The isomorphism problem for cominuscule Schubert varieties

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Abstract

Cominuscule flag varieties generalize Grassmannians to other Lie types. Schubert varieties in cominuscule flag varieties are indexed by posets of roots labeled long/short. These labeled posets generalize Young diagrams. We prove that Schubert varieties in potentially different cominuscule flag varieties are isomorphic as varieties if and only if their corresponding labeled posets are isomorphic, generalizing the classification of Grassmannian Schubert varieties using Young diagrams by the last two authors. Our proof is type-independent.

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1 Introduction

Cominuscule flag varieties correspond to algebraic varieties that admit the structure of a compact Hermitian symmetric space and have been studied extensively due their shared properties with Grassmannians [1–3, 11, 15, 17]. These varieties come in five infinite families and two exceptional types and are determined by a pair $(\mathcal{D}, \gamma)$ of a Dynkin diagram $\mathcal{D}$ of a reductive Lie group and a cominuscule simple root γ. See Table 1 for a classification of cominuscule flag varieties. Let X denote the cominuscule flag variety corresponding to $(\mathcal{D}, \gamma)$ and R denote the root system of the Dynkin

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Table 1 Cominuscule flag varieties

Cominuscule roots are denoted by the filled-in circles. This table is a modification of [5, Table 1]

diagram $\mathcal{D}$. Set $\mathcal{P}_X := \{\alpha \in R : \alpha \geq \gamma\}$ with the partial order $\alpha \leq \beta$ if $\beta - \alpha$ is a non-negative sum of simple roots, and give $\mathcal{P}_X$ a labeling of long/short roots. By [5]*Theorem 2.4, Schubert varieties in X are indexed by lower order ideals in $\mathcal{P}_X$, generalizing the fact that Schubert varieties in a Grassmannian are indexed by Young diagrams.

Our main result Theorem 1 is a combinatorial criterion for distinguishing isomorphism classes of Schubert varieties coming from cominuscule flag varieties.

Theorem 1 *Let $X_\lambda \subseteq X$ and $Y_\mu \subseteq Y$ be cominuscule Schubert varieties indexed by lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$, respectively. Then X_λ and Y_μ are algebraically isomorphic if and only if λ and μ are isomorphic as labeled posets.*

For illustrative examples of Theorem 1, see Sect. 2.

Since Grassmannians are cominuscule flag varieties, Theorem 1 extends the work of Țarigradschi and Xu in [16], where they prove two Grassmannian Schubert varieties
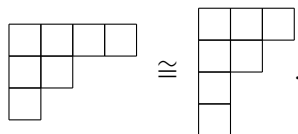
are isomorphic if and only if their Young diagrams are the same or the transpose of each other. Other related works include Richmond and Slofstra's characterization of the isomorphism classes of Schubert varieties coming from complete flag varieties in [14] using Cartan equivalence. However, they also note that Cartan equivalence is neither necessary nor sufficient to distinguish Schubert varieties in partial flag varieties. A class of smooth Schubert varieties in type A partial flag varieties are classified by Develin, Martin, and Reiner in [6]. Yet many Schubert varieties are singular, with the first example being the Schubert divisor in the Grassmannian $\mathrm{Gr}(2, 4)$.

We discuss preliminaries in Sect. 3, and then in Sect. 4, we prove Theorem 1 and illustrate it with examples. Our proof is type-independent and employs several new techniques. To show that the labeled poset λ depends only on the isomorphism class of the Schubert variety X_λ, we construct it from the effective cone in the Chow group of X_λ and intersection products of classes in this cone with the unique effective generator of the Picard group of the variety. To prove the converse, we embed each Schubert variety X_λ in a "minimal" cominuscule flag variety uniquely determined by the labeled poset λ. Assuming that the labeled posets $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$ are isomorphic, we construct an explicit isomorphism between the Dynkin diagrams of the "minimal" cominuscule flag varieties determined by λ and μ. The corresponding flag variety isomorphism identifies the Schubert varieties X_λ and Y_μ.

2 Examples of Theorem 1

For the following examples, recall that cominuscule Schubert varieties are indexed by lower order ideals in $\mathcal{P}_X$. Examples of $\mathcal{P}_X$ are illustrated in Table 2, where each element in $\mathcal{P}_X$ is drawn as a box, and boxes decorated with an "s" correspond to short roots. The partial order on boxes is given by $\alpha \leq \beta$ if and only if α is weakly north-west of β. In particular, lower order ideals are given by subsets of boxes that are closed under moving to the north and west.

Example 2 As illustrated below, transposing a Young diagram does not change the poset structure:


Therefore, two Grassmannian Schubert varieties are isomorphic if their indexing Young diagrams are the transpose of each other. Geometrically, this is related to the isomorphism $\mathrm{Gr}(m, m + k) \cong \mathrm{Gr}(k, m + k)$, which comes from the reflection symmetry of the A_{m+k-1} Dynkin diagram:
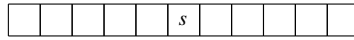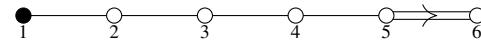
Table 2 The labeled poset $\mathcal{P}_X$ for a cominuscule flag variety X. Each element in $\mathcal{P}_X$ is drawn as a box, and boxes decorated with an "s" correspond to short roots. The partial order on boxes is given by $\alpha \leq \beta$ if and only if α is weakly north-west of β. This table is a modification of [2, Table 1]
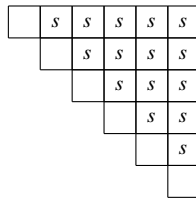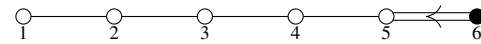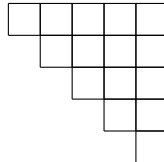
Grassmannian Gr(3, 7) of type A
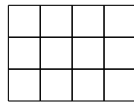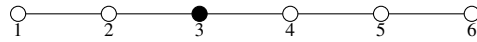
Odd quadric $Q^{11} \subset \mathbb{P}^{12}$

Lagrangian Grassmannian LG(6, 12)

Max. orthog. Grassmannian OG(6, 12)
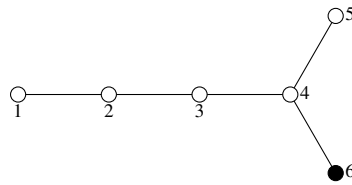
Even quadric $Q^{10} \subset \mathbb{P}^{11}$

Table 2 continued

Cayley Plane E_6/P_6

Freudenthal variety E_7/P_7
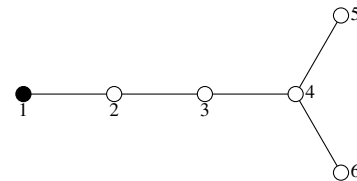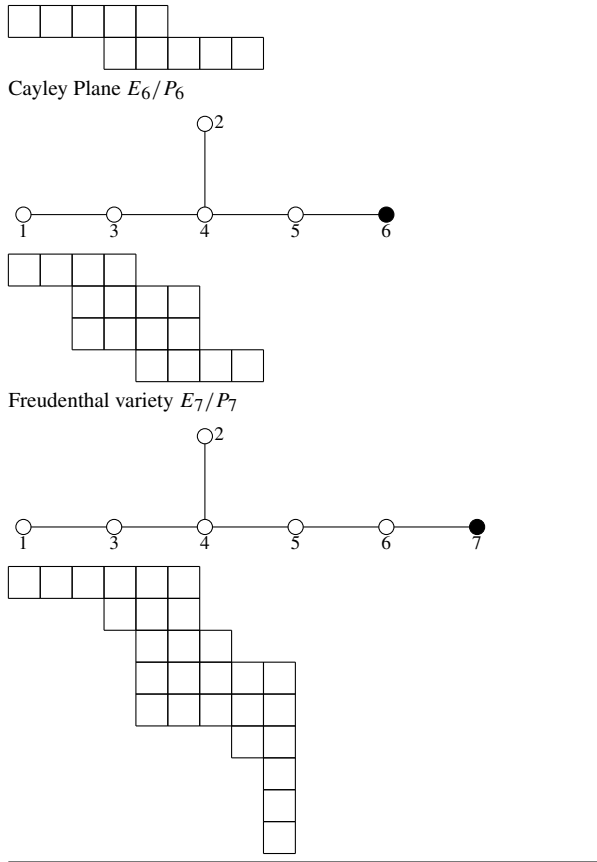
Example 3

$$\mathcal{P}_{Q^6} = \vcenter{\hbox{\includegraphics{}}} \cong \vcenter{\hbox{\includegraphics{}}} = \mathcal{P}_{\mathrm{OG}(4,8)},$$

therefore, $Q^6 \cong \mathrm{OG}(4, 8)$. This isomorphism comes from the rotation symmetry of the D_4 Dynkin diagram:
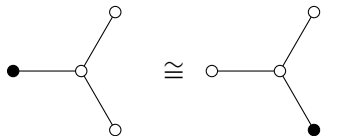
Example 4 Using Table 2, it is not hard to see that if a Grassmannian Schubert variety is isomorphic to a non-type A cominuscule Schubert variety, then they are both iso-

morphic to a projective space. Indeed, in order to fit inside a $\mathcal{P}_X$ of another type, the lower order ideal is forced to be a chain.

As a special case, we also see that any Schubert curve in any cominuscule flag variety is isomorphic to $\mathbb{P}^1$. In fact, any Schubert curve in any flag variety is isomorphic to $\mathbb{P}^1$, which follows from the more general statements that Schubert varieties are rational normal projective varieties and that $\mathbb{P}^1$ is the only rational normal projective curve.

Example 5 The Schubert divisor in Q^3 is not isomorphic to $\mathbb{P}^2$, because the labeling of their posets does not match:

$$\boxed{\ \ |\ s\ } \ncong \boxed{\ \ |\ \ }\ .$$

We can also see it geometrically, as the Schubert divisor in Q^3 is singular.

Example 6 The quadric Q^3 embeds in $\mathrm{LG}(n, 2n)$ ($n \geq 3$) as a Schubert variety, as illustrated by
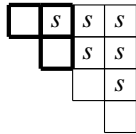
Example 7 The quadric Q^{10} embeds in E_7/P_7 as a Schubert variety, as illustrated by
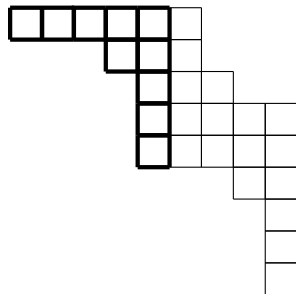
Example 8 There are two non-isomorphic 6-dimensional Schubert varieties in E_6/P_6, given by the two order ideals illustrated below.
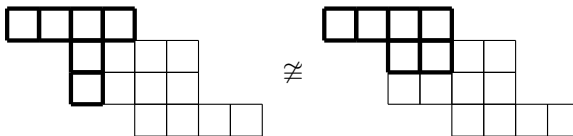
Example 9 While $\mathcal{P}_{\mathrm{LG}(n,2n)}$ and $\mathcal{P}_{\mathrm{OG}(n+1,2n+2)}$ are isomorphic as posets, this isomorphism does not preserve the labeling of long/short roots (see illustration below). As a result, $\mathrm{LG}(n, 2n)$ and $\mathrm{OG}(n+1, 2n+2)$ do not contain isomorphic Schubert varieties

of dimension greater than one.
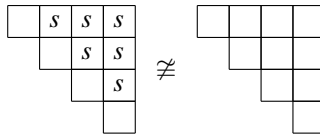
3 Preliminaries

Let G be a complex reductive linear algebraic group. We fix subgroups $T \subset B \subset G$, where T is a maximal torus and B is a Borel subgroup. With this setup, $T \subset G$ determines a root system R of G, with corresponding Weyl group $W := N(T)/T$, and B determines a set of simple roots $\Delta \subseteq R$. The set of roots decomposes into positive and negative roots $R = R^+ \sqcup R^-$, with R^+ being non-negative sums of simple roots. The Weyl group W is generated by the set of simple reflections

$$S := \{s_\alpha : \alpha \in \Delta\}.$$

To each subset $I \subseteq S$ one can associate a Weyl subgroup $W_I := \langle s : s \in I \rangle \subseteq W$, a parabolic subgroup $P_I = BW_I B \subseteq G$ and the corresponding (partial) flag variety $X = G/P_I$. Schubert varieties in X are indexed by W^I, the set of minimal length coset representatives of W/W_I. Explicitly, for $w \in W^I$, the Schubert variety

$$X_w := \overline{BwP_I/P_I}$$

Fig. 1 The Bruhat poset W^I when $X = \mathrm{Gr}(2, 4)$. Permutations in W^I are denoted using one-line notation, and next to each is the corresponding lower order ideal in $\mathcal{P}_X$. Join-irreducible elements of W^I are the ones underlined, and the generator of the corresponding principal lower order ideal in $\mathcal{P}_X$ is decorated with a ⋆
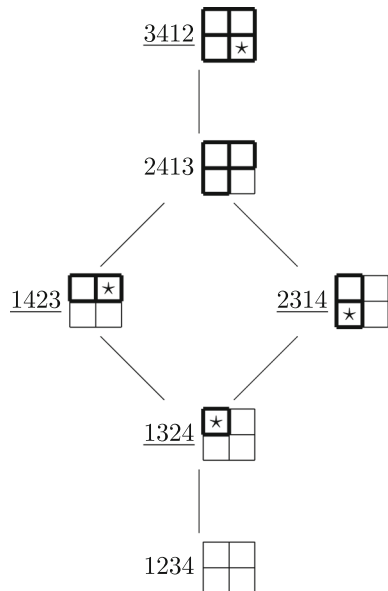
has dimension the Coxeter length of w, denoted $\ell(w)$. Moreover, for any $u \in W^I$, we have $X_u \subseteq X_w$ if and only if $u \leq w$ in Bruhat order.

From now on, X is a cominuscule flag variety. In other words, $I = S \setminus \{s_\gamma\}$, where γ is a cominuscule simple root, i.e., γ appears with coefficient 1 in the highest root of R. Cominuscule roots are illustrated by filled-in circles in Tables 1 and 2.

Recall that

$$\mathcal{P}_X := \{\alpha \in R : \alpha \geq \gamma\}$$

inherits the usual partial order on roots, i.e., $\alpha \leq \beta$ if $\beta - \alpha$ is a non-negative sum of simple roots, and in addition, we give $\mathcal{P}_X$ a labeling of long/short roots.

In [12], Proctor proves that W^I is a distributive lattice under the induced Bruhat partial order from W. Birkhoff's representation theorem implies there is a bijection between W^I and the set of lower order ideals in $\mathcal{P}_X$. In particular, the join-irreducible elements of W^I are identified with principal lower order ideals of $\mathcal{P}_X$ and hence with $\mathcal{P}_X$ itself. See Fig. 1 for an illustration when $X = \mathrm{Gr}(2,4)$. Explicitly, to each $w \in W^I$ we associate its inversion set

$$\lambda(w) := \{\alpha \in R^+ : w.\alpha < 0\}, \tag{1}$$

viewed as a sub-poset of $\mathcal{P}_X$. It is well known that $\ell(w) = |\lambda(w)|$. Moreover, the following proposition was proved in [17, Proposition 2.1 and Lemma 2.2] and [5, Theorem 2.4 and Corollary 2.6]:

Proposition 10 *(Thomas–Yong, Buch–Samuel) For any $w \in W^I$, the inversion set $\lambda(w)$ is a lower order ideal in $\mathcal{P}_X$. Moreover:*

(1) *The map $w \mapsto \lambda(w)$ is a bijection between W^I and the set of lower order ideals in $\mathcal{P}_X$.*
(2) *For any $u \in W^I$, we have $u \leq w$ in Bruhat order if and only if $\lambda(u) \subseteq \lambda(w)$.*
(3) *If $\alpha \in \lambda(w)$ and $\lambda(w) \setminus \{\alpha\}$ is a lower order ideal, then $ws_\alpha \in W^I$ and $\lambda(ws_\alpha) = \lambda(w) \setminus \{\alpha\}$, where $s_\alpha \in W$ is the reflection corresponding to α.*

Notation 11 Given a lower order ideal $\lambda \subseteq \mathcal{P}_X$, we will write w_λ for the element of W^I corresponding to λ in Proposition 10. We also use $X_\lambda := X_{w_\lambda}$ to denote the corresponding Schubert variety.

In Sect. 4.2, we will use a map $\delta : \mathcal{P}_X \to \Delta$ defined in [2] as follows.

Definition 12 For $\alpha \in \mathcal{P}_X$, let λ_α be the principal lower order ideal generated by α. Let $\delta(\alpha) = -w_{\lambda_\alpha}.\alpha \in R^+$. Then $s_{\delta(\alpha)} = w_{\lambda_\alpha} s_\alpha w_{\lambda_\alpha}^{-1}$ has length 1 [2]*Section 4.1. Therefore, $\delta(\alpha) \in \Delta$.

The following lemma is a restatement of [5]*Corollary 2.10. See also [2]*Section 4.1.

Lemma 13 *(Buch–Samuel) Let $\lambda \subseteq \mathcal{P}_X$ be a lower order ideal and $\beta_1, \beta_2, \ldots, \beta_\ell$ be the boxes it contains written in an increasing order. Then $s_{\delta(\beta_\ell)} \cdots s_{\delta(\beta_2)} s_{\delta(\beta_1)}$ is a reduced decomposition of w_λ. Moreover, every reduced decomposition of w_λ can be obtained in this way.*

4 Proof of Theorem 1

In this section, we prove each direction of Theorem 1 separately.

4.1 Forward direction: the isomorphism class of X_λ determines the labeled poset λ

Let $i_\lambda : X_\lambda \hookrightarrow X$ denote the embedding of a Schubert subvariety into a cominuscule flag variety $X = G/P_I$. The definition of the labeled poset λ depends on the root system of the reductive group G and hence on the embedding $i_\lambda : X_\lambda \hookrightarrow X$. The goal of this section is to show that λ (as a labeled poset) can be constructed using only the variety structure of X_λ and is therefore intrinsic to the isomorphism class of X_λ. We prove the following proposition which states the "forward" direction of Theorem 1.

Proposition 14 *Let $X_\lambda \subseteq X$ and $Y_\mu \subseteq Y$ be cominuscule Schubert varieties indexed by lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$, respectively. If X_λ and Y_μ are algebraically isomorphic, then λ and μ are isomorphic as labeled posets.*

Our primary tools come from the intersection theory of algebraic varieties (see [8] for more details). Let $\mathrm{Pic}(X_\lambda)$ and $A_*(X_\lambda)$ denote the Picard and Chow groups of X_λ. It is well known that these groups are algebraic invariants of X_λ. Recall that the k-th Chow group $A_k(Z)$ of a scheme Z is the free abelian group on the k-dimensional subvarieties of Z modulo rational equivalence. When Z is a normal variety, the Picard group $\mathrm{Pic}(Z)$ can be identified with the subgroup of $A_{\dim(Z)-1}(Z)$ generated by classes of locally principal divisors (note that all Schubert varieties are normal). Our aim is to construct the labeled poset λ from the intersection class map or intersection product [8, Definition 2.3]:

$$\mathrm{Pic}(X_\lambda) \times A_*(X_\lambda) \to A_*(X_\lambda).$$

If $(\sigma, \tau) \in \mathrm{Pic}(X_\lambda) \times A_*(X_\lambda)$, we denote the image of the intersection product by $\sigma \cdot \tau$. Next, we consider the effective cone of a scheme:

Definition 15 Let Z be a scheme. The *effective cone* in the Chow group $A_*(Z)$ is the semigroup in $A_*(Z)$ generated by the classes of closed subvarieties of Z.

Since the flag variety X is cominuscule, there is a unique Schubert variety of codimension 1 in X, called the Schubert divisor. Its class, denoted D, is the unique effective generator of the Picard group $\mathrm{Pic}(X) \subseteq A_*(X)$. Recall that $i_\lambda : X_\lambda \hookrightarrow X$ is a closed embedding of varieties and let $i_\lambda^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X_\lambda)$ denote the induced map on Picard groups. Lemma 16 below follows from [10, Proposition 6] (see also [4, Proposition 2.2.8 part (ii)]).

Lemma 16 *For any non-empty lower order ideal $\lambda \subseteq \mathcal{P}_X$, the map $i_\lambda^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X_\lambda)$ is an isomorphism.*

Since D is effective and generates $\mathrm{Pic}(X)$, Lemma 16 implies that $i_\lambda^*(D)$ is the unique effective generator of $\mathrm{Pic}(X_\lambda)$. Recall from Proposition 10 that we have lower order ideals $\mu \subseteq \lambda$ if and only if $w_\mu \leq w_\lambda$ in Bruhat order. Hence, we have $\mu \subseteq \lambda$ if

and only if $X_\mu \subseteq X_\lambda$. We write $[X_\mu]$ for the class of X_μ in $A_*(X_\lambda)$. It is well known that the classes $\{[X_\mu]\}_{\mu \subseteq \lambda}$ form an integral basis of $A_*(X_\lambda)$. Lemma 17 below is a special case of [7, Corollary of Thereom 1] and allows us to identify Schubert classes (the effective cone) in $A_*(X_\lambda)$.

Lemma 17 *(Fulton–MacPherson–Sottile–Sturmfels)* The Schubert classes $[X_\mu]$ such that $\mu \subseteq \lambda$ are exactly the minimal elements in the extremal rays of the effective cone in $A_*(X_\lambda)$.

We shall see later that the poset structure of λ can be recovered from the intersection products $i_\lambda^*(D) \cdot [X_\mu]$. Let $(i_\lambda)_* : A_*(X_\lambda) \to A_*(X)$ denote the proper push-forward on Chow groups. By the projection formula ([8, Proposition 2.5 (c)]), we have

$$(i_\lambda)_*(i_\lambda^*(D) \cdot [X_\mu]) = D \cdot (i_\lambda)_*([X_\mu]).$$

Since $(i_\lambda)_*$ is injective, the product $i_\lambda^*(D) \cdot [X_\mu]$ in $A_*(X_\lambda)$ can be computed via the product $D \cdot (i_\lambda)_*([X_\mu])$ in $A_*(X)$. By [8, Example 19.1.11], the Chow group $A_*(X)$ can be identified with the homology group $H_*(X)$, with $(i_\lambda)_*([X_\mu])$ corresponding to the homology class of the Schubert variety $X_\mu \subseteq X$. By [8, Proposition 19.1.2] we have that the intersection product $D \cdot (i_\lambda)_*([X_\mu])$ can be identified with a cap product. Since X is a smooth complex variety, the Poincaré duality further identifies the intersection product with the cup product of cohomology classes corresponding to D and $(i_\lambda)_*([X_\mu])$. This cup product is given by the Chevalley formula [9, Lemma 8.1]. Using these identifications, we restate the Chevalley formula for cominuscule flag varieties (and hence Schubert varieties):

Lemma 18 *(Fulton–Woodward) Let X be a cominuscule flag variety with corresponding cominuscule simple root γ. For any lower order ideals $\mu \subseteq \lambda \subseteq \mathcal{P}_X$, let $[X_\mu]$ denote the class of X_μ in $A_*(X_\lambda)$. Then*

$$i_\lambda^*(D) \cdot [X_\mu] = \sum \frac{(\gamma, \gamma)}{(\alpha, \alpha)} [X_{\mu \setminus \{\alpha\}}]$$

where the sum is over all positive roots α such that $\mu \setminus \{\alpha\}$ is a lower order ideal in $\mathcal{P}_X$. Here $(\cdot, \cdot)$ denotes the usual inner product.

Observe that Lemma 18 reinterprets the Chevalley formula as a degree lowering operator since intersection product with divisors is a map from $A_k(X_\lambda)$ to $A_{k-1}(X_\lambda)$. This is opposite to the standard presentation of the Chevalley formula as a degree raising operator in cohomology.

Example 19 By Lemma 18, the following calculations hold for $X = \mathrm{LG}(3, 6)$. We refer to Table 1 for the poset $\mathcal{P}_X$.

$$D \cdot \left[X_{\boxed{\,\,\square\,s\,s\,}} \right] = 2 \left[X_{\boxed{\,\,\square\,s\,}} \right] + \left[X_{\boxed{\,\square\,s\,s\,}} \right],$$

$$D \cdot \left[X_{\boxed{\,\square\,s\,}} \right] = \left[X_{\boxed{\,\square\,s\,}} \right],$$

$$D \cdot \left[X_{\boxed{\,s}} \right] = 2 \left[X_{\square} \right].$$

Remark 20 Note that a coefficient 2 occurs whenever the removed box (root) is short. Otherwise the coefficient is 1. This is due to the fact that cominuscule flag varieties only appear in Dynkin types that are at most "doubly laced" (see Table 2).

Proof of Proposition 14 Let $\tilde{X}$ be a variety that is algebraically isomorphic to a cominuscule Schubert variety. Let $\tilde{E} := \{E_1, \ldots, E_k\} \subseteq A_*(\tilde{X})$ denote the set of minimal elements in the extremal rays of the effective cone. By Lemma 17, these classes form the Schubert basis of $A_*(\tilde{X})$. Lemma 16 implies there is a unique effective generator of $\mathrm{Pic}(\tilde{X})$ which we denote by Z. For any $E_i \in \tilde{E}$, consider the intersection product

$$Z \cdot E_i = \sum_j c_{ij} E_j.$$

Define a partial order on the set $\tilde{E}$ via the covering relations

$$E_j < E_i \text{ if and only if } c_{ij} \neq 0.$$

If $\tilde{X} \simeq X_\lambda$ and $i_\lambda : X_\lambda \hookrightarrow X = G/P_I$ is an embedding of a Schubert subvariety into a cominuscule flag variety, then Z corresponds to $i_\lambda^*(D)$ under the identification $\mathrm{Pic}(\tilde{X}) \simeq \mathrm{Pic}(X_\lambda)$. Lemma 18 implies that the poset $\tilde{E}$ is isomorphic to the set of lower order ideals in $\mathcal{P}_X$ that is contained in λ, ordered by inclusion. Hence, $\tilde{E}$ can be identified with the Bruhat interval $\{u \in W^I : u \leq w_\lambda\}$ via Proposition 10. Let $\tilde{\lambda}$ denote the sub-poset of join-irreducible elements in $\tilde{E}$. Our discussions in Sect. 3 imply that $\tilde{\lambda}$ is poset isomorphic to λ. Hence, the poset is independent of the embedding $i_\lambda : X_\lambda \hookrightarrow X$.

We finish the proof by showing that the labeling of long/short roots can also be recovered from the Chevalley formula. Let $E_i \in \tilde{\lambda}$ and hence E_i is join-irreducible in the poset $\tilde{E}$. First, if E_i is the unique minimal element in $\tilde{\lambda}$, then we label E_i as long (this corresponds to the cominuscule simple root). Otherwise, Lemma 18 implies that

$$Z \cdot E_i = c_{ij} E_j$$

for some unique $E_j \in \tilde{E}$ with $c_{ij} \neq 0$. If $c_{ij} = 1$, then we label E_i as long. If $c_{ij} \neq 1$, then we label E_i as short. If $\tilde{X} \simeq X_\lambda$, then Lemma 18 implies that this labeling of $\tilde{\lambda}$ corresponds to the labeling of long/short roots in λ.

In conclusion, the labeled poset $\tilde{\lambda}$ only depends on the isomorphism class of $\tilde{X}$. In particular, if two cominuscule Schubert varieties X_λ and Y_μ are algebraically isomorphic, then $\lambda \simeq \mu$ as labeled posets. $\qquad\square$

4.2 Converse direction: the labeled poset λ determines the isomorphism class of X_λ

Let $X = G/P_I$ be a cominuscule flag variety and $\lambda \subseteq \mathcal{P}_X$ be a lower order ideal. In this section, we prove that the poset λ and its labeling of long/short roots determine the isomorphism class of X_λ. More precisely, we prove the following proposition, which states the "converse" direction of Theorem 1.

Proposition 21 *Let $X_\lambda \subseteq X$ and $Y_\mu \subseteq Y$ be cominuscule Schubert varieties indexed by lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$, respectively. If λ and μ are isomorphic as labeled posets, then X_λ and Y_μ are algebraically isomorphic.*

Our strategy is to embed X_λ in a "minimal" flag variety X' determined by the labeled poset λ.

Recall that S is the set of simple reflections defined in Sect. 3.

Definition 22 The *support* of λ is defined as

$$S(\lambda):=\{s \in S : s \le w_\lambda\}.$$

Equivalently, $S(\lambda)$ is the set of simple reflections appearing in any reduced decomposition of w_λ.

Every reduced decomposition of w_λ, and in particular, $S(\lambda)$, can be read out from the poset λ [2, Section 4]. The variety X' is constructed using $S(\lambda)$ as follows. Let G' be the reductive subgroup of $P_{S(\lambda)}$ with Weyl group $W':=W_{S(\lambda)}$ and $P':=G' \cap P_I$ be the reductive subgroup of G' corresponding to $I':=I \cap S(\lambda)$. Set $X':=G'/P'$. Note that $w_\lambda \in W'^{I'}$.

Lemma 23 below is a restatement of [13, Lemma 4.8].

Lemma 23 Richmond–Slofstra *The inclusion $X' \hookrightarrow X$ induces an isomorphism $X'_{w_\lambda} \to X_\lambda$.*

Let Y be another cominuscule flag variety and $\mu \subseteq \mathcal{P}_Y$ be a lower order ideal. Next, we show that a labeled poset isomorphism between λ and μ induces an isomorphism between X' and Y', which restricts to an isomorphism between X_λ and Y_μ. We shall see that X' and Y' are cominuscule and that this isomorphism is given by an isomorphism of their Dynkin diagrams.

In the following, let $\mathcal{D}_X$ be the Dynkin diagram of X with vertex set Δ_X.

Definition 24 The diagram $\mathcal{D}_X^\lambda$ is defined to be the full subgraph of $\mathcal{D}_X$ with vertex set

$$\Delta_X^\lambda:=\{\alpha \in \Delta_X : s_\alpha \in S(\lambda)\}.$$

Definition 25 Let a *Dynkin chain* in $\mathcal{P}_X$ be a chain $\pi \subseteq \mathcal{P}_X$ such that:

(1) the set π is a lower order ideal;

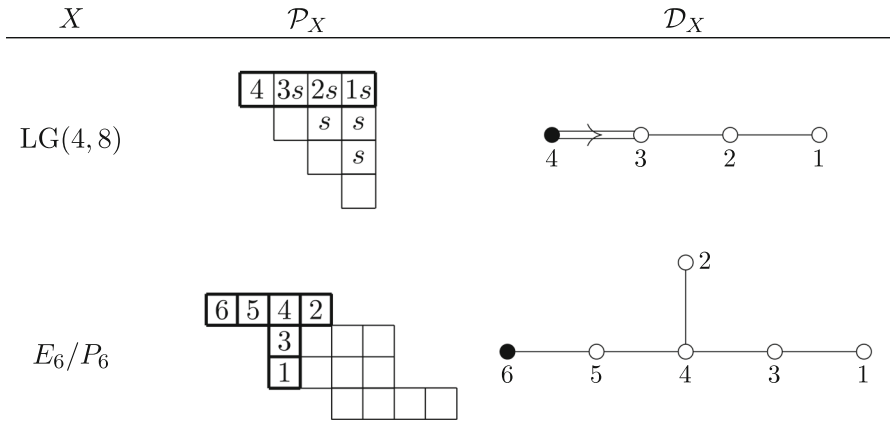| X	$\mathcal{P}_X$	$\mathcal{D}_X$

Fig. 2 We highlight $\mathcal{P}_X^\Delta \subset \mathcal{P}_X$ with a bold border and label its boxes by the images of δ

(2) the lengths of roots in π are weakly decreasing.

The lower order ideal $\mathcal{P}_X^\Delta \subseteq \mathcal{P}_X$ is defined to be the union of all Dynkin chains in $\mathcal{P}_X$.

In the proof of Lemma 26, we shall see that Dynkin chains in $\mathcal{P}_X$ correspond to paths in $\mathcal{D}_X$ starting from the cominuscule root γ. Examples of $\mathcal{P}_X^\Delta$ are illustrated in Fig. 2.

Lemma 26 *The restriction $\delta : \mathcal{P}_X^\Delta \to \Delta_X$ is a bijection.*

Proof Note that the cominuscule root γ is the unique minimal root in $\mathcal{P}_X$ and $\delta(\gamma) = \gamma$. Also, since γ is a long root, it does not obstruct condition (2) in Definition 25 and is an element of any non-empty Dynkin chain. Moreover, if $\gamma, \beta_1, \ldots, \beta_{m-1}, \beta_m$ are distinct simple roots along a path in $\mathcal{D}_X$, then

$$\gamma < (\gamma + \beta_1) < \cdots < (\gamma + \beta_1 + \cdots + \beta_m)$$

is a Dynkin chain in $\mathcal{P}_X$ and $\delta(\gamma + \beta_1 + \cdots + \beta_m) = \beta_m$. This proves surjectivity.

To prove injectivity, note that for β, β' in $\mathcal{P}_X$, if $\delta(\beta) = \delta(\beta')$, then β and β' are comparable [2, Remark 4.2(b)]. Since Dynkin chains are lower order ideals (Definition 25 part (1)), it suffices to prove that each Dynkin chain in $\mathcal{P}_X^\Delta$ maps injectively to Δ_X. Let $\gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_k$ be a Dynkin chain in $\mathcal{P}_X^\Delta$. (While we will not need it, we must have $\gamma_0 = \gamma$ because a Dynkin chain is a lower order ideal and γ is the unique minimal root in $\mathcal{P}_X$.) By [2, Remark 4.2(c)] we have that $\delta(\gamma_0), \delta(\gamma_1), \ldots, \delta(\gamma_k)$ forms a walk on $\mathcal{D}_X$. For a contradiction, assume this walk is not a path. Since $\mathcal{D}_X$ does not contain any cycles, we can assume without loss of generality that $\delta(\gamma_{i-1}) = \delta(\gamma_{i+1})$ for some i, where $0 < i < k$. By definition,

$$\delta(\gamma_j) = -w_j.\gamma_j \text{ for all } j,$$

where $w_j := w_{\lambda_{\gamma_j}}$. By Lemma 13, we have

$$w_i = s_{\delta(\gamma_i)} w_{i-1} \quad \text{and} \quad w_{i+1} = s_{\delta(\gamma_{i+1})} s_{\delta(\gamma_i)} w_{i-1}.$$

Therefore,

$$
\begin{aligned}
\gamma_{i-1} &= -w_{i-1}^{-1}.\delta(\gamma_{i-1}) = -w_{i-1}^{-1}.\delta(\gamma_{i+1}), \\
\gamma_i &= -w_{i-1}^{-1} s_{\delta(\gamma_i)}.\delta(\gamma_i) = w_{i-1}^{-1}.\delta(\gamma_i), \\
\gamma_{i+1} &= -w_{i-1}^{-1} s_{\delta(\gamma_i)} s_{\delta(\gamma_{i+1})}.\delta(\gamma_{i+1}) = w_{i-1}^{-1} s_{\delta(\gamma_i)}.\delta(\gamma_{i+1}).
\end{aligned}
\tag{2}
$$

Note that $(\delta(\gamma_j), \delta(\gamma_j)) = (\gamma_j, \gamma_j)$ for all j. Hence, using 2 we have

$$(\delta(\gamma_{i-1}), \delta(\gamma_{i-1})) = (\delta(\gamma_{i+1}), \delta(\gamma_{i+1})),$$

and by the decreasing condition on lengths (Definition 25), we must have

$$(\delta(\gamma_{i-1}), \delta(\gamma_{i-1})) = (\delta(\gamma_i), \delta(\gamma_i)) = (\delta(\gamma_{i+1}), \delta(\gamma_{i+1})).$$

Since the simple roots $\delta(\gamma_{i+1})$ and $\delta(\gamma_i)$ are adjacent in the Dynkin diagram and of the same length,

$$\langle \delta(\gamma_{i+1}), \delta(\gamma_i) \rangle := \frac{2(\delta(\gamma_{i+1}), \delta(\gamma_i))}{(\delta(\gamma_i), \delta(\gamma_i))} = -1.$$

As a consequence,

$$s_{\delta(\gamma_i)}.\delta(\gamma_{i+1}) = \delta(\gamma_{i+1}) - \langle \delta(\gamma_{i+1}), \delta(\gamma_i) \rangle \delta(\gamma_i) = \delta(\gamma_{i+1}) + \delta(\gamma_i). \tag{3}$$

By 2 and 3, we have

$$\gamma_{i+1} - \gamma_i = w_{i-1}^{-1}.\delta(\gamma_{i+1}) = -\gamma_{i-1},$$

which is a contradiction.   □

Corollary 27 *Let $\lambda \subseteq \mathcal{P}_X$ be a lower order ideal. Then $\delta(\lambda \cap \mathcal{P}_X^\Delta) = \Delta_X^\lambda$, and $\mathcal{D}_X^\lambda$ is a connected Dynkin diagram.*

Proof By [2]*Remark 4.2.(b), for each $\alpha \in \Delta_X$, there is a unique minimal root $\beta \in \mathcal{P}_X$ such that $\delta(\beta) = \alpha$. From Lemma 26 we see that $\beta \in \mathcal{P}_X^\Delta$. It follows that $\delta(\lambda) = \delta(\lambda \cap \mathcal{P}_X^\Delta)$. By Lemma 13, we have $\delta(\lambda) = \Delta_X^\lambda$. Therefore, $\delta(\lambda \cap \mathcal{P}_X^\Delta) = \Delta_X^\lambda$. Note that $\lambda \cap \mathcal{P}_X^\Delta$ is exactly the union of Dynkin chains contained in λ, and by the proof of Lemma 26, the map δ sends each Dynkin chain to a path in $\mathcal{D}_X$ starting from γ. Therefore, $\mathcal{D}_X^\lambda$ is connected.   □

Remark 28 Corollary 27 implies that X' is the cominuscule flag variety given by the pair $(\mathcal{D}_X^\lambda, \gamma)$.

The last ingredient is Proposition 29, a purely combinatorial result. Geometrically, it implies that the "minimal" cominuscule flag varieties for Schubert varieties with isomorphic labeled posets are isomorphic.

Proposition 29 *Let $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$ be lower order ideals. Then every labeled poset isomorphism between λ and μ induces a graph isomorphism between $\mathcal{D}_X^\lambda$ and $\mathcal{D}_Y^\mu$ that identifies reduced decompositions of w_λ and w_μ.*

Proof Let $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$ be lower order ideals and $f : \lambda \to \mu$ a labeled poset isomorphism. Note that f restricts to a labeled poset isomorphism $f : \lambda \cap \mathcal{P}_X^\Delta \to \mu \cap \mathcal{P}_Y^\Delta$, since $\lambda \cap \mathcal{P}_X^\Delta$ is exactly the union of Dynkin chains contained in λ. Define a map $\psi_f : \Delta_X^\lambda \to \Delta_Y^\mu$ by $\psi_f(\alpha) = \delta \circ f \circ \delta^{-1}(\alpha)$, where $\delta^{-1} : \Delta_X \to \mathcal{P}_X^\Delta$ is the inverse of $\delta : \mathcal{P}_X^\Delta \to \Delta_X$. Then ψ_f is a bijection.

By the proof of Lemma 26, simple roots α and β are adjacent vertices in $\mathcal{D}_X$ if and only if either $\delta^{-1}(\alpha) < \delta^{-1}(\beta)$ or $\delta^{-1}(\beta) < \delta^{-1}(\alpha)$ is a covering relation. Since δ preserves root lengths, the map ψ_f can be extended to a graph isomorphism $\mathcal{D}_X^\lambda \to \mathcal{D}_Y^\mu$.

The identification of reduced decompositions now follows from Lemma 13. □

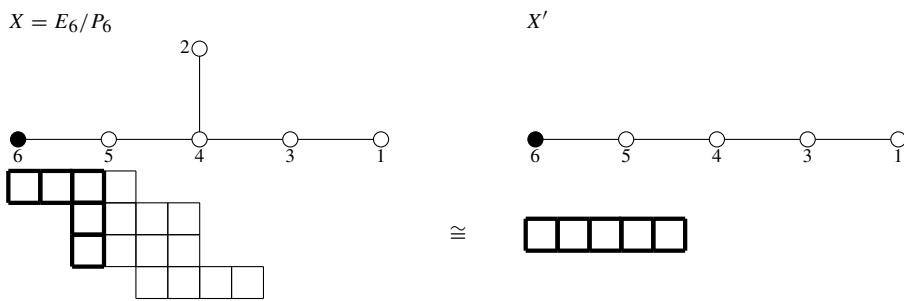**Proof of Proposition 21** Let γ_X and γ_Y denote the cominuscule simple roots corresponding to the cominuscule flag varieties X and Y. Let X' and Y' denote the cominuscule flag varieties given by the pairs $(\mathcal{D}_X^\lambda, \gamma_X)$ and $(\mathcal{D}_Y^\mu, \gamma_Y)$. Proposition 29 implies the cominuscule flag varieties X' and Y' are isomorphic. By Lemma 23, this isomorphism restricts to an isomorphism between X_λ and Y_μ, upon identifying them with Schubert varieties in X' and Y', respectively. □

We illustrate the above process with Example 30 and Example 31 below.

Example 30 Let $X = E_6/P_6$ and λ be the lower order ideal depicted on the left below. Then $S(\lambda) = \{s_2, s_3, s_4, s_5, s_6\}$, where s_i is the simple reflection corresponding to the simple root labeled by i. Therefore, the pair $(\mathcal{D}_X^\lambda, \gamma)$ is as depicted on the right, isomorphic to that of Q^8, showing $X' \cong Q^8$, and $X_\lambda \cong X'_{\lambda'}$, where λ' is the lower order ideal depicted on the right below.
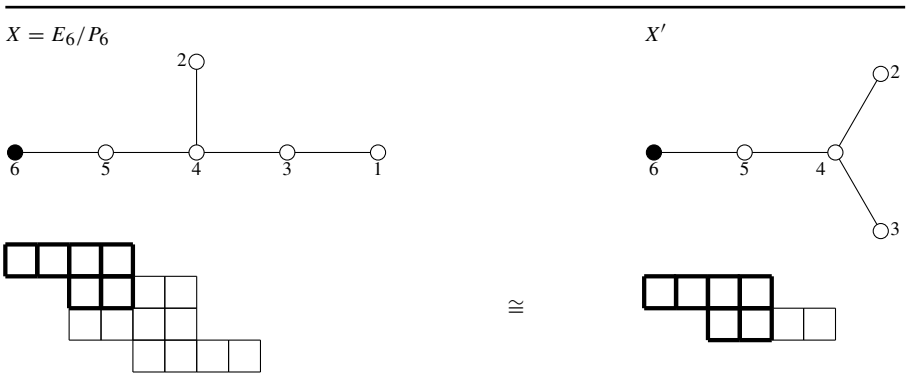
Example 31 Let $X = E_6/P_6$ and λ be the lower order ideal depicted on the left below. Then $S(\lambda) = \{s_1, s_3, s_4, s_5, s_6\}$, where s_i is the simple reflection corresponding to the simple root labeled by i. Therefore, the pair $(\mathcal{D}_X^\lambda, \gamma)$ is as depicted on the right, isomorphic to that of $\mathbb{P}^5$, showing $X_\lambda \cong X' \cong \mathbb{P}^5$.

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