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Families of relatively exact Lagrangians, free loop spaces and generalised homology

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Abstract

We prove that (under appropriate orientation conditions, depending on R) a Hamiltonian isotopy ψ^1 of a symplectic manifold (M, ω) fixing a relatively exact Lagrangian L setwise must act trivially on $R_*(L)$, where R_* is some generalised homology theory. We use a strategy inspired by that of Hu et al. (Geom Topol 15:1617–1650, 2011), who proved an analogous result over $\mathbb{Z}/2$ and over $\mathbb{Z}$ under stronger orientation assumptions. However the differences in our approaches let us deduce that if L is a homotopy sphere, $\psi^1|_L$ is homotopic to the identity. Our technical set-up differs from both theirs and that of Cohen et al. (in: Algebraic topology, Springer, Berlin, 2019) and Cohen (in: The Floer memorial volume, Birkhäuser, Basel). We also prove (under similar conditions) that $\psi^1|_L$ acts trivially on $R_*(\mathcal{L}L)$, where $\mathcal{L}L$ is the free loop space of L. From this we deduce that when L is a surface or a $K(\pi, 1)$, $\psi^1|_L$ is homotopic to the identity. Using methods of Lalonde and McDuff (Topology 42:309–347, 2003), we also show that given a family of Lagrangians all of which are Hamiltonian isotopic to L over a sphere or a torus, the associated fibre bundle cohomologically splits over $\mathbb{Z}/2$.

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1 Introduction

1.1 Background

Let (M^{2n}, ω) be a symplectic manifold of dimension $2n$, and $L^n \subseteq M$ a Lagrangian submanifold. Let ψ^t be a Hamiltonian isotopy of M such that its time-1 flow preserves L setwise, $\psi^1(L) = L$. Its restriction to L is then a self-diffeomorphism of L. We can consider the *Hamiltonian monodromy group* $\mathcal{G}_L \subseteq \mathrm{Diff}(L)$ of diffeomorphisms of L arising in this way. A natural question is:

Question 1.1 *What is* $\mathcal{G}_L$*?*

An elementary argument using the Weinstein neighbourhood theorem shows that if f and g are isotopic diffeomorphisms of L and g lies in $\mathcal{G}_L$, then f does too. This implies that $\mathcal{G}_L$ is a union of connected components in $\mathrm{Diff}(L)$, and hence to study $\mathcal{G}_L$, it suffices to study its image in the mapping class group $\pi_0\,\mathrm{Diff}(L)$.

The subgroup $\mathcal{G}_L$ was first studied by Yau in [30], who proved the following theorem.

Theorem 1.2 ([30]) *Let T and T' be the standard monotone Clifford and Chekanov tori in $\mathbb{C}^2$ with the same monotonicity constant. Then in both cases $L = T$ or T', $\pi_0\mathcal{G}_L \cong \mathbb{Z}/2$.*

However, there is no isomorphism $H_1(T) \cong H_1(T')$ which respects the $\mathbb{Z}/2$ action on both and also preserves the Maslov index homomorphism.

This provides a new proof that these two Lagrangians are not Hamiltonian isotopic.

Question 1.1 has been studied in few other places: by Hu, Lalonde and Leclercq in [19], by Mangolte and Welschinger in [22], by Evans and Rizell in [27], by Ono in

[25], by Varolgunes in [29], by Keating in [20], and by Augustynowicz, J. Smith and Wornbard in [6]. Out of these, only Hu-Lalonde-Leclercq and Evans-Rizell focus on the case when L is exact and embedded, which is where we will focus.

We now assume (for the rest of the paper) the following:

Assumption 1.3 1. M is a product of symplectic manifolds which are either compact or Liouville.
2. L is compact, connected and relatively exact, i.e. ω vanishes on the relative homotopy group $\pi_2(M, L)$.
3. ψ^t is compactly supported for all t and constant in t near 0 and 1.

The first two assumptions constrain the behaviour of the holomorphic curves that we will consider. Note that given ψ^t not satisfying the third, we can always deform it (by multiplying the Hamiltonian by an appropriate cut-off function) so that it does, without changing $\psi^1|_L$, so this assumption does not lose any generality.

$\mathcal{G}_L$ was studied by Hu, Lalonde and Leclercq in this more restrictive setting in [19], where they proved the following:

Theorem 1.4 ([19])

1. *$\psi^1|_L$ acts as the identity on $H_*(L; \mathbb{Z}/2)$.*
2. *If L admits a relative spin structure in M which ψ^1 preserves, $\psi^1|_L$ acts as the identity on $H_*(L; \mathbb{Z})$.*

It follows from Yau's results that relative exactness is a necessary condition here.

We will recover Theorem 1.4 as a special case of Theorem 1.8. Note that in the case that L is a homotopy sphere, this theorem does not give any information on the homotopy class of $\psi^1|_L$: if $\psi^1|_L$ were not homotopic to the identity, it would not preserve any orientation of L and hence would not preserve any relative spin structure.

Our goals will be to weaken the orientation assumptions, extend this result to some other generalised cohomology theories, and extend this result to the free loop space of L. From this, we will fully compute $\mathcal{G}_L$ in the case that $n = 2$. Our technical set-up will be very different to that of Hu, Lalonde and Leclercq, but our general approach is inspired by theirs.

Remark 1.5 Hu, Lalonde and Leclercq prove this using Morse and quantum cohomology (as in [7]), and one possible approach to extending Theorem 1.4 to other generalised cohomology theories would be to recreate Hu, Lalonde and Leclercq's proof using the methods of Cohen, Jones and Segal [8, 10]. However one may need to require stronger orientation hypotheses than Assumption 1.7.

Remark 1.6 If L not assumed to be relatively exact, Theorem 1.4 fails, as seen in Yau's example ([30]). However if L is monotone, the failure for Theorem 1.4 to hold is understood- it can be seen in the failure of a certain element in $HF^*(L)$ to lie in the centre of $HF^*(L)$ (when L is exact, $HF^*(L) \cong H^*(L)$ is always commutative)- see [6, Remark 4.2] and [29] for further discussions on this point.

It would be interesting to see if a similar phenomenon holds in the setting of other generalised cohomology theories, though there is not yet a construction of a Lagrangian Floer cohomology ring for non-relatively exact Lagrangians in the literature.

1.2 Extensions to generalised homology theories

We will define a space $\mathcal{D}_0$, which is roughly the space of smooth maps $D^2 \to M$ sending 1 to L, with moving Lagrangian boundary conditions on the rest of the boundary. This admits a map $\pi : \mathcal{D}_0 \to L$ by evaluating at 1. We will construct a virtual vector bundle called the *index bundle*, Ind, on any finite CW complex in $\mathcal{D}_0$ (compatible with restriction). After picking a (generic) almost complex structure J on M and some specific choices of moving Lagrangian boundary conditions, the tangent bundle of the moduli space of J-holomorphic discs with these boundary conditions will be stably isomorphic to the restriction of Ind. In particular, L embeds into $\mathcal{D}_0$ as the space of constant discs, with Ind restricting to TL.

We fix some ring spectrum R, and will recall definitions of ring spectra and R-orientability in Sect. 2. We will make the following assumption, and in Sect. 4.2, find conditions under which it holds.

Assumption 1.7 The virtual vector bundle $\mathrm{Ind} - \pi^* TL$ is R-orientable.

We will show:

Theorem 1.8 *Under Assumption 1.7, the map*

$$\psi^1|_L : \Sigma^\infty_+ L \wedge R \to \Sigma^\infty_+ L \wedge R$$

is homotopic to the identity as a map of R-modules.

Remark 1.9 Our proof of this will use a minimal amount of technical machinery: we will only use standard Gromov compactness results and standard transversality results, and we will not need any form of gluing. It is possible to prove this without invoking any transversality results, and instead only use the fact that certain operators are Fredholm, using ideas in [18]; see [17, Remark 2.13].

From standard duality theory for spectra (Corollary 2.14), we deduce from Theorem 1.8:

Corollary 1.10 *Under Assumption 1.7,*

1. *$\psi^1|_L$ induces the identity map on $R_*(L)$.*
2. *$\psi^1|_L$ induces the identity map on $R^*(L)$.*

We will deduce Theorem 1.8 from the following sequence of lemmas.

In Sect. 4, we will define a moduli space $\mathcal{P}$ of holomorphic discs in M, with moving boundary conditions, lying naturally inside $\mathcal{D}_0$. Its tangent space $T\mathcal{P}$ will be stably isomorphic to Ind. We will prove that it satisfies the following lemmas:

Lemma 1.11 *$\mathcal{P}$ is a closed smooth manifold of dimension n.*

and

Lemma 1.12 *The following diagram commutes up to homotopy:*

$$\begin{tikzcd} & L \arrow[dd, "\psi^1"] \\ \mathcal{P} \arrow[ur, "\pi"] \arrow[dr, "\pi"'] & \\ & L \end{tikzcd}$$

Furthermore, by varying the boundary conditions in a 1-parameter family, we will construct a cobordism $\pi : \mathcal{W} \to L$ from $Id : L \to L$ to $\pi : \mathcal{P} \to L$ over L, such that $T\mathcal{W} - \pi^* TL$ will be R-orientable under Assumption 1.7. A standard application of the Pontryagin–Thom construction (see Sect. 2) will prove the following lemma:

Lemma 1.13 *The induced map $\pi : \Sigma^\infty_+ \mathcal{P} \wedge R \to \Sigma^\infty_+ L \wedge R$ admits a section of R-module maps (up to homotopy of R-module maps).*

Proof This follows from the construction of $\mathcal{W}$ in Section 4, along with Lemma 2.24. □

Proof of Theorem 1.8 Let s be the section from Lemma 1.13. Then from Lemma 1.12, we have that

$$\psi^1 \simeq \psi^1 \circ \pi \circ s \simeq \pi \circ s \simeq Id$$

as maps of R-modules $\Sigma^\infty_+ L \wedge R \to \Sigma^\infty_+ L \wedge R$. □

In Sect. 4.2, we will show the following:

Proposition 1.14 *Assumption 1.7 holds when:*

1. *$R = H\mathbb{Z}/2$ representing mod-2 singular homology.*
2. *$R = H\mathbb{Z}$ representing integral singular homology, and $w_2(L)(x) = 0$ whenever x is a homology class represented by a 2-torus $S^1 \times \partial D^2$ in L which can be extended to a solid torus $S^1 \times D^2$ in M.*
3. *$R = KU$ representing complex K-theory, and L admits a spin structure.*
4. *$R = KO$ representing real K-theory, and TL admits a stable trivialisation over a 3-skeleton of L which extends (after applying $\cdot \otimes \mathbb{C}$) to a stable trivialisation of TM over a 4-skeleton of M.*
5. *R is a complex-orientable generalised cohomology theory and L is stably parallelisable.*
6. *$R = \mathbb{S}$ is the sphere spectrum and there is a real vector bundle E over X, along with stable isomorphisms $TM \cong E \otimes \mathbb{C}$ and $TL \cong E|_L$ compatible with each other.*

Remark 1.15 Our methods show that when $R = H\mathbb{Z}$, Assumption 1.7 holds if L is spin. We will deduce the stronger statement (2) from a theorem of Georgieva ([15, Theorem 1.1]).

Corollary 1.16 *When L is a homotopy sphere, $\psi^1|_L$ is homotopic to the identity.*

Proof L admits a spin structure, so ψ^1 acts as the identity on integral homology and so is homotopic to the identity. □

It is nontrivial to find examples of self-diffeomorphisms of spin manifolds which act trivially on integral cohomology but non-trivially on complex K-theory, so we provide an example in Appendix A, following a suggestion of Randal-Williams ([26]). We furthermore provide an example in Appendix B of a self-diffeomorphism of a manifold which acts trivially on integral cohomology but non-trivially on real K-theory.

1.3 Extensions to the free loop space

Fix a ring spectrum R. We will prove an analogue of Theorem 1.8 for the free loop space $\mathcal{L}L$ of L, the space of maps $S^1 \to L$. Once again, we will need to make an assumption in order to orient our moduli spaces.

Theorem 1.17 *Assume Assumption 5.13 holds. Then the induced map*

$$\psi^1|_L : \Sigma^\infty_+ \mathcal{L}L \wedge R \to \Sigma^\infty_+ \mathcal{L}L \wedge R$$

is homotopic to the identity, as maps of R-modules.

Idea of proof To prove this, we will define $\mathcal{L}_1$ to be the space of free loops in the mapping torus L_{ψ^1} of $\psi^1|_L$ which have winding number one over S^1, and whose basepoint lies in the fibre of the mapping torus over the basepoint in S^1. ψ determines an automorphism $\Psi^1 : \mathcal{L}_1 \to \mathcal{L}_1$, which one should think of as parallel transporting once around this mapping torus.

We will construct a map

$$p \circ ([\mathcal{N}] \cdot) : \bigvee_{j \in \mathbb{Z}} \Sigma^{\infty+j}_+ \mathcal{L}_1 \wedge R \to \Sigma^\infty_+ \mathcal{L}L \wedge R$$

This will use a moduli space of holomorphic discs with moving boundary conditions $\mathcal{N}$, as well as Cohen and Jones' version of the Chas-Sullivan product, constructed similarly to [9]. We will then show:

1. Ψ^1 acts as the identity on $\mathcal{L}_1$ up to homotopy.
2. $p \circ ([\mathcal{N}] \cdot)$ intertwines the actions of Ψ^1 and $\psi^1|_L$.
3. $p \circ ([\mathcal{N}] \cdot)$ admits a section up to homotopy.

Together, these will imply Theorem 1.17. □

Corollary 1.18 *Assume Assumption 5.13 holds. Then the map*

$$\left(\psi^1|_L\right)_* : R_*(\mathcal{L}L) \to R_*(\mathcal{L}L)$$

is the identity.

Proposition 1.19 *Assumption 5.13 holds when*

1. $R = H\mathbb{Z}/2$.
2. $R = H\mathbb{Z}$ *and* L *admits a spin structure which is preserved by* ψ^1.
3. $R = KU$ *and* L *admits a spin structure which is preserved by* ψ^1.
4. R *is any complex-orientable generalised cohomology theory and* TL *admits a homotopy class of stable trivialisations which is preserved under* $\psi^1|_L$.

From this, we can obtain strong restrictions on $\mathcal{G}_L$ for certain L that we could not get from Theorem 1.8:

Corollary 1.20 *1. If* L *is a* $K(\pi_1(L), 1)$, $\psi^1|_L$ *is homotopic to the identity.*
2. If $n = 2$, $\psi^1|_L$ *is isotopic to the identity, and hence* $\mathcal{G}_L$ *is trivial.*

Proof Recall that a diffeomorphism of a closed surface is homotopic to the identity iff it is isotopic to the identity. Therefore to prove (2), it suffices to prove (1) along with the special cases $L = S^2$ and $\mathbb{RP}^2$.

It follows from Corollary 1.16 that if $L = S^2$, then $\psi^1|_L$ is isotopic to the identity. Furthermore, since the mapping class group of $\mathbb{RP}^2$ is trivial, we can assume that L is a $K(\pi_1(L), 1)$.

Note that $H_0(\mathcal{L}L; \mathbb{Z}/2)$ is the free $\mathbb{Z}/2$-module generated by the set of homotopy classes of free loops in L. Therefore by Theorem 1.17, $\psi^1|_L$ acts as the identity on this set of generators. Then by [11, Theorem 2.4], $\psi^1|_L$ must be homotopic to the identity. □

1.4 Extensions to other bases

In this subsection and Sect. 6, we restrict to the case $R = H\mathbb{Z}/2$.

Definition 1.21 Given a fibre bundle $E \twoheadrightarrow B$, we say it *c-splits* if the inclusion of a fibre into E induces an injection on mod-2 singular homology, for any fibre of this bundle.

If the fibre F has $H^*(F; \mathbb{Z}/2)$ finite-dimensional (as will be the case below), the Leray-Hirsch theorem says that this implies that

$$H^*(E; \mathbb{Z}/2) \cong H^*(B; \mathbb{Z}/2) \otimes H^*(F; \mathbb{Z}/2)$$

We define Lag_L to be the space of (unparametrised) Lagrangian submanifolds of M which are Hamiltonian isotopic to L. There is a natural fibre bundle $\mathcal{E}$ over Lag_L, where the fibre over K in Lag_L is K. This is naturally a subbundle of $Lag_L \times M$. Given a map $\gamma : S^1 \to Lag_L$, Theorem 1.4 implies that the pullback bundle $\gamma^*\mathcal{E}$ c-splits. We will use techniques in [21] to deduce the following from Theorem 1.4:

Theorem 1.22 *Let* X *be a finite CW complex that admits a map* $f : (S^1)^i \to X$ *which is surjective on mod-2 homology (such as a product of spheres), and let* $\gamma : X \to Lag_L$ *be any map. Then the pullback bundle* $\gamma^*\mathcal{E}$ *c-splits.*

2 Spectra and Pontryagin–Thom maps

We assume all spaces that we work with are Hausdorff, paracompact and homotopy equivalent to CW complexes. For an unbased space X, we will write X_+ for X with a disjoint basepoint added.

We will use the category of spectra described in [28], whose construction and properties we will sketch here.

2.1 Spectra

Definition 2.1 A *spectrum* X is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of based spaces along with *structure maps* $\Sigma X_n \to X_{n+1}$.

A *function* f from X to Y is a family of based maps of spaces $f_n : X_n \to Y_n$ such that the square

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\Sigma f_n} & \Sigma Y_n \\ \downarrow & & \downarrow \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

commutes.

We will not define a morphism of spectra (see [28, §II.1] for details), but any function of spectra is a morphism of spectra, and all morphisms of spectra that we will directly construct will arise in this way.

Remark 2.2 Rudyak actually defines a spectrum in a slightly different way but shows the two are equivalent in [28, Lemma II.1.19].

Definition 2.3 Given a spectrum X and a based space Z, there is a spectrum $X \wedge Z$ with $(X \wedge Z)_n = X_n \wedge Z$.

A *homotopy* between two functions $f, g : X \to Y$ is a function

$$H : X \wedge [0, 1]_+ \to Y$$

restricting to f over $\{0\}_+$ and to g over $\{1\}_+$.

One can extend this definition to homotopies of morphisms of spectra, as in [28, Definition II.1.9], and for homotopic morphisms f and g we write $f \simeq g$. Then for spectra X and Y, we define $[X, Y]$ to be the set of homotopy classes of morphisms $X \to Y$. There is a natural structure of an abelian group on this set.

From this definition of a homotopy we can define a notion of *(homotopy) equivalence* as with spaces, which we denote by $\simeq$.

There is a natural functor Σ^∞ from based spaces to Sp sending a space X to the spectrum with $(\Sigma^\infty X)_n = \Sigma^n X$, with structure maps the identity. We write Σ^∞_+ for the functor from unbased spaces to Sp sending X to $\Sigma^\infty X_+$. These both send homotopies to homotopies. We will denote $\Sigma_+^\infty\{*\}$ by $\mathbb{S}$, and call this the *sphere spectrum*. This has $\mathbb{S}_n = S^n$, justifying the name.

For any N in $\mathbb{Z}$, there is an endofunctor Σ^N of Sp which sends X to the spectrum $\Sigma^N X$ with

$$(\Sigma^N X)_n = \begin{cases} X_{n+N} & \text{if } n+N \geq 0 \\ \{*\} & \text{Otherwise} \end{cases}$$

This satisfies $\Sigma^N \Sigma^M X \simeq \Sigma^{N+M} X$ for all X, and $[X, Y] = [\Sigma^N X, \Sigma^N Y]$ for all X, Y. We write $\Sigma_+^{\infty+N} X$ for $\Sigma^N \Sigma_+^\infty X$.

Definition 2.4 Given a family $\{X_\lambda\}_{\lambda\in\Lambda}$ of spectra, we can define their *wedge product* to be the spectrum $\bigvee\limits_\lambda X_\lambda$ with

$$\left(\bigvee_\lambda X_\lambda\right)_n = \bigvee_\lambda X_n$$

Lemma 2.5 ([28, Proposition II.1.16]) *For any spectrum Y, there are natural isomorphisms*

$$\left[\bigvee_\lambda X_\lambda, Y\right] \cong \prod_\lambda [X_\lambda, Y]$$

and

$$\left[Y, \bigvee_\lambda X_\lambda\right] \cong \bigoplus_\lambda [Y, X_\lambda]$$

We can extend the smash product $\wedge$ to two spectra, functorially in each argument, as in [2]. We will not need the explicit construction but will use some of its properties:

Theorem 2.6 ([28, Theorems II.2.1 and II.2.2]) *There are equivalences*

1. $(X \wedge Y) \wedge Z \simeq X \wedge (Y \wedge Z)$
2. $X \wedge Y \simeq Y \wedge X$
3. $X \wedge \mathbb{S} \simeq X$
4. $\Sigma X \wedge Y \simeq \Sigma(X \wedge Y)$

such that all natural diagrams made up of these equivalences commute up to homotopy.

Definition 2.7 A *ring spectrum* R is a spectrum equipped with a unit morphism $\eta : \mathbb{S} \to R$ and product morphism $\mu : R \wedge R \to R$, satisfying appropriate associativity and unitality conditions up to homotopy.

A *(right) module* over a ring spectrum is defined similarly.

A *map* of R-modules $X \to Y$ is a map of spectra $X \to Y$ which commutes with the actions of R on X and Y up to homotopy.

A *homotopy* of R-module maps between f and $g : X \to Y$ is a homotopy of spectra $X \wedge [0,1]_+ \to Y$ between f and g, which is also a map of R-modules.

Example 2.8 The sphere spectrum $\mathbb{S}$ is naturally a ring spectrum, and every spectrum is naturally a module spectrum over it.

Example 2.9 For a ring spectrum R, there is a functor from spectra to (right) R-module spectra, given by $\cdot \wedge R$.

Definition 2.10 We define the *stable homotopy groups* of X to be

$$\pi_i X := [\Sigma^i \mathbb{S}, X]$$

This is covariantly functorial in X.

Given a spectrum R, we define $R_i(X)$ to be $\pi_i(X \wedge R)$ and $R^i(X)$ to be $[X, \Sigma^i R]$. These are functorial in X, covariantly and contravariantly respectively.

Given a space Z, we define $R_*(Z)$ to be $R_*(\Sigma^\infty_+ Z)$ and $R^*(Z)$ to be $R^*(\Sigma^\infty_+ Z)$. These are functorial in Z, covariantly and contravariantly respectively.

By Brown's representability theorem ([16, Theorem 4E.1]), for an abelian group G there is a (unique up to homotopy equivalence) spectrum HG such that HG_* and HG^* are homology and cohomology with co-efficients in G respectively, and when G is a ring these are in fact ring spectra. Similarly there are (unique up to homotopy equivalence) ring spectra KO and KU such that KO^* and KU^* are real and complex K-theory respectively.

2.2 Duality

We say a spectrum X is *finite* if it is of the form $X = \Sigma^{\infty+i} Y$ for Y a based finite CW complex, and i in $\mathbb{Z}$. Finite spectra admit duals in the following sense.

Lemma 2.11 ([28, Corollary II.2.9]) *For a finite spectrum X, there is another finite spectrum $X^\vee$, unique up to natural homotopy equivalence, along with natural isomorphisms*

$$[X, E] \cong [\mathbb{S}, X^\vee \wedge E]$$

and

$$[X^\vee, E] \cong [\mathbb{S}, X \wedge E]$$

for any other spectrum E. These induce further natural isomorphisms

$$E_*(X) \cong E^{-*}(X).$$

We will also use a base-change isomorphism. Let R be a ring spectrum, with unit map $i : \mathbb{S} \to R$.

Lemma 2.12 *Let X be a spectrum and M an R-module, with R-action $\mu : M \wedge R \to M$. Then there is a natural isomorphism*

$$[X, M] \cong [X \wedge R, M]^R$$

where $[\cdot, \cdot]^R$ denotes homotopy classes of R-module maps.

Proof Let $\alpha : [X, M] \to [X \wedge R, M]^R$ send ϕ to the composition

$$X \wedge R \xrightarrow{\phi \wedge R} M \wedge R \xrightarrow{\mu} M$$

and let $\beta : [X \wedge R, M]^R \to [X, M]$ send ψ to the composition

$$X \xrightarrow{X \wedge i} X \wedge R \xrightarrow{\psi} M.$$

Then α and β are inverses to each other. □

From these two lemmas we deduce the following corollary.

Corollary 2.13 *For finite spectra X and Y, there is an isomorphism*

$$[X \wedge R, Y \wedge R]^R \cong [Y^\vee \wedge R, X^\vee \wedge R]^R.$$

Proof We have isomorphisms

$$\begin{aligned} [X \wedge R, Y \wedge R]^R &\cong [X, Y \wedge R] \\ &\cong [Y^\vee \wedge X, R] \\ &\cong [Y^\vee, X^\vee \wedge R] \\ &\cong [Y^\vee \wedge R, X^\vee \wedge R]^R. \end{aligned}$$

The first and fourth isomorphisms are from Lemma 2.12, and the second and third isomorphisms are from Lemma 2.11. □

Corollary 2.14 *Let X be a finite spectrum and $f : X \to X$ be a map such that $f \wedge R : X \wedge R \to X \wedge R$ is homotopic to the identity as a map of R-modules. Then f induces the identity map on both $R_*(X)$ and $R^*(X)$.*

Proof $R_*(X) = \pi_*(X \wedge R)$, so f induces the identity map on $R_*(X)$.

When $X = Y$, the isomorphism of Corollary 2.13 sends the identity to the identity, so the induced map of R-modules $X^\vee \wedge R \to X^\vee \wedge R$ is homotopic to the identity. But the action of this map on homotopy groups is the same as the action of f on $R^*(X)$. □

2.3 Thom spectra

Let $\xi : E \to X$ be a vector bundle of rank n over a space X. We let DE and SE denote the unit disc and unit sphere bundles of E respectively, with respect to some choice of metric.

Definition 2.15 We define the *Thom space* of E, denoted X_u^E, to be the quotient DE/SE. This is a based space with basepoint given by the image of SE.

We define the *Thom spectrum* of E, denoted X^E, to be the spectrum $\Sigma^\infty X_u^E$.

If E' is a virtual vector bundle which can be written as $E' \cong E - \mathbb{R}_X^m$ for some m (in particular, this includes any virtual vector bundle pulled back from one over a compact space), we define the *Thom spectrum* of E' to be $\Sigma^{\infty - m} X_u^E$.

Remark 2.16 None of these depend on the choice of metric up to homotopy equivalence, and furthermore the Thom spectrum only depends on the stable isomorphism class of the virtual vector bundle by [28, Lemma IV.5.14].

Now let $\xi : E \to X$ be a virtual vector bundle of rank n.

Definition 2.17 E is *orientable* with respect to R if there is a morphism $U : X^E \to \Sigma^n R$, called the *Thom class*, whose restriction to a fibre (which is equivalent to a copy of $\Sigma^n \mathbb{S}$) represents plus or minus the unit in R.

An *orientation* is a homotopy class of such morphisms.

It follows from Remark 2.16 that being R-orientable only depends on the stable isomorphism class of E.

Lemma 2.18 *[[28, Proposition V.1.10 and Examples V.1.23]]*

1. *Any virtual vector bundle is canonically oriented with respect to $H\mathbb{Z}/2$.*
2. *A virtual vector bundle is orientable with respect to $H\mathbb{Z}$ iff it is orientable in the usual sense, and there is a natural bijection between $H\mathbb{Z}$-orientations and orientations in the usual sense.*
3. *A trivialisation of a virtual vector bundle induces a natural orientation with respect to any R.*
4. *If E is oriented with respect to R and $f : Y \to X$ is any map, the pullback bundle f^*E admits a natural R-orientation.*
5. *If F is another virtual vector bundle over X and any two of E, F and $E \oplus F$ are R-oriented, then the third admits a natural R-orientation.*

We will need a stable version of the Thom isomorphism theorem:

Theorem 2.19 ([28, Theorem V.1.15 and Exercise V.1.28]) *An R-orientation of E induces an equivalence of R-module spectra*

$$\Sigma_+^{\infty+n} X \wedge R \simeq X^E \wedge R$$

More generally, if $F \to X$ is another virtual vector bundle, then an R-orientation of E induces an equivalence of R-module spectra

$$\Sigma^n X^F \wedge R \simeq X^{E \oplus F} \wedge R$$

2.4 Pontryagin–Thom collapse maps

Let $i : X \hookrightarrow Y$ be a smooth embedding of manifolds with X compact, such that $i(\partial X) \subseteq \partial Y$. Let ν be the normal bundle of i, with unit disc bundle $D\nu$ and unit sphere bundle $S\nu$ (with respect to some choice of metric). Furthermore we fix a tubular neighbourhood of X, i.e. we pick an embedding $j : D\nu \hookrightarrow Y$ extending i, and not touching ∂Y apart from over ∂X.

We will denote the one-point compactification of Y by Y_∞, viewed as a based space with basepoint at infinity. If Y is already compact then this is Y_+.

Definition 2.20 We define the *Pontryagin–Thom collapse map* $i_{!,u} : Y_\infty \to X_u^\nu$ by

$$i_{!,u}(x) = \begin{cases} j^{-1}(x) & \text{if } x \in j(D\nu) \\ S\nu & \text{otherwise} \end{cases}$$

We denote the stabilisation $\Sigma^\infty i_{!,u}$ by $i_!$.

The map $i_{!,u}$ is continuous, and both the space X_u^ν and the map $i_{!,u}$ are independent of the choices of metric and tubular neighbourhood up to homotopy.

Now let $f : X^m \to Y^n$ be any smooth map of manifolds.

Definition 2.21 We define the *stable normal bundle* of f, ν_f, to be the virtual vector bundle $f^*TY - TX$.

When f is an embedding, this is stably isomorphic to the normal bundle of the embedding, defined in the usual sense.

Now assume X and Y are closed, and choose a smooth embedding

$$i : X \hookrightarrow \mathbb{R}^N$$

for some N. Now $\nu_f \oplus \mathbb{R}^N$ is stably isomorphic to the normal bundle $\nu_{f\times i}$ of the embedding

$$f \times i : X \hookrightarrow Y \times \mathbb{R}^N$$

We can consider the map

$$(f \times i)_! : \Sigma^\infty Y_\infty \simeq \Sigma^{\infty - N}\left(Y \times \mathbb{R}^N\right)_\infty \to \Sigma^{-N} X^{\nu_{f\times i}} \simeq X^{\nu_f}$$

Lemma 2.22 *Up to homotopy, the morphism $\Sigma^\infty Y_\infty \to X^{\nu_f}$ does not depend on the choice of i or N.*

Proof First, observe that if we compose i with the standard embedding $\mathbb{R}^N \hookrightarrow \mathbb{R}^{N+1}$, the resulting map is the same up to suspension.

Assume i, i' are two choices of embedding $X \hookrightarrow \mathbb{R}^N$. Pick an embedding

$$I : X \times [0, 1] \hookrightarrow \mathbb{R}^N \times [0, 1]$$

over $[0, 1]$ restricting to i and i' over 0 and 1 respectively, increasing N if necessary as above. Then $(f \times I)_{!,u}$ provides a homotopy between $(f \times i)_{!,u}$ and $(f \times i')_{!,u}$. □

Now that we know the desuspended map $\Sigma^\infty Y_\infty \to X^{\nu_f}$ doesn't depend on these choices (up to homotopy), we denote it by $f_!$.

We fix a ring spectrum R. Given an R-orientation of ν_f, we consider the following composition, which is a map of R-modules:

$$p_f : \Sigma^\infty Y_\infty \wedge R \xrightarrow{f_!} X^{\nu_f} \wedge R \xrightarrow{\simeq} \Sigma^{\infty+d}_+ X \wedge R \xrightarrow{f} \Sigma^{\infty+d} Y_\infty \wedge R$$

where $d = n - m$, and the middle map is from the Thom isomorphism theorem. By construction, this factors through $\Sigma^{\infty+d}_+ X \wedge R$.

Now suppose that $n = m$ (so $d = 0$), Y is compact (so $Y_\infty = Y_+$), and there is a cobordism W from X to Y along with a map $F : W \to Y$ restricting to the identity on Y and to f on X, as in the following diagram:

$$\begin{tikzcd}
X \arrow[dr, hook] \arrow[drr, "f"] & & \\
& W \arrow[r, "F"] & Y \\
Y \arrow[ur, hook] \arrow[urr, "Id"'] & &
\end{tikzcd}$$

Remark 2.23 We will find ourselves in this situation in the proof of Theorem 1.8, with $Y = L$, $X = \mathcal{P}$ a moduli space of suitable holomorphic discs with moving Lagrangian boundary conditions, and $\mathcal{W}$ a moduli space of moving boundary conditions varying in a family of such boundary conditions.

We assume that ν_F admits an orientation with respect to R, which, by restriction, induces one for ν_f.

Lemma 2.24 *After applying $\Sigma^\infty_+ \cdot \wedge R$, f admits a section of R-module maps up to homotopy (of R-module maps).*

Proof We will show that the map p_f constructed above is an equivalence, then $f_! \circ p_f^{-1}$ will be the desired section.

We pick an embedding

$$I : W \hookrightarrow \mathbb{R}^N \times [0, 1]$$

for some N, such that $I^{-1}\left(\mathbb{R}^N \times \{0\}\right) = X$ and $I^{-1}\left(\mathbb{R}^N \times \{1\}\right) = Y$. Then choosing an orientation of ν_F and performing the above construction to F gives us a homotopy from p_f to p_{Id}, where p_{Id} is constructed similarly to p_f but for $Id : Y \to Y$, using some choice of trivialisation on the trivial vector bundle over Y. But p_{Id} is a composition of three equivalences and hence p_f is an equivalence.

All the maps and homotopies are constructed either by applying $\cdot \wedge R$ to a map of spectra or by applying the Thom isomorphism theorem over R, so they are all R-module maps. □

Corollary 2.25 *The map $f_* : R_*X \to R_*Y$ is split surjective.*

It follows from [28, Lemma II.2.4 and Theorem V.2.3] that

Corollary 2.26 *If W is R-orientable (and hence X and Y are too), the map $f^* : R^*Y \to R^*X$ is injective.*

We can also define Pontryagin–Thom collapse maps without smoothness assumptions. Let $i : X \hookrightarrow Y$ be an embedding of spaces.

Definition 2.27 We say X admits a *tubular neighbourhood* in Y with *normal bundle* ν if there is a vector bundle ν over X and an open neighbourhood U of X in Y, such that there is a homeomorphism $\phi : U \to DE$ between U and the unit disc bundle DE of ν (for some choice of fibrewise metric on ν), sending X to the zero section.

Definition 2.28 Let $E \to Y$ be another vector bundle. We define the *Pontryagin–Thom collapse map* $i_{!,u} : Y_u^E \to X_u^{\nu \oplus E}$ by

$$(x, v) \mapsto \begin{cases} (\phi(x), v) & \text{if } x \in U \\ S(\nu \oplus E) & \text{otherwise} \end{cases}$$

We write $i_!$ for the induced map on Thom spectra.

This definition extends to the case when E is a virtual vector bundle of the form $E' - \mathbb{R}^N$, as we can apply this construction to E' and desuspend. Similarly to before, up to homotopy $i_! := \Sigma^\infty i_{!,u}$ then only depends on the virtual vector bundle up to stable isomorphism.

Example 2.29 When $X \hookrightarrow Y$ is a proper embedding of smooth manifolds, X always admits a tubular neighbourhood in Y, and we recover the earlier construction.

2.5 Fundamental classes

Let X^n be a closed manifold of dimension n. Choose an embedding

$$i : X \hookrightarrow \mathbb{R}^N \hookrightarrow S^N$$

for some N, with normal bundle ν. Then as virtual vector bundles, there is a natural isomorphism $\nu - \mathbb{R}^N \cong -TX$. Consider the map $i_{!,u} : S^N \to X^\nu$. Stabilising and desuspending gives us a well-defined (up to homotopy) map of spectra

$$[X] : \mathbb{S} \to X^{-TX}$$

Given a ring spectrum R and an R-orientation of TX (which induces one on $-TX$), the Thom isomorphism theorem gives us a well-defined (up to homotopy) map of spectra

$$[X] : \mathbb{S} \to \Sigma_+^{\infty - n} X \wedge R$$

representing an element $[X]$ in $R_n(X)$. More generally, given a space Z, a map of spaces $f : X \to Z$, a vector bundle $E \to Z$, and an R-orientation of $TX - f^*E$ (which we assume to have rank d), we can use the Thom isomorphism theorem to obtain a well-defined (up to homotopy) map of spectra

$$\mathbb{S} \xrightarrow{[X]} X^{-TX} \wedge R \simeq \Sigma^{-d} X^{-f^*E} \wedge R \to \Sigma^{-d} Z^{-E} \wedge R$$

which we also denote by $[X]$. Here the middle homotopy equivalence is the Thom isomorphism.

We call all of these maps $[X]$ *fundamental classes*.

Lemma 2.30 *Let X and Y be closed manifolds, and let $f : X \hookrightarrow Y$ be an embedding. Then the following diagram commutes:*

$$\begin{tikzcd} \mathbb{S} \arrow[d, "{[Y]}"] \arrow[dr, "{[X]}"] & \\ Y^{-TY} \arrow[r, "f_!"] & X^{-TX} \end{tikzcd}$$

Proof Let $i : Y \hookrightarrow \mathbb{R}^N$ be an embedding. Then $i \circ f : X \hookrightarrow \mathbb{R}^N$ is an embedding too, and using this embedding to construct the unstable Pontryagin–Thom collapse map, we get a commutative diagram (of spaces):

$$\begin{tikzcd} S^N \arrow[d, "i_{!,u}"] \arrow[dr, "(i\circ f)_{!,u}"] & \\ Y^{\nu_i} \arrow[r, "f_{!,u}"] & X^{\nu_{i\circ f}} \end{tikzcd}$$

where ν_i and $\nu_{i\circ f}$ are the normal bundles of i and $i \circ f$ respectively.

Stabilising this diagram gives the required diagram. □

Lemma 2.31 *Let Y be another closed manifold of dimension n. Suppose $f : X \to Z$ and $g : Y \to Z$ are two maps, such that there is a cobordism W from X to Y, and a map $F : W \to Z$ extending f and g, as shown below.*

$$\begin{tikzcd} X \arrow[dr, hook] \arrow[drr, "f"] & & \\ & W \arrow[r, "F"] & Z \\ Y \arrow[ur, hook] \arrow[urr, "g"'] & & \end{tikzcd}$$

Let $E \to Z$ be a vector bundle and assume there is an R-orientation of $TW - F^*E$, which we assume to have rank $d + 1$. Then the two fundamental classes $[X], [Y]$ in $R_d(Z^{-E})$ (defined using the orientations of $TX - f^*E$ and $TY - g^*E$ given by restricting the orientation of $TW - F^*E$) agree.

Proof Pick an embedding

$$I : W \hookrightarrow \mathbb{R}^N \times [0, 1]$$

for some N, such that $W^{-1}\left(\mathbb{R}^N \times \{0\}\right) = X$ and $W^{-1}\left(\mathbb{R}^N \times \{1\}\right) = Y$. Then the composition

$$\mathbb{S} \wedge [0, 1]_+ \xrightarrow{I_!} W^{\mathbb{R}-TW} \wedge R \simeq \Sigma^{-d} W^{-F^*E} \to \Sigma^{-d} Y^{-E}$$

where the middle map is the Thom isomorphism, defines a homotopy between $[X]$ and $[Y]$. □

Remark 2.32 Let $X = \bigsqcup_{i \geq 0} X^i$ be a manifold consisting of components X^i of dimension i. If X is compact (in particular implying that only finitely many components X^i are non-empty), we can consider the fundamental classes of these independently and obtain a fundamental class $[X] := \sum_i [X^i]$ in $\bigoplus_i R_i\left(X^{-TX}\right)$. We will use this later in Sect. 5.

3 Index bundles and their orientations

3.1 Cauchy–Riemann operators

Let B be a space and $A \subseteq B$ some subspace.

Definition 3.1 A *bundle pair* over the pair (B, A) is a complex vector bundle E over B along with a totally real subbundle F of $E|_A$. This is written as

$$(E, F) \to (B, A)$$

We call E the *complex part* and F the *real part*.

We can perform certain operations on bundle pairs, analogously to vector bundles.

- Given a real vector bundle G over B, we define $(E, F) \otimes G$ to be the bundle pair

$$(E \otimes_{\mathbb{R}} G, F \otimes_{\mathbb{R}} G|_A)$$

- Given $(E', F') \to (B, A)$ another bundle pair, we define $(E, F) \oplus (E', F')$ to be the bundle pair

$$(E \oplus E', F \oplus F')$$

- A *virtual bundle pair* is a formal difference $(E, F) - (E', F')$ of two bundle pairs, and we say two virtual bundle pairs $(E, F) - (E', F')$ and $(G, H) - (G', H')$ are *stably isomorphic* if there is an isomorphism of bundle pairs

$$(E, F) \oplus (G', H') \oplus (\mathbb{C}^N_B, \mathbb{R}^N_A) \cong (E', F') \oplus (G, H) \oplus (\mathbb{C}^N_B, \mathbb{R}^N_A)$$

for some N.

- The *rank* of a bundle pair (E, F) is $\mathrm{Rank}_{\mathbb{C}} E$, and the *(virtual) rank* of a virtual bundle pair $(E, F) - (E', F')$ is $\mathrm{Rank}_{\mathbb{C}} E - \mathrm{Rank}_{\mathbb{C}} E'$.
- Given a bundle pair $(E, F) \to (B, A)$ and a map of pairs $f : (B', A') \to (B, A)$, we define the bundle pair $f^*(E, F) \to (B', A')$ to be the bundle pair (f^*E, f^*F).
- We define a *section* of (E, F) to be a smooth section of E whose restriction to A lies in F. We write $\Gamma(E, F)$ for the space of sections.

Similarly to the case of real or complex vector bundles, when B is a finite CW complex and A a subcomplex, every virtual bundle pair over (B, A) is stably isomorphic to $(E, F) - (\mathbb{C}^N_B, \mathbb{R}^N_A)$ for some (E, F) and some N. Similarly the pullbacks of a bundle pair under homotopic maps are isomorphic.

We will usually take (B, A) to be $(D^2, \partial D^2) \times X$ for some space X, and we will assume this to be the case for the rest of Sect. 3.

For the rest of Sect. 3.1, we will assume that X is a point, so (E, F) is a bundle pair over $(D^2, \partial D^2)$.

Definition 3.2 A *(real) Cauchy–Riemann operator* on (E, F) is an $\mathbb{R}$-linear first order differential operator

$$D : \Gamma(E, F) \to \Omega^{0,1}(E)$$

satisfying the Leibniz rule:

$$D(f\eta) = (\bar{\partial} f)\eta + f D\eta$$

for f in $C^\infty(D^2, \mathbb{R})$ and η in $\Gamma(E, F)$.

Lemma 3.3 ([23, Remark C.1.2]) *The space of Cauchy–Riemann operators on (E, F) is contractible (and in fact, convex).*

For a choice of Hermitian metric on E and a real number $q > 2$ (which we fix), a Cauchy–Riemann operator induces an operator

$$\hat{D} : W^{1,q}(E, F) \to W^q\left(\Lambda^{0,1} T^* D^2 \otimes E\right)$$

where these spaces are appropriate Sobolev completions of the above spaces of smooth sections. By [23, Theorem C.1.10], this operator $\hat{D}$ is Fredholm, and in fact $\mathrm{Ker}\, \hat{D} = \mathrm{Ker}\, D$.

3.2 Index bundles

Let X be a finite CW complex, and let

$$(E, F) \to (D^2, \partial D^2) \times X$$

be a bundle pair.

By Lemma 3.3, we can choose a Cauchy–Riemann operator D_x on

$$\left(E|_{D^2\times\{x\}}, F|_{\partial D^2\times\{x\}}\right)$$

for each x, varying continuously in x. The space of such choices is contractible.

Definition 3.4 Assume $\hat{D}_x$ is surjective for all x. The *index bundle* of (E, F) is the (real) vector bundle $\text{Ind}(E, F)$ over X, with fibre at a point x given by $\text{Ker } \hat{D}_x$.

If $\hat{D}_x$ is not always surjective, we can still define the index bundle, following [4]. Since X is compact, we can find, for some finite N, a continuous family of linear maps

$$\phi_x : \mathbb{R}^N \to W^q\left(\Lambda^{0,1}T^*D^2 \otimes E|_{D^2\times\{x\}}\right)$$

for x in X, such that the *stabilised operator* $T_x := \hat{D}_x + \phi_x$

$$T_x : W^{1,q}\left(E|_{D^2\times\{x\}}, F|_{\partial D^2\times\{x\}}\right) \oplus \mathbb{R}^N \to W^q\left(\Lambda^{0,1}T^*D^2 \otimes E|_{D^2\times\{x\}}\right)$$

is surjective. In this situation we call ϕ a *stabilisation* of *rank* N. We then define $\text{Ind}(E, F) + \mathbb{R}^N_X$ to be the vector bundle with fibre over x given by $\text{Ker } T_x$.

Lemma 3.5 *The vector bundle* $\text{Ind}(E, F)$ *is well-defined up to stable isomorphism. Furthermore this stable isomorphism can be chosen in a way that is unique up to weakly contractible choice.*

Proof Firstly we note that stabilising ϕ by adding another copy of $\mathbb{R}$ which is sent to 0 does not change the index bundle up to canonical isomorphism.

We choose a metric on E, and note that the space of such choices is contractible. This defines a $W^{0,2}$ inner product on $W^{1,q}(E, F)$.

Given two choices of family of Cauchy–Riemann operators and stabilisation (D, ϕ) and (D', ϕ'), by the above we can assume that both stabilisations are of the same rank. If (D, ϕ) and (D', ϕ') have distance less than 1 with respect to the operator norm, orthogonal projection defines an isomorphism between their kernels. Note that since the space of such operators is convex, this open ball is contractible.

In general, we can pick a path in the space of such (D, ϕ), and iterate this process along the path. Because the spaces of such pairs (D, ϕ) (up to stabilisation) are weakly contractible, the uniqueness result follows from a similar argument. □

Remark 3.6 We needed compactness of X for there to exist a family of linear maps ϕ_x as above. If X was not compact, the $\hat{D}_x$ may not have a uniform upper bound on the rank of its cokernel, in which case such ϕ_x cannot exist.

If X was not assumed to be compact, we could construct the index bundle on compact subsets $Y \subseteq X$ and take colimits in an appropriate manner. This would require a longer discussion and is not required for our purposes.

We will require some elementary properties of index bundles.

Lemma 3.7 *If X' is another finite CW complex and $f : X' \to X$ is any map, then there is a natural isomorphism of virtual vector bundles*

$$f^*\mathrm{Ind}(E, F) \cong \mathrm{Ind}\, f^*(E, F)$$

where we write $f^*(E, F)$ for $(Id_{D^2} \times f)^*(E, F)$.

Proof All of the choices made to define the index bundle are compatible under pullbacks. □

Lemma 3.8 *If (E, F) and (E', F') are bundle pairs on $(D^2, \partial D^2) \times X$, then there is a natural isomorphism of virtual vector bundles*

$$\mathrm{Ind}(E, F) \oplus \mathrm{Ind}(E', F') \cong \mathrm{Ind}(E \oplus E', F \oplus F')$$

Proof All of the choices made to define the index bundle are compatible under direct sums. □

Corollary 3.9 *A stable isomorphism between (E, F) and (E', F') induces a stable isomorphism between* $\mathrm{Ind}(E, F)$ *and* $\mathrm{Ind}(E', F')$.

Lemma 3.10 *Let $G \to X$ be a real virtual vector bundle over X. Then there is a canonical (up to weakly contractible choice) stable isomorphism*

$$(\mathrm{Ind}(E, F)) \otimes G \cong \mathrm{Ind}\left((E, F) \otimes G\right)$$

Proof We assume G is an actual vector bundle, and observe that the general case follows from Lemma 3.8.

Let D be a family of Cauchy–Riemann operators on (E, F), and ϕ a stabilisation of rank N, corresponding to the family of stabilised operators T. We pick another vector bundle G' over X, along with an isomorphism $G \oplus G' \cong \mathbb{R}^M_X$ for some M.

For each x in X, we consider the Cauchy–Riemann operator

$$D_x \otimes Id_{G_x} : \Gamma\left(\left(E|_{D^2\times\{x\}}, F|_{\partial D^2\times\{x\}}\right) \otimes G_x\right) \to \Omega^{0,1}\left(E_{D^2\times\{x\}} \otimes G_x\right)$$

along with the stabilisation of rank NM

$$\theta_x : \mathbb{R}^{NM} \cong (\mathbb{R}^N \otimes G_x) \oplus (\mathbb{R}^N \otimes G'_x) \to W^q\left(\Lambda^{0,1}T^*D^2 \otimes E|_{D^2\times\{x\}} \otimes G_x\right)$$

sending $(a \otimes u, b \otimes v)$ to $\phi_x(a) \otimes u$.

Then $\mathrm{Ind}\left((E, F) \otimes G\right)_x \oplus \mathbb{R}^{NM}$ is equal to $\mathrm{Ker}\, S_x$ by definition, where S_x is the stabilised operator

$$\left(\widehat{D_x \otimes Id_{G_x}}\right) \oplus \theta_x$$

But Ker S_x is isomorphic to

$$((\mathrm{Ker}\, T_x) \otimes G_x) \oplus \left(\mathbb{R}^N \otimes G'_x\right)$$

which is isomorphic to

$$\left((\mathrm{Ind}(E, F))_x \otimes G_x\right) \oplus \mathbb{R}^{NM}$$

A similar argument to Lemma 3.5 shows that this isomorphism is canonical up to weakly contractible choice. □

There is a standard bundle pair of rank 1

$$H = (\mathbb{C}, \delta) \to (D^2, \partial D^2)$$

where $\delta(z) = \sqrt{z}\mathbb{R}$ for z in ∂D^2.

We can pick a trivialisation $E \cong \mathbb{C}^n_{D^2 \times \{x\}}$ over $D^2 \times \{x\}$ for a point x in X. Then $F|_{\partial D^2 \times \{x\}}$ determines a loop in the space of totally real subspaces of $\mathbb{C}^n$, which is isomorphic to $U(n)/O(n)$. There are isomorphisms

$$\pi_1 U(n)/O(n) \cong \mathbb{Z}$$

for all n, compatibly with stabilisation in n. There are two such choices of isomorphism, and we fix the one such that $H \oplus (\mathbb{C}^{n-1}_{D^2}, \mathbb{R}^{n-1}_{\partial D^2})$ is sent to 1.

Definition 3.11 We define the *Maslov index* of (E, F), denoted $\mu(E, F)$, to be the image of F under the isomorphism $\pi_1 U(n)/O(n) \to \mathbb{Z}$ after choosing a trivialisation of E over a point in X. This is well-defined on each path component of X, i.e. it is a map $\pi_0 X \to \mathbb{Z}$.

Using the index theorem, one can compute the virtual rank of the index bundle.

Lemma 3.12 ([23, Theorem C.1.10]) *The virtual rank of* $\mathrm{Ind}(E, F)$ *is*

$$n + \mu(E, F)$$

We note that therefore $\mathrm{Ind}\, H \cong \mathbb{R}^2$. We fix such an isomorphism (forever).

3.3 Orientations

If (E, F) is a virtual bundle pair over $(D^2, \partial D^2) \times Y$ for some (possibly non-compact) space Y, for any map from a finite CW complex X to Y, $\mathrm{Ind}(E, F)|_X$ is well-defined, and furthermore this is compatible with restriction.

Definition 3.13 An *R-orientation* on $\mathrm{Ind}(E, F)$ is a choice of R-orientation on $\mathrm{Ind}(E, F)|_X$ for all X, compatible with restriction. We say $\mathrm{Ind}(E, F)$ is *R-orientable* if such a choice exists.

Our goal in this section will be to establish the following two propositions. These will be used later to find conditions under which the tangent bundles to the moduli spaces constructed in Sects. 4.1 and 5.3 are R-orientable for various ring spectra R.

Proposition 3.14 *A stable trivialisation of* (E, F) *induces a stable trivialisation of* $\mathrm{Ind}(E, F)$*, and hence an* R*-orientation for any ring spectrum* R*, compatibly with direct sums and pullbacks.*

Proof We write $(\mathbb{C}^k, \mathbb{R}^k)_X$ for the bundle pair $(\mathbb{C}^k_{D^2\times X}, \mathbb{R}^k_{\partial D^2\times X})$. By Corollary 3.9, it suffices to prove the result for $(E, F) = (\mathbb{C}^n, \mathbb{R}^n)_X$. Over every point x in the base X, we pick the standard Cauchy–Riemann operator

$$\bar{\partial} : \Gamma\left(\mathbb{C}^n, \mathbb{R}^n\right) \to \Omega^{0,1}\left(\mathbb{C}^n\right)$$

on sections over the disc $D^2 \times \{x\}$.

Evaluation at any fixed point z in ∂D^2 defines a map $\mathrm{Ker}\,\bar{\partial} \to \mathbb{R}^n$, which is an isomorphism (and which does not depend on z). Then because the index of this Cauchy–Riemann operator is n, (the Sobolev completion of) this Cauchy–Riemann operator must be surjective. Therefore $\mathrm{Ker}\,\bar{\partial} = \mathrm{Ind}(\mathbb{C}^n, \mathbb{R}^n)_X$ and so this gives us the required trivialisation of $\mathrm{Ind}(E, F)$. □

The following proposition seems well-known to experts, but to the author's knowledge a proof has not yet appeared in the literature.

Proposition 3.15 *A stable trivialisation of* F *induces a stable complex structure on* $\mathrm{Ind}(E, F)$*, compatibly with direct sums and pullbacks.*

Proof Stabilising if necessary, we can assume that we have a choice of (unstable) trivialisation $F \cong \mathbb{R}^n_{\partial D^2\times X}$.

This trivialisation of F induces a trivialisation of $E|_{\{1\}\times X}$, which (since the disc is contractible) naturally extends to a trivialisation $E \cong \mathbb{C}^n_{D^2\times X}$ over $D^2 \times X$. This is compatible with the trivialisation of F over $\{1\} \times X$, but not necessarily over the rest of $\partial D^2 \times X$.

By identifying ∂D^2 with S^1, we identify $\partial D^2 \times X/(\{1\} \times X)$ with the suspension ΣX_+. There is then an induced map $f : \Sigma X_+ \to U/O$, sending y to the the totally real subspace F_y of $E_y \cong \mathbb{C}^n$. Here U/O is naturally identified with the colimit as $N \to \infty$ of totally real subspaces of $\mathbb{C}^N$, where a matrix A in $U(N)/O(N)$ corresponds to the totally real subspace $A(\mathbb{R}^N) \subseteq \mathbb{C}^N$.

The group of based homotopy classes of maps $\Sigma X_+ \to U/O$ classifies virtual bundle pairs over $(D^2, \partial D^2) \times X$ equipped with a stable trivialisation over $\{1\} \times X$, up to stable isomorphism and stabilisation (i.e. direct summing a copy of $(\mathbb{C}, \mathbb{R})_X$).

Claim 3.16 *The trivialisation of* F *induces a lift* $\tilde{f} : \Sigma X_+ \to U$*, fitting into the commutative diagram*

$$\begin{tikzcd} & U \arrow[d, "q"] \\ \Sigma X_+ \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & U/O \end{tikzcd}$$

where $q : U \to U/O$ is the quotient map.

Proof of claim We will construct $\tilde{f} : \Sigma X_+ \to U(n)$, then stabilising in n will give the desired map.

Let $\phi : \mathbb{R}^n_{\partial D^2 \times X} \xrightarrow{\cong} F \subseteq \mathbb{C}^n_{\partial D^2 \times X}$ be the trivialisation (which we assume to be orthogonal), let $a_1, \ldots, a_n$ be the standard basis of $\mathbb{C}^n$ and let $b_1, \ldots, b_n$ be the standard basis of $\mathbb{R}^n$.

We then define $\tilde{f}(y)$ to be the unitary matrix that sends a_i to $\phi(b_i)$ for all i. □

There is a natural commutative diagram, from Bott periodicity (using the isomorphisms in [3]):

$$\begin{tikzcd} \left[X_+, BU \times \mathbb{Z}\right] \arrow[r] \arrow[d, "g", "\cong"'] & \left[X_+, BO \times \mathbb{Z}\right] \arrow[d, "h", "\cong"'] \\ \left[\Sigma X_+, U\right] \arrow[r, "q_*"] & \left[\Sigma X_+, U/O\right] \end{tikzcd}$$

f lives in the bottom right group and $\tilde{f}$ lives in the bottom left group of the above diagram.

There is a map

$$I : \left[\Sigma X_+, U/O\right] \to \left[X_+, BO \times \mathbb{Z}\right]$$

sending a map r to the index bundle of the bundle pair determined by r.

Claim 3.17 *The composition $I \circ h : [X_+, BO \times \mathbb{Z}] \to [X_+, BO \times \mathbb{Z}]$ is the identity map.*

In particular, this implies that the map I is an isomorphism. This fact, along with its relation to orientations on moduli spaces of holomorphic curves, was first observed by de Silva in [12].

Proof of claim Let H be the bundle pair of rank 1 over $(D^2, \partial D^2)$ defined in Sect. 3.2. It admits a canonical trivialisation over $\{1\}$. Then h is defined to send (the classifying map for) a virtual vector bundle G over X to (the classifying map for) the bundle pair

$$H \otimes G - (\mathbb{C}, \mathbb{R})_X \otimes G$$

equipped with its canonical stable trivialisation over $\{1\} \times X$.

Then by the properties of index bundles developed in Sect. 3.2, we have isomorphisms

$$\begin{aligned}\operatorname{Ind}(h(G)) &\cong \operatorname{Ind}\left(H \otimes G - (\mathbb{C}, \mathbb{R})_X \otimes G\right)\\ &\cong \operatorname{Ind}(H) \otimes G - \operatorname{Ind}(\mathbb{C}, \mathbb{R})_X \otimes G\\ &\cong \mathbb{R}^2_X \otimes G - \mathbb{R}_X \otimes G\\ &\cong G\end{aligned}$$

as required. □

Then since $h^{-1}(f)$ is the classifying map of $\operatorname{Ind}(E, F)$, $g^{-1}(\tilde{f})$ is the choice of stable complex structure that we desired. Note that all choices we made were compatible with direct sums and pullbacks. □

4 The moduli spaces $\mathcal{P}$ and $\mathcal{W}$

4.1 Construction of the moduli spaces

Let C be a convex domain in the upper half plane in $\mathbb{C}$, with smooth boundary ∂C containing 0. Let $f : \partial C \to [0, 1]$ be a smooth map sending 0 to 0, and let J be an ω-tame almost complex structure on M which is convex at infinity.

Definition 4.1 We define $\mathcal{D}_{f,C}$ to be the space of smooth maps $u : C \to M$ such that for all z in ∂C, $u(z)$ lies in $\psi^{f(z)}(L)$.

We define π to be the natural evaluation map $\pi : \mathcal{D}_{f,C}(J) \to L$ sending u to $u(0)$.

Definition 4.2 We define $\mathcal{U}_{f,C}(J)$ to be the space of maps u in $\mathcal{D}_{f,C}$ which are J-holomorphic, namely

$$J \circ du = du \circ j$$

where j is the complex structure on $C \subseteq \mathbb{C}$.

We call the triple (C, f, J) the *moduli data*.

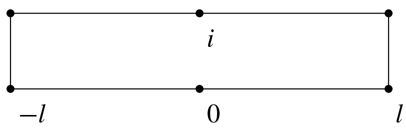
Let G be a fixed convex domain in $\mathbb{C}$ with smooth boundary, such that both line segments $(-\eta, \eta) \subset \mathbb{R}$ and $i + (-\eta, \eta) \subset i + \mathbb{R}$ are contained in ∂G for some $\eta > 0$. Let $G_\pm$ be $G \cap \mathbb{C}_{\pm \mathrm{Re} \geq 0}$.

For $l \geq 0$, define Z_l to be $[0, 1]i + [-l, l]$, and G_l to be

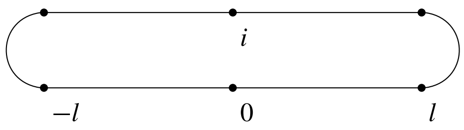
$$Z_l \cup (G_+ + l) \cup (G_- - l)$$

as shown below.

Z_l:



G_l:



We now define a 1-parameter family of moduli data (C_l, f_l, J_l) as follows.

$$C_l := \begin{cases} G_{l-1} & \text{if } l \geq 1 \\ G_0 & \text{if } l \leq 1 \end{cases}$$

$$f_l(z) := \begin{cases} \operatorname{Im} z & \textit{if } l \geq 1 \\ l \operatorname{Im} z & \textit{if } l \leq 1 \end{cases}$$

and J_l is defined to be some choice of ω-tame almost complex structure (convex at infinity if M is non-compact) on M which is independent of l (though we will apply an l-dependent generic perturbation to J_l soon).

As a 1-parameter family of domains in $\mathbb{C}$ and maps to $[0, 1]$, C_l and f_l are continuous everywhere in l and vary smoothly in l except at 1, so we pick a small smoothing of these families supported near $l = 1$, by reparametrising in the l direction.

Relative exactness of L in M gives us a uniform energy bound.

Lemma 4.3 *There exists some A in $\mathbb{R}$ such that for any $l \geq 0$ and any smooth map u in $\mathcal{D}_{f_l, C_l}$, the topological energy $\int_{C_l} u^*\omega$ is bounded above by A.*

The proof is a minor adaptation of that of [18, Lemma 2]. We remark that if the isotopy ψ^t were through symplectomorphisms instead of Hamiltonian diffeomorphisms, the proof of this lemma would fail.

Proof We choose $H : M \times [0, 1] \to M$ a time-dependent Hamiltonian generating ψ^t, meaning that for x in M and t in $[0, 1]$,

$$\omega\left(\cdot, \frac{d}{dt}(\psi^t)(x)\right) = dH_t(x)$$

and we choose H to be constant outside compact subsets of M.

Choose $l \geq 0$. There is some λ in $[0, 1]$ such that $f_l(z) = \lambda \mathrm{Im}\, z$ for all z in ∂C_l. Choose u in $\mathcal{D}_{f_l, C_l}$, and define $w : C_l \to M$ by

$$w(s + it) := (\psi^{\lambda t})^{-1}(u(s + it))$$

Then w sends ∂C_l to L so by relative exactness of L, we must have $\int_{C_l} w^*\omega = 0$.

The topological energy of u is

$$\begin{aligned}
\int_{C_l} u^*\omega &= \int\limits_{s+it\in C_l} \omega(\partial_s u, \partial_t u) ds\, dt \\
&= \int\limits_{s+it\in C_l} \omega\left(d\psi^{\lambda t}(\partial_s w), d\psi^{\lambda t}(\partial_t w) + \frac{d}{dt}(\psi^{\lambda t})(w)\right) ds\, dt \\
&= \int\limits_{s+it\in C_l} \omega(\partial_s w, \partial_t w) ds\, dt + \int\limits_{s+it\in C_l} \omega\left(d\psi^t(\partial_s w), \lambda\frac{d}{dt}(\psi^t)(w)\right) ds\, dt \\
&= \int_{C_l} w^*\omega + \lambda \int\limits_{s+it\in C_l} dH_t\left(d\psi^t(\partial_s w)\right) ds\, dt \\
&= 0 + \lambda \int_{t=0}^1 H_t\left(\psi^t(w(s_{max}(t) + it))\right) - H_t\left(\psi^t(w(s_{min}(t) + it))\right) dt \\
&\leq \max_{t,x} H_t(x) - \min_{t,x} H_t(x) =: A
\end{aligned}$$

where $s_{max}(t)$ is the largest s such that $s + it$ lies in C_l, and $s_{min}(t)$ is the smalles s such that $s + it$ lies in C_l. $\square$

We now pick a Riemannian metric on L, with injectivity radius ε and distance function d. Using this energy bound, [18, Proposition 3] then directly implies the following result, which should be viewed as a form of Gromov compactness. Roughly, Hofer shows that for large l, these moduli spaces, when restricted to Z_l, live close to holomorphic strips with boundary on L and $\psi^1(L)$, which we know are constant since $L = \psi^1(L)$ is relatively exact.

Lemma 4.4 *There is some $c > 1$ such that for all $l \geq c$ and u in $\mathcal{U}_{f_l, C_l}(J_l)$,*

$$d(u(i), u(0)) < \varepsilon$$

Proof We apply [18, Proposition 3], to the Lagrangians L and itself. In this setting, for $l > 1$, what Hofer calls $\Omega_{J_l}(L, L)$ is the space of finite-energy J_l-holomorphic strips with boundary sent to L, and so consists only of constant maps to L, and in particular does not depend on l.

Then take the open neighbourhood U of $\Omega_{J_l}(L, L)$ to be the space of maps $u : \mathbb{R} \times [0, 1] \to M$ such that $d(u(i), u(0)) < \varepsilon$ (where we equip these mapping spaces with the weak C^∞ Whitney topology). [18, Proposition 3] says that there is some $c > 1$ such that for all $l \geq c$ and u in $\mathcal{U}_{f_l, C_l}(J_l)$, $r_l(u)$ lies in U, where Hofer defines

$r_l(u) : \mathbb{R} \times [0, 1] \to M$ so that in particular $r_l(u)$ and u agree when restricted to neighbourhoods of $i[0, 1]$, and so in particular u also lies in U. □

Standard techniques in [23] show that for a generic (l-dependent) perturbation of J_l, $\mathcal{W}' := \bigsqcup_{l \geq 0} \mathcal{U}_{f_l, C_l}(J_l)$ is a smooth manifold with boundary $\mathcal{U}_{f_0, C_0}(J_0)$, which consists only of constant maps $C_0 \to L$ and hence $\pi : \mathcal{U}_{f_0, C_0}(J_0) \to L$ is a diffeomorphism. If we choose the perturbation small enough then Lemma 4.4 still holds, by Lemma 4.3 along with Gromov compactness.

Let $\mathcal{U}_l = \mathcal{U}_{f_l, C_l}(J_l)$, and let $p : \mathcal{W}' \to \mathbb{R}$ send $\mathcal{U}_l$ to l.

Lemma 4.5 *Each $\mathcal{U}_l$ is compact and p is a proper map.*

Proof We will show each $\mathcal{U}_l$ is compact, a similar argument shows that p is a proper map.

Let u_i be a sequence in $\mathcal{U}_l$. Gromov compactness as in [14] implies that this contains a convergent subsequence, but only after precomposing with a sequence of holomorphic automorphisms of the domain.

Let $\tilde{M}$ be $M \times \mathbb{C}$, with almost complex structure J_l times the standard complex structure on $\mathbb{C}$. Let $\tau : \tilde{M} \to \mathbb{C}$ be the projection map onto the second factor. Let $\tilde{L}$ be

$$\bigcup_{z \in \partial C_l} \psi^{f_l(z)}(L) \times \{z\}$$

a totally real submanifold of $\tilde{M}$.

We let $\tilde{\mathcal{U}_l}$ be the moduli space of holomorphic discs in $\tilde{M}$ with boundary in $\tilde{L}$. There is an embedding $\theta : \mathcal{U}_l \hookrightarrow \tilde{\mathcal{U}_l}$ sending u to the map $\tilde{u}$ which sends z to $(u(z), z)$. This clearly is a bijection with the subset of $\tilde{\mathcal{U}_l}$ consisting of sections of $\tau|_{\tau^{-1} C_l}$.

For u in $\mathcal{U}_l$, the energy of $\tilde{u}$ is equal to the energy of u plus the area of C_l, so in particular by Lemma 4.3, the energies of the $\tilde{u}_i$ are uniformly bounded. Therefore by [14, Theorem 1.1], there is a subsequence (which we will continue to call $\tilde{u}_i$, by abuse of notation) and a sequence of holomorphic automorphisms ϕ_i of C_l, such that $\tilde{u}_i \circ \phi_i$ lives in $\tilde{\mathcal{U}_l}$ for all i, and $\tilde{u}_i \circ \phi_i$ converges to some v in $\tilde{\mathcal{U}_l}$.

$\tau \circ \tilde{u}_i \circ \phi_i = \phi_i$ also converges to $\phi := \tau \circ v$ which is a holomorphic automorphism of C_l, so $\tilde{u}_i$ converges to $v \circ \phi^{-1}$, which lies in the image of θ and so $v = \tilde{u}$ for some u in $\mathcal{U}_l$. Therefore u_i converges to u in $\mathcal{U}_l$. □

We pick $c' > c$ (where c is from Lemma 4.4), a regular value of p. We then define $\mathcal{W}$ to be the path component of $\mathcal{U}_0 = L$ in $p^{-1}[0, c']$, and $\mathcal{P}$ to be

$$\mathcal{W} \cap \mathcal{U}_{c'}$$

Then, by construction, $\pi : \mathcal{W} \to L$ is a cobordism from $Id : L \to L$ to $\pi : \mathcal{P} \to L$ over L. Furthermore whilst $\mathcal{W}'$ may have had different path components of different dimensions, $\mathcal{W}$ is a manifold (with boundary) of dimension $n + 1$, and $\mathcal{P}$ is a manifold of dimension n.

Proof of Lemma 1.12 We define $\pi' : \mathcal{P} \to L$ to send u to $u(i)$. Then by Lemma 4.4, this is homotopic to π. We now write down a homotopy

$$H : \mathcal{P} \times [0, 1] \to L$$

from $\psi^1 \circ \pi$ to π'.

Let $\gamma : [0, 1] \to \partial C_{c'}$ be some fixed path from 0 to i. For u in $\mathcal{P}$ and t in $[0, 1]$, we define $H(u, t)$ to be

$$\psi^1 \left(\left(\psi^{\operatorname{Im} \gamma(t)} \right)^{-1} (u(\gamma(t))) \right)$$

□

Remark 4.6 [18, Theorem 1] shows that even without any genericity assumption on the almost complex structure, the map $\pi : \mathcal{P} \to L$ induces an injective map

$$H^*(L; \mathbb{Z}/2) \cong \check{H}^*(L; \mathbb{Z}/2) \to \check{H}^*(\mathcal{P}; \mathbb{Z}/2)$$

where $\check{H}^*$ denotes Čech cohomology. The statement of Lemma 4.4 still holds and hence so does Lemma 1.12. From this it follows that $\psi^1|_L$ induces the identity map on $H^*(L; \mathbb{Z}/2)$.

This can be adapted to other generalised cohomology theories too, as in [17].

4.2 Orientations on the moduli space

In this subsection, we will prove Proposition 1.14.

We define $\mathcal{D}_0$ to be the space of triples (f, C, u), where C is a convex domain in the upper half plane in $\mathbb{C}$ with smooth boundary ∂C containing 0, $f : \partial C \to [0, 1]$ is a smooth map sending 0 to 0, and u is an element of $\mathcal{D}_{f,C}$. Note that because the choices of C and f were contractible, each inclusion $\mathcal{D}_{f,C} \hookrightarrow \mathcal{D}_0$ is a homotopy equivalence.

There is a tautological smooth fibre bundle over $\mathcal{D}_0$ with fibre over (f, C, u) given by C. The structure group of this bundle is the group of orientation-preserving diffeomorphisms of D^2 which send 1 to itself. This group is contractible, so we can pick a trivialisation Φ from $D^2 \times \mathcal{D}_0$ to this bundle, sending 1 in D^2 to 0 in each C. Note the space of such choices is contractible.

For each (f, C), we choose some extension $\tilde{f}$ of f to the entirety of C. Since for each (f, C) the space of such extensions is contractible, we can make some continuous choice over the whole of $\mathcal{D}_0$, and furthermore this choice is unique up to contractible choice.

There is then an evaluation map

$$ev : \left(D^2, \partial D^2 \right) \times \mathcal{D}_0 \to (M, L)$$

sending (z, u) in $D^2 \times \mathcal{D}_{f,C}$ to

$$\left(\psi^{\tilde{f}(\Phi(z))}\right)^{-1}(u(\Phi(z)))$$

Because the space of choices for the extensions $\tilde{f}$ was contractible, this defines ev uniquely up to a contractible space of choices.

There is a bundle pair over $\left(D^2, \partial D^2\right) \times \mathcal{D}_0$ given by $ev^*(TM, TL)$. For a map from a finite CW complex $g : X \to \mathcal{D}_0$, we write Ind for the index of the pullback of this bundle pair along g. Note that the virtual rank of Ind may be different on different components of X if the evaluation map lands in different components of $\mathcal{D}_0$.

Because the linearisation of the Cauchy–Riemann equation along its zero set is a Cauchy–Riemann operator, if J is regular, $T\mathcal{U}_{f,C}(J)$ is stably isomorphic to the index bundle Ind, and similarly $T\mathcal{W}$ is stably isomorphic to Ind $\oplus \mathbb{R}$.

Fix X a finite CW complex and $g : X \to \mathcal{D}_0$ some map. In the rest of this section we will find conditions under which Ind is R-orientable over X for various ring spectra R.

Proposition 4.7 *If there is a real vector bundle E over M, along with stable isomorphisms $TM \cong E \otimes \mathbb{C}$ and $TL \cong E|_L$ compatible with each other, then* Ind $- \pi^*TL$ *is stably trivial and hence R-orientable for any ring spectrum R.*

Proof We observe that the bundle pair $ev^*(TM, TL) - ev^*(E \otimes \mathbb{C}, E)$ is stably trivial, and that $ev^*(E \otimes \mathbb{C}, E) \cong q^*\pi^*(E \otimes \mathbb{C}, E)$ where $q : (D^2, \partial D^2) \times X \to X$ collapses the disc to a point. Therefore $\mathrm{Ind}(ev^*(E \otimes \mathbb{C}, E)) \cong \pi^*E \cong \pi^*TL$.

Then Ind $- \pi^*TL$ is stably isomorphic to $\mathrm{Ind}(ev^*(TM, TL) - ev^*(E \otimes \mathbb{C}, E))$, which is the index bundle of a stably trivial bundle pair, which is stably trivial by Proposition 3.14. □

Proposition 4.8 *If there is a stable trivialisation $TL \cong \mathbb{R}^n_{L_3}$ over the 3-skeleton of L which (after applying $\cdot \otimes \mathbb{C}$) extends to a stable trivialisation $TM \cong \mathbb{C}^n_{M_4}$ over the 4-skeleton of M, then* Ind *and hence* Ind $- \pi^*TL$ *are KO-orientable.*

Proof We pick a 3-skeleton L_3 of L, a 4-skeleton M_4 of M containing L_3, and a 2-skeleton X_2 of X. Then the map of pairs $ev : (D^2, \partial D^2) \times X_2 \to (M, L)$ can be homotoped to land in (M_4, L_3), and we can apply Proposition 3.14 to see that Ind admits a stable trivialisation over a 2-skeleton. Then by [5, Theorem 12.3], Ind is KO-orientable. □

Lemma 4.9 *If there is a stable trivialisation $TL \cong \mathbb{R}^n_{L_{i+1}}$ over the $(i+1)$-skeleton of L,* Ind *admits a stable complex structure over the i-skeleton of X.*

Proof The real part of the bundle pair $(ev^*TM, ev^*TL) \to (D^2, \partial D^2) \times X$ is pulled back from a map $\partial D^2 \times X \to L$. Letting X_i be some choice of i-skeleton of X, $ev : \partial D^2 \times X_i \to L$ can be homotoped to have image in the $(i+1)$-skeleton of L, so by Proposition 3.15, Ind admits a stable complex structure over X_i. □

Corollary 4.10 *1. If L admits a spin structure,* Ind *and* Ind $- \pi^*TL$ *are orientable.*

2. *If L admits a spin structure,* Ind *and* Ind $-\pi^*TL$ *are KU-orientable.*

Proof 1. If L admits a spin structure, TL is stably trivial over the 2-skeleton of L, and so Ind admits a complex structure over the 1-skeleton of X and hence is orientable.

2. A vector bundle over L is stably trivial over an i-skeleton L_i iff the classifying map $L \to BO$ is nullhomotopic when restricted to L_i. Since $\pi_3 BO = 0$, if $L_2 \to BO$ is nullhomotopic, so is $L_3 \to BO$. So if L admits a spin structure, TL admits a stable trivialisation over any 3-skeleton. Therefore Ind admits a spin^c structure, i.e. it admits a stable complex structure over any 2-skeleton. Then by [5, Theorem 12.3], Ind is KU-orientable.

□

We use the computation of w_1 of the index bundle of a bundle pair from [15] to weaken the hypothesis of Corollary 4.10(1).

Lemma 4.11 *If $w_2(L)(x) = 0$ whenever x is a homology class represented by a 2-torus $S^1 \times \partial D^2$ in L which extends to a solid torus $S^1 \times D^2$ in M, then* Ind $-\pi^*TL$ *is orientable.*

Proof $w_1(\text{Ind} - \pi^*TL)$ vanishes by [15, Theorem 1.1]. □

Together, the results in this section combine to form Proposition 1.14.

5 Monodromy in the looped case

5.1 General set-up

We define $L_{\psi^1} \subseteq S^1 \times M$ to be the space of pairs (t, x), where t lies in $S^1 = \mathbb{R}/\mathbb{Z}$ and x lies in $\psi^t(L)$. We let $q : L_{\psi^1} \to S^1$ be the projection map to the first co-ordinate. This is a model for the mapping torus of $\psi^1|_L$. We denote its vertical tangent space by $T^v L_{\psi^1}$.

The Hamiltonian flow induces a natural family of maps which we denote by $\Psi^t : L_{\psi^1} \to L_{\psi^1}$ for t in $\mathbb{R}$, which live over the identity map on S^1 for t in $\mathbb{Z}$. More explicitly, pick s in S^1, x in $\psi^s(L)$ and t in $\mathbb{R}$, and write $t + s = m + \tau$, where m is in $\mathbb{Z}$ and τ is in $[0, 1)$. Then we define $\Psi^t(s, x)$ to be

$$\left(\tau, \psi^\tau \circ (\psi^1)^{\circ m} \circ (\psi^s)^{-1}(x)\right)$$

Then Ψ^0 is the identity map, Ψ^t is continuous in t and $\Psi^t \circ \Psi^s = \Psi^{t+s}$ for all t, s in $\mathbb{R}$. Furthermore $\Psi^1|_{q^{-1}(\{0\})} = \psi^1|_L$. Ψ^t should be thought of as parallel transporting by a distance t around the base of the fibre bundle $L_{\psi^1} \twoheadrightarrow S^1$.

The free loop space $\mathcal{L}L_{\psi^1}$ of L_{ψ^1} is naturally a fibre bundle over L_{ψ^1}. We define $\mathcal{L}$ to be the restriction of this to the fibre over 0 in S^1. Then $\mathcal{L}$ is naturally a fibre bundle over L and splits into a disjoint union

$$\mathcal{L} = \bigsqcup_{j \in \mathbb{Z}} \mathcal{L}_j$$

where $\mathcal{L}_j$ is the space of loops in $\mathcal{L}$ with winding number j about S^1.

Ψ^1 induces a map $\mathcal{L} \to \mathcal{L}$ which preserves winding numbers, $\Psi^1(\mathcal{L}_i) = \mathcal{L}_i$ for all i, and we will also write Ψ^1 for this restriction.

We define $\mathcal{L}'_j$ to be subspace of $\mathcal{L}_j$ given by loops γ such that the composition $S^1 \to L_{\psi^1} \to S^1$ is given by $z \mapsto jz$, and $\mathcal{L}'$ to be the disjoint union of all $\mathcal{L}'_j$. Then Ψ^1 sends $\mathcal{L}'$ to $\mathcal{L}'$.

Lemma 5.1 *The inclusion $\mathcal{L}' \hookrightarrow \mathcal{L}$ is a homotopy equivalence.*

Proof We first define a continuous map $f : \mathcal{L}_j \to \{\text{continuous maps } [0,1] \to \mathbb{R}\}$. For all γ in $\mathcal{L}_j$, $f(\gamma)$ is uniquely determined by the following properties.

1. $f(\gamma)(0) = 0$
2. $f(\gamma)(1) = j$
3. The composition

$$[0,1] \to S^1 \xrightarrow{\gamma} L_{\psi^1} \xrightarrow{q} S^1$$

agrees with the composition

$$[0,1] \xrightarrow{f(\gamma)} \mathbb{R} \to S^1$$

We define a map $F : \mathcal{L} \to \mathcal{L}'$ by

$$F(\gamma)(t) = \Psi^{jt - f(\gamma)(t)}(\gamma(t))$$

for t in $[0,1]$. Then this is a homotopy inverse to the inclusion $\mathcal{L}' \hookrightarrow \mathcal{L}$. □

By Lemma 5.1, $\mathcal{L}'_0 \xrightarrow{\simeq} \mathcal{L}_0$, and we see that the action of Ψ^1 on $\mathcal{L}_0$ restricts to the action of $\psi^1|_L$ on $\mathcal{L}L = \mathcal{L}'_0$.

Therefore to study the action of $\psi^1|_L$ on $\mathcal{L}L$, it suffices to study the action of Ψ^1 on $\mathcal{L}_0$.

Lemma 5.2 Ψ^1 *acts as the identity on $\mathcal{L}_{\pm 1}$ up to homotopy.*

Proof Note by Lemma 5.1 it suffices to prove Ψ^1 acts as the identity on $\mathcal{L}'_{\pm 1}$.

We define $H : \mathcal{L}'_{\pm 1} \times [0,1] \to \mathcal{L}'_{\pm 1}$ by

$$H(\gamma, s)(t) = \Psi^{\pm s}(\gamma(t - s))$$

Then $H(\cdot, 0)$ is the identity and $H(\cdot, 1)$ is $\Psi^{\pm 1}$. In the case of $+1$, H is the required homotopy, and in the case of -1, $\Psi^1 \circ H$ is the required homotopy. □

5.2 A Chas-Sullivan type product

In this section we construct a product structure on the Thom spectrum $\mathcal{L}^{-TL}$ (defined shortly). Under Hypothesis 5.13, the moduli spaces $\mathcal{M}$ and $\mathcal{N}$ (constructed in Sect. 5.3) will have fundamental classes living in $R_*(\mathcal{L}^{-TL})$. This structure will be crucial in proving Theorem 1.17.

We first note that by [24], all spaces considered so far in this section are homotopy equivalent to CW complexes, and furthermore they are all Hausdorff and paracompact.

We denote the pullback of the virtual vector bundle $-TL \to L$ to $\mathcal{L}$ also by $-TL$. We will define ring and module structures on the spectra $\mathcal{L}^{-TL}$ and $\mathcal{L}$, following [9]. These will be needed in Sects. 5.6 and 5.7 to prove Theorem 1.17.

We have a commutative diagram:

$$\begin{tikzcd} \mathcal{L}_j \times_L \mathcal{L}_k \arrow[r, hook, "\Delta"] \arrow[d, hook, "c"] & \mathcal{L}_j \times \mathcal{L}_k \\ \mathcal{L}_{j+k} & \end{tikzcd}$$

where $\mathcal{L}_j \times_L \mathcal{L}_k$ is the fibre product of $\mathcal{L}_j$ and $\mathcal{L}_k$ over L, Δ is the natural inclusion map into $\mathcal{L}_j \times \mathcal{L}_k$, and c is a concatenation map: given (γ, δ) in $\mathcal{L}_j \times_L \mathcal{L}_k$ and t in $[0, 1]$, we define

$$c(\gamma, \delta)(t) = \begin{cases} \gamma(2t) & \text{if } t \le \frac{1}{2} \\ \delta(2t-1) & \text{if } t \ge \frac{1}{2} \end{cases}$$

Then $\mathcal{L}_j \times_L \mathcal{L}_k$ admits a tubular neighbourhood in $\mathcal{L}_j \times \mathcal{L}_k$ with normal bundle TL, and by applying Definition 2.28 with the virtual vector bundles $-TL \boxplus -TL$ and $-TL \boxplus 0$ over $\mathcal{L}_j \times \mathcal{L}_k$, we obtain two maps, both of which we denote by μ:

$$\mu : \mathcal{L}_j^{-TL} \wedge \mathcal{L}_k^{-TL} \xrightarrow{\Delta_!} \left(\mathcal{L}_j \times_L \mathcal{L}_k\right)^{-TL} \xrightarrow{c} \mathcal{L}_{j+k}^{-TL}$$
$$\mu : \mathcal{L}_j^{-TL} \wedge \Sigma^\infty_+ \mathcal{L}_k \xrightarrow{\Delta_!} \Sigma^\infty_+ \left(\mathcal{L}_j \times_L \mathcal{L}_k\right) \xrightarrow{c} \Sigma^\infty_+ \mathcal{L}_{j+k}$$

which fit together to give maps

$$\mu : \mathcal{L}^{-TL} \wedge \mathcal{L}^{-TL} \to \mathcal{L}^{-TL}$$
$$\mu : \mathcal{L}^{-TL} \wedge \Sigma^\infty_+ \mathcal{L} \to \Sigma^\infty_+ \mathcal{L}$$

It follows from the coassociativity of Δ that these define a homotopy associative product on $\mathcal{L}^{-TL}$, and a left module action of this on $\Sigma^\infty_+ \mathcal{L}$.

Let $i : L \hookrightarrow \mathcal{L}$ be the inclusion of constant loops into $\mathcal{L}_0$. The composition

$$[L] : \mathbb{S} \xrightarrow{[L]} L^{-TL} \xrightarrow{i} \mathcal{L}^{-TL}$$

defines a unit for $\mathcal{L}^{-TL}$, similarly to [9]. Together, we have

Proposition 5.3 *$\mathcal{L}^{-TL}$ is a ring spectrum, and $\Sigma^\infty_+\mathcal{L}$ is a left module over it.*

Furthermore if R is a ring spectrum, $\mathcal{L}^{-TL} \wedge R$ is a ring spectrum, and $\Sigma^\infty_+\mathcal{L} \wedge R$ is a left module over it.

It follows from construction that

Lemma 5.4 *All of the maps μ constructed in this subsection commute with the map Ψ^1 up to homotopy.*

Remark 5.5 For R-homology classes x and y in the R-homology of $\mathcal{L}^{-TL}$ or $\Sigma^\infty_+\mathcal{L}$, we will often write $x \cdot y$ for $\mu(x, y)$ when it is unambiguous to do so.

5.3 The moduli spaces $\mathcal{M}, \mathcal{N}$ and $\mathcal{Q}$

We fix G a convex domain in $\mathbb{C}$ with smooth boundary, such that ∂G contains $(-\eta, \eta)$ and $(-\eta, \eta) + i$ for some $\eta > 0$, and G is symmetric in the lines $i\mathbb{R}$ and $\frac{i}{2} + \mathbb{R}$. We define

$$D_+ = (\mathbb{R}_{\leq 0} + i[0, 1]) \cup G \sqcup \{-\infty\}$$

and

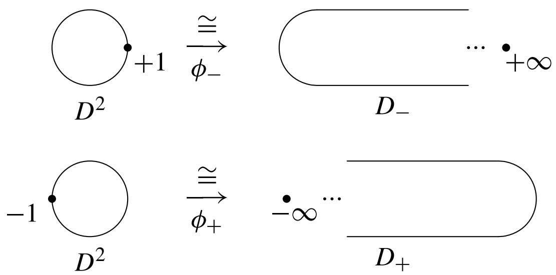
$$D_- = (\mathbb{R}_{\geq 0} + i[0, 1]) \cup G \sqcup \{+\infty\}$$

viewed as the one-point compactifications of half-infinite strips. Both are compact Riemann surfaces with boundary, biholomorphic to a disc.

We fix biholomorphisms $\phi_\pm : D^2 \to D_\pm$ which send ∓ 1 to $\mp\infty$, satisfying

$$\phi_-(z) = i - \phi_+(-z)$$

for all z, as shown below.



Definition 5.6 We define $\mathcal{R}_\pm$ to be the space of smooth maps $w : D_\pm \to M$ such that for all $z \neq \mp\infty$ in $\partial D_\pm$, $w(z)$ lies in $\psi^{\operatorname{Im} z}(L)$.

Remark 5.7 Since $D_\pm$ are compact, the topological energy $\int w^*\omega$ of any w in $\mathcal{R}_\pm$ is finite.

Fix J an ω-tame almost complex structure on M which is convex at infinity.

Definition 5.8 We define $\mathcal{M}$ to be the moduli space of J-holomorphic maps in $\mathcal{R}_-$, and $\mathcal{N}$ to be the moduli space of J-holomorphic maps in $\mathcal{R}_+$.

There are natural evaluation maps $\pi : \mathcal{R}_\pm \to L$ sending u to $u(\mp\infty)$. We define $\mathcal{Q}$ to be the fibre product $\mathcal{N} \times_L \mathcal{M}$ with respect to these evaluation maps.

Remark 5.9 Morally $\mathcal{Q}$ is the limit as $l \to \infty$ of $\mathcal{P}$.

For generic J the moduli spaces $\mathcal{M}$ and $\mathcal{N}$ are naturally smooth manifolds, and a similar argument to Sect. 4 shows that they are compact. Furthermore, for generic J the two maps $\pi : \mathcal{M} \to L$ and $\pi : \mathcal{N} \to L$ are transverse, and hence $\mathcal{Q}$ is also a compact smooth manifold. We assume we have chosen our J sufficiently generically to satisfy all of this.

$\mathcal{M}$ and $\mathcal{N}$ may have different connected components which have different dimensions.

However, note that since both of these moduli spaces are compact, they only have finitely many non-empty components.

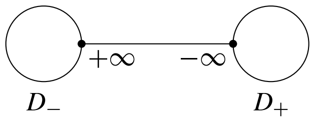
5.4 Bundle pairs on the moduli spaces

We define Σ to be the space

$$(D_- \cup [-1, +1] \cup D_+) \,/\, \sim$$

where $+\infty \sim -1$ and $+1 \sim -\infty$, as shown below (identifying $D_\pm$ with discs, using $\phi_\pm$).

Σ:



We define $\partial\Sigma \subseteq \Sigma$ to be the subspace

$$(\partial D_- \cup [-1, +1] \cup \partial D_+) \,/\, \sim$$

There are natural collapse maps

$$\xi_\pm : (\Sigma, \partial\Sigma) \to (D_\pm, \partial D_\pm)$$

which collapse $D_\mp$ and $[-1, 1]$ to $\mp\infty$, and are the identity on $D_\pm$.

We define a submanifold $\tilde{L}_{\psi^1} \subseteq \mathbb{C} \times M$ to be the space of pairs (z, x) such that $|z| = 1$ and if $z = 1$ then x lies in L, otherwise x lies in

$$\psi^{\operatorname{Im} \phi_-(z)}(L)$$

This is diffeomorphic to L_{ψ^1}. We let $T^v\tilde{L}_{\psi^1}$ be its vertical tangent bundle.

There are natural maps

$$ev_{\pm} : (D_{\pm}, \partial D_{\pm}) \times \mathcal{R}_{\pm} \to \left(\mathbb{C} \times M, \tilde{L}_{\psi^1}\right)$$

defined by

$$ev_{+}(z, w) = \left(-\phi_{+}^{-1}(i + \bar{z}), w(z)\right)$$

and

$$ev_{-}(z, w) = \left(\phi_{-}^{-1}(z), w(z)\right)$$

We define a map

$$C : (\Sigma, \partial\Sigma) \times \mathcal{Q} \to \left(\mathbb{C} \times M, \tilde{L}_{\psi^1}\right)$$

by

$$C(z, (v, u)) := \begin{cases} (ev_{-}(z, u)) & \text{if } z \in D_{-} \\ (ev_{+}(z, v)) & \text{if } z \in D_{+} \\ (1, u(+\infty)) & \text{if } z \in [-1, +1] \end{cases}$$

We will consider the bundle pairs

$$\begin{aligned} \xi_{\pm}^{*} ev_{\pm}^{*}\left(TM, T^v\tilde{L}_{\psi^1}\right), \\ C^{*}\left(TM, T^v\tilde{L}_{\psi^1}\right) \end{aligned}$$

and

$$\xi_{\pm}^{*}\pi^{*}\left(TM, T^v\tilde{L}_{\psi^1}\right)$$

From the construction, we see that the fibres of the complex parts of these bundle pairs are given as follows.

$$\left(\xi_-^* ev_-^* TM\right)_{(z,(v,u))} = \begin{cases} T_{u(z)}M & \text{if } z \in D_- \\ T_{u(+\infty)}M & \text{if } z \in D_+ \\ T_{u(+\infty)}M & \text{if } z \in [-1,1] \end{cases}$$

$$\left(\xi_+^* ev_+^* TM\right)_{(z,(v,u))} = \begin{cases} T_{u(+\infty)}M & \text{if } z \in D_- \\ T_{v(z)}M & \text{if } z \in D_+ \\ T_{u(+\infty)}M & \text{if } z \in [-1,1] \end{cases}$$

$$\left(C^* TM\right)_{(z,(v,u))} = \begin{cases} T_{u(z)}M & \text{if } z \in D_- \\ T_{v(z)}M & \text{if } z \in D_+ \\ T_{u(+\infty)}M & \text{if } z \in [-1,1] \end{cases}$$

$$\left(\xi_\pm^* \pi^* TM\right)_{(z,(v,u))} = T_{u(+\infty)}M$$

Similarly, the fibres of the real parts are given as follows, where z now lies in $\partial\Sigma$.

$$\left(\xi_-^* ev_-^* T^v \tilde{L}_{\psi^1}\right)_{(z,(v,u))} = \begin{cases} T_{u(z)}\psi^{\operatorname{Im} z}(L) & \text{if } z \in \partial D_- \\ T_{u(+\infty)}L & \text{if } z \in \partial D_+ \\ T_{u(+\infty)}L & \text{if } z \in [-1,1] \end{cases}$$

$$\left(\xi_+^* ev_+^* T^v \tilde{L}_{\psi^1}\right)_{(z,(v,u))} = \begin{cases} T_{u(+\infty)}L & \text{if } z \in \partial D_- \\ T_{v(z)}\psi^{\operatorname{Im} z}(L) & \text{if } z \in \partial D_+ \\ T_{u(+\infty)}L & \text{if } z \in [-1,1] \end{cases}$$

$$\left(C^* T^v \tilde{L}_{\psi^1}\right)_{(z,(v,u))} = \begin{cases} T_{u(z)}\psi^{\operatorname{Im} z}(L) & \text{if } z \in \partial D_- \\ T_{v(z)}\psi^{\operatorname{Im} z}(L) & \text{if } z \in \partial D_+ \\ T_{u(+\infty)}L & \text{if } z \in [-1,1] \end{cases}$$

$$\left(\xi_\pm^* \pi^* T^v \tilde{L}_{\psi^1}\right)_{(z,(v,u))} = T_{u(+\infty)}L$$

Lemma 5.10 *There is an isomorphism of bundle pairs F from*

$$\xi_-^* ev_-^* \left(TM, T^v \tilde{L}_{\psi^1}\right) \oplus \xi_+^* ev_+^* \left(TM, T^v \tilde{L}_{\psi^1}\right)$$

to

$$C^* \left(TM, T^v \tilde{L}_{\psi^1}\right) \oplus \xi_+^* \pi^* \left(TM, T^v \tilde{L}_{\psi^1}\right)$$

Proof We define $F_{(z,(v,u))}$ explicitly as follows.

If z lies in D_-, then

$$F_{(z,(v,u))} : T_{u(z)}M \oplus T_{u(+\infty)}M \to T_{u(z)}M \oplus T_{u(+\infty)}M$$

is given by the identity.

If z lies in D_+, then

$$F_{(z,(v,u))} : T_{u(+\infty)}M \oplus T_{v(z)}M \to T_{v(z)}M \oplus T_{u(+\infty)}M$$

is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to the direct sum decomposition above.

If z lies in $[-1, +1]$, then

$$F_{(z,(v,u))} : T_{u(+\infty)}M \oplus T_{u(+\infty)}M \to T_{u(+\infty)}M \oplus T_{u(+\infty)}M$$

is given by the matrix

$$\begin{pmatrix} \cos\left(\frac{z+1}{4}\pi\right) & \sin\left(\frac{z+1}{4}\pi\right) \\ -\sin\left(\frac{z+1}{4}\pi\right) & \cos\left(\frac{z+1}{4}\pi\right) \end{pmatrix}$$

with respect to the decomposition above.

Note that this map F respects the totally real subbundles when z lies in ∂D. $\square$

Note that

$$\operatorname{Ind} \pi^* \left(TM, T^v\tilde{L}_{\psi^1}\right)$$

is isomorphic to π^*TL, over both $\mathcal{R}_\pm$.

Now pick a "pinch" map $r : \left(D^2, \partial D^2\right) \to (\Sigma, \partial\Sigma)$ such that the compositions $\xi_\pm \circ r$ induce maps of degree 1 from ∂D^2 to $\partial D_\pm$. This description determines r up to homotopy, but for convenience we will make a more specific choice of r later.

Corollary 5.11 *There is an isomorphism of bundle pairs r^*F from*

$$r^*\xi_-^*ev_-^*\left(TM, T^v\tilde{L}_{\psi^1}\right) \oplus r^*\xi_+^*ev_+^*\left(TM, T^v\tilde{L}_{\psi^1}\right)$$

to

$$r^*C^*\left(TM, T^v\tilde{L}_{\psi^1}\right) \oplus r^*\xi_+^*\pi^*\left(TM, T^v\tilde{L}_{\psi^1}\right)$$

5.5 Gluing and orientations of the moduli spaces

The proof of gluing in [13] shows that there is a diffeomorphism $\mathcal{Q} \cong \mathcal{P}_{l+1}$ for sufficiently large l, so $\mathcal{W}$ can be viewed as a bordism from L to $\mathcal{Q}$. Here we take

$\mathcal{P}_{l+1}$ and $\mathcal{W}$ as in Sect. 4. [13] uses in an important way the fact that there is no disc bubbling (which follows from relative exactness of L).

The vertical tangent bundle of this cobordism restricted to $\mathcal{Q}$ is then naturally isomorphic to

$$\operatorname{Ind} r^* C^* \left(TM, T^v \tilde{L}_{\psi^1} \right)$$

Remark 5.12 $\operatorname{Ind} r^* C^* \left(TM, T^v \tilde{L}_{\psi^1} \right)$ is isomorphic to the index bundle Ind constructed in Sect. 4.2. For the purposes of brevity we will therefore refer to it as Ind for the rest of Sect. 5.

Therefore an R-orientation of Ind induces an R-orientation of $T\mathcal{Q}$.

By construction, $T\mathcal{M}$ and $T\mathcal{N}$ are naturally stably isomorphic to

$$\operatorname{Ind} r^* \xi_{\mp}^* ev_{\mp}^* \left(TM, T^v \tilde{L}_{\psi^1} \right)$$

respectively. Therefore R-orientations on both of these as well as TL induce one on $T\mathcal{Q}$, since $\mathcal{Q} = \mathcal{N} \times_L \mathcal{M}$ implies that

$$T\mathcal{Q} = T\mathcal{N} + T\mathcal{M} - \pi^* TL$$

Assumption 5.13 There are R-orientations of $T\mathcal{N}$, $T\mathcal{M}$, Ind and L, such that the two induced R-orientations on $T\mathcal{Q}$ above agree under the isomorphism r^*F constructed in Lemma 5.10. Furthermore the restriction of the R-orientation of Ind to the space of constant maps L is the given R-orientation of L.

Lemma 5.14 *Under Assumption 5.13, the cobordism $\mathcal{W}$ admits an R-orientation, restricting to the given ones on L and $\mathcal{Q}$.*

Proof The cobordism $\mathcal{W}$ has tangent bundle stably isomorphic to $\operatorname{Ind} \oplus \mathbb{R}$, and therefore the R-orientation on Ind induces an R-orientation on $\mathcal{W}$. By the last part of Assumption 5.13, this R-orientation restricts on the space of constant maps L to the given R-orientation of L.

$\mathcal{Q}$ now admits two R-orientations: one coming from the R-orientation on $\mathcal{W}$, which has $\mathcal{Q}$ as a boundary component (by identifying $\mathcal{P}_{l+1}$ with $\mathcal{Q}$ via the Ekholm-Smith diffeomorphism), and the one coming from the R-orientations on $T\mathcal{M}$, $T\mathcal{N}$ and TL under the identification

$$T\mathcal{Q} = T\mathcal{N} + T\mathcal{M} - \pi^* TL$$

It will therefore suffice to check that the natural bundle isomorphism which comes from gluing

$$\operatorname{Ind} \cong T\mathcal{Q} \cong T\mathcal{N} + T\mathcal{M} - \pi^* TL$$

from above agrees with the one induced by the isomorphism F from Lemma 5.10, up to homotopy of bundle isomorphisms. Assumption 5.13 then tells us that the R-orientations on both sides agree. To carry this out we need to open up both of these isomorphisms.

We will recall a sketch of how Ekholm and Smith, in [13], construct a diffeomorphism from $\mathcal{Q}$ to $\mathcal{P}_{l+1}$ (where $\mathcal{P}_{l+1}$ is defined as in Sect. 4). Fix some very large $l > 0$. We define $D_{\pm,l}$ to be the subset of $D_{\pm}$ given by

$$D_{\pm} \cap \{\pm\mathrm{Re} \geq -(l-1)\}$$

which we view as subsets of $G_l = C_{l+1}$ (G_l, C_{l+1} defined as in Sect. 4) by translating by $\pm l$. We let H be the closure of the complement in G_l of the union of these two regions:

$$H := [-1, 1] + i[0, 1]$$

Ekholm and Smith construct a pre-gluing embedding

$$PG : \mathcal{Q} \hookrightarrow \mathcal{D}_{f_{l+1}, C_{l+1}}$$

sending (v, u) to a map $(u\#_c v) : C_{l+1} \to M$ which agrees with u and v on $D_{\pm,l}$ respectively, and is C^1-small when restricted to H. Roughly, this is constructed by picking a metric on M, picking l large enough so that outside $D_{\pm,l}$ the images of both u and v lie inside a single geodesically convex neighbourhood of M, and cutting them off with a bump function.

We view $T\mathcal{Q}$ as lying inside

$$\Gamma\left(r^*\xi_+^* ev_+^*\left(TM, T^v\tilde{L}_{\psi^1}\right) \oplus r^*\xi_-^* ev_-^*\left(TM, T^v\tilde{L}_{\psi^1})\right)\right)$$

and $T\mathcal{P}_{l+1}$ as lying inside

$$\Gamma\left(r^*C^*\left(TM, T^v\tilde{L}_{\psi^1}\right)\right)$$

Note that we can do this without introducing stabilisations because our moduli spaces are cut out transversely, and so our choices of almost complex structure induce surjective Cauchy–Riemann operators.

Using a Newton-Picard iteration, Ekholm and Smith show that there is a diffeomorphism $\rho : PG(\mathcal{Q}) \to \mathcal{P}_{l+1}$, such that PG and $\rho \circ PG$ are C^1- close. Therefore the derivative $d(\rho \circ PG)$ sends a pair of sections (s_+, s_-) lying in $T\mathcal{Q}$, to a section s lying in $T\mathcal{P}_{l+1}$, such that s is C^0-small on H and C^0-close to u and v when restricted to $D_{\pm,l}$.

We make some choice of "pinch" map $r : C_{l+1} \to \Sigma$ which sends ∂C_{l+1} to $\partial\Sigma$, and which restricts to the identity on $D_{\pm,l}$. Note that $ev \simeq C \circ r$ on C_{l+1}.

By construction, we see that the map induced by r^*F on sections of these bundle pairs sends a pair of sections $(s_\pm)$, whose evaluations at $\mp\infty$ agree, to a section s, whose restrictions to $D_{\pm,l}$ agree with $s_\pm$ respectively. Consider the composition A:

$$\begin{array}{c} T\mathcal{Q} \\ \big\downarrow \\ \Gamma\left(r^*\xi_-^* ev_-^*\left(TM, T^v\tilde{L}_{\psi^1}\right) \oplus r^*\xi_+^* ev_+^*\left(TM, T^v\tilde{L}_{\psi^1}\right)\right) \\ \big\downarrow r^*F \\ \Gamma\left(r^*C^*\left(TM, T^v\tilde{L}_{\psi^1}\right) \oplus r^*\xi_+^*\pi^*\left(TM, T^v\tilde{L}_{\psi^1}\right)\right) \\ \big\downarrow \text{Projection to the first factor} \\ \Gamma\left(r^*C^*\left(TM, T^v\tilde{L}_{\psi^1}\right)\right) \\ \big\downarrow \text{Orthogonal projection} \\ T\mathcal{P}_{l+1} \end{array}$$

where we weight the metric on the bundle pair to be small on H. Note that since r^*F respects the direct sum decomposition away from H, if the weight is sufficiently small on H, any Cauchy–Riemann operator on the bundle pair in either the second or third term which respects the direct sum decomposition is of distance less than 1 to a Cauchy–Riemann operator on the other which respects the direct sum. Therefore the map $T\mathcal{Q} \to T\mathcal{P}_{l+1}$ coming from Lemma 3.5 agrees up to homotopy with A, and so under Assumption 5.13, A respects the R- orientations.

A is C^0-close to $d(\rho \circ PG)$ and so the two maps are homotopic isomorphisms of vector bundles, which is what we wanted. □

Lemma 5.15 *If $T^v\tilde{L}_{\psi^1}$ admits a stable trivialisation over an $(i+1)$-skeleton of $\tilde{L}_{\psi^1}$, then the induced stable trivialisations over an i-skeleton $\mathcal{Q}_i$ of $\mathcal{Q}$ of the bundle pairs appearing in Corollary 5.11 agree up to homotopy, under r^*F.*

Proof Homotoping our maps if necessary, we can assume that our maps

$$\left(D^2, \partial D^2\right) \times \mathcal{Q} \to \left(\mathbb{C} \times M, \tilde{L}_{\psi^1}\right)$$

send $\partial D^2 \times \mathcal{Q}_i$ to an $(i+1)$-skeleton of $\tilde{L}_{\psi^1}$.

Then the two stable trivialisations are related by a map $\eta : \partial D^2 \times \mathcal{Q}_i \to O$, where O is the infinite orthogonal group. We will use the choice of r from the proof of Lemma 5.14 for convenience, noting that this is unique up to homotopy.

From the construction of F, for any y in $\mathcal{Q}$ and x in $r|_{\partial D^2}^{-1}\partial D_-$, $\eta(x, y)$ is given by the identity matrix. For x in $r|_{\partial D^2}^{-1}\partial D_+$, $\eta(x, y)$ is given by (the stabilisation of) the block matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then along the top part of ∂D^2, $\eta(\cdot, y)$ is given by some homotopy between these two matrices, and travelling in the opposite direction along the bottom produces the reverse of this homotopy. Therefore for a fixed y, $\eta(\cdot, y)$ is a contractible loop. However these loops are independent of y, so the contractions can be chosen to be independent of y too, and so η is nullhomotopic. □

Then from Proposition 3.15 along with the fact that $\pi_3 BO = 0$, it follows that

Corollary 5.16 *1. If $T^v L_{\psi^1}$ admits a stable trivialisation over a 2-skeleton of L_{ψ^1}, then Assumption 5.13 holds for $R = H\mathbb{Z}$.*

2. If $T^v L_{\psi^1}$ admits a stable trivialisation over a 2-skeleton of L_{ψ^1}, then Assumption 5.13 holds for $R = KU$.

3. If $T^v L_{\psi^1}$ admits a stable trivialisation, then Assumption 5.13 holds when R is any complex-oriented cohomology theory.

Proof Assume $T^v \tilde{L}_{\psi^1}$ admits a stable trivialisation over an $(i+1)$-skeleton. Note that since $\pi_3 BO = 0$, if this holds for $i = 2$, it holds for $i = 3$ too.

We use that $T\mathcal{Q}$ is naturally stably isomorphic to Ind, and $T\mathcal{M}$ and $T\mathcal{N}$ are naturally stably isomorphic to

$$\operatorname{Ind} r^* \xi_\mp^* ev_\mp^* \left(TM, T^v \tilde{L}_{\psi^1}\right)$$

respectively. Since these all are index bundles of bundle pairs pulled back via some map to $(M \times \mathbb{C}, \tilde{L}_{\psi^1})$, these maps can be homotoped to send an i-skeleton of the moduli space times ∂D^2 to an $(i+1)$-skeleton of $\tilde{L}_{\psi^1}$. Therefore this stable trivialisation induces a stable complex structure on each an i-skeleton of these moduli spaces, by Proposition 3.15. We then use the fact that a complex structure over a 1-skeleton induces an orientation and a complex structure over a 2-skeleton induces a KU-orientation.

We then apply Proposition 5.15 to see that these induced stable complex structures agree under the isomorphism $r^* F$.

Finally, use the fact that a stable complex structure over a 1-skeleton induces an orientation, and one over a 2-skeleton induces a KU-orientation. □

Proof of Proposition 1.19 We will show that if TL admits a homotopy class of stable trivialisations over an i-skeleton L_i of L which is preserved by ψ^1 and can be extended to an $(i+1)$-skeleton L_{i+1}, then $T^v L_{\psi^1}$ admits a stable trivialisation over an i-skeleton of L_{ψ^1}. Then Corollary 5.16 will imply the result.

Let $\tilde{\psi}^1 : L \to L$ be a map homotopic to $\psi^1|_L$ which sends L_i to itself. Then L_{ψ^1} is homotopy equivalent to

$$L \times [0, 1]/(0, x) \sim (1, \tilde{\psi}^1(x))$$

and an $(i+1)$-skeleton of this is given by

$$(L_i \times [0, 1]) \cup L_{i+1}$$

Then by assumption, $T^v L_{\psi^1}$ admits a stable trivialisation over this $(i+1)$-skeleton. $\square$

5.6 Composition of the moduli spaces

We now assume Assumption 5.13 holds throughout the rest of Sect. 5.

Fix some diffeomorphism $\tilde{L}_{\psi^1} \cong L_{\psi^1}$ covering some orientation-reversing diffeomorphism $\partial D^2 \cong S^1$ sending $1 \in \partial D^2$ to $0 \in S^1 = \mathbb{R}/\mathbb{Z}$. There are natural evaluation maps

$$S^1 \times \mathcal{R}_\pm \xrightarrow{\exp(-2\pi i\cdot)} \partial D^2 \times \mathcal{R}_\pm \xrightarrow{\phi_\pm(\mp\cdot)} \partial D_\pm \times \mathcal{R}_\pm \xrightarrow{ev_\pm} \tilde{L}_{\psi^1} \xrightarrow{\cong} L_{\psi^1}$$

whose adjoints define maps

$$\sigma_\pm : \mathcal{R}_\pm \to \mathcal{L}_{\mp 1}$$

Therefore $\mathcal{Q}$ admits a natural evaluation map

$$\sigma : \mathcal{Q} \to \mathcal{L}_0$$

sending (v, u) to the concatenation $c(\sigma_-(v), \sigma_+(u))$. Then choices of R-orientations of L and all the moduli spaces allow us to use $\sigma_\pm$ and σ to define fundamental classes $[\mathcal{M}]$, $[\mathcal{N}]$ and $[\mathcal{Q}]$ in $\bigoplus_j R_j\left(\mathcal{L}_{\pm 1}^{-TL}\right)$ and $\bigoplus_j R_j\left(\mathcal{L}_0^{-TL}\right)$ respectively, as in Sect. 2.5.

Our goal in this subsection is to prove the following two lemmas.

Lemma 5.17

$$[\mathcal{N}] \cdot [\mathcal{M}] = [\mathcal{Q}]$$

in $\bigoplus_j R_j\left(\mathcal{L}_0^{-TL}\right)$.

Lemma 5.18

$$[\mathcal{Q}] = [L]$$

in $\bigoplus_j R_j\left(\mathcal{L}_0^{-TL}\right)$.

From these, we deduce

Lemma 5.19 *The composition*

$$\Sigma_+^\infty \mathcal{L}_0 \wedge R \xrightarrow{[\mathcal{M}]\cdot} \bigvee_j \Sigma_+^{\infty+j} \mathcal{L}_1 \wedge R \xrightarrow{[\mathcal{N}]\cdot} \bigvee_j \Sigma_+^{\infty+j} \mathcal{L}_0 \wedge R \xrightarrow{p} \Sigma_+^\infty \mathcal{L}_0 \wedge R$$

is an equivalence, where $p : \bigvee_j \Sigma_+^{\infty+j}\mathcal{L}_0 \to \Sigma_+^{\infty}\mathcal{L}_0$ *is the natural projection map.*

Proof of Lemma 5.17 Consider the following diagram.

$$\begin{tikzcd}
\mathbb{S} \arrow[d, "{[\mathcal{N}]\wedge[\mathcal{M}]}"] \arrow[dr, "{[\mathcal{Q}]}"] & \\
\mathcal{N}^{-T\mathcal{N}} \wedge R \wedge \mathcal{M}^{-T\mathcal{M}} \wedge R \arrow[r, "i_!"] \arrow[d, "\mathrm{Thom}"] & \mathcal{Q}^{-(T\mathcal{N}+T\mathcal{M}-\pi^*TL)} \wedge R \arrow[d, "\mathrm{Thom}"] \\
\bigvee_j \Sigma^j \mathcal{N}^{-\pi^*TL} \wedge R \wedge \mathcal{M}^{-\pi^*TL} \wedge R \arrow[r, "i_!'"] \arrow[d, "\sigma_- \wedge \sigma_+"] & \bigvee_j \Sigma^j \mathcal{Q}^{-\pi^*TL} \wedge R \arrow[d, "\sigma"] \\
\bigvee_j \Sigma^j \mathcal{L}_{-1}^{-TL} \wedge R \wedge \mathcal{L}_{+1}^{-TL} \wedge R \arrow[r, "\mu"] & \bigvee_j \Sigma^j \mathcal{L}_0^{-TL} \wedge R
\end{tikzcd}$$

The two arrows labelled Thom are the isomorphisms from the Thom isomorphism theorem for our choices of R-orientations on $T\mathcal{M} - \pi^*TL$ and $T\mathcal{N} - \pi^*TL$ followed by inclusion into this wedge product, and $i_!$ and $i_!'$ are obtained by applying Definition 2.28 to the embedding

$$\mathcal{Q} \hookrightarrow \mathcal{N} \times \mathcal{M}$$

using the virtual vector bundles $-T\mathcal{N} \boxplus -T\mathcal{M}$ and $-\pi^*TL \boxplus -\pi^*TL$ respectively.

The top triangle commutes by Lemma 2.30, the middle square commutes by naturality of the Thom isomorphism, and the bottom square commutes by construction of the product map μ. Therefore the entire diagram commutes.

The composition down along the left and across is given by:

$$\mu \circ (\sigma_- \wedge \sigma_+) \circ \mathrm{Thom} \circ [\mathcal{N}] \wedge [\mathcal{M}] = [\mathcal{N}] \cdot [\mathcal{M}]$$

whereas composition down along the right is given by

$$\sigma \circ \mathrm{Thom} \circ [\mathcal{Q}] = [\mathcal{Q}]$$

so the result follows. □

Proof of Lemma 5.18 By Lemma 5.14, there exists an R-orientable cobordism $\mathcal{W}$ from $\mathcal{Q}$ to L, with respect to the given R-orientations on both ends. Furthermore, evaluating along the boundary allows us to extend the maps $\sigma : \mathcal{Q} \to \mathcal{L}_0$ and $L \hookrightarrow \mathcal{L}_0$ to the entirety of $\mathcal{W}$. Therefore the result follows from Lemma 2.31.

□

5.7 Proof of Theorem 1.17

Proof of Theorem 1.17 Lemma 5.4 implies that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\Sigma^\infty_+ \mathcal{L}_0 \wedge R \arrow[r, "{[\mathcal{M}]_!}"] \arrow[d, "\Psi^1"] & \bigvee_j \Sigma^{\infty+j}_+ \mathcal{L}_1 \wedge R \arrow[r, "{[\mathcal{N}]_!}"] \arrow[d, "\Psi^1"] & \bigvee_j \Sigma^{\infty+j}_+ \mathcal{L}_0 \wedge R \arrow[r, "p"] \arrow[d, "\Psi^1"] & \Sigma^\infty_+ \mathcal{L}_0 \wedge R \arrow[d, "\Psi^1"] \\
\Sigma^\infty_+ \mathcal{L}_0 \wedge R \arrow[r, "{\Psi^1_*[\mathcal{M}]_!}"] & \bigvee_j \Sigma^{\infty+j}_+ \mathcal{L}_1 \wedge R \arrow[r, "{\Psi^1_*[\mathcal{N}]_!}"] & \bigvee_j \Sigma^{\infty+j}_+ \mathcal{L}_0 \wedge R \arrow[r, "p"] & \Sigma^\infty_+ \mathcal{L}_0 \wedge R
\end{tikzcd}$$

By Lemma 5.2, the second vertical arrow is homotopic to the identity, and the horizontal arrows along the top are all homotopic to the corresponding horizontal arrows along the bottom. Furthermore Lemma 5.19 tells us that the composition along the top (and similarly along the bottom) is an equivalence. It follows that the map

$$\Psi^1 : \Sigma^\infty_+ \mathcal{L}_0 \wedge R \to \Sigma^\infty_+ \mathcal{L}_0 \wedge R$$

is homotopic to the identity. Note that all maps and homotopies we used here were R-linear. □

6 Families over other bases

Our goal in this section will be to prove Theorem 1.22. In this section, we only consider mod-2 singular homology. We assume all spaces in this section are homotopy equivalent to CW complexes. We will use the following pair of purely topological lemmas:

Lemma 6.1 ([21, Lemma 4.3]) *Fix fibre bundles $A \hookrightarrow B \twoheadrightarrow C$ and $F \hookrightarrow E \twoheadrightarrow B$. Assume that the fibre bundles*

$$E|_A \hookrightarrow E \twoheadrightarrow C$$

and

$$F \hookrightarrow E|_A \twoheadrightarrow A$$

both c-split. Then the fibre bundle $F \hookrightarrow E \twoheadrightarrow B$ c-splits.

Proof The inclusion of a fibre $F \hookrightarrow E$ is the composition $F \hookrightarrow E|_A$ and $E|_A \hookrightarrow E$. The hypothesis of the lemma is that both these maps induce injections on mod-2 singular homology. □

Lemma 6.2 ([21, Lemma 4.1(ii)]) *Let $E \twoheadrightarrow B$ be a fibre bundle, and $f : B' \to B$ a map which is surjective on mod-2 singular homology. Assume that the pullback bundle $f^*E \twoheadrightarrow B'$ c-splits. Then $E \twoheadrightarrow B$ c-splits.*

Proof of Theorem 1.22 We first assume that $X = (S^1)^i$.

The case $i = 0$ is trivial and the case $i = 1$ follows from Theorem 1.4 along with an application of the Mayer-Vietoris sequence. We proceed by induction, and assume we know the result for $i - 1$, for all M and L as in the statement of Theorem 1.4.

We let $E = \gamma^* \mathcal{E}$. Then we have two fibre bundles

$$L \hookrightarrow E \twoheadrightarrow (S^1)^i$$

and

$$S^1 \hookrightarrow (S^1)^i \twoheadrightarrow (S^1)^{i-1}$$

where the inclusion map is inclusion to the first factor and the projection map is projection to all the other factors.

The fibre bundle $E|_{S^1} \twoheadrightarrow S^1$ c-splits by Theorem 1.4. So by Lemma 6.1, it suffices to show the fibre bundle given by the composition

$$E \twoheadrightarrow (S^1)^i \twoheadrightarrow (S^1)^{i-1}$$

c-splits.

For τ in $(S^1)^{i-1}$, we let γ_τ be the map $S^1 \to Lag_L$ given by $\gamma_\tau(t) = \gamma(t, \tau)$.

We perturb the map $\gamma : S^1 \times (S^1)^{i-1}$ so that for each τ in $(S^1)^{i-1}$, $\gamma_\tau(t)$ is constant in t in a neighbourhood of 0.

Claim 6.3 *There exists a smooth map $H : S^1 \times (S^1)^{i-1} \to \mathbb{R}$ such that for τ in $(S^1)^{i-1}$, the Hamiltonian $H_\tau : M \times S^1 \to \mathbb{R}$ generates the Hamiltonian isotopy of Lagrangians γ_τ.*

More explicitly, this means that the Hamiltonian flow of H_τ applied to the Lagrangian $\gamma_\tau(0)$ is γ_τ.

Proof of Claim 6.3 For each (t_0, τ) in $S^1 \times (S^1)^{i-1}$, let $V_{t_0,\tau}$ be the space of compactly supported smooth maps $f : M \to \mathbb{R}$ such that the Hamiltonian flow of f applied to $\gamma_\tau(t_0)$ agrees with $\gamma_\tau(t_0 + \cdot)$ to first order.

Together these determine a fibre bundle over $S^1 \times (S^1)^{i-1}$. Each $V_{t_0,\tau}$ is convex and hence contractible, so we can choose some smooth section of this fibre bundle. This gives us the map $H \times S^1 \times (S^1)^{i-1} \to \mathbb{R}$ that we require.

□

We now use this family of Hamiltonians H_τ to construct a suspension of each Lagrangian isotopy γ_τ, as follows. We define a map

$$f : E \hookrightarrow M \times T^*S^1 \times (S^1)^{i-1}$$

by

$$f(x) = (x, t, H_\tau(x, t), \tau)$$

where x lies in the fibre over (t, τ) in $S^1 \times (S^1)^{i-1}$, and we identify T^*S^1 with $S^1 \times \mathbb{R}$ in the usual way. Note that $E \to (S^1)^{i-1}$ is a fibre bundle.

A direct computation shows that for each τ, this gives a relatively exact Lagrangian in $M \times T^*S^1$, therefore this realises $E \twoheadrightarrow (S^1)^{i-1}$ as a family of relatively exact Lagrangians in $M \times T^*S^1$, and therefore by the induction hypothesis it c-splits.

Therefore by Lemma 6.1, the fibre bundle $E \twoheadrightarrow (S^1)^i$ c-splits.

For the general case, we consider the composition $(S^1)^i \xrightarrow{f} X \xrightarrow{\gamma} Lag_L$. Since $(f \circ \gamma)^*\mathcal{E} \twoheadrightarrow (S^1)^i$ c-splits, by Lemma 6.2, $\gamma^*\mathcal{E} \twoheadrightarrow X$ also c-splits. □

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A An example in complex *K*-Theory

Our goal in this section is to prove the following.

Proposition A.1 *In all sufficiently high dimensions n, there exists a closed n-dimensional manifold V such that:*

1. *V is stably parallelisable (and hence spin).*
2. *V admits a self-diffeomorphism $\theta : V \to V$, which acts as the identity on integral cohomology $H^*(V)$, but does not act as the identity on $K^*(V)$, where K^* denotes complex K-theory.*
3. *If n is odd, we can take V to be a simply-connected rational homology sphere.*

Corollary A.2 *If L is diffeomorphic to V, then θ does not lie in $\mathcal{G}_L$.*

The rest of the section will be dedicated to a proof of Proposition A.1, following a suggestion of Randal-Williams ([26]). We will construct these in sufficiently high odd dimensions and then observe that taking a product with S^1 provides the desired examples in all sufficiently high even dimensions.

Let q be a positive integer, and p an odd prime. Let Y be the homotopy mapping cone of the map $S^{2q} \to S^{2q}$ of degree p.

Lemma A.3 *We have the following:*

1.

$$\tilde{K}^i(Y) \cong \begin{cases} 0 & i \text{ even} \\ \mathbb{Z}/p & i \text{ odd} \end{cases}$$

2. When i is odd,

$$K_i(Y) \cong 0$$

3.

$$\tilde{H}^i(Y) \cong \begin{cases} \mathbb{Z}/p & i = 2q+1 \\ 0 & \text{otherwise} \end{cases}$$

4.

$$\tilde{H}_i(Y) \cong \begin{cases} \mathbb{Z}/p & i = 2q \\ 0 & \text{otherwise} \end{cases}$$

5. For all i,

$$\tilde{H}^i(Y; \mathbb{Q}) = 0$$

Proof Follows from the long exact sequence of a cone. (2) also uses the decomposition

$$K_i(Y) \cong \tilde{K}_i(Y) \oplus K_i(\text{point})$$

□

By [1, Theorem 1.7], for sufficiently large q, there is a map

$$g : \Sigma^{2(p-1)}Y \to Y$$

which induces an isomorphism on $\tilde{K}^*$, but must be 0 on $\tilde{H}^*$ and $\tilde{H}_*$ for degree reasons. Furthermore we can choose q to be sufficiently large that Y is simply connected.

Let $Z = \Sigma^{2(p-1)}Y \vee Y$, and we define $f : Z \to Z$ to be the composition

$$Z = \Sigma^{2(p-1)}Y \vee Y \xrightarrow{\text{Pinch}\vee Id} \Sigma^{2(p-1)}Y \vee \Sigma^{2(p-1)}Y \vee Y \xrightarrow{Id\vee g\vee Id} \Sigma^{2(p-1)}Y \vee Y = Z$$

where the first map is the pinching map, which uses the fact that

$$\Sigma^{2(p-1)}Y = \Sigma\left(\Sigma^{2(p-1)-1}Y\right)$$

and the second map sends the first copy of $\Sigma^{2(p-1)}Y$ to itself, the second copy of $\Sigma^{2(p-1)}Y$ to Y (using g), and sends Y to itself.

Lemma A.4 *Let E be a generalised homology or cohomology theory. Let A be a connected finite CW complex, and $h : A \to A$ some map.*

Then h acts as the identity on $E(A)$ if and only if h acts as the identity on $\tilde{E}(A)$.

Proof There is a canonical splitting

$$E(A) \cong \tilde{E}(A) \oplus E(\text{point})$$

h acts diagonally with respect to this splitting and always acts as the identity on $E(\text{point})$. □

Lemma A.5 *Let $f : Z \to Z$ be the map constructed above.*

1. *f does not act as the identity on $K^*(Z)$.*
2. *f acts as the identity on $H^*(Z)$ and $H_*(Z)$. Therefore by Whitehead's and Hurewicz' theorems, since Z is simply connected, f is a homotopy equivalence.*

Proof With respect to the decomposition

$$\tilde{K}^*(Z) \cong \tilde{K}^*\left(\Sigma^{2(p-1)}Y\right) \oplus \tilde{K}^*(Y)$$

we see that f^* is given by the block matrix

$$\begin{pmatrix} Id & 0 \\ g^* & Id \end{pmatrix}$$

and so f does not act as the identity on $\tilde{K}^*(Z)$, and hence not on $K^*(Z)$ either, by Lemma A.5. A similar argument shows the analogous results for $H^*(Z)$ and $H_*(Z)$. □

Now if Z were a closed manifold with the right properties and f were a diffeomorphism, we would be done, but unfortunately this is not the case.

Pick an embedding $Z \hookrightarrow \mathbb{R}^N$ for some large even $N > 4(p+q)$. Let W be a regular neighbourhood of Z with smooth boundary such that W deformation retracts to Z. Then W is a compact N-dimensional manifold with boundary, has one handle for each cell of Z, is simply connected, and for sufficiently large N, its boundary ∂W is also simply connected. The tangent bundle TW is trivial since W is a codimension 0 submanifold of $\mathbb{R}^N$, and so $T\partial W$ is also stably trivial.

Since $Z \simeq W$, f induces a map $f : W \to W$, well-defined up to homotopy. If N was chosen to be sufficiently large, f can be homotoped to an orientation-preserving embedding $e : W \hookrightarrow W$, which we can assume has image which does not touch ∂W.

Lemma A.6 *e is homotopic to an orientation-preserving diffeomorphism $\phi : W \to W$.*

Proof The complement $C := W \setminus e(W \setminus \partial W)$ is a cobordism from $e(\partial W)$ to ∂W. All three of C, $e(\partial W)$ and ∂W are simply connected, and since e is a homotopy equivalence, by excision (applied to the interior of $e(W)$) the inclusion $e(\partial W) \hookrightarrow C$ is a homology equivalence.

By Poincaré-Lefschetz duality, the inclusion $\partial W \hookrightarrow C$ is also a homology equivalence, and so by the h-cobordism theorem C is a trivial cobordism.

We fix a trivialisation of C, meaning a diffeomorphism $C \cong \partial W \times [0,1]_s$ sending ∂W to $\partial W \times \{1\}$, where s is the co-ordinate on $[0,1]$. Extend ∂_s to a vector field P on the whole of W, and let ρ be its time-1 flow.

Then $\rho \circ e$ is isotopic to e as an embedding, but is itself a diffeomorphism. Since e was orientation-preserving, $\rho \circ e$ also is. □

Now we let $V = \partial W$ and $\theta = \phi|_{\partial W}$.

Lemma A.7 *V is a rational homology sphere.*

Proof By Lemma A.3, Y and hence Z and W are rationally homology equivalent to a point. Then the result follows from the exact sequence of a pair (using Poincaré duality):

$$H_i(W;\mathbb{Q}) \to H^{N-i}(W;\mathbb{Q}) \to H_{i-1}(V;\mathbb{Q}) \to H_{i-1}(W;\mathbb{Q}) \to H^{N-i+1}(W;\mathbb{Q})$$

□

The following two lemmas complete the proof of Proposition A.1.

Lemma A.8 *θ acts as the identity on $H^*(V)$.*

Proof Note that since ϕ acts as the identity on $H_*(W)$ (by Lemma A.5), by Poincaré duality, ϕ also acts as the identity on $H^*(W, \partial W)$.

From Lemma A.3, we see that

$$\tilde{H}^i(W) \cong \begin{cases} \mathbb{Z}/p & i = 2q+1 \text{ or } 2(p+q-1)+1 \\ 0 & \text{otherwise} \end{cases}$$

and since W is orientable, by Poincaré duality

$$H^i(W, \partial W) \cong H_{N-i}(W) \cong \begin{cases} \mathbb{Z} & i = N \\ \mathbb{Z}/p & i = N-2q \text{ or } N-2(p+q-1) \\ 0 & \text{otherwise} \end{cases}$$

Therefore by the long exact sequence of the pair $(W, \partial W)$, for all i either the restriction map $\tilde{H}^i(W) \to \tilde{H}^i(\partial W)$ is an isomorphism or the boundary map $\tilde{H}^i(\partial W) \to H^{i+1}(W, \partial W)$ is an isomorphism. Both of these maps are compatible with the actions of ϕ and θ, so because ϕ acts as the identity on $H^*(W, \partial W)$ and on $H^*(W)$, the result follows. □

Lemma A.9 *θ does not act as the identity on $K^*(V)$.*

Proof We will show that the restriction map $\tilde{K}^i(W) \to \tilde{K}^i(V)$ is injective for all i, then the result follows from Lemmas A.5 and A.4.

By the long exact sequence of a pair, the kernel of the restriction map is the image of

$$K^i(W, \partial W) \to \tilde{K}^i(W)$$

TW is trivial and hence oriented with respect to K^*, so by Atiyah duality

$$K^i(W, \partial W) \cong K_{N-i}(W)$$

Therefore using the decomposition

$$\tilde{K}^*(W) \cong \tilde{K}^*\left(\Sigma^{2(p-1)}Y\right) \oplus \tilde{K}^*(Y)$$

and Lemma A.3, we see that if i is even then $\tilde{K}^i(W) = 0$, and if i is odd then $K_{N-i}(W) = 0$. In either case this implies that the restriction map is injective in degree i. □

B An example in real K-Theory

Let $U = S^3 \times S^2$. In this section we use the Hopf action of S^3 on S^2 to construct a simple diffeomorphism $U \to U$, and prove the following.

Proposition B.1 *There is a self-diffeomorphism $\zeta : U \to U$ which acts as the identity on integral homology but not as the identity on (the homology theory associated to) real K-theory KO_*.*

Then Theorem 1.8 implies

Corollary B.2 *If L is diffeomorphic to U and the condition of Proposition 1.14(4) holds, then ζ does not lie in $\mathcal{G}_L$.*

We view S^3 as the unit quaternions with unit $e \in S^3$, and we identify S^2 with the quotient S^3/S^1 of S^3 by the right action of the unit complex numbers. Let

$$\eta : S^3 \to S^3/S^1 = S^2$$

be the quotient map and let

$$\mu : S^3 \times S^3 \to S^3$$

be the product map. μ descends to a map

$$f : S^3 \times S^2 \to S^2$$

which defines a left action of S^3 on S^2. We then define the diffeomorphism ζ to be the map $U \to U$ sending (x, y) to

$$(x, f(x, y)).$$

Proposition B.1 then follows from Lemmas B.3 and B.4.

Lemma B.3 *ζ acts as the identity on the integral homology of U.*

Proof We first note that by the Künneth theorem,

$$H_i(U) \cong \begin{cases} \mathbb{Z} & \text{if } i \in \{0, 2, 3, 5\} \\ 0 & \text{otherwise.} \end{cases}$$

First note that ζ acts as the identity on $H_0(U)$ as U is path-connected.

The following diagram commutes

$$\begin{tikzcd} \{e\} \times S^2 \arrow[d, hook, "\iota"] \arrow[dr, hook, "\iota"] & \\ U \arrow[r, "\zeta"] & U \end{tikzcd}$$

where ι is the natural inclusion map. All maps in this diagram induce isomorphisms on H_2 so it follows that ζ acts as the identity on $H_2(U)$.

Similarly the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "\zeta"] \arrow[d, "p"] & U \arrow[dl, "p"] \\ S^3 & \end{tikzcd}$$

where $p : U \to S^3$ is the projection map onto the first coordinate. All maps in this diagram induce isomorphisms on H_3 so it follows that ζ acts as the identity on $H_3(U)$.

By the universal coefficients theorem, ζ acts as the identity on $H^2(U)$ and $H^3(U)$. Therefore since the cup product $H^2(U) \otimes H^3(U) \to H^5(U)$ is surjective, ζ acts as the identity on $H^5(U)$ and hence also on $H_5(U)$. □

Lemma B.4 *ζ does not act as the identity on $KO_3(U)$.*

Proof It follows from [3] that the map induced by η

$$\eta_* : \mathbb{Z} \cong \tilde{KO}_3(S^3) \to \mathbb{Z}/2 \cong \tilde{KO}_3(S^2)$$

is the quotient map $\mathbb{Z} \to \mathbb{Z}/2$.

Stably, U is homotopy equivalent to a wedge product

$$U \simeq \Sigma^2\mathbb{S} \vee \Sigma^3\mathbb{S} \vee \Sigma^5\mathbb{S}$$

From this we see that

$$\begin{aligned} \tilde{KO}_3(U) &\cong KO_3(\Sigma^2\mathbb{S}) \oplus KO_3(\Sigma^3\mathbb{S}) \oplus KO_3(\Sigma^5\mathbb{S}) \\ &\cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus 0 \end{aligned}$$

and with respect to this decomposition, ζ_* is given by the matrix

$$\begin{pmatrix} Id_{\mathbb{Z}} & 0 \\ r & Id_{\mathbb{Z}/2} \end{pmatrix}$$

where $r : \mathbb{Z} \to \mathbb{Z}/2$ is the quotient map. This is not the identity so ζ does not act as the identity on $\tilde{KO}_3(U)$. By Lemma A.4, ζ does not act as the identity on $KO_3(U)$. □

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