. From this description, we have $r^*(\mathrm{SL}(2, p^f), \mathbb{F}_{p^{fj'}}, n) = f^{\pi(n)}$, where as before $\pi(n)$ is the number of prime factors of n.

On the other hand, consider $\mathbb{F}_{p^j}$ with j coprime to f. By [28, Proposition 5.4.6], and the description of the absolutely irreducible representations of $\mathrm{SL}(2, p)$ in [20, Section 8] (compare with the proof of Theorem E), there is one absolutely irreducible representation of $\mathrm{SL}(2, p^f)$ in each of dimensions $1^f, 2^f, \ldots, p^f$, and no others.

Now we can apply these upper bounds to count absolutely irreducible representations for the whole of G. Fix a finite field $\mathbb{F}_{q^j}$, q prime, and count absolutely irreducible representations in dimension n.

As for Theorem E, we write such a representation as a tensor product of two absolutely irreducible representations, one of $\mathrm{SL}(2, q^f)^{q^f}$ of dimension m and one of $H = \prod_{p^f \leq 2n/m+1, p \neq 2,3,q} \mathrm{SL}(2, p^f)^{p^f}$ of dimension n/m; when $q = 2$ or 3, we simply assume $m = 1$. We see that H contains $\sum_{p^f \leq 2n/m+1, p \neq 2,3,q} p^f \leq (2n/m + 1)(n/m + 1)$ special linear direct factors, each with at most $8n/m + 13$ absolutely irreducible representations in any dimension $\leq n/m$; we conclude that

$$r^*(H, \mathbb{F}_{q^j}, n/m) \leq ((2n/m + 1)(n/m + 1)(8n/m + 13))^{\log_2(n/m)} = 2^{O(\log_2(n/m)^2)}.$$

When j is coprime to f, as for Theorem E, we get $r^*(\mathrm{SL}(2, q^f)^{q^f}, \mathbb{F}_{q^j}, l^f) \leq (q^f)^{\pi(l)}$; meanwhile $r^*(\mathrm{SL}(2, q^f)^{q^f}, q^j, m) = 0$ for other values of m.

When $j = fj'$ for an integer j', each $\mathrm{SL}(2, q^f)$ has at most $f^{\pi(m)}$ absolutely irreducible representations over $\mathbb{F}_{q^{fj'}}$ in dimension m, giving the upper bound $r^*(\mathrm{SL}(2, q^f)^{q^f}, \mathbb{F}_{q^{fj'}}, m) \leq (q^f f)^{\pi(m)}$.

We now split $\log(\zeta_G) = \Sigma_1 + \Sigma_2$ into two sums, and consider the convergence of each separately, where

$$\Sigma_1 = \sum_{q \in \mathcal{P}} \sum_{n=1}^{\infty} \sum_{j : \gcd(j,f)=1} \frac{r^*(G, q^j, n)}{j} q^{-snj} |\mathbb{P}^{n-1}(\mathbb{F}_{q^j})|,$$

$$\Sigma_2 = \sum_{q \in \mathcal{P}} \sum_{n=1}^{\infty} \sum_{j'=1}^{\infty} \frac{r^*(G, q^{fj'}, n)}{fj'} q^{-snfj'} |\mathbb{P}^{n-1}(\mathbb{F}_{q^{fj'}})|.$$

First we deal with Σ_1. Write $n = m(n/m)$ as above. We may assume m has the form l^f. We consider separately the case where $l = 1$: the reader may verify it converges for all $s > 1$. So we need only check convergence for

$$\Sigma_1' = \sum_{q\in\mathcal{P}} \sum_{j:\gcd(j,f)=1} \sum_{l\geq 2} \sum_{l'=1}^{\infty} \frac{r^*(G, \mathbb{F}_{q^j}, l^f l')}{j} q^{-sl^f l' j} |\mathbb{P}^{l^f l'-1}(\mathbb{F}_{q^j})|,$$

where we write n as $l^f l'$. The approximations above give

$$\Sigma_1' \leq \sum_{q\in\mathcal{P}} \sum_{\gcd(j,q)} \sum_{l\geq 2} \sum_{l'=1}^{\infty} 2^{O(\log_2(l')^2)} q^{f\pi(l)} q^{(1-s)l^f l' j-1}.$$

Now we sum over all l, l' and j such that $l^f l' j = k$ to get

$$\Sigma_1' \leq \sum_{q\in\mathcal{P}} \sum_{k\geq 2^f} 2^{O(\log_2(k)^2)} q^{\pi(k)+(1-s)k-1}$$

(because $k \geq l^f \geq 2^f$) – which, as for Theorem E, converges when $\pi(k)+(1-s)k < 0$ for all $k \geq 2^f$, or equivalently (using $\pi(k) \leq \log_2(k)$) when $s > \max_{k\geq 2^f}(\pi(k)/k) + 1 = 1 + f/2^f$.

Now we will deal with Σ_2. From the bounds above, we get $r^*(G, \mathbb{F}_{q^{fj'}}, n) \leq 2^{O(\log_2(n)^2)}(q^f f)^{\pi(n)}$, and we conclude

$$\Sigma_2 \leq \sum_{q\in\mathcal{P}} \sum_{n,j'=1}^{\infty} 2^{O(\log_2(n)^2)} q^{f\pi(n)+(1-s)nfj'-f}.$$

Replacing nj' with k as in the proof of Theorem E, by the arguments above, this converges for s such that $f(\pi(k) + (1-s)k - 1) < -1$ for all k, which holds when $s > \pi(k)/k - (f-1)/(fk) + 1$.

This function has a maximum when $k = 4$, so Σ_2 converges when $s > 3/2 - (f-1)/(4f)$. For all primes $f \geq 5$, we find

$$a(G) \leq \max\left(\frac{3}{2} - \frac{f-1}{4f}, 1 + \frac{f}{2^f}\right) = \frac{3}{2} - \frac{f-1}{4f} < \frac{3}{2}.$$

We obtain a corresponding lower bound by looking at representations of dimension 4. We have $r^*(G, \mathbb{F}_{q^f}, 4) \geq q^{2f}$ for $q \geq 5$, so

$$\log(\zeta_G)(s) \geq \sum_{q\in\mathcal{P}} \frac{q^{2f}}{f} q^{-4fs} \frac{q^{4f}-1}{q^f-1} \geq \sum_{q\in\mathcal{P}} \frac{q^{(5-4s)f}}{f},$$

which diverges when $(5-4s)f \geq -1$, or equivalently when $s \leq 3/2 - (f-1)/(4f)$. Therefore $a(G) = 3/2 - (f-1)/(4f)$.

On the other hand, when we extend G by C_f acting diagonally by Frobenius automorphisms on each $\mathrm{SL}(2, p^f)$, each $\mathrm{SL}(2, p^f) \rtimes C_f$ has an absolutely irreducible representation of dimension $2f$ over $\mathbb{F}_p$. So we get $\log(\zeta_{G \rtimes C_f})(s) \geq \sum_q q^f q^{-2fs} \frac{q^{2f}-1}{q-1} \geq \sum_q q^{f-2fs+2f-1}$, which diverges when $3f - 2fs \geq 0$, or equivalently $s \leq 3/2$. □

8 Weil zeta functions of finite groups

In this section we prove Theorem F.

Proof of Theorem F Since $\mathbb{Q}[G]$ is a semisimple $\mathbb{Q}$-algebra, we may write

$$\mathbb{Q}[G] \cong \prod_{i=1}^{r} A_i$$

where A_i is a simple $\mathbb{Q}$-algebra with centre $K_i = Z(A_i)$. Let $\mathcal{O}_i$ denote the ring of algebraic integers in K_i. Put $n_i^2 = \dim_{K_i} A_i$. Note that $\dim_{K_i} A_i \leq \dim_{\mathbb{Q}} \mathbb{Q}[G] = |G|$ and so $n_i \leq \sqrt{|G|}$. We have $r = |\mathrm{Irr}(G, \mathbb{Q})|$ and the rational irreducible representations of G correspond to the simple modules of the algebras A_i. Let χ_i be the character of the simple rational representation of $\mathbb{Q}[G]$, which factors through A_i. Let $m(\chi_i)$ denote the Schur index. Then $\chi_i(1) = n_i m(\chi_i)[K_\chi : \mathbb{Q}]$; i.e. $n_i = \frac{\chi_i(1)}{[K_{\chi_i}:\mathbb{Q}]m(\chi)}$.

We say that a prime number p in *unramified* in $\mathbb{Z}[G]$, if the following hold

(i) p is unramified in K_i for all $i \in \{1, \ldots, r\}$,
(ii) for all i the algebra A_i splits at all primes $\mathfrak{p}$ of $\mathcal{O}_i$ dividing p
(iii) $\mathbb{Z}_p[G]$ is a maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$.

The integral group ring $\mathbb{Z}[G]$ is a $\mathbb{Z}$-order in $\mathbb{Q}[G]$. For almost all prime numbers p the completion $\mathbb{Z}_p[G]$ is a maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$; this follows from the existence of maximal orders and [42, (11.6)]. We deduce that almost all primes are unramified in $\mathbb{Z}[G]$; see [39, (8.4)] and [42, (32.1)]. If p is unramified, then

$$\mathbb{Q}_p[G] \cong \prod_{i=1}^{r} \prod_{\mathfrak{p}|p} M_{n_i}(K_{i,\mathfrak{p}}).$$

By [42, (17.3)] the maximal R-orders in $M_n(F)$ are conjugate to $M_n(R)$ if R is a complete discrete valuation ring with quotient field F. In particular, we deduce for unramified p that

$$\mathbb{Z}_p[G] \cong \prod_{i=1}^{r} \prod_{\mathfrak{p}|p} M_{n_i}(\mathcal{O}_{i,\mathfrak{p}})$$

and so

$$\mathbb{F}_p[G] \cong \prod_{i=1}^{r} \prod_{\mathfrak{p}|p} M_{n_i}(\mathcal{O}_i/\mathfrak{p}).$$

Let $N(\mathfrak{p}) = |\mathcal{O}_i/\mathfrak{p}|$ be the norm of the prime ideal $\mathfrak{p} \subseteq \mathcal{O}_i$ and write $N(\mathfrak{p}) = p^{f(\mathfrak{p})}$. We recall that $r^*(M_{n_i}(\mathcal{O}_i/\mathfrak{p}), \mathbb{F}_{p^k}, n)$ vanishes unless $n = n_i$ and k is a multiple of $f(\mathfrak{p})$, in which case $r^*(M_{n_i}(\mathcal{O}_i/\mathfrak{p}), \mathbb{F}_{p^k}, n_i) = f(\mathfrak{p})$. This allows us to calculate the Weil representation zeta function of G up to ramified primes. The correction amounts to multiplication of a finite number of factors of the form (1.3). We obtain

$$\begin{aligned}
\zeta_G(s) &\sim \prod_{i=1}^{r} \exp\left(\sum_{\mathfrak{p}\subseteq\mathcal{O}_i} \sum_{k=1}^{\infty} \frac{r_n^*(M_{n_i}(\mathcal{O}_i/\mathfrak{p}), \mathbb{F}_{p^k})}{k} p^{-sn_ik} |\mathbb{P}^{n_i-1}(\mathbb{F}_{p^k})| \right) \\
&= \prod_{i=1}^{r} \exp\left(\sum_{\mathfrak{p}\subseteq\mathcal{O}_i} \sum_{k'=1}^{\infty} \frac{f(\mathfrak{p})}{k' f(\mathfrak{p})} p^{-sn_ik' f(\mathfrak{p})} |\mathbb{P}^{n_i-1}(\mathbb{F}_{p^{k' f(\mathfrak{p})}})| \right) \\
&= \prod_{i=1}^{r} \prod_{\mathfrak{p}\subseteq\mathcal{O}_i} \exp\left(\sum_{k'=1}^{\infty} \frac{1}{k'} N(\mathfrak{p})^{-sn_ik'} \left(N(\mathfrak{p})^{k'(n_i-1)} + \cdots + N(\mathfrak{p})^{k'} + 1 \right) \right) \\
&= \prod_{i=1}^{r} \prod_{\mathfrak{p}\subseteq\mathcal{O}_i} \prod_{j=0}^{n_i-1} \left(1 - N(\mathfrak{p})^{-sn_i+j}\right)^{-1} = \prod_{i=1}^{r} \zeta_{K_i}^{\#n_i}(s).
\end{aligned}$$

It is a classical result of Hecke that the Dedekind zeta function ζ_K of a number field K admits a meromorphic extension to $\mathbb{C}$ with a simple pole at $s = 1$. The poles of $\zeta_K^{\#n}(s)$ are thus $\frac{1}{n}, \frac{2}{n}, \ldots, 1$. The rational correction factors (1.3) satisfy $a_k \leq \sqrt{|G|}$ and thus can have poles at non-negative rational numbers which are also bounded from above by $1 - \sqrt{|G|^{-1}}$. □

9 Virtually abelian groups

In this section we prove Theorem G.

Proof of Theorem G Let $A \trianglelefteq G$ be an abelian normal subgroup of finite index $d = |G : A|$ in G. Then every absolutely irreducible representation of G has dimension at most d. For every $n \in \{1, 2, \ldots, d\}$ there is a moduli variety M_n defined over $\mathbb{F}_p$ such that the closed points $M_n(\mathbb{F}_{p^k})$ are in bijective correspondence with the isomorphism classes of absolutely irreducible representations of G over the field $\mathbb{F}_{p^k}$; see e.g. [31, Theorem 6.23]. Define $V_n = M_n \times \mathbb{P}^{n-1}$. Then

$$|V_n(\mathbb{F}_{p^k})| = r^*(G, \mathbb{F}_{p^k}, n) \cdot |\mathbb{P}^{n-1}(\mathbb{F}_{p^k})|$$

hold for all $k \in \mathbb{N}$. This implies that

$$\zeta_{G,p}(s) = \sum_{n=1}^{d} Z(V_n, s)$$

where $Z(V_n, s)$ denotes the local Hasse–Weil zeta function of V_n. The local Hasse–Weil zeta function is a rational function in p^{-s} by [14]. □

The following is probably well-known and it was communicated to the authors by Alexander Moretó.

Lemma 9.1 *[[38]] Let K be a field of characteristic $p > 0$. Then there exists a real-valued function f such that if G is a finite group such that all irreducible representations over K have dimension at most n, then G has a characteristic p-abelian subgroup A such that $|G : A| \leq f(n)$.*

It follows that the groups to which the proof of Theorem G can be applied are exactly the finitely generated virtually abelian groups. However, the class of UBERG groups with rational local factors is larger; for instance, it contains the lamplighter group (see Appendix A:). It would be interesting to have a description of the class of groups for which the rationality result holds.

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Appendix A: Examples

In this section we will collect various examples of Weil representation zeta functions that we have calculated. Some examples have appeared in this article; others can be obtained by following the steps of Example A.5. Finally, we also calculate the Weil representation zeta functions for the lamplighter groups $C_2 \wr \mathbb{Z}$ and $C_3 \wr \mathbb{Z}$. In this section $\zeta(s)$ will denote the Riemann zeta function.

A.1 Abelian groups

Since one-dimensional representations are always absolutely irreducible, ζ_G can be easily calculated for abelian groups by counting homomorphisms into $\mathbb{F}_q^{\times}$ for every q.

Example A.1 [C_p cyclic group]

$$\zeta_{C_p}(s) = \zeta(s) \cdot \prod_p \prod_{\chi \in \mathbb{F}_p^\times} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \zeta(s) \cdot \zeta_{\mathbb{Q}(\eta)}(s) \cdot \left(1 - \frac{1}{p^s}\right)$$

where η denotes a primitive pth root of unity.

It is instructive to compare this with Theorem F and its proof in Sect. 8. The cyclic group C_p has exactly two irreducible rational representations: the trivial representation and a representation of dimension $p-1$ (multiplication with η on $\mathbb{Q}(\eta)$). This gives the decomposition $\mathbb{Q}[C_p] \cong \mathbb{Q} \times \mathbb{Q}(\eta)$ and in fact, (after inverting the prime p) one has $\mathbb{Z}[1/p][C_p] = \mathbb{Z}[1/p] \times \mathbb{Z}[1/p, \eta]$.

Example A.2 [$\mathbb{Z}^r$ free abelian group] We have seen in Sect. 5.1 that

$$\zeta_{\hat{\mathbb{Z}}^r}(s) = \prod_{i=0}^{r} \zeta(s-i)^{(-1)^{r-i}\binom{r}{i}}.$$

A.2 Symmetric groups

In Theorem F we gave a formula for the zeta function of a finite group up to rational factors. An exact formula for ζ_G requires a concise understanding of the modular representation theory of G. The modular representation theory of the symmetric group S_n is well-studied and the description of the absolutely irreducible representations of S_n as quotients $S^\mu/(S^\mu \cap (S^\mu)^\perp)$ of the Specht modules S^μ given in [25, Theorem 4.9] can be used to compute an exact formula for ζ_{S_n} for small values of n.

Example A.3 [S_4 symmetric group]

$$\zeta_{S_4}(s) = \zeta(s)^2 \zeta^{\#2}(s) \zeta^{\#3}(s)^2 (1-2^{-s}) \prod_{k \in \{2,3\}} \prod_{j=0}^{k-1} (1 - 3^{j-ks}).$$

We can also give a more concise formula for the Weil representation zeta function up to rational factors for primes below n. Given two meromorphic functions f, g on $\mathbb{C}$ we write $f \sim_n g$ if there is a rational function h in $\{p^{-s} \mid p \leq n,\ p \text{ prime}\}$ such that $fh = g$.

Example A.4 [S_n symmetric group] Let $n \geq 2$ be an integer. Then

$$\zeta_{S_n}(s) \sim_n \prod_{\chi \in \mathrm{Irr}(S_n, \mathbb{C})} \prod_{j=0}^{\chi(1)-1} \zeta(\chi(1)s - j).$$

It follows from Theorem F that the Weil representation zeta function $\zeta_{S_n}(s)$ has a pole of order $P(n)$ at $s = 1$, where $P(n)$ denotes the number of partitions of n.

A.3 Virtually abelian groups

For virtually abelian groups ζ_G can be calculating by inducing irreducible representations from an abelian normal subgroup. We explain this for $\mathbb{Z} \wr C_2$. All the examples of this section can be calculated in essentially the same way.

Example A.5 [$\mathbb{Z} \wr C_2$ wreath product] Consider the group $G = \mathbb{Z} \wr C_2 = \mathbb{Z}^2 \rtimes C_2$, where C_2 swaps the two copies of $\mathbb{Z}$. Then

$$\zeta_{\mathbb{Z} \wr C_2}(s) \sim \frac{\zeta(s-1)^2}{\zeta(s)^2} \cdot \frac{\zeta(2s)\zeta(2s-3)}{\zeta(2s-1)\zeta(2s-2)}.$$

It is easy to see that $G^{\mathrm{ab}} \cong C_2 \times \mathbb{Z}$. Therefore, if $2 \nmid q$, $r^*(G, \mathbb{F}_q, 1) = 2(q-1)$ and, if $2 \mid q$, $r^*(G, \mathbb{F}_q, 1) = q-1$.

For degree two representations, consider two distinct one-dimensional irreducible representations χ_1, χ_2 of $\mathbb{Z}$ over $\mathbb{F}_q$, then the induced representation $\rho(\chi_1, \chi_2) = \mathrm{Ind}_{\mathbb{Z}^2}^G(\chi_1 \otimes \chi_2)$ is an absolutely irreducible representation of G and the number of such representations is $\frac{1}{2}(q-1)(q-2)$ (the order of χ_1, χ_2 does not matter). However, it is possible that the field of definition of the induced representation ρ is smaller than $\mathbb{F}_q$ and this is actually one of the main difficulties that arise for virtually abelian groups. In the case at hand, there could be a representation $\rho : G \to \mathrm{GL}_2(\mathbb{F}_q)$ such that $\rho|_{\mathbb{Z}^2}$ is diagonalisable over $\mathbb{F}_{q^2}$, but not over $\mathbb{F}_q$. More precisely, given $\chi_1, \chi_2 \colon \mathbb{Z} \to \mathbb{F}_{q^2}$, then $\rho(\chi_1, \chi_2)$ is defined over $\mathbb{F}_q$ exactly if χ_1 and χ_2 are conjugate under $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. The number of representations of this type is $\frac{1}{2}(q^2-q)$ and

$$r^*(G, \mathbb{F}_q, 2) = \frac{1}{2}(q-1)(q-2) + \frac{1}{2}(q^2-q) = q^2 - 2q + 1;$$

we note that the factor $\frac{1}{2}$ disappeared.

Calculating the Weil representation zeta function $\zeta_G(s)$:

$$\exp\left(\sum_{p \neq 2}\left(\sum_{j=1}^{\infty} \frac{2(p^j-1)}{j} p^{-sj} + \sum_{j=1}^{\infty} \frac{p^{2j} - 2p^j + 1}{j} p^{-2sj}(p^j+1)\right)\right) \cdot$$
$$\exp\left(\sum_{j=1}^{\infty} \frac{2^j-1}{j} p^{-sj} + \sum_{j=1}^{\infty} \frac{2^{2j} - 2 \cdot 2^j + 1)}{j} p^{-2sj}(2^j+1)\right)$$
$$= \left(1 - \frac{1}{2^{s-1}}\right)\left(1 - \frac{1}{2^s}\right)^{-1} \frac{\zeta(s-1)^2}{\zeta(s)^2} \cdot \frac{\zeta(2s)\zeta(2s-3)}{\zeta(2s-1)\zeta(2s-2)}.$$

Example A.6 [D_∞ infinite dihedral group]

$$\zeta_{D_\infty}(s) \sim \zeta(s)^4 \cdot \frac{\zeta(2s-2)}{\zeta(2s-1)\zeta(2s)^2}$$

Example A.7 [$\mathbb{Z} \wr C_3$ wreath product]

$$\zeta_{\mathbb{Z}\wr C_3}(s) \sim \frac{\zeta(s-1)^3}{\zeta(s)^3} \cdot \frac{\zeta(3s-3)\zeta(3s-2)}{\zeta(3s)}$$

Example A.8 [$BS(1,-1)$ metabelian Baumslag-Solitar group]

$$\zeta_{BS(1,-1)}(s) \sim \frac{\zeta(s-1)^2}{\zeta(s)^2} \cdot \frac{\zeta(2s)^2\zeta(2s-3)}{\zeta(2s-1)\zeta(2s-2)^2}$$

A.4 Two lamplighter groups

Let G be the lamplighter group $C_2 \wr \mathbb{Z}$. Let $N = \bigoplus_{i\in\mathbb{Z}} C_2$ denote the base of the wreath product. We determine the number $r^*(G, \mathbb{F}_q, n)$ absolutely irreducible n-dimensional representations of G over the finite field $\mathbb{F}_q$.

Proposition A.9 *Let q be a prime power.*

(i) *If q is even, then all irreducible $\mathbb{F}_q$-representations of G are* 1*-dimensional and* $r^*(G, \mathbb{F}_q, 1) = q - 1$.
(ii) *If q is odd, then*

$$r^*(G, \mathbb{F}_q, n) = (q-1)\frac{F(n)}{n}$$

where $F(n) = \sum_{d|n} 2^d \mu(n/d)$.

Before we can prove the proposition, we study the function F.

Lemma A.10 *Let $n \in \mathbb{N}$. Then $F(n) = \sum_{d|n} 2^d \mu(n/d)$ is the number of sequences in $\{\pm 1\}^{\mathbb{Z}}$ which have stabilizer $n\mathbb{Z}$ under the Bernoulli shift action.*

Proof For every divisor d of n let $X(d)$ denote the subset of $\{\pm 1\}^{\mathbb{Z}}$ consisting of sequences with stabilizer $d\mathbb{Z}$ under the shift action of $\mathbb{Z}$. Define $F(d) = |X(d)|$; we will show that this function satisfies the formula. The union $\bigcup_{d|n} X(d)$ consists of all sequences which are stabilized by $n\mathbb{Z}$. Since the first n entries determine such a sequence completely, we deduce

$$2^n = \sum_{d|n} |X(d)| = \sum_{d|n} F(d). \tag{A.1}$$

The Möbius inversion formula implies that

$$F(n) = \sum_{d|n} 2^d \mu(n/d).$$

□

Proof of Proposition A.9 Let V be an absolutely irreducible representation of G over $\mathbb{F}_q$. If q is even, then $V_{|N}$ is trivial, since N is a 2-group and $V_{|N}$ is semisimple. Every irreducible representation over $\mathbb{F}_q$ factors through the infinite cyclic quotient. In particular, the representations are 1-dimensional and $r_1^*(G, q) = q - 1$.

Assume that q is odd. Then $V_{|N}$ decomposes as a sum of irreducible $\mathbb{F}_q[N]$-modules. We note that every irreducible N-module is absolutely irreducible, one dimensional and of the form

$$\chi_v((\sigma_i)_{i\in\mathbb{Z}}) = \prod_i v_i^{\sigma_i}$$

for some sequence $v = (v_i)_{i\in\mathbb{Z}} \in \{\pm 1\}^{\mathbb{Z}}$. As $V_{|N}$ is finite dimensional and consists of a single orbit of a representation χ_v, the sequence v needs to be periodic with some period n. The representation factors through $G_n := \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} C_2 \rtimes \mathbb{Z}$. The central subgroup $n\mathbb{Z}$ in G_n acts with a character $\psi : n\mathbb{Z} \to \mathbb{F}_q^\times$ and by Clifford Theory

$$V \cong \mathrm{Ind}_{N\times n\mathbb{Z}}^G(\chi_v \otimes \psi).$$

In particular, V has dimension n. Conversely, every irreducible representation is of this form. A sequence $v \in \{\pm 1\}^{\mathbb{Z}}$ is periodic with period n exactly if χ_v has stabilizer $n\mathbb{Z}$ in the infinite cyclic subgroup of G. By Lemma A.10, the number of such sequences G is $F(n)$. For every character $\psi : n\mathbb{Z} \to \mathbb{F}_q$, $\mathrm{Ind}_{N\times n\mathbb{Z}}^G(\chi_v \otimes \psi)$ is an absolutely irreducible representation of G defined over $\mathbb{F}_q$. Since the orbit of χ_v contains exactly n elements, we obtain

$$r^*(G, q, n) = (q-1)\frac{F(n)}{n}.$$

□

Theorem A.11 *Let G be the lamplighter group $C_2 \wr \mathbb{Z}$. Then*

$$\zeta_G(s) = \left(\frac{1-2^{-s}}{1-2^{1-s}}\right) \cdot \prod_{p>2} \left(\frac{1-2p^{-s}}{1-2p^{1-s}}\right).$$

The abscissa of convergence is $a(G) = 2$.

Proof For $p = 2$ the local zeta function is

$$\log \zeta_{G,2}(s) = \log(1-2^{-s}) - \log(1-2^{1-s})$$

using the same calculation as for the infinite cyclic group. Let p be an odd prime. Then $\log \zeta_{G,p}(s)$ equals

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{r_n^*(G, p^k)}{k} p^{-skn} |\mathbb{P}^{n-1}(\mathbb{F}_{p^k})| = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{(p^k-1)F(n)}{nk} p^{-skn} |\mathbb{P}^{n-1}(\mathbb{F}_{p^k})|$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(p^{nk}-1)F(n)}{nk}p^{-skn} = \sum_{m=1}^{\infty}\sum_{n|m}\frac{(p^m-1)F(n)}{m}p^{-sm}$$
$$= \sum_{m=1}^{\infty}\frac{(p^m-1)}{m}p^{-sm}\left(\sum_{n|m}F(n)\right) = \sum_{m=1}^{\infty}\frac{(p^m-1)2^m}{m}p^{-sm}$$
$$= \log(1-2p^{-s}) - \log(1-2p^{1-s}).$$

Note that we used (A.1) in the chain of equalities above. □

The calculation for the lamplighter group $C_3 \wr \mathbb{Z}$ is very similar, albeit more involved, and here we only report the result of our calculation.

Theorem A.12 *Let G be the lamplighter group $C_3 \wr \mathbb{Z}$. Then*

$$\zeta_G(s) = \frac{1-3^{-s}}{1-3^{1-s}}\prod_{p\equiv 1 \bmod 3}\frac{1-3p^{-s}}{1-3p^{1-s}}\prod_{p\equiv 2 \bmod 3}\frac{(1-3p^{-2s})(1+p^{1-s})}{(1-3p^{2-2s})(1+p^{-s})}.$$

In particular, the abscissa of convergence of $C_3 \wr \mathbb{Z}$ is 2.

Appendix B: Comparing two zeta functions

What is the effect of using absolutely irreducible representations in the definition of the zeta function? That is, UBERG can be measured in terms of the growth of irreducible representations, instead of absolutely irreducible ones. What would happen if we defined the Weil representation zeta function in those terms?

Let η_G be the complex function defined by

$$\log(\eta_G)(s) = \sum_{p\text{ prime}}\sum_{n=1}^{\infty}\frac{r_n(G,\mathbb{F}_{p^j})}{j}p^{-snj}|\mathbb{P}^{n-1}(\mathbb{F}_{p^j})|,$$

i.e. just replacing r^* with r in the definition of ζ_G.

Clearly $\zeta_G(s) \leq \eta_G(s)$ for real s, where they both converge. But in general, they need not have the same abscissa of convergence. We omit the proof of the following theorem for conciseness, as it follows the lines of the proof of Theorem E.

Theorem B.1 *Let $G = \prod_{p\text{ prime}\geq 3} SL(2,p^p)^{p^p}$. Then η_G has abscissa of convergence $\geq 3/2$, while ζ_G has abscissa of convergence $\leq 11/8$.*

As we saw in Sects. 8 and 9, for several groups we have a nice form for the Weil representation zeta function, with properties like meromorphic continuation and rationality of local factors. In contrast, the next example illustrates that the zeta function $\eta_G(s)$ defined via irreducible representations can be 'wild' even for virtually abelian groups.

Example B.2 Let $G = \mathbb{Z} \wr C_2$. By repeating word for word the calculations in example A.5 with *all* irreducible representations, we get

$$\eta_G(s) \sim \frac{\zeta(s-1)^2}{\zeta(s)^2} \cdot \frac{\zeta(2s)}{\zeta(2s-2)} \sqrt{\frac{\zeta(2s-3)}{\zeta(2s-1)}}.$$

As we can see, this function does not have rational local factors and its analytic properties are more difficult to understand compared to the ones of $\zeta_G(s)$.

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