



SL_2 tilting modules in the mixed case

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Abstract

Using the non-semisimple Temperley–Lieb calculus, we study the additive and monoidal structure of the category of tilting modules for SL_2 in the mixed case. This simultaneously generalizes the semisimple situation, the case of the complex quantum group at a root of unity, and the algebraic group case in positive characteristic. We describe character formulas and give a presentation of the category of tilting modules as an additive category via a quiver with relations. Turning to the monoidal structure, we describe fusion rules and obtain an explicit recursive description of the appropriate analog of Jones–Wenzl projectors.

Keywords Tilting modules in the mixed case · Diagrammatic algebra · Temperley–Lieb algebras and categories · Fusion rules · Braided structures

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1 Introduction

Let \mathbb{k} be a field of characteristic p , fix a non-zero element $q \in \mathbb{k}^*$, and let \mathbb{K} be an algebraically closed field containing \mathbb{k} . Tilting modules for SL_2 , the reductive group $SL_2(\mathbb{K})$ if $q = \pm 1$ or Lusztig's divided power quantum group for \mathfrak{sl}_2 if $q \neq \pm 1$, are among the most well-studied objects in representation theory. In this paper, we use diagrammatic methods to study monoidal categories of tilting modules in the mixed case, i.e. for arbitrary (\mathbb{k}, q) . As a modern day perspective, the mixed case can be thought of as a simultaneous generalization of the case of the complex quantum group (at a root of unity), the case of the algebraic group in positive characteristic, as well as the classical semisimple situation.

Tilting modules form a monoidal category, so one can ask questions concerning objects, morphisms, and how these behave under the tensor product. Concentrating on objects and their characters is the classical approach in representation theory. Recently, the focus has shifted towards understanding morphisms between tilting modules, especially from a monoidal perspective, which has been driven by work from quantum topology and categorification. A more thorough understanding of the associated diagrammatic and combinatorial model that underpins the behavior of these tilting modules, known as the Temperley–Lieb category, was a key ingredient in recent progress.

In this paper, we let $\mathbf{Tilt}^{\mathbb{k}, q}$ for arbitrary (\mathbb{k}, q) denote the monoidal category obtained by idempotent completion from the Temperley–Lieb category $\mathbf{TL}^{\mathbb{k}, q}$ (see Remark 2.15). We study $\mathbf{Tilt}^{\mathbb{k}, q}$ with a focus on the behavior of its objects and morphisms with respect to its monoidal structure, which is a natural progression of previous work [52, 53]. The main results of this paper are contained within Sects. 3 and 4 and can be summarized as follows.

In Sect. 3B, we define *mixed JW projectors* E_{v-1} in $\mathbf{TL}^{\mathbb{k}, q}$ for $v \in \mathbb{N}$ and show that they correspond to indecomposable tilting modules $T(v-1)$ of highest weight $v-1$. These idempotents have been constructed independently in [41] and are a

simultaneous generalization of the classical Jones–Wenzl (short: JW) projectors [36, 54], the projectors of Goodman–Wenzl [32], and the pJones–Wenzl projectors of Burrull–Libedinsky–Sentinelli [14].

In Sect. 3C, we study morphisms between mixed JW projectors in $\mathbf{TL}^{k, q}$ and obtain a presentation of $\mathbf{Tilt}^{k, q}$ as an additive category by generators and relations. Specifically, we exhibit $\mathbf{Tilt}^{k, q}$ as the category of projective modules for the path algebra of a quiver with relations explicitly described in Theorem 3.21, which can be interpreted as the (semi-infinite) Ringel dual of SL_2 .

In Sect. 4, we turn to the monoidal structure and study fusion rules for $\mathbf{Tilt}^{k, q}$ and their categorified analogs in $\mathbf{TL}^{k, q}$. Classically, fusion rules express the structure constants for the representation ring, i.e. the decomposition multiplicities of tensor products of modules, such as $T(v) \otimes T(w)$, into indecomposable modules. On the categorified level, one is interested in explicitly describing the projection and inclusion maps realizing such decompositions. In the Temperley–Lieb context this means decomposing the tensor products $E_v \otimes E_w$ of projectors into idempotents that project onto the indecomposable summands predicted by the fusion rule.

A famous example is the recursion for the classical Jones–Wenzl projectors

$$\boxed{v-1} = \boxed{v} + \frac{1}{g_*} \cdot \begin{array}{c} \boxed{v-1} \\ \text{---} \\ \boxed{v-2} \\ \text{---} \\ \boxed{v-1} \end{array} \text{ , } \quad \frac{1}{g_*} = -\frac{[v-1]_q}{[v]_q} \tag{1.1}$$

which describes the decomposition $T(v-1) \otimes T(1) \cong T(v) \oplus T(v-2)$ whenever all involved tilting modules are simple. In fact, the Jones–Wenzl recursion (1.1) is often taken as (part of) the definition of the Jones–Wenzl projectors.

In Theorem 4.8 we establish decompositions analogous to (1.1) in the mixed setting of $\mathbf{TL}^{k, q}$. These provide a recursive description of the mixed JW projectors, which appear to be new in this generality, even new when specialized to the positive characteristic or complex quantum group cases, cf.[12]. As an example, we show an instance going beyond (1.1), which describes a decomposition $T(v-1) \otimes T(1) \cong T(v) \oplus T(v-2)$ with summands that need not be simple:

$$\boxed{v-1} = \boxed{v} + \left(\frac{1}{g_*} \cdot \begin{array}{c} \boxed{v-1} \\ \text{---} \\ \boxed{v-2} \\ \text{---} \\ \boxed{v-1} \end{array} - \frac{f_*}{g_*} \cdot \begin{array}{c} \boxed{v-1} \\ \text{---} \\ \boxed{v-2} \\ \text{---} \\ \boxed{v-1} \end{array} \right) \tag{1.2}$$

(Here and throughout the paper we use colored boxes to encode mixed JW projectors corresponding to tilting modules that need not be simple.) The middle part of the rightmost diagram in (1.2) corresponds to a nilpotent endomorphism of $T(v-2)$. In particular, if $T(v-2)$ is simple, then the rightmost diagram is zero and we recover (1.1). In general, however, the decompositions provided by Theorem 4.8 are more complex than suggested by the example (1.2). In particular, arbitrarily many summands can appear, with multiplicities up to two.

A bit of historical background and other works

Tilting modules for SL_2 have played a crucial role in representation theory and low-dimensional topology, even before their introduction by Donkin [20] and Ringel [43]. Let us recall parts of this story.

In the semisimple case, $\mathbf{Tilt}^{k,q}$ is well-understood on the level of objects and morphisms: The characters are given by Weyl's character formula and the fusion rules by the Clebsch–Gordan rule. On the morphism level, $\mathbf{Tilt}^{k,q}$ was given a diagrammatic presentation early on by Rumer–Teller–Weyl [45] using what is nowadays called the Temperley–Lieb algebra or category $\mathbf{TL}^{k,q}$. This diagrammatic presentation, in its quantum version, lies at the heart of constructions and calculations for the Jones-type invariants of links and 3-manifolds via Jones–Wenzl projectors and recoupling theory, see e.g. [38].

In the complex quantum group case, many of our results have previously appeared in the literature. The fusion rules on the object level in this case have certainly been known since the end of the 1980s, but are a bit hard to track down, see however [21] for a slightly later reference. The category $\mathbf{TL}^{k,q}$ plays a major role as it provides the diagrammatic and integral model of $\mathbf{Tilt}^{k,q}$. (While we do not know an explicit exposition, this can be deduced from [22].) The appropriate analog of JW projectors in this case was defined by Goodman–Wenzl [32], the Ringel dual quiver was computed in [10], and (parts of) recoupling theory was developed under the umbrella of non-semisimple 3-manifold invariants, see e.g. [12] or [18].

Historically speaking, the characteristic p case came long before the complex quantum group case, for example, due to its relationship to projective modules of the finite group $SL_2(\mathbb{F}_p)$. On the level of objects, the characteristic p case has been intensively studied throughout the literature, see e.g. [4, 16, 19, 20, 23, 24]. Of particular importance, are Steinberg's and Donkin's tensor product formulas, which give the characters of simple and tilting modules. However, not much appears to be known about fusion of objects beyond special cases, e.g. coming from studying Verlinde quotients, see for example [1] or [13], or the situation of the finite group $SL_2(\mathbb{F}_p)$, see for example [17]. On the morphism level, the use of $\mathbf{TL}^{k,q}$ is crucial, specifically in the recent work of Burrull–Libedinsky–Sentinelli [14] that introduced the p JW projectors, which was a main ingredient to find the quiver with relations and the center of $\mathbf{Tilt}^{k,q}$ – see [52] and [53]. When it comes to fusion for morphisms, our results are new.

In the (strictly) mixed case, most of the results in the present paper are new. See, however, e.g. [21] and [1] for character and fusion formulas, and [41] for their (independent) construction of the mixed JW projectors.

Further directions

A potential application concerns quotients of $\mathbf{Tilt}^{k,q}$, especially in the characteristic p and the strictly mixed cases in which the category $\mathbf{Tilt}^{k,q}$ has infinitely many \otimes -ideals. This is in stark contrast to the semisimple and complex quantum group cases, where one has no or only one non-trivial \otimes -ideal. The strictly mixed cases turn out to be very appealing in two directions. First, in generalizing e.g. the results of [13]

to the mixed situation, where $\mathbf{Tilt}^{k, \mathfrak{q}}$ may have universal properties that are similar to those studied in the characteristic \mathfrak{p} case. Second, it is easy to see that $\mathbf{Tilt}^{k, \mathfrak{q}^{1/2}}$ has a non-degenerate braiding in the mixed case. This is particularly interesting from the viewpoint of non-semisimple $3d$ TQFT, where one could try to apply the strategy of modified traces from e.g. [30] and [31], to obtain new non-semisimple $3d$ TQFTs.

The fusion rules for $\mathbf{Tilt}^{k, \mathfrak{q}}$ are also of importance in physics (from which its name arose), whose study so far has been focused on the semisimple and complex quantum group cases. In fact, this was one motivation to develop the Temperley–Lieb calculus [38] and its variations, which appear under different names in the physics- and mathematics-oriented literature. For example, idempotent truncations by tensor products of classical JW projectors are studied under the names valenced Temperley–Lieb algebras in [29] and symmetric webs in [44]. (See also [49] for a discussion using the $\mathfrak{p}\ell$ JW projectors.) Other recent work concerns the non-semisimple complex quantum group case and its relation to mathematical physics, see e.g. [39], but a non-semisimple recoupling theory along the lines of [38] seems largely undeveloped.

Finally, the algorithm given in [35] to compute \mathfrak{p} -Kazhdan–Lusztig basis elements of affine type A_1 played a key role in [14], and one could hope that this is a two-way street. For example, via quantum Satake [26] and the approach in [46] it might be possible to study analogs of mixed Kazhdan–Lusztig bases.

Further remarks

Before we start with the main bulk of the paper, we remark:

- (a) We tried to make the exposition of this paper self-contained by introducing all relevant concepts and definitions, many of which are identical or very similar to those in [52]. Some results that we need, for example Theorem 3.21, can then be proved analogously as in [52]. Instead of repeating these proofs here, we decided to only include detailed commentary about the necessary changes. In summary, while some of our results here depend on those of [52], a reader who wants to skip the proofs does not need to be familiar with that paper.
- (b) The comparison of the various (non-)semisimple JW projectors in the literature and their relation to this paper is as follows.
 - Our mixed projectors agree (modulo conventions in illustrations) with the projectors constructed independently in [41].
 - In the semisimple case, they agree with the classical JW projectors, and over \mathbb{C} and at a root of unity they agree with the projectors from [32], again modulo conventions.
 - For $\mathfrak{p} = \ell < \infty$ and $\mathfrak{q} = 1$, our projectors agree with the projectors from [14].

In each case, constructing the projectors and proving that they are well-defined, e.g. Theorem 3.18, require non-trivial numerical data. In our approach, these are the tilting characters, see Sect. 3A. There are two other ways to get equivalent numerical data: First, one could use the Soergel category for affine type A_1 and the \mathfrak{p} -Kazhdan–Lusztig basis as in [14]. This works for the quantum parameter being 1, but for a general quantum parameter the situation is more complicated. A second method is to

calculate the simple multiplicities within the projective cover of the trivial Temperley–Lieb module, as done in [41] (which is a follow-up of [50] where the decomposition numbers of the Temperley–Lieb algebra are computed). This works in the mixed case. However, a crucial upshot of the tilting characters approach taken in this paper is that it might generalize beyond SL_2 , e.g. see [47] and [48] for the complex quantum group case, which has been very explicitly worked out in [51].

2 Preliminaries

In this section we introduce necessary $p\ell$ -adic notation, and recall how tilting modules of SL_2 and the Temperley–Lieb calculus are related.

2A Basics of $p\ell$ -adic expansions

Let \mathbb{k} denote a field, and fix an invertible element $q \in \mathbb{k}$ throughout. We also let v denote a formal variable. For $x \in \mathbb{k}$ and $a \in \mathbb{N}$ we consider the quantum numbers:

$$[0]_x = 0, \quad [a]_x = x^{-(a-1)} + x^{-(a-3)} + \dots + x^{a-3} + x^{a-1}, \quad [-a]_x = -[a]_x.$$

Definition 2.1 The **mixed characteristic** of the pair (\mathbb{k}, q) is $mchar(\mathbb{k}, q) = (p, \ell)$, where $p \in \mathbb{N} \cup \{\infty\}$ denotes the additive order of 1 in \mathbb{k} , and $\ell \in \mathbb{N} \cup \{\infty\}$ is minimal such that $[\ell]_q = 0$ for $q \neq \pm 1$, and $\ell = p$ for $q = \pm 1$.

Note that p is a prime number, if finite, but ℓ can be any element in $\mathbb{N}_{\geq 2} \cup \{\infty\}$. Moreover, for finite ℓ the equation $[\ell]_q = \frac{(1-q^{-\ell})(q^\ell+1)}{q-q^{-1}} = 0$ implies that $q^\ell = \pm 1$. Conversely, the order $n = ord(q)$ of the root of unity q , if finite, determines ℓ and the signs q^ℓ and $(-q)^\ell$ as follows:

$$\begin{aligned} \ell &= \begin{cases} n & \text{if } n \equiv 1 \pmod{2}; \\ n/2 & \text{if } n \equiv 0 \pmod{2}, \end{cases} \\ q^\ell &= \begin{cases} 1 & \text{if } n \equiv 1 \pmod{2}; \\ -1 & \text{if } n \equiv 0 \pmod{2}, \end{cases} \\ (-q)^\ell &= \begin{cases} -1 & \text{if } n \equiv 0, 1, 3 \pmod{4}; \\ 1 & \text{if } n \equiv 2 \pmod{4}. \end{cases} \end{aligned}$$

The signs q^ℓ and $(-q)^\ell$ will appear in (3.10).

Example 2.2 The examples for (\mathbb{k}, q) that the reader should keep in mind are:

- (a) The **integral case**, where the pair is $(\mathbb{Z}[v^{\pm 1}], v)$. Beware that here \mathbb{k} is not a field, and we will always treat this case separately.
- (b) The **semisimple case**, where p is arbitrary and $\ell = \infty$. Explicit examples include $(\mathbb{Q}(v), v)$, and in fact $(\mathbb{k}(v), v)$ for any field \mathbb{k} .

- (c) The **complex quantum group case** (at a root of unity), where $p = \infty$ and $\ell < \infty$. For example, one could take $\mathbb{k} = \mathbb{C}$ with $\mathfrak{q} = \exp(\pi i/\ell)$ or $\mathfrak{q} = \exp(2\pi i/\ell)$, the former for all possible ℓ and the latter for odd ℓ .
- (d) The **characteristic p case**, where $p = \ell < \infty$ and $\mathfrak{q} = 1$, e.g. $(\mathbb{F}_p, 1)$ or $(\mathbb{F}_{p^k}, 1)$.
- (e) The **(strictly) mixed cases** are all other cases, i.e. $p < \infty, \ell < \infty$ with $p \neq \ell$. An explicit example is the pair $(\mathbb{F}_7, 2)$ for which the mixed characteristic is $(7, 3)$.

For the rest of this paper, with the exception of concrete examples, we fix a pair $(\mathbb{k}, \mathfrak{q})$ of mixed characteristic (p, ℓ) . The numbers p and ℓ will play a crucial role in this paper, e.g. via $p\ell$ -**adic expansions**:

Definition 2.3 Set $p^{(0)} = 1$, and for $i \in \mathbb{N}$ let $p^{(i)} = p^{i-1}\ell$. For any $v \in \mathbb{N}$ we write $[a_j, \dots, a_0]_{p,\ell} = \sum_{i=0}^j a_i p^{(i)} = v$ with $a_j \neq 0$. The digits are from the sets $a_i \in \{0, \dots, p-1\}$ for $i > 0$, and $a_0 \in \{0, \dots, \ell-1\}$. The higher digits are declared to be zero: $a_{>j} = 0$.

Conversely, any tuple $(b_j, \dots, b_0) \in \mathbb{Z}^{j+1}$ defines an integer $[b_j, \dots, b_0]_{p,\ell} = \sum_{i=0}^j b_i p^{(i)} \in \mathbb{Z}$. Here we explicitly allow negative digits.

The $p\ell$ -adic expansion of a natural number v as defined above is clearly unique: a_0 is uniquely determined as the remainder of v upon division by ℓ , and the remaining digits $[a_j, \dots, a_1]_p$ are determined by the usual p -adic expansion of the quotient $\frac{v-a_0}{\ell}$. The **leading digit** a_j and the **zeroth digit** a_0 will play slightly different roles than the other digits. If an index $i \geq 0$ is implicit from the context, then the symbol $p \vee \ell$ refers to p if $i > 0$ and to ℓ if $i = 0$.

Remark 2.4 We will repeatedly encounter the **law of small primes**, *losp* for short: we see special behavior in cases when relevant digits are (close to) 0 modulo $p \vee \ell$. For large characteristics such cases are exceptions, while for small ones they are the rule.

The following is taken from [52], but for $p\ell$ -adic expansions.

Definition 2.5 If $v = [a_j, \dots, a_0]_{p,\ell} \in \mathbb{N}$ has only a single non-zero digit, then v is called an **eve**. The set of eves is denoted by *Eve*. If $v \notin \text{Eve}$, then the **mother** m_v of v is obtained by setting the rightmost non-zero digit of v to zero.

Assume that $v \notin \text{Eve}$ has k non-zero, non-leading digits. We will also consider the set $A(v) = \{m_v, m_v^2 = m_{m_v}, \dots, m_v^{k-1}, [a_j, 0, \dots, 0]_{p,\ell} = m_v^\infty\}$ of (matrilineal) **ancestors** of v , whose size gen_v is called the **generation** of v . By convention, $A(v) = \emptyset$ and $\text{gen}_v = 0$ for $v \in \text{Eve}$. The **support** $\nabla \text{supp}(v) \subset \mathbb{N}$ is the set of the 2^{gen_v} integers of the form $w = [a_j, \pm a_{j-1}, \dots, \pm a_0]_{p,\ell}$.

We emphasise that every $v \notin \text{Eve}$ has an associated eve $m_v^k = m_v^\infty$ with k as in Definition 2.5, and $\text{gen}_v = k$. We think of the generation and the ancestry chart as a measure of the complexity of the associated SL_2 -modules. For example, in Proposition 3.3 we will see that a tilting module is simple if and only if its ρ -shifted highest weight is an eve.

Example 2.6 In the semisimple case where $\ell = \infty$, every $v \in \mathbb{N}_0$ is an eve and has no ancestors. In the complex quantum group case where $\mathfrak{p} = \infty$ and $\ell < \infty$, every $v \in \mathbb{N}_0$ is either an eve or of generation 1. In the other cases, the generation can be any number in \mathbb{N}_0 . For example, $68 = [68]_{\infty, \infty} = [66, 2]_{\infty, 3} = [1, 2, 5]_{7, 7} = [3, 1, 2]_{7, 3}$ has generation 0, 1, 2 and 2 in the respective mixed characteristics. In mixed characteristic $(7, 3)$, we have $A(68) = \{66 = [3, 1, 0]_{7, 3}, 63 = [3, 0, 0]_{7, 3}\}$ and $\nabla\text{supp}(68 = [3, 1, 2]_{7, 3}) = \{68 = [3, 1, 2]_{7, 3}, 64 = [3, 1, -2]_{7, 3}, 62 = [3, -1, 2]_{7, 3}, 58 = [3, -1, -2]_{7, 3}\}$.

The elements w in the support $\nabla\text{supp}(v)$ of $v \in \mathbb{N}$ can be described by the sets of indices of digits of v , which are negated (or “reflected”) to obtain an expression for w . To obtain a bijection between elements in $\nabla\text{supp}(v)$ and sets of indices, we enforce certain admissibility conditions on the latter:

Definition 2.7 For $S \subset \mathbb{N}_0$ a finite set, we consider partitions $S = \bigsqcup_i S_i$ of S into subsets S_i of consecutive integers that we call **stretches**. For the rest of the definition, we let $S = \bigsqcup_i S_i$ be the coarsest such partition into stretches.

The set S is called **down-admissible** for $v = [a_j, \dots, a_0]_{\mathfrak{p}, \ell}$ if the following conditions hold:

- (i) $a_{\min(S_i)} \neq 0$ for every i , and
- (ii) if $s \in S$ and $a_{s+1} = 0$, then $s + 1 \in S$.

If $S \subset \mathbb{N}_0$ is down-admissible for $v = [a_j, \dots, a_0]_{\mathfrak{p}, \ell}$, then we define its **downward reflection** along S as

$$v[S] = [a_j, \epsilon_{j-1}a_{j-1}, \dots, \epsilon_0 a_0]_{\mathfrak{p}, \ell}, \quad \epsilon_k = \begin{cases} 1 & \text{if } k \notin S; \\ -1 & \text{if } k \in S. \end{cases}$$

Conversely, S is **up-admissible** for $v = [a_j, \dots, a_0]_{\mathfrak{p}, \ell}$ if the following conditions hold:

- (i) $a_{\min(S_i)} \neq 0$ for every i , and
- (ii) if $s \in S$ and $a_{s+1} = \mathfrak{p} - 1$, then $s + 1 \in S$.

If $S \subset \mathbb{N}_0$ is up-admissible for $v = [a_j, \dots, a_0]_{\mathfrak{p}, \ell}$, then we define its **upward reflection** along S as

$$v(S) = [a'_{r(S)}, \dots, a'_0]_{\mathfrak{p}, \ell}, \quad a'_k = \begin{cases} a_k & \text{if } k \notin S, k - 1 \notin S; \\ a_k + 2 & \text{if } k \notin S, k - 1 \in S; \\ -a_k & \text{if } k \in S, \end{cases}$$

where we extend the digits of v by $a_h = 0$ for $h > j$ if necessary, and $r(S)$ is the biggest integer such that $a'_k \neq 0$.

Any down- or up-admissible set S has a unique finest partition into down- or up-admissible sets, each of which consists of consecutive integers and which we call **minimal down- or up-admissible stretches**, respectively.

A stretch $\{k, k - 1, \dots, l + 1, l\}$ is *minimal down-admissible* if and only if

$$(a_{k+1}, a_k, \dots, a_{l+1}, a_l) = (a_{k+1}, 0, \dots, 0, a_l) \quad \text{with } a_{k+1} \neq 0, a_l \neq 0.$$

It is *minimal up-admissible* if and only if

$$(a_{k+1}, a_k, \dots, a_{l+1}, a_l) = (a_{k+1}, p - 1, \dots, p - 1, a_l) \quad \text{with } a_{k+1} \neq p - 1, a_l \neq 0.$$

Very often (unless *losp* applies), the minimal stretches will just be singleton sets $\{i\}$ specifying a single digit in which we reflect. We also tend to omit the set brackets of down- or up-admissible sets if no confusion can arise, e.g. we write $v[i]$ instead of $v[\{i\}]$.

For $v \in \mathbb{N}$, a finite set $S \subset \mathbb{N}_0$ is down-admissible if and only if it is up-admissible for $v[S]$, and in this case $vS = v$. For a representation-theoretic interpretation of the admissibility conditions see Remark 3.1: Note that $\bigvee \text{supp}(v) = \{v[S] \mid S \text{ down-admissible}\}$.

Definition 2.8 If S is up-admissible for $v \in \mathbb{N}$, then we denote by $\overline{S} \subset \mathbb{N}_0$ the down-admissible hull of S , the smallest down-admissible set containing S , if it exists.

Note that \overline{S} is only defined for up-admissible S , which excludes stretches with rightmost digit zero. The singleton containing the leading digit is always up-admissible and its down-admissible hull does not exist.

Remark 2.9 The above admissibility condition is taken from [52, Definition 2.8]. The whole discussion after [52, Definition 2.8] works *verbatim*. Explicit examples appear there and in [53, Example 2.9].

2B Tilting modules and their diagrams

Let $\mathbb{k} \subset \mathbb{K}$ denote an algebraically closed field containing \mathbb{k} . We use the symbol SL_2 to denote the reductive group SL_2 over \mathbb{K} if $\mathfrak{q} = \pm 1 \in \mathbb{K}$ and Lusztig’s divided power quantum group (using the conventions from [6]) associated to \mathfrak{sl}_2 for other values of \mathfrak{q} . We will identify dominant integral weights of SL_2 with \mathbb{N}_0 and weights with \mathbb{Z} in the usual way.

We consider finite-dimensional (left) SL_2 -modules of type 1 over \mathbb{K} . (It is common practice to restrict to SL_2 -modules of type 1 out of convenience – see e.g. [34, Section 5.2] for details.) These form an abelian, \mathbb{K} -linear category $\mathbf{fdMod}^{\mathbb{K}, \mathfrak{q}}$: = $SL_2\text{-fdMod}^{\mathbb{K}, \mathfrak{q}}$, for which we additionally choose a monoidal and a pivotal structure using the comultiplication of SL_2 , the antipode of SL_2 and the analog of the involution ω from [34, Lemma 4.6]. The category $\mathbf{fdMod}^{\mathbb{K}, \mathfrak{q}}$ contains four families of highest weight modules of particular interest for our purpose, all parameterized by \mathbb{N}_0 .

Remark 2.10 Here and in the following sections, we often write the highest weights of these modules as $v - 1$ for $v \in \mathbb{N}$. This puts an emphasis on the quantity v , the ρ -shifted highest weight, which will play a greater role than the highest weight itself.

The first two families are formed by the *Weyl* modules $\Delta(v - 1)$ and the *dual Weyl* modules $\nabla(v - 1)$. These do not depend on the mixed characteristic in the sense that they can be defined integrally, i.e. for $(\mathbb{Z}[\mathbb{v}^{\pm 1}], \mathbb{v})$. Their characters are given by the Weyl character formula.

The other two families of modules are formed by the *simple* modules $\mathbb{L}(v - 1)$ and the *indecomposable tilting* modules $\mathbb{T}(v - 1)$. These modules do not admit a construction independent of the mixed characteristic. Their characters are given by Proposition 3.3 below.

Let $\mathbf{Tilt}^{\mathbb{K}, \mathbb{q}} := \mathbb{SL}_2\text{-Tilt}^{\mathbb{K}, \mathbb{q}}$ be the full subcategory of $\mathbf{fdMod}^{\mathbb{K}, \mathbb{q}}$ whose objects are direct sums of $\mathbb{T}(v - 1)$ for $v \in \mathbb{N}$. We also write $\mathbb{T}(z)$ for $z < 0$, which is zero by convention. The category $\mathbf{Tilt}^{\mathbb{K}, \mathbb{q}}$ is additive, idempotent closed, Krull–Schmidt (i.e. there is a unique decomposition into indecomposables, and an object is indecomposable if and only if its endomorphism ring is local), \mathbb{K} -linear, and pivotal (restricting the structures from $\mathbf{fdMod}^{\mathbb{K}, \mathbb{q}}$ to $\mathbf{Tilt}^{\mathbb{K}, \mathbb{q}}$). It is the main object under study in this paper and called the *category of tilting modules* of \mathbb{SL}_2 .

Remark 2.11 Classically $\mathbf{Tilt}^{\mathbb{K}, \mathbb{q}}$ would be defined as the full subcategory of $\mathbf{fdMod}^{\mathbb{K}, \mathbb{q}}$ whose objects have Weyl and dual Weyl filtrations, and its closure under tensor product would be a theorem. The above definition is equivalent to the classical one for \mathbb{SL}_2 , because the sole fundamental representation, $\mathbb{T}(1)$, is tilting. Thus, all indecomposable tiltings appear as direct summands of tensor powers thereof. This may fail for other types in small characteristic.

Generally these four types of modules (Weyl, dual Weyl, simple, and indecomposable tilting) for a fixed highest weight are distinct from one another. If, however, any two are isomorphic e.g. $\mathbb{T}(v - 1) \cong \nabla(v - 1)$, then it follows that all four types of modules of the same highest weight are isomorphic. An example is $\mathbb{T}(0) \cong \Delta(0) \cong \nabla(0) \cong \mathbb{L}(0) \cong \mathbb{K}$, which is the monoidal unit of $\mathbf{Tilt}^{\mathbb{K}, \mathbb{q}}$ and which we denote by 1.

Remark 2.12 Let us comment on the references for the above commentary as well as some of the material below, using the terminology from Example 2.2. In the semisimple case, $\mathbf{Tilt}^{\mathbb{K}, \mathbb{q}}$ is equivalent to $\mathbf{fdMod}^{\mathbb{K}, \mathbb{q}}$, a semisimple category endowed with the classical combinatorics of $\mathbb{SL}_2(\mathbb{C})$, which is covered in many textbooks. Otherwise, $\mathbf{Tilt}^{\mathbb{K}, \mathbb{q}}$ is non-semisimple and we refer to [6] and [2] in the complex quantum group case, to [43] and [20] in the characteristic p case, and to [5] as well as [21] and [1] in the mixed case. A summary for tilting modules can also be found in [7].

The diagrammatic incarnation of $\mathbf{Tilt}^{\mathbb{K}, \mathbb{q}}$ is sometimes called the *Temperley–Lieb category* (abbreviated to TL category) and can be defined as follows. Let $\mathbf{TL}^{\mathbb{Z}[\mathbb{v}^{\pm 1}], \mathbb{v}}$ denote the $\mathbb{Z}[\mathbb{v}^{\pm 1}]$ -linear category with objects indexed by $m \in \mathbb{N}_0$, and with morphisms from m to n being $\mathbb{Z}[\mathbb{v}^{\pm 1}]$ -linear combinations of unoriented string diagrams drawn in a horizontal strip $\mathbb{R} \times [0, 1]$ between m marked points on the lower boundary $\mathbb{R} \times \{0\}$ and n marked points on the upper boundary $\mathbb{R} \times \{1\}$, considered up to planar isotopy relative to the boundary and the relation that a circle evaluates to $-[2]_{\mathbb{v}}$. The category $\mathbf{TL}^{\mathbb{Z}[\mathbb{v}^{\pm 1}], \mathbb{v}}$ is (strict) monoidal with \otimes given by horizontal concatenation and admits a (strict) pivotal structure given by cups and caps (the duality maps), and all objects are self-dual.

Proposition 2.14 *If \mathbb{q} has a square root in \mathbb{K} , then the functor $\mathcal{D}^{\mathbb{K},\mathbb{q}}$ from Proposition 2.13 is an equivalence of braided categories. \square*

Remark 2.15 Note that Proposition 2.13 allows us to identify the additive Karoubi closure of $\mathbf{TL}^{\mathbb{K},\mathbb{q}}$ with $\mathbf{Tilt}^{\mathbb{K},\mathbb{q}}$. Motivated by this, we will also denote the additive Karoubi closure of $\mathbf{TL}^{\mathbb{k},\mathbb{q}}$ by $\mathbf{Tilt}^{\mathbb{k},\mathbb{q}}$ in case \mathbb{k} is not necessarily an algebraically closed field, and the objects $\mathbb{T}(v - 1) \in \mathbf{Tilt}^{\mathbb{k},\mathbb{q}}$ are defined as the images of primitive idempotents under $\mathcal{D}^{\mathbb{k},\mathbb{q}}$. With this notation, the functor $\mathcal{D}^{\mathbb{k},\mathbb{q}}$ is the universal embedding of $\mathbf{TL}^{\mathbb{k},\mathbb{q}}$ into its Karoubi closure, and we omit it from our notation without confusion.

Remark 2.16 At this point, the reader is warned that $\mathbf{Tilt}^{\mathbb{k},\mathbb{q}}$ may not be equivalent to the category of tilting modules over \mathbb{k} (as defined via (dual) Weyl filtrations), even in semisimple cases, if \mathbb{k} is finite – see [11, Section 5].

In the semisimple situation of $\mathbf{TL}^{\mathbb{k}(v),v}$, the primitive idempotents that are mapped to the indecomposable tilting modules $\mathbb{T}(v - 1)$ in $\mathbb{T}(1)^{\otimes(v-1)}$ are the well-known Jones–Wenzl projectors. Since $\mathbb{T}(v - 1)$ is a simple module in this case, we will call these idempotents *simple Jones–Wenzl projectors* (simple JW projectors for short), which also allows us to distinguish them from their non-simple analogs. All we need to know about these projectors is summarized in the following proposition – see e.g. [38] for a proof.

Proposition 2.17 *For all $v \in \mathbb{N}$, there exists a unique idempotent $\tilde{e}_{v-1} \in \text{End}_{\mathbf{TL}^{\mathbb{k}(v),v}}(v - 1)$, which is invariant under duality ($\tilde{e}_{v-1}^\dagger = (\tilde{e}_{v-1})^{\leftrightarrow} = \tilde{e}_{v-1}$ (this implies that the following relations hold under their mirror images as well) and satisfies:*

$$\boxed{\begin{array}{c} w-1 \\ \hline v-1 \end{array}} = \boxed{v-1}, \quad (2.2) \quad \boxed{\begin{array}{c} k \\ \hline v-1 \end{array}} = 0, \quad (2.3) \quad \boxed{v-1}^k = (-1)^k \frac{[v]_v}{[v-k]_v} \boxed{v-1-k}. \quad (2.4)$$

Here we use the usual box notation for these projectors, where a number k next to a strand means k parallel strands. The projector \tilde{e}_{w-1} in (2.2) and the cup (respectively, a cap under duality) in (2.3) can be placed at arbitrary positions. The idempotent \tilde{e}_{v-1} satisfies the recursion in (1.1). \square

In Definition 2.18 and Convention 2.19 we will define various different bases of morphism spaces in Temperley–Lieb categories. The first example is given by *integral bases*: sets of crossingless matchings (a.k.a. *Temperley–Lieb diagrams*) $B_{v-1,w-1}^{int}$ of $v + w - 2$ points. These are integral in the sense that they provide isomorphisms $\text{Hom}_{\mathbf{TL}^{\mathbb{Z}[v^{\pm 1}],v}}(v - 1, w - 1) \cong \mathbb{Z}[v^{\pm 1}]B_{v-1,w-1}^{int}$ of $\mathbb{Z}[v^{\pm 1}]$ -modules. Second, $\mathbf{TL}^{\mathbb{k},\mathbb{q}}$ has *projector bases* $B_{v-1,w-1}^{\mathbb{q}}$ given by decomposing $\mathbb{T}(1)^{\otimes(v-1)}$ into indecomposable summands. (For $\mathbf{TL}^{\mathbb{k}(v),v}$, a basis of the form $B_{v-1,w-1}^v$ is an *Artin–Wedderburn basis* since these summands are simple.) We stress that these bases are not unique unless one specifies further properties that these should satisfy. The existence of these bases follows from abstract theory (see [9]) and all of these are cellular and related by unitriangular basis change matrices. To construct these bases explicitly we can use the *light ladder strategy* (see [25] and [9]).

Definition 2.18 Fix a family of morphisms $G_{v-1} \in \text{End}_{\mathbf{TL}^{k,q}}(v-1)$ for $v \in \mathbb{N}$. Then for each $F \in \text{Hom}_{\mathbf{TL}^{k,q}}(w-1, v-1)$ we define morphisms $\tilde{\varepsilon}_1(F) \in \text{Hom}_{\mathbf{TL}^{k,q}}(w, v)$ and (provided $v > 1$) $\tilde{\varepsilon}_{-1}(F) \in \text{Hom}_{\mathbf{TL}^{k,q}}(w, v-2)$ by sending:

$$F = \begin{array}{|c|} \hline v-1 \\ \hline \boxed{F} \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline G_v \\ \hline \boxed{F} \\ \hline \end{array} = \tilde{\varepsilon}_1(F), \quad F = \begin{array}{|c|} \hline v-1 \\ \hline \boxed{F} \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline G_{v-2} \\ \hline \boxed{F} \\ \hline \end{array} \cup = \tilde{\varepsilon}_{-1}(F).$$

For any path π in the positive Weyl chamber, considered as a finite sequence of ± 1 whose partial sums are non-negative, we associate a **down morphism** $\delta(\pi)$ by using the operators $\tilde{\varepsilon}_{\pm 1}$ in order specified by π , starting with F being the empty diagram. Similarly, we define an **up morphism** $\nu(\pi)$ as $\delta(\pi)^\dagger$. For a pair (π, π') , we define an element $c_{\pi, \pi'}^\lambda := \nu(\pi')\delta(\pi)$ whenever it makes sense to do so, i.e. for $\delta(\pi) \in \text{Hom}_{\mathbf{TL}^{k,q}}(v-1, \lambda)$ and $\nu(\pi') \in \text{Hom}_{\mathbf{TL}^{k,q}}(\lambda, w-1)$.

Convention 2.19 We will use the light ladder strategy from Definition 2.18 in several different contexts. The associated down and up morphisms are consistently distinguished throughout this paper by the following convention in notation.

- (a) For $G_{v-1} = \text{id}_{v-1}$, which works for any ground ring (in particular for $(\mathbb{Z}[\mathbb{v}^{\pm 1}], \mathbb{v})$), we obtain the integral bases $B_{v-1, w-1}^{\text{int}}$ for morphism spaces. We reserve the following notation for these morphisms: $G_{v-1} = \text{id}_{v-1} \implies d(\pi), u(\pi)$.
- (b) For $G_{v-1} = \tilde{\varepsilon}_{v-1}$ and working over $(\mathbb{k}(\mathbb{v}), \mathbb{v})$ we get the Artin–Wedderburn basis. The associated morphisms will be denoted with tilde symbols: $G_{v-1} = \tilde{\varepsilon}_{v-1} \implies \tilde{d}(\pi), \tilde{u}(\pi)$.
- (c) For $G_{v-1} = E_{v-1}$, i.e. for the projectors constructed in Sect. 3B for non-semisimple situations, we will use capital letters: $G_{v-1} = E_{v-1} \implies D(\pi), U(\pi)$. These are specializations of morphisms that one gets for $G_{v-1} = \bar{\varepsilon}_{v-1}$ (with $\bar{\varepsilon}_{v-1}$ as in Definition 3.12 below), and we will use an overline in this situation: $G_{v-1} = \bar{\varepsilon}_{v-1} \implies \bar{d}(\pi), \bar{u}(\pi)$.

Definition 2.20 A family of morphisms $G_{v-1} \in \text{End}_{\mathbf{TL}^{k,q}}(v-1)$ for $v \in \mathbb{N}$ is **left-aligned** if

$$G_{v-1}(G_{w-1} \otimes \text{id}_{v-w}) = (G_{w-1} \otimes \text{id}_{v-w})G_{v-1} = G_{v-1} \quad \text{for } 1 \leq w \leq v,$$

and **right-aligned** if

$$G_{v-1}(\text{id}_{v-w} \otimes G_{w-1}) = (\text{id}_{v-w} \otimes G_{w-1})G_{v-1} = G_{v-1} \quad \text{for } 1 \leq w \leq v.$$

We draw morphisms from a left-aligned family as boxes with a bar at the left-hand side, and vice versa for right-aligned. Using this notation, the two conditions in Definition 2.20 read:

$$\begin{array}{|c|} \hline G_{w-1} \\ \hline \boxed{G_{v-1}} \\ \hline \end{array} = \boxed{G_{v-1}} = \begin{array}{|c|} \hline G_{v-1} \\ \hline \boxed{G_{w-1}} \\ \hline \end{array}, \quad \begin{array}{|c|} \hline G_{w-1} \\ \hline \boxed{G_{v-1}} \\ \hline \end{array} = \boxed{G_{v-1}} = \begin{array}{|c|} \hline G_{v-1} \\ \hline \boxed{G_{w-1}} \\ \hline \end{array}.$$

Note that left- and right-aligned families of morphisms are always idempotents, by the $v = w$ case of the defining relation.

Remark 2.21 The families of identity morphisms id_{v-1} are both left- and right-aligned and so are simple JW projectors by (2.2). However, in the mixed case the corresponding projectors E_{v-1} form a family that is only left-aligned, see Example 3.13. (Of course, there are also right-aligned versions $(E_{v-1})^{\leftrightarrow}$.) This asymmetry will play an important role within our setup. For example, in Definition 2.18 we presented a version of the light ladders strategy that favors left-aligned families of projectors, which will be significant when discussing fusion rules for morphisms.

3 Additive structure

In this section, we explain the additive structure of the category of tilting modules. Some of the results in this section are well-known, while others generalize results from [14] and [52]. We have also added a few new observations.

3A Character formulas

The Weyl and dual Weyl modules have classical Weyl characters, i.e. $\chi_{\Delta(v-1)} = \chi_{\nabla(v-1)} = [v]_v$, which we view as elements of $\mathbb{N}_0[v^{\pm 1}]$ where the coefficient of v^k is the dimension of the weight space of weight k . Each $\mathbb{T}(v-1)$ has a (dual) Weyl filtration and we denote the **(dual) Weyl multiplicities** by $(\mathbb{T}(v-1) : \Delta(w-1)) = (\mathbb{T}(v-1) : \nabla(w-1))$.

Remark 3.1 The purpose of the admissibility conditions on finite sets $S \subset \mathbb{N}_0$ from Definition 2.7 is so that for $v \in \mathbb{N}$ we have bijections

$$\begin{aligned} & \{S \subset \mathbb{N}_0 \mid S \text{ is down-admissible for } v\} \\ & \rightarrow \{w \in \mathbb{N} \mid (\mathbb{T}(v-1) : \Delta(w-1)) = 1\}, \quad S \mapsto v[S]; \\ & \{S \subset \mathbb{N}_0 \mid S \text{ is up-admissible for } v\} \\ & \rightarrow \{w \in \mathbb{N} \mid (\mathbb{T}(w-1) : \Delta(v-1)) = 1\}, \quad S \mapsto v(S). \end{aligned}$$

Moreover, each (dual) Weyl module has a filtration by simple modules, and we denote the corresponding **simple multiplicities** by $[\Delta(v-1) : \mathbb{L}(w-1)] = [\nabla(v-1) : \mathbb{L}(w-1)]$. These have a similar description as the Weyl multiplicities:

Definition 3.2 Let $v = [a_j, \dots, a_0]_{\mathfrak{p}, \ell}$. The **L-support** $\text{Lsupp}(v) \subset \mathbb{N}$ is defined as follows.

- (a) If $a_i \neq 0, \mathfrak{p} - 1$ for all $j > i > 0$, then for all $v[S] \in \nabla \text{supp}(v)$ we set

$$v[S]^{\mathbb{L}} = v[S] - 2 \sum_{S_i, i>0} \mathfrak{p}^{\min(S_i)} \quad \text{and} \quad \text{Lsupp}(v) = \{v[S]^{\mathbb{L}} \mid S\}.$$

Here i denotes the index for possible non-leading digits.

- (b) Otherwise, there is a recursive description of $\text{Lsupp}(v)$ as in [52, Section 5B], working with $\mathfrak{p} \vee \ell$ instead of \mathfrak{p} .

One can check that $\nabla\text{supp}(v)$ is always of order 2^{gen_v} , while the size of $\text{Lsupp}(v)$ can be different (for example when losp applies and digits are zero).

Proposition 3.3 *Let $v = [a_j, \dots, a_0]_{\mathfrak{p}, \ell}$.*

- (a) *We have*

$$\begin{aligned}
 (\mathbb{T}(v-1) : \Delta(w-1)) &= \begin{cases} 1 & \text{if } w \in \nabla\text{supp}(v); \\ 0 & \text{else,} \end{cases} \\
 [\Delta(v-1) : \text{L}(w-1)] &= \begin{cases} 1 & \text{if } w \in \text{Lsupp}(v); \\ 0 & \text{else.} \end{cases}
 \end{aligned}$$

Thus, the tilting characters are

$$\chi_{\mathbb{T}(v-1)} = \sum_{w \in \nabla\text{supp}(v)} \chi_{\Delta(w-1)} = \sum_{w \in \nabla\text{supp}(v)} [w]_v,$$

from which the simple characters can be obtained by inverting the identities

$$\chi_{\Delta(v-1)} = \sum_{w \in \text{Lsupp}(v)} \chi_{\text{L}(w-1)}.$$

- (b) *We have a version of (Brauer–Humphreys or BGG) reciprocity, i.e. if $a_i \neq 0, \mathfrak{p} - 1$ for all $j > i > 0$, then*

$$\begin{aligned}
 (\mathbb{T}(v-1) : \Delta(w-1)) &= [\Delta(v-1) : \text{L}(w^{\perp} - 1)] \\
 &= \begin{cases} 1 & \text{if } w = v[S] \text{ and } w^{\perp} = v[S]^{\perp}; \\ 0 & \text{if } w \neq v[S] \text{ and } w^{\perp} \neq v[S]^{\perp}. \end{cases}
 \end{aligned}$$

In particular, $(\mathbb{T}(v-1) : \Delta(w-1)) = [\Delta(v-1) : \text{L}(w-1)]$ for $v = [a, b]_{\mathfrak{p}, \ell}$.

Proof The Weyl multiplicities are known – see [21, Section 3.4] for the potentially first written account in the mixed case. The simple multiplicities can be obtained by direct calculation using the simple characters in (3.2) below. The reciprocity follows immediately from these. □

Two remarkable results describing the structure of objects in $\mathbf{fdMod}^{\mathbb{K}, \mathbb{q}}$ are Donkin’s (3.1) and Steinberg’s (3.2) **tensor product formulas**, which we recall in the following proposition. Both formulas describe modules of highest weight $v - 1$ in terms of tensor products of **Frobenius–Lusztig twisted** modules of lower weight, following the $\mathfrak{p}\ell$ -adic expansion of v . The i th Frobenius–Lusztig twist will be denoted by $(-)^{\mathfrak{p}^{(i)}}$. It acts as the Frobenius twist on digits a_i for $i > 0$ and as its quantum analog on the zeroth digit. Furthermore, we will accompany the two famous tensor product formulas with a third one. To this end, we note that we can naively apply $(-)^{a_i \mathfrak{p}^{(i)}}$ to weight spaces, although we lose the module structure for $a_i \neq 1$.

Proposition 3.4 Let $v = [a_j, \dots, a_0]_{\mathfrak{p}, \ell}$ and $v - 1 = [b_j, \dots, b_0]_{\mathfrak{p}, \ell}$.

(a) We have

$$\mathbb{T}(v - 1) \cong \mathbb{T}(a_j - 1)^{\mathfrak{p}^{(j)}} \otimes \bigotimes_{a_i} \mathbb{T}(a_i + \mathfrak{p} \vee \ell - 1)^{\mathfrak{p}^{(i)}}, \tag{3.1}$$

where the monoidal product runs over all non-leading digits of v . Thus, $\chi_{\mathbb{T}(v-1)} = [a_j]_{\mathfrak{v}\mathfrak{p}^{(j)}} \prod_{a_i} ([a_i + \mathfrak{p} \vee \ell]_{\mathfrak{v}\mathfrak{p}^{(i)}} + [-a_i + \mathfrak{p} \vee \ell]_{\mathfrak{v}\mathfrak{p}^{(i)}})$.

(b) We have

$$\mathbb{L}(v - 1) \cong \bigotimes_{b_i \neq 0} \mathbb{L}(b_i)^{\mathfrak{p}^{(i)}}, \tag{3.2}$$

where the monoidal product runs over all (non-zero) digits of $v - 1$. Thus, $\chi_{\mathbb{L}(v-1)} = \prod_{b_i \neq 0} [b_i + 1]_{\mathfrak{v}\mathfrak{p}^{(i)}}$.

(c) We have an isomorphism of \mathbb{Z} -graded vector spaces

$$\mathbb{T}(v - 1) \cong \mathbb{T}(\mathfrak{m}_{\mathfrak{v}}^{\infty} - 1) \otimes \bigotimes_{a_i \neq 0} \mathbb{T}(1)^{(a_i \mathfrak{p}^{(i)})}, \tag{3.3}$$

where the monoidal product runs over all non-zero and non-leading digits of v . Thus, $\chi_{\mathbb{T}(v-1)} = [\mathfrak{m}_{\mathfrak{v}}^{\infty}]_{\mathfrak{v}} \prod_{a_i \neq 0} [2]_{\mathfrak{v} a_i \mathfrak{p}^{(i)}}$.

(d) (3.3) can be realized as an isomorphism of SL_2 -modules if all non-zero digits a_i are equal to 1. In this case, $\mathbb{T}(v - 1)$ is a tensor product of simple modules.

Proof For the tensor product formulas (3.1) and (3.2), see [1, Proposition 5.2] (to be precise, the above is [53, Proposition 4.7] adjusted to mixed characteristic) and [5, Theorem 1.10] for the mixed versions. We will give a diagrammatic proof of the (apparently new) character formula in (c) in Proposition 3.20 below. For the final statement, by (c), it suffices to observe that $\bigotimes_{a_i \neq 0} \mathbb{T}(1)^{(a_i \mathfrak{p}^{(i)})}$ is simple by (3.2), which implies that the right-hand side of (3.3) is tilting by the mixed characteristic analog of [13, Lemma 3.3]. \square

Example 3.5 Recall from Example 2.6 that $\nabla \mathrm{supp}(68 = [3, 1, 2]_{7,3}) = \{68, 64, 62, 58\}$.

- (a) For 68 we get $\chi_{\mathbb{T}(68-1)} = [68]_{\mathfrak{v}} + [64]_{\mathfrak{v}} + [62]_{\mathfrak{v}} + [58]_{\mathfrak{v}}$, as well as $\chi_{\mathbb{T}(68-1)} = [3]_{\mathfrak{v}2^1} ([8]_{\mathfrak{v}3} + [6]_{\mathfrak{v}3}) ([5]_{\mathfrak{v}} + [1]_{\mathfrak{v}})$ and $\chi_{\mathbb{T}(68-1)} = [63]_{\mathfrak{v}} [2]_{\mathfrak{v}3} [2]_{\mathfrak{v}2}$.
- (b) From $\nabla \mathrm{supp}(68)$ we obtain $\mathbb{L} \mathrm{supp}(68) = \{68, 64, 58, 48\}$, since we need to adjust $62 = [3, -1, 2]_{7,3}$ to $62 - 2 \cdot 7 = 48$. We thus get $\chi_{\mathbb{L}(68-1)} = [4]_{\mathfrak{v}2^1} [2]_{\mathfrak{v}3} [2]_{\mathfrak{v}}$, using $67 = [3, 1, 1]_{7,3}$, and $\chi_{\mathbb{L}(68-1)} = [68]_{\mathfrak{v}} - \chi_{\mathbb{L}(64-1)} - \chi_{\mathbb{L}(58-1)} - \chi_{\mathbb{L}(48-1)}$.

Note that part (d) of Proposition 3.4 implies a remarkable appearance of losp :

Corollary 3.6 All indecomposable tilting modules are tensor products of simple modules in characteristic $\mathfrak{p} = 2$.

The tilting modules $\mathbb{T}(v - 1)$ for $v \in \mathrm{Eve}$ will also be called *eves*. By Proposition 3.3 these are the only simple tilting modules, i.e. $\mathbb{T}(v - 1) \cong \mathbb{L}(w - 1)$ if and only if $v = w \in \mathrm{Eve}$. The *prime eves* are those where $v = \mathfrak{p}^{(i)}$, and they play a special role in the theory of tilting modules.

Definition 3.9 Suppose that $S = \{s_k > \dots > s_1 > s_0\}$ is down-admissible for v and $S' = \{s'_j > \dots > s'_1 > s'_0\}$ is up-admissible for v . Then we define **simple trapezes and loops**

$$\begin{aligned} \boxed{s} &:= \tilde{e}_{v[S]-1} \tilde{d}_S := \tilde{e}_{v[S]-1} \tilde{d}_{s_0} \cdots \tilde{d}_{s_k} \text{id}_{v-1}, & \boxed{s'} &:= \tilde{u}_{S'} \tilde{e}_{v-1} := \text{id}_{v(S')-1} \tilde{u}_{s'_j} \cdots \tilde{u}_{s'_0} \tilde{e}_{v-1}, \\ \boxed{s} & & \boxed{s} &:= \tilde{L}_{v-1}^S := \tilde{u}_S \tilde{e}_{v[S]-1} \tilde{d}_S, \end{aligned}$$

which we also define for the other two variations from Convention 2.19, with the appropriate adjustment of notation.

Note that in all cases loops carry an idempotent $G_{v[S]-1}$ in the center and down and up morphisms carry this idempotent on their thin end.

Remark 3.10 Using Convention 2.19, we can give an alternative description of the simple trapezes. If S is down-admissible for $v = [a_j, \dots, a_0]_{p,\ell}$, then we define a sign sequence

$$\pi_S(v) = \underbrace{\dots + \dots}_{a_j p^{(j)-1}} \underbrace{+ \dots + \dots}_{a_{j-1} p^{(j-1)}} \dots \underbrace{\epsilon_0 \dots \epsilon_0}_{a_0 p^{(0)}} \in \{+, -\}^{v-1}, \quad \epsilon_i = \begin{cases} - & \text{if } i \in S; \\ + & \text{if } i \notin S. \end{cases}$$

That is, $\pi_S(v)$ is a concatenation of signs with multiplicity given by the digits of v . We get

$$\boxed{s} = \tilde{d}(\pi_S(v)), \quad \boxed{s} = \tilde{u}(\pi_S(v)).$$

Note the subtle, but important difference that $\tilde{d}(\pi_S(v))$ includes an idempotent $\tilde{e}_{v[S]-1}$ on the left, while \tilde{d}_S does not. As a consequence, composites of morphisms of type $\tilde{d}(\pi)$ are automatically zero, while the morphisms of type \tilde{d}_S can be composed in interesting ways. Remarkably, this distinction disappears when considering analogs of such morphisms built from mixed projectors, see Proposition 3.19.

Example 3.11 For $v = [a, b, c]_{p,\ell}$ we have:

$$\boxed{\emptyset} = \text{I}, \quad \boxed{\{0\}} = \text{I} \cup \text{hook}_c, \quad \boxed{\{1\}} = \text{I} \cup \text{hook}_{bp^{(1)}c}, \quad \boxed{\{1, 0\}} = \text{I} \cup \text{hook}_{bp^{(1)}c} \cup \text{hook}_c.$$

Recall that we use $v = [a_j, \dots, a_0]_{p,\ell}$ and write $p^{(i)} = p^{i-1} \ell$ for $i > 0$. For $v \in \mathbb{N}$ and $s \in \mathbb{N}_0$ let $a_{v,s}$ denote the youngest ancestor of v whose s th digit is zero. (When $s = -1$, we define $a_{v,-1} = v$.) For each down-admissible S for v we let

$$\lambda_{v,S} = \prod_{s \in S} (-1)^{a_s p^{(s-1)}} \frac{[a_{v,s-1}[S]]_v}{[a_{v,s}[S]]_v} \in \mathbb{k}(v).$$

Definition 3.12 generalizes [52, Definition 2.22] and Lemma 3.16 below generalizes [14, Proposition 3.3] to the mixed case.

Definition 3.12 For $v \in \mathbb{N}$ the **semisimple pℓJW projector** $\bar{e}_{v-1} \in \text{End}_{\mathbb{T}\mathbb{L}^{\mathbb{k}(v),v}}(v-1)$ is defined to be

$$\boxed{v-1} := \bar{e}_{v-1} := \sum_{v[S] \in \nabla \text{supp}(v)} \lambda_{v,S} \tilde{L}_{v-1}^S = \sum_{v[S] \in \nabla \text{supp}(v)} \lambda_{v,S} \cdot \boxed{s} \quad (3.5)$$

The choice of name for \bar{e}_{v-1} is because the associated tilting module is a direct sum of simple tilting modules, i.e.

$$\bar{\mathbb{T}}(v-1) \cong \bigoplus_{v[S] \in \nabla \text{supp}(v)} \mathbb{T}(v[S]-1) \cong \bigoplus_{v[S] \in \nabla \text{supp}(v)} \Delta(v[S]-1). \quad (3.6)$$

Note that $m\text{char}(\mathbb{k}(v), v) = (\mathfrak{p}, \infty)$, so (3.5) is well-defined. In Theorem 3.18 we will see that the semisimple pℓJW projectors can be base changed to $(\mathbb{k}, \mathfrak{q})$ with $m\text{char}(\mathbb{k}, \mathfrak{q}) = (\mathfrak{p}, \ell)$.

Example 3.13 By construction, $(\bar{e}_{v-1})^\dagger = \bar{e}_{v-1}$. However, $(\bar{e}_{v-1})^{\leftrightarrow} \neq \bar{e}_{v-1}$ in general (we will address this in Lemma 3.17), as can be seen by the following example in characteristic $\mathfrak{p} = 3$:

$$\boxed{3} = \left| \right| \left| + \frac{1}{2} \cdot \frown \right| \neq \left| \right| \left| + \frac{1}{2} \cdot \right| \frown = \boxed{3}$$

Remark 3.14 Further concrete examples for projector expansions (3.5) can be found in [52, Examples 2.20 and 2.23]. Relative to the treatment there, we allow the following two generalizations. First, the p-adic expansions should be replaced by the pℓ-adic expansions, and \mathfrak{p}^k therein by $\mathfrak{p}^{(k)}$. Second, all coefficients use quantum numbers instead of integers.

The following is reproduced from [52, Section 3B], adjusting the scalars.

Lemma 3.15

(a) Suppose that S and S' are down-admissible for v . Then we have

$$\tilde{e}_{v[S]-1} \tilde{d}_S \tilde{u}_{S'} \tilde{e}_{v[S']-1} = \begin{array}{c} \boxed{v[S]-1} \\ \hline S \\ \hline S' \\ \hline \boxed{v[S']-1} \end{array} = \delta_{S,S'} \lambda_{v,S'}^{-1} \cdot \boxed{v[S']-1} = \delta_{S,S'} \lambda_{v,S'}^{-1} \tilde{e}_{v[S']-1}.$$

Here $\delta_{S,S'}$ denotes the Kronecker delta, so $\delta_{S,S'} = 1$ if $S = S'$ and zero otherwise.

(b) Suppose S is down-admissible for v , and $S' = \{s, \dots, s' - 1\}$ is a minimal down-admissible stretch for v . Then we have

$$\boxed{s} \overset{s'}{\curvearrowright} = \begin{cases} (-1)^{a_S \mathfrak{p}^{(s)}} \frac{[a_{v,S}[S]]_v}{[a_{v,s-1}[S]]_v} \cdot \boxed{s \setminus S'} & \text{if } s \in S, s' \notin S; \\ \boxed{s \cup S'} & \text{if } s \notin S, s' \in S; \\ 0 & \text{otherwise.} \end{cases}$$

(c) Suppose that $S' = \{s, \dots, s' - 1\}$ is the smallest minimal down-admissible stretch for v and let S be down-admissible for $a_{v,s} = m_v$. Then we have

$$\begin{aligned} \left[\begin{array}{c} \curvearrowright^{s'} \\ s \end{array} \right] &= \begin{cases} \left[\begin{array}{c} s \\ s \cup s' \\ s \end{array} \right] = \tilde{u}_s \tilde{e}_{v[S \cup S']-1} \tilde{d}_{S \cup S'} & \text{if } s' \notin S; \\ \left[\begin{array}{c} s \cup s' \\ s \end{array} \right] = \tilde{u}_{S \cup S'} \tilde{e}_{v[S]-1} \tilde{d}_S & \text{if } s' \in S. \end{cases} \end{aligned}$$

Proof Word-for-word as in [52, Lemmas 3.7, 3.8 and 3.9]. □

Lemma 3.16 *The semisimple pℓJW projectors can be expanded as*

$$\boxed{v-1} = \sum_{m_v[S] \in \mathbb{V}\text{supp}(m_v)} \lambda_{m_v, S} \left(\begin{array}{c} a_s p^{(s)} \\ s \\ v[S]-1 \\ s \\ a_s p^{(s)} \end{array} + (-1)^{a_s p^{(s)}} \frac{[v[S][s]]_v}{[m_v[S]]_v} \cdot \begin{array}{c} a_s p^{(s)} \\ s \\ v[S][s]-1 \\ s \\ a_s p^{(s)} \end{array} \right), \tag{3.7}$$

where a_s is the first non-zero digit of v . As a consequence, for any ancestor m_v^j of v , the projector \bar{e}_{v-1} absorbs $\bar{e}_{m_v^j-1}$ when left-aligned as in Definition 2.20.

Proof With the properties listed in Lemma 3.15, the proof follows verbatim as in [52, Lemma 2.24] and [14, Proposition 3.3]. □

The next statement of this section enables us to relate the left and right versions of the pℓJW projectors. Let $v = [a_j, \dots, a_0]_{p,\ell}$, as usual, and let S_{v-1} denote the symmetric group on $v - 1$ letters. Assuming the existence of square roots, we can use (2.1) to define $g := g(v - 1)$ to be the positive braid lift of the longest element of S_{v-1} (the positive half twist, a Garside element), and $r := r(a_j, \dots, a_0)$ to be the positive braid lift of a shortest coset representative for $S_{v-1} / (S_{a_j p^{(j)}-1} \times S_{a_{j-1} p^{(j-1)}} \times \dots \times S_{a_0})$.

Lemma 3.17 *Assume that v has a square root, i.e. we are working in $\mathbb{k}(v^{1/2})$, we have*

$$\begin{array}{c} g^{-1} \\ \boxed{v-1} \\ g \end{array} = \begin{array}{c} r^{-1} \\ \boxed{v-1} \\ r \end{array} = \boxed{v-1}.$$

Proof A straightforward consequence of two facts: conjugation with the half twist g acts as the involution $(-)^{\leftrightarrow}$ on the integral basis (and thus on everything) and JW projectors absorb crossings up to scalars $v^{\pm 1/2}$. □

The semisimple pℓJW projectors \bar{e}_{v-1} are defined over $\mathbb{F}_p(v)$, but the algorithm to construct them generates coefficients (with respect to the integral basis) that we can view as elements of $\mathbb{Q}(v)$, and we will do this below. We write $sp_{p,\ell}(-)$ for the specialization of morphisms to \mathbb{k} , if it exists.

Theorem 3.18 *We have that*

$$E_{v-1} := \boxed{v-1} := sp_{p,\ell}(\bar{e}_{v-1}) \in \text{End}_{\mathbf{T}\mathbb{k},\mathbb{q}}(v - 1)$$

is a well-defined idempotent whose coefficients are elements of $\mathbb{F}_p(\mathbb{q})$. Moreover, $\mathcal{D}^{\mathbb{k},\mathbb{q}}(E_{v-1}) = \text{id}_{\mathbf{T}(v-1)}$, and $\mathbf{Tilt}^{\mathbb{k},\mathbb{q}}$ is Krull–Schmidt.

In particular, under the equivalence induced by $\mathcal{D}^{\mathbb{K}, \mathbb{q}}$ (see Proposition 2.13) the image of the idempotent E_{v-1} is mapped to $T(v-1)$. We call the E_{v-1} **mixed JW projectors**.

Proof We start by explaining the lifting strategy. Assume that we are in the mixed cases (the other cases are easier and omitted). Let $\bar{\delta} = -[2]_{\mathbb{q}} = -\mathbb{q} - \mathbb{q}^{-1}$, the circle value. Let $p(\delta)$ be the minimal polynomial in $\mathbb{F}_{\mathbb{p}}[\delta]$ satisfied by δ . We lift $p(\delta)$ to $\mathbb{Z}[\delta]$, and denote this lift by the same symbol. Let L denote the localization of $\mathbb{Z}[\delta]$ at the maximal ideal $m = (p(\delta), \mathfrak{p})$. (That m is maximal in $\mathbb{Z}[\delta]$ follows since (\mathfrak{p}) is maximal in \mathbb{Z} and $p(\delta)$ is irreducible in $\mathbb{F}_{\mathbb{p}}[\delta]$.) Then we have that mL is a maximal ideal in L , and the completion of L is a complete Noetherian local domain \mathbb{L} , and its field of fractions F is a characteristic zero field. The residue field is contained in \mathbb{k} . The triple $(F, \mathbb{L}, \mathbb{k})$ satisfies $\mathbb{k} \leftarrow \mathbb{L}/m\mathbb{L} \leftarrow \mathbb{L} \hookrightarrow F$, and which allows comparison of idempotents similarly to the \mathfrak{p} -adic case and the classical theory of idempotent lifts. In fact, by construction of the TL category and our idempotents, we will work in the subfield $\mathbb{Q}(\delta)$ of F , which we immediately extend to $\mathbb{Q}(v)$ along $\delta = -v - v^{-1}$.

As above we assume that we are in the mixed cases (the other cases are easier and omitted). Note that we could have defined \bar{e}_{v-1} directly over $\mathbb{Q}(\delta)$, and then the idempotents over $\mathbb{Q}(v)$ and F arise by extending scalars. These settings are all semisimple, so the projector combinatorics over $\mathbb{Q}(\delta)$ and F is the same as in $\mathbb{Q}(v)$. We will also implicitly extend scalars from the residue field to \mathbb{k} , and we will do both in the proof to be consistent with the above lifting strategy.

Note that \bar{e}_{v-1} has the correct character, namely $\chi_{\text{Im}\bar{e}_{v-1}} = \chi_{T(v-1)}$. By Lemma 3.16, $\bar{e}_{v-1} \in \text{End}_{\mathbf{Tilt}^F}(T(1)^{\otimes(v-1)})$ is an idempotent and it absorbs the tensor product $\bar{e}_{m_v-1} \otimes \text{id}_{v-m_v}$ of the idempotent for the mother with extra strands. Now we claim there is exactly one idempotent in $\text{End}_{\mathbf{Tilt}^F}(T(1)^{\otimes(v-1)})$ with this property and the correct character. To see that this is true let us denote by $\bar{T}(m_v-1) \in \mathbf{Tilt}^F$ the direct sum of Weyl modules with the correct character. Now $\bar{T}(m_v-1) \otimes T(1)^{\otimes(v-m_v)} \in \mathbf{Tilt}^F$ contains each Weyl factor of $T(v-1)$ exactly once, see Lemma 4.2, so there is exactly one idempotent in $\text{End}_{\mathbf{Tilt}^F}(T(1)^{\otimes(v-1)})$ with the correct character and absorption property, and the claim follows.

Now let \mathbb{L} be the completion as described above. There are specialization maps and functors

$$\begin{array}{ccc}
 \mathbb{L} & & \text{End}_{\mathbf{Tilt}^{\mathbb{L}}}(T(1)^{\otimes(v-1)}) \\
 \swarrow & & \swarrow \mathcal{F}_{\mathbb{k}} \quad \searrow \mathcal{F}_F \\
 \mathbb{k} & & \text{End}_{\mathbf{Tilt}^{\mathbb{k}}}(T(1)^{\otimes(v-1)}) \quad \text{End}_{\mathbf{Tilt}^F}(T(1)^{\otimes(v-1)})
 \end{array}$$

Next, we show that \bar{e}_{v-1} can be lifted to $\text{End}_{\mathbf{Tilt}^{\mathbb{L}}}(T(1)^{\otimes(v-1)})$ and its specialization to \mathbb{k} projects to $T(v-1)$. To this end, we use induction over the ancestry of v , with the case of $v \in \text{Eve}$ being clear since $T(v-1) \cong \Delta(v-1)$ in these cases. So let \bar{e}_{m_v} be liftable and let $l_F(\bar{e}_{m_v})$ be its lift. Induction implies

$$\mathcal{F}_{\mathbb{k}}(l_F(\bar{e}_{m_v})) = \text{id}_{T(m_v-1)}.$$

We further know that $\mathbb{T}(v-1)$ is a direct summand of $\mathbb{T}(m_v-1) \otimes \mathbb{T}(1)^{\otimes(v-m_v)} \in \mathbf{Tilt}^k$, so, there is some projector $E' \in \text{End}_{\mathbf{Tilt}^k}(\mathbb{T}(1)^{\otimes(v-1)})$ projecting to this summand, which absorbs the idempotent corresponding to the mother tensor product with strands. By idempotent lifting, cf. [40, Theorem 21.31], we can pull E' back to \mathbf{Tilt}^L giving us another projector $l_k(E')$. Pushing this forward gives a projector $\mathcal{F}_F(l_k(E'))$ in the semisimple case with the correct character and absorption property. However, as we have seen, such a projector is unique and thus, has to be \bar{e}_{v-1} . Hence, we get that $l_k(E')$ is a lift of \bar{e}_{v-1} .

Thus, we can specialize \bar{e}_{v-1} to $E_{v-1} = E'$, and the claims about the coefficients and $E_{v-1} = \text{id}_{\mathbb{T}(v-1)}$ follow by construction of \bar{e}_{v-1} . The Krull–Schmidt property then follows inductively as the above constructs all highest weight projectors, and the claim about the coefficients being in $\mathbb{F}_p(\mathfrak{q})$ is evident by the construction of the projectors. \square

Diagrammatically, the three types of projectors are distinguished as follows:

$$\tilde{e}_{v-1} = \boxed{v-1}, \quad \bar{e}_{v-1} = \begin{cases} \boxed{v-1} \\ \boxed{v-1} \end{cases} \text{ for } v \in \text{Eve}, \quad E_{v-1} = \begin{cases} \boxed{v-1} \\ \boxed{v-1} \end{cases} \text{ for } v \in \text{Eve}. \tag{3.8}$$

The middle and the rightmost projector have the same character, but \bar{e}_{v-1} corresponds to a direct sum of simple tilting modules in the semisimple setting, cf. (3.6), and E_{v-1} corresponds to the indecomposable $\mathbb{T}(v-1)$. We will use the middle projectors to deduce properties of the right projectors. Moreover, as illustrated in (3.8), we also use white boxes for eves to indicate that these satisfy the same diagrammatic properties as the simple JW projectors.

We stress again that the non-semisimple projectors do not have a left-right-symmetry, and their properties do not have such a symmetry either. For the remainder of the paper, each cup and cap in the illustrations is a parallel bundle of cups and caps, depending on S or respectively S' , or a plain number. (We also omit to illustrate these if no confusion can arise.)

We have the following generalizations of Proposition 2.17, called *classical absorption*, *non-classical absorption*, *shortening* and *partial trace*.

Proposition 3.19

- (a) The projectors \bar{e}_{v-1} form a left-aligned family in the sense of Definition 2.20.
- (b) Let S be a down-admissible stretch for v . Then we have

where the top boxes are labeled $v[S]-1$, and the small box is labeled by $a_{v,S}-1$ for $a_{v,S}$ being the youngest ancestor of v for which all digits indexed by elements of S are zero.

(c) For $v = [a_j, \dots, a_0]_{p,\ell} \notin \text{Eve}$ let $w = [a_k, \dots, a_0]_{p,\ell}$ for some $k < j$. Then we have

$$\boxed{v-1}^w = (-1)^w \prod_{\substack{a_i \neq 0 \\ i < j}} [2]_{q^{a_i p^{(i)}}} \cdot \boxed{m_v^x - 1},$$

where the product runs over all non-zero digits of w , and $m_v^x = v - w$ is the corresponding ancestor. (Note that for $q^\ell = \pm 1$ and $i > 0$ we have $(-1)^{a_i p^{(i)}} [2]_{q^{a_i p^{(i)}}} = (-q)^{a_i p^{(i)}} 2$.) For $v = [a_j, 0, \dots, 0]_{p,\ell} \in \text{Eve}$, $v \geq \ell$ and $k \leq v$ such that $v - k = [b_i, 0, \dots, 0]_{p,\ell} \in \text{Eve}$ with $i < j$ we additionally have

$$\boxed{v-1}^k = 0 = k \boxed{v-1} \in \text{End}_{\mathbf{TL}^{k,q}}(v - k).$$

(A special case of this is the trace down to the empty diagram.)

Proof All except the final statement can be shown as in [52, Propositions 3.11, 3.13 and 3.14]. The final statement follows by using (2.4) and observing that the projector after taking partial trace satisfies $\tilde{e}_{v-k} = \bar{e}_{v-k}$ and the zero obtained by (2.4) annihilates it. □

The projectors \bar{e}_{v-1} typically do not form a right-aligned family. Example 3.13 gives a counterexample to right-aligned absorption of $\text{id}_1 \otimes \bar{e}_2$ into \bar{e}_3 .

Recall the definition of the **categorical dimension** $\text{dim}_{\mathcal{C}}$ of objects in a pivotal category \mathcal{C} , see e.g. [28, Definition 4.7.1] (the categorical dimension is the trace of the identity therein).

Proposition 3.20 For $v = [a_j, \dots, a_0]_{p,\ell}$ we have

$$\text{dim}_{\mathbf{TL}^{k,q}}(\mathbb{T}(v - 1)) = (-1)^{v-1} \sum_{S \in \nabla \text{supp}(v)} [v[S]]_q = (-1)^{v-1} [m_v^\infty]_q \prod_{a_i \neq 0} [2]_{q^{a_i p^{(i)}}},$$

where the product runs over all non-zero and non-leading digits of v .

Proof The categorical dimension in $\mathbf{TL}^{k,q}$ is given by closing pictures in the usual way, and for the first equality we calculate

$$\begin{aligned} \boxed{v-1} &= \sum_{S \in \nabla \text{supp}(v)} \lambda_{v,S} \cdot \begin{array}{c} \text{---} S \text{---} \\ \text{---} v[S]-1 \text{---} \\ \text{---} S \text{---} \end{array} = \sum_{S \in \nabla \text{supp}(v)} \lambda_{v,S} \cdot \begin{array}{c} \text{---} S \text{---} \\ \text{---} S \text{---} \\ \text{---} v[S]-1 \text{---} \end{array} \\ &= \sum_{S \in \nabla \text{supp}(v)} \boxed{v[S]-1}. \end{aligned}$$

Observing that $(-1)^{v[S]-1} = (-1)^{v-1}$ for all $S \in \nabla\text{supp}(v)$, the first equality follows by classical theory, see e.g. [38, Section 9.5]. The second equation follows then from Proposition 3.19. \square

Note that the categorical dimension of $\mathbb{T}(v - 1)$ is an element of the underlying field, but interpreted in $\mathbb{N}_0[v^{\pm 1}]$ we obtain the character $\chi_{\mathbb{T}(v-1)}$.

3C Tilting modules as an additive category

Let us define a (locally unital) \mathbb{k} -algebra via

$$\mathbb{Z}^{k,q} = \bigoplus_{v,w \in \mathbb{N}} \text{Hom}_{\mathbf{Tilt}^{k,q}}(\mathbb{T}(v - 1), \mathbb{T}(w - 1)).$$

Let $\mathbf{Proj}\text{-}\mathbb{Z}^{k,q}$ denote the category of finitely generated, projective (right) $\mathbb{Z}^{k,q}$ -modules. By construction we obtain, as instance of *Ringel duality* (semi-infinite in the sense of [15]), that

$$\mathcal{F} : \mathbf{Tilt}^{k,q} \rightarrow \mathbf{Proj}\text{-}\mathbb{Z}^{k,q}, \mathbb{T} \mapsto \bigoplus_{v \in \mathbb{N}} \text{Hom}_{\mathbf{Tilt}^{k,q}}(\mathbb{T}(v - 1), \mathbb{T})$$

is an equivalence of additive, \mathbb{k} -linear categories, sending indecomposable tiltings to indecomposable projectives. Let us describe $\mathbb{Z}^{k,q}$ explicitly.

By construction, morphisms in $\text{Hom}_{\mathbf{Tilt}^{k,q}}(\mathbb{T}(v - 1), \mathbb{T}(w - 1))$ are given by flanking TL morphisms with E_{v-1} from the bottom and with E_{w-1} from the top, and the primitive idempotents (which are local units) are the E_{v-1} for $v \in \mathbb{N}$. Other morphisms, called *mixed trapezes and loops*, are diagrammatically given by the analog of Definition 3.9: if S and S' are down- and up-admissible for v , respectively, and assuming that S and S' are minimal admissible stretches of consecutive integers, then we define

$$\begin{array}{c}
 \text{E}_S \text{ (trapeze)} : = E_{v[S]-1} D_S E_{v-1} = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \\
 \text{E}_{S'} \text{ (trapeze)} : = E_{v(S)-1} U_{S'} E_{v-1} = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \\
 \end{array}
 \tag{3.9}$$

These are the generators of $\mathbb{Z}^{k,q}$, and (up to loop) the respective minimal stretches are singleton sets $S = \{i\}$, reflecting along the i th digit. The corresponding S -labeled cups and caps in (3.9) consist of $a_i \rho^{(i)}$ parallel strands. We also define the loops $L_{v-1}^S := E_{v-1} U_S E_{v[S]-1} D_S E_{v-1}$. Finally, note that these morphisms can be defined more generally for any down- and up-admissible stretches, but then their diagrammatic incarnations can involve multiple stretch-labeled cups and caps.

To describe the relations between expressions in the generating morphisms, we will use the same scalars (depending on the digits) as in [52, Section 3A], namely

$$\begin{aligned}
 f(a) &= \begin{cases} (-q)^{(a+1)\ell} \cdot \frac{-2}{a} & \text{if } 1 \leq a \leq p-2; \\ 0 & \text{if } a = 0 \text{ or } a = p-1, \end{cases} \\
 g(a) &= \begin{cases} (-q)^\ell \left(\frac{a+1}{a}\right) & \text{if } 1 \leq a \leq p-1; \\ (-q)^\ell 2 & \text{if } a = 0, \end{cases} \tag{3.10}
 \end{aligned}$$

$$\begin{aligned}
 f_S E_{v-1} &= f(a_{\max(S)+1}) E_{v-1}, & g_S E_{v-1} &= g(a_{\max(S)+1}) E_{v-1}, \\
 h_S E_{v-1} &= g(a_{\max(S)+1} - 1) E_{v-1}.
 \end{aligned}$$

In fact, as we will see later, these scalars can be seen as (inverses of higher order) **local intersection forms** in the language of [25].

We obtain the mixed characteristic version of [52, Theorem 3.2]:

Theorem 3.21 *The algebra $Z^{k,q}$ is generated by E_{v-1} for $v \in \mathbb{N}$, and elements $D_S E_{v-1}$ and $U_{S'} E_{v-1}$, where S and S' denote minimal down- and up-admissible stretches for v , respectively. These generators are subject to the following complete set of relations. (As before, we omit idempotents from the notation if they can be recovered from the given data.)*

(1) **Idempotents.**

$$\begin{aligned}
 E_{v-1} E_{w-1} &= \delta_{v,w} E_{v-1}, \\
 E_{v[S]-1} D_S E_{v-1} &= E_{v[S]-1} D_S = D_S E_{v-1}, \\
 E_{v(S')-1} U_{S'} E_{v-1} &= E_{v(S')-1} U_{S'} = U_{S'} E_{v-1}.
 \end{aligned}$$

(2) **Containment.** *If $S' \subset S$, then we have*

$$D_S D_S E_{v-1} = 0, \quad U_S U_{S'} E_{v-1} = 0.$$

(3) **Far-commutativity.** *If $d(S, S') > 1$, then*

$$\begin{aligned}
 D_S D_{S'} E_{v-1} &= D_{S'} D_S E_{v-1}, & D_S U_{S'} E_{v-1} &= U_{S'} D_S E_{v-1}, \\
 U_S U_{S'} E_{v-1} &= U_{S'} U_S E_{v-1}.
 \end{aligned}$$

(4) **Adjacency relations.** *If $d(S, S') = 1$ and $S' > S$, then*

$$\begin{aligned}
 D_{S'} U_S E_{v-1} &= D_{S \cup S'} E_{v-1}, & D_S U_{S'} E_{v-1} &= U_{S' \cup S} E_{v-1}, \\
 D_{S'} D_S E_{v-1} &= U_S D_{S'} h_S E_{v-1}, & U_S U_{S'} E_{v-1} &= h_S U_{S'} D_S E_{v-1}.
 \end{aligned}$$

(5) **Overlap relations.** *If $S' \geq S$ with $S' \cap S = \{s\}$ and $S' \not\subset S$, then we have*

$$D_{S'} D_S E_{v-1} = U_{\{s\}} D_S D_{S' \setminus \{s\}} E_{v-1}, \quad U_S U_{S'} E_{v-1} = U_{S' \setminus \{s\}} U_S D_{\{s\}} E_{v-1}.$$

(6) **Zigzag.**

$$D_S U_S E_{v-1} = U_{\bar{S}} D_{\bar{S}} \mathfrak{g}_S E_{v-1} + U_T U_{\bar{S}} D_{\bar{S}} D_T \mathfrak{f}_S E_{v-1}.$$

Here, if the down-admissible hull \bar{S} , or the smallest minimal down-admissible stretch T with $T > \bar{S}$ does not exist, then the involved symbols are zero by definition.

The elements of the form

$$E_{w-1} U_{S'_{i_1}} \cdots U_{S'_{i_0}} D_{S_{i_0}} \cdots D_{S_{i_k}} E_{v-1}, \tag{Basis}$$

with $S'_{i_1} > \cdots > S'_{i_0}$, and $S_{i_0} < \cdots < S_{i_k}$, form a basis for $E_{w-1} Z^{k, \mathfrak{q}} E_{v-1}$.

Any word $E_{w-1} X E_{v-1}$ in the generators of $Z^{k, \mathfrak{q}}$ can be rewritten as a linear combination of basis elements from (Basis) using only the above relations.

(Complete)

Proof This is analogous to the ten page proof of the characteristic p case in [52]. The proof given therein splits into the following steps which one can copy up to some adjustments detailed below.

- (a) The proof starts with [52, Lemma 3.6] which proves (Complete). The arguments given therein work *verbatim*.
- (b) The proof continues with [52, Section 3B] listing properties of the p JW projectors. The $p\ell$ -versions of these properties are given above.
- (c) Then (Basis) is proven, see [52, Section 3C]. The arguments given therein work again *verbatim*.
- (d) Then some of the relations are proven in the following order. Relation (1) is clear, relation (2) is proven in [52, Lemma 3.22], relation (3) is proven in [52, Lemma 3.23], and the first pair of the relations (4) is proven in [52, Lemma 3.24]. All of these proofs go through *verbatim*.
- (e) The gist is the simultaneous proof of the first pair of the relations (4), relation (5), relation (6), and the identification of $\text{End}_{\mathbf{T}\mathbf{L}^{k, \mathfrak{q}}}(E_{v-1})$, which are [52, Lemmas 3.25–3.28]. The arguments given in [52] carry over to the mixed case, but some subtle changes need to be made as detailed below.

As mentioned above, the proofs given in [52] need some adjustment due to e.g. the appearance of signs in \mathfrak{f} and \mathfrak{g} from (3.10). We record the necessary modification to the numerical arguments used in [52].

First of all, the scalars $\lambda_{v, S}$ and partial trace formulas for the various JW projectors now involve fractions of quantum numbers. Moreover, a few signs that have started their lives as $-1 = (-1)^{p^i}$ now have to be replaced by $(-1)^{p^{(i)}} = (-1)^\ell$ when $i > 0$. This concerns the sign of the fraction of quantum numbers in [52, (4-2)] (this replacement leads to the desired interpretation in terms of \mathfrak{g}), the sign in q from [52, (4-8)] should be $(-1)^{p^{(s)}}$, which balances against the sign of $\lambda_{w, R}$ in the following display. Further, in the *Proof, which caveat* for [52, Lemma 4.9], the signs $(-1)^{w-u}$ and $(-1)^{w+1-u}$

are to be replaced with $(-q)^{w-u}$ and $(-q)^{w+p^{(i)}-u}$, which is again compatible with \mathfrak{f} and \mathfrak{g} as desired. The vanishing of q' from [52, (4-4)] follows using a similar argument using quantum numbers. Finally, the zigzag relations are established by an inductive argument based on the case of generation 2, which is proved exactly as outlined in [52, Lemma 4.8].

□

Note that the non-idempotent generators of $\mathbb{Z}^{k,q}$ are given by down and up morphisms for minimal stretches, a.k.a. singleton sets if we ignore losp (we will write e.g. D_i instead of $D_{\{i\}}$ for these to simplify notation). By using the relations, e.g. Theorem 3.21.(4), one obtains down and up morphisms for more general stretches.

Example 3.22 For the complex quantum group case the only possible stretch is $S = \{0\}$, which is down-admissible unless $a_0 = 0$, where we note that $\overline{\{0\}}$ does not exist in this case (and so T does not exist either). The only relevant relations are the ones in Theorem 3.21.(1) and

$$D_0D_0E_{v-1} = 0, \quad U_0U_0E_{v-1} = 0, \quad D_0U_0E_{v-1} = \begin{cases} \mathfrak{g}(a_1)U_0D_0E_{v-1} & \text{if } a_0 \neq 0; \\ 0 & \text{if } a_0 = 0, \end{cases}$$

and $\mathbb{Z}^{k,q}$ has connected components corresponding to (scaled) zigzag algebras for each $v < \ell$, and single vertices for $v = [a_1, 0]_{\infty,\ell}$. Thus, we recover [10, Theorem 3.12].

Example 3.23 One can show the useful relation that

$$D_SU_S D_S E_{v-1} = 0, \quad E_{v-1}U_S D_S U_S = 0, \tag{3.11}$$

for any down-admissible stretch S , cf. [52, (3-13)]. In particular, loops square to zero.

The following proposition gives explicit versions of the cellular bases constructed in [3] and [9]. To state it recall from [33] that the crossingless matching basis of $\mathbf{TL}^{\mathbb{Z}[v^{\pm 1}],v}$ together with $(\mathbb{N}_0, <)$ and $(-)^{\dagger}$ endows $\mathbf{TL}^{\mathbb{Z}[v^{\pm 1}],v}$ with the structure of a (strictly object adapted) cellular category. (We refer to [55, Definition 2.1] and [27, Definition 2.4] for the terminology.)

Proposition 3.24 *Let B denote the set given by the elements in Theorem 3.21.(Basis). Moreover, let \tilde{b} and \bar{b} denote the respective sets obtained from the analogous expressions based on the respective projectors \tilde{e}_{v-1} and \bar{e}_{q-1} .*

- (a) *The sets \tilde{b} and \bar{b} give bases for the hom-spaces in $\mathbf{Tilt}^{k(v),v}$, while the set B gives a basis for the hom-spaces in $\mathbf{Tilt}^{k,q}$.*
- (b) *All of the aforementioned bases are unitriangularly equivalent to the crossingless matching bases (with respect to $(\mathbb{N}_0, <)$).*
- (c) *All of the aforementioned bases together with $(\mathbb{N}_0, <)$ and $(-)^{\dagger}$ endow $\mathbf{Tilt}^{k(v),v}$, respectively $\mathbf{Tilt}^{k,q}$, with the structure of a (strictly object adapted) cellular category.*

Proof Theorem 3.21 shows that \bar{b} and B give bases of the respective hom-spaces, and the former is unitriangularly equivalent to \tilde{b} , by construction. Moreover, \tilde{b} is unitriangularly equivalent to the crossingless matching basis, and a base change that is unitriangular with respect to the cell order preserves all structures defining a (strictly object adapted) cellular category. \square

Finally, we record a useful consequence of Theorem 3.21.(Basis).

Lemma 3.25 *Suppose that $v, w \in \mathbb{N}$ are such that $\nabla\text{supp}(v) \cap \nabla\text{supp}(w) = \emptyset$. We have*

$$\text{Hom}_{\text{Tilt}^{k,q}}(\mathbb{T}(v - 1), \mathbb{T}(w - 1)) \cong E_{w-1} Z^{k,q} E_{v-1} = \{0\}.$$

In particular, this holds true if the zeroth digit b_0 of w satisfies $b_0 \neq a_0$ and $b_0 \neq \ell - a_0$, or $b_0 = a_0 \neq \frac{\ell}{2}$ but $p > 2$ and the parity of the sum of the remaining digits of v and w is different.

Proof The first part of the statement is clear by Theorem 3.21.Basis. The first condition for when $\nabla\text{supp}(v) \cap \nabla\text{supp}(w) = \emptyset$ is immediate from the definitions as the corresponding $p\ell$ -adic expansions of the elements of $\nabla\text{supp}(v)$ and $\nabla\text{supp}(w)$ have to agree on the zeroth digit. For the second condition note that $b_0 = a_0 \neq \frac{\ell}{2}$ ensures that every element of $\nabla\text{supp}(v)$ is distinguished from the elements of $\nabla\text{supp}(w)$ by its zeroth digit or by the parity of the sum of higher digits (here we use that p is odd). \square

Note that the conditions given at the end of Lemma 3.25 are in general only sufficient to ensure $\nabla\text{supp}(v) \cap \nabla\text{supp}(w) = \emptyset$.

Remark 3.26 The statement in Lemma 3.25 is known as a **Weyl factor overlap criterion** in the theory of tilting modules and follows from **Ext-vanishing**, see e.g. [9, Section 2B], using the integrality of these statements which follows from [42]. One can see Lemma 3.25 as an explicit incarnation of these (general) facts about tilting modules.

3D More partial trace formulas

The next lemma deals with partial traces that do not reach an ancestor. Hence, they are complementary to part (c) of Proposition 3.19.

Lemma 3.27 *Let $v = [a_j, \dots, a_k, 0, \dots, 0]_{p,\ell}$ with $k > 0$, and $a_k \neq 0$. Suppose that $w = (p - a_i)p^{(i)}$ for some $1 \leq i < k$. Then we have*

$$\boxed{v-1} \text{ with loop } w = \begin{array}{c} \boxed{x-1} \\ \boxed{v+w-1} \\ \boxed{x-1} \end{array} \text{ with loop} = D_S U_S E_{x-1},$$

where $x = (v + w)[S] = [a_j, \dots, a_k - 1, p - 1, \dots, p - 1, a_i, 0, \dots, 0]_{p,\ell}$ with $S = \{i, \dots, k - 1\}$. The same holds for $0 < w' = \ell - a_0 < \ell$, where $x' = (v + w')[S'] = [a_j, \dots, a_k - 1, p - 1, \dots, p - 1, a_0]_{p,\ell}$ and $S = \{0, \dots, k - 1\}$.

Proof This follows from projector absorption and shortening where we have used that S is the minimal down-admissible stretch of $v + w$. The formula for w' and x' can be proven *verbatim*. □

We can now use the zigzag relation from Theorem 3.21.(6) to simplify $D_S U_S E_{x-1}$ further. To use this relation, recall that if the down-admissible hull \overline{S} , or the smallest minimal down-admissible stretch T with $T > \overline{S}$ does not exist, then the involved symbols are zero by definition.

Proposition 3.28 *Retain notation as in Lemma 3.27. We have*

$$\boxed{v-1}^w = g(a_k - 1) \cdot U_{\overline{S}} D_{\overline{S}} E_{x-1} + f(a_k - 1) \cdot U_T U_{\overline{S}} D_{\overline{S}} D_T E_{x-1}.$$

Moreover, the formula holds for w' and x' .

Note that the f -terms in Proposition 3.28 vanish if $a_k = 1$. On the other hand, the g -terms can be zero only if no admissible hull \overline{S} exists. Hence, the whole partial trace vanishes if and only if $a_k = a_j = 1$, i.e. if and only if $v = p^{(k)}$ is a prime eve.

We will state more partial trace formulas in Theorem 4.8 later on.

4 Monoidal structure

In this section, we study $\mathbf{Tilt}^{k, q}$ as a monoidal category. In the semisimple and complex quantum group cases, the results in this section appear throughout the literature. For example, see the remarks in this section for a small (and somewhat biased) collection of references.

4A Fusion rules

We start by recalling the well-known fusion rules for tilting modules lying in the fundamental alcove, in which tilting modules are simple. Note that in the semisimple case, i.e. when $\ell = \infty$, the fundamental alcove is the whole of \mathbb{N}_0 .

Lemma 4.1 *For $1 \leq v, w \leq \ell$ and $v + w - 2 < \ell$, we have*

$$T(v - 1) \otimes T(w - 1) \cong \bigoplus_{i=1}^{\min(v, w)} T(v + w - 2i). \tag{4.1}$$

Let us write x' to denote x , if x is odd, and $x - 1$, if x is even. For $1 \leq v, w \leq \ell$ and $v + w - 2 \geq \ell$, we have

$$T(v - 1) \otimes T(w - 1) \cong \bigoplus_{i=1}^{\ell - \max(v, w)} T([|v - w| - 1 + 2i]_{p, \ell} - 1) \oplus \bigoplus_{i=0}^{((v+w-\ell')-1)/2} T([1, v + w - 1 - \ell - 2i]_{p, \ell} - 1), \tag{4.2}$$

where, for later use, we indicate the $\mathfrak{p}\ell$ -adic expansions of the occurring terms. (The first direct sum in (4.2) is empty if $\max(v, w) = \ell$.)

Proof The first part is classical; the second part is easy using Proposition 3.3. \square

The summands with highest weights in the fundamental alcove appear in the first direct sum in (4.2). The second direct sum collects all remaining summands. If $v + w$ and ℓ have the same parity, then each of the summands are of generation one, otherwise there exists a simple summand $\mathbb{T}(\ell - 1)$. Equation (4.1) is the **Clebsch–Gordan rule**.

We note the following consequence, used in the proof of Theorem 3.18:

Lemma 4.2 For all $v, w \in \mathbb{N}$ such that $v \notin \text{Eve}$, suppose that $(\mathbb{T}(v - 1) : \Delta(w - 1)) = 1$. Then

$$(\mathbb{T}(m_v - 1) \otimes \mathbb{T}(1)^{\otimes(v-m_v)} : \Delta(w - 1)) = 1.$$

Proof This follows a character argument: we compute the Weyl factors of $\mathbb{T}(m_v - 1) \otimes \mathbb{T}(1)^{\otimes(v-m_v)}$ by repeatedly raising or lowering the corresponding highest weights of these Weyl factors by ± 1 since

$$\Delta(0) \otimes \Delta(1) \cong \Delta(1), \quad \Delta(n) \otimes \Delta(1) \cong \Delta(n + 1) \oplus \Delta(n - 1) \text{ if } n \neq 0. \quad (4.3)$$

In other words, we multiply the character of $\mathbb{T}(m_v - 1)$ as it appears in part (a) of Proposition 3.3 by $[2]_v^{v-m_v}$. Now observe that any Weyl factor of $\mathbb{T}(v - 1)$ is uniquely obtained from the summand $[w]_v$ that appears in the character of $\mathbb{T}(m_v - 1)$, and so the statement follows. \square

Definition 4.3 We define the **tail-length** $\text{tl}(v)$ of $v = [a_j, \dots, a_0]_{\mathfrak{p}, \ell}$ to be the maximal $k \in \mathbb{N}_0$ such that $a_0 = \ell - 1$ and $a_i = \mathfrak{p} - 1$ for all $k > i > 0$. If $a_0 \neq \ell - 1$, then $\text{tl}(v) = 0$.

The next fusion rule, which goes beyond the fundamental alcove, involves tensoring with the monoidal generator $\mathbb{T}(1)$.

Proposition 4.4 Let $v = [a_j, \dots, a_0]_{\mathfrak{p}, \ell}$. We have

$$\mathbb{T}(v - 1) \otimes \mathbb{T}(1) \cong \mathbb{T}(v) \oplus \bigoplus_{i=0}^{\text{tl}(v)} \mathbb{T}(v - 2\mathfrak{p}^{(i)})^{\oplus x_i},$$

$$x_i = \begin{cases} 0 & \text{if } a_i = 0 \text{ or } i = j \text{ and } a_j = 1; \\ 2 & \text{if } a_i = 1; \\ 1 & \text{if } a_i > 1. \end{cases}$$

Proof A character computation based on (3.3). \square

Let us comment on the qualitative differences between the cases found in Proposition 4.4. For the sake of exposition, we consider $\ell \geq 4$. Focusing on the zeroth digit, we see:

$$\mathbb{T}(v - 1) \otimes \mathbb{T}(1) \cong \begin{cases} \mathbb{T}(v) & \text{if } a_0 = 0; \\ \mathbb{T}(v) \oplus \mathbb{T}(v - 2)^{\oplus 2} & \text{if } a_0 = 1; \\ \mathbb{T}(v) \oplus \mathbb{T}(v - 2) & \text{if } a_0 \in \{2, \dots, \ell - 2\}; \\ \text{generation drop case} & \text{if } a_0 = \ell - 1. \end{cases}$$

The **generation drop case** occurs when the zeroth digit is maximal, in which case additional direct summands may appear. Recall that $\mathbb{T}(v - 1)$ has 2^{gen_v} Weyl factors. Under tensoring with $\mathbb{T}(1) \cong \Delta(1)$, most of them produce two new Weyl factors by (4.3). In total, $\mathbb{T}(v - 1) \otimes \mathbb{T}(1)$ will have 2^{gen_v+1} or $2^{\text{gen}_v+1} - 1$ Weyl factors. Observe that we are guaranteed to find a direct summand $\mathbb{T}(v)$ in $\mathbb{T}(v - 1) \otimes \mathbb{T}(1)$. Now we have three cases depending on whether the generation increases, stays constant or drops, which are precisely the respective cases $a_0 = 0$, $a_0 \in \{1, \dots, \ell - 2\}$ and $a_0 = \ell - 1$, as above. In the first case, $\mathbb{T}(v)$ exhausts all newly generated Weyl factors, so it is the only summand that appears. In the second case, it exhausts roughly half of all Weyl factors, so one expects a further summand to appear. In the generation drop case, we have $\text{gen}_{v+1} = \text{gen}_v - \text{tl}(v)$. Hence $\mathbb{T}(v)$ only accounts for a small proportion of the Weyl modules, and we expect several other tilting summands to appear.

Proposition 4.4 immediately implies the following appearance of $\mathbb{1}$.

Proposition 4.5 *Let $d \in \mathbb{N}$. If $m\text{char}(\mathbb{k}, \mathfrak{q}) = (2, 2)$, then $\mathbb{1}$ is never a direct summand of $\mathbb{T}(1)^{\otimes 2d}$, whereas if $m\text{char}(\mathbb{k}, \mathfrak{q}) = (3, 3)$, then $\mathbb{1}$ appears exactly once.*

Note the contrast to the semisimple situation, where the multiplicities of $\mathbb{1}$ in tensor products $\mathbb{T}(1)^{\otimes 2d}$ are given by the Catalan numbers, which grow exponentially.

Proof Let us prove the (harder) case $m\text{char}(\mathbb{k}, \mathfrak{q}) = (3, 3)$ by induction on d . The case $d = 1$ is just $\mathbb{T}(1)^{\otimes 2} \cong \mathbb{1} \oplus \mathbb{T}(2)$. For $d > 1$, we observe that $\mathbb{T}(v) \otimes \mathbb{T}(1)$ for $v > 1$ will never contain a summand below 2 by Proposition 4.4. Hence, we are done since the summand with the second lowest highest weight in $\mathbb{T}(1)^{\otimes (d-1)}$ is $\mathbb{T}(2)$, by induction. □

Remark 4.6 Questions about the structure constants of the representation ring have been studied for the finite group $\text{SL}_2(\mathbb{F}_{p^k})$ for a long time. (The connection to our setup is to embed $\text{SL}_2(\mathbb{F}_{p^k})$ into $\text{SL}_2(\overline{\mathbb{F}}_{p^k})$ via fixed points under the Frobenius twist.) For example, Lemma 4.1 and [23, Lemma 5] is used in [17, Section 3] to find the finite group analog of fusion rules.

4B Categorized fusion rules for tensoring with the vector representation

The fusion rule from Proposition 4.4 describes the multiplicities of indecomposable tilting modules in the tensor product $\mathbb{T}(v - 1) \otimes \mathbb{T}(1)$. In this section, we consider the refined problem of describing the morphisms that project onto such summands

using the Temperley–Lieb calculus. Specifically, in Theorem 4.8 we will decompose the idempotent $E_{v-1} \otimes \text{id}_1$ into a sum of orthogonal, primitive idempotents factoring through E_v as well as the other $E_{v-2p(i)}$ predicted by Proposition 4.4. Conversely, such a decomposition can also be read as a recursive description of the mixed JW projector E_v in terms of mixed JW projectors of lower order.

For the following definition, we use scalars determined by evaluating the functions g_q and f_x on digits. On all digits, except for the zeroth one, we use (3.10). For the zeroth digit, we instead use:

$$\begin{aligned}
 f_x(a) &= \begin{cases} (-1)^{a+1} \cdot \frac{-2}{[a]_x} & \text{if } 1 \leq a \leq \ell - 2; \\ 0 & \text{if } a = 0 \text{ or } a = \ell - 1, \end{cases} \\
 g_x(a) &= \begin{cases} -\frac{[a+1]_x}{[a]_x} & \text{if } 1 \leq a \leq \ell - 1; \\ -[2]_x & \text{if } a = 0. \end{cases} \tag{4.4}
 \end{aligned}$$

Armed with this notation, we now define the morphisms that will feature in the decomposition of $E_{v-1} \otimes \text{id}_1$ into orthogonal, primitive idempotents.

Definition 4.7 Let $v = [a_j, \dots, a_0]_{p,\ell}$ and $0 \leq i \leq \text{tl}(v)$. If $a_i = 1$ and $i \neq j$, then we define

$$A_v^i := \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}, \quad w = p^{(i)}.$$

Here the caps and cups have thickness $w = p^{(i)}$, and are thus admissible. If $a_i = 1$ and $i = j$, then we declare $A_v^i = 0$ (the diagram makes no sense in this case since $v < 2w$). We will also consider the reflected morphism $(A_v^i)^\dagger$ along a horizontal line.

For $a_i > 1$ and $i \neq j$, we consider

$$B_v^i := \frac{1}{g_q(a_i-1)} \cdot \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} - \frac{f_q(a_i)}{g_q(a_i-1)} \cdot \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}, \quad w = p^{(i)}.$$

Here the caps and cups are of thickness $a_i p^{(i)}$ and are thus admissible. If $i = j$, then we use the same formula to define B_v^i , except we omit the second summand.

The *categorified fusion rule* for $\mathbb{T}(v - 1) \otimes \mathbb{T}(1)$ is now given by the following theorem.

Theorem 4.8 Let $v = [a_j, \dots, a_0]_{p,\ell}$.

- (a) We have the following decomposition of $E_{v-1} \otimes \text{id}_1$ into a sum of orthogonal, primitive idempotents.

$$\boxed{v-1} = \boxed{v} + \sum_{i=0}^{\text{tl}(v)} P_v^i \quad \text{where } P_v^i = \begin{cases} 0 & \text{if } a_i = 0; \\ A_v^i + (A_v^i)^\dagger & \text{if } a_i = 1; \\ B_v^i & \text{if } a_i > 1. \end{cases} \tag{4.5}$$

For $a_i = 1$, both the summands A_v^0 and $(A_v^0)^\dagger$ are orthogonal, primitive idempotents.

- (b) Further, we have the following **partial trace rules**. Let $0 \leq i \leq \text{tl}(v)$, $a_i \neq 0$, and $w = \mathfrak{p}^{(i)}$. For the case $i = j$, we additionally assume $a_j > 1$. Then we have

$$\boxed{v-w} \Big] \Big] = \mathfrak{g}_q(a_i - 1) \cdot \boxed{v-2w} + \mathfrak{f}_q(a_i - 1) \cdot L_{v-2w}^i. \tag{4.6}$$

(If $a_i = 1$ or $i = j$, then the second summand is zero. Even though L_{v-2w}^0 is not defined on its own, this is meaningful in both cases because $\mathfrak{f}_q(0) = 0$ or $v - 2w + 1$ is an eve.)

Example 4.9 For $v = [4, 1, 6, 6, 6, 10]_{7,11}$, we have $\text{tl}(v) = 4$ and

$$E_{v-1} \otimes \text{id}_1 = E_v + B_v^0 + B_v^1 + B_v^2 + B_v^3 + (A_v^4 + (A_v^4)^\dagger).$$

For $v = [1, 1, 1, 1]_{2,2}$, we have $\text{tl}(v) = 4$ and

$$E_{v-1} \otimes \text{id}_1 = E_v + (A_v^0 + (A_v^0)^\dagger) + (A_v^1 + (A_v^1)^\dagger) + (A_v^2 + (A_v^2)^\dagger),$$

where we note $A_v^3 = (A_v^3)^\dagger = 0$ since the leading digit is $a_3 = 1$, and $P_v^4 = 0$ since $a_4 = 0$. The occurrence of multiple pairs $A_v^i + (A_v^i)^\dagger$ is an instance of losp . For $\ell \neq 2$ and $\mathfrak{p} \neq 2$ we encounter at most one pair of the form $A_v^i + (A_v^i)^\dagger$ since $a_i = 1$ implies that $\text{tl}(v) \leq i$.

Remark 4.10 The fusion rule (1.1) can be used to express classical JW projectors in terms of JW projectors of lower order. Analogously, Theorem 4.8 gives a recursion of $\mathfrak{p}\ell\text{JW}$ projectors in terms of $\mathfrak{p}\ell\text{JW}$ projectors of lower order. This is in contrast to the defining description in (3.7), which uses classical JW projectors.

Remark 4.11 In the complex quantum group case and for $v \leq 2\ell - 2$, the fusion rule (4.5) can be deduced from [12, Lemma 3.2]. The three cases of their rule reflect the trichotomy of $a_0 = 0$, $a_0 = 1$, and $a_0 > 1$.

Remark 4.12 We do not know a good partial trace formula of type (4.6) in the case $a_i = 1$, not even for $i = 0$ and $w = 1$. (One can write down a formula using (4.5), of course.) We expect this formula to be more complicated, because it deals with a generation increase (on comparison with the increased complexity of the fusion rule when the generation drops).

Proof of Theorem 4.8 The proof proceeds by induction on v . To do so, we split the statement of the theorem into the following two types of assertions.

$\mathbf{F}(v)$ denotes: *The categorified fusion rules are given by (4.5) for v .*

$\mathbf{PT}(v)$ denotes: *The partial trace rules (4.6) hold for v .*

The former makes sense for all $v \geq 1$ and the latter for all $v \geq 2$. We will also write $\mathbf{F}(< v)$ to express the assertion that $\mathbf{F}(w)$ holds for all $1 \leq w < v$, and similarly for $\mathbf{PT}(< v)$.

The base cases for the inductive argument are given by $\mathbf{F}(1)$, which is immediate.

The induction step will be accomplished by two arguments that we separate into two distinct statements below. Lemma 4.13 shows the implication $\mathbf{F}(< v) \implies \mathbf{PT}(v)$ for all $v \geq 2$. Lemma 4.14 shows the implication $\mathbf{PT}(v) \implies \mathbf{F}(v)$ for all $v \geq 2$. Induction then shows that both assertions hold for all relevant values of v . \square

We now turn to the two lemmas that form the heart of the proof of Theorem 4.8.

Lemma 4.13 *We have $\mathbf{F}(< v) \implies \mathbf{PT}(v)$ for all $v \geq 2$.*

Proof We first consider $i = 0$, where we have $w = 1$ and assume $a_0 \neq 0$, i.e. we aim to prove

$$\boxed{v-1} = \mathfrak{g}_q(a_0 - 1) \cdot \boxed{v-2} + \mathfrak{f}_q(a_0 - 1) \cdot L_{v-2}^0. \tag{4.7}$$

To verify this, we will use the fusion rule for $v - 1$ in reverse to expand the projector E_{v-1} . If $a_0 = 1$, then we have $E_{v-1} = E_{v-2} \otimes \text{id}_1$. The claimed statement follows since the circle value is $-[2]_q = \mathfrak{g}_q(0)$ and the second term is zero by definition. If $\ell = 2$, then we are done. Thus, we suppose $\ell > 2$ from now on.

We consider the case $a_0 = 2$, where the fusion rule involves $A_{v-1}^0 + (A_{v-1}^0)^\dagger$:

The first term in each of the brackets is L_{v-2}^0 ; the second is $(L_{v-2}^0)^2$, which is zero by (3.11). Since $\mathfrak{g}_{\mathbb{q}}(1) = -[2]_{\mathbb{q}}$ and $\mathfrak{f}_{\mathbb{q}}(1) = -2$, the claim follows for $a_0 = 2$.

Next, we consider $a_0 \in \{3, \dots, \ell - 1\}$. Here we use $-[2]_{\mathbb{q}} + \frac{[a_0-2]_{\mathbb{q}}}{[a_0-1]_{\mathbb{q}}} = -\frac{[a_0]_{\mathbb{q}}}{[a_0-1]_{\mathbb{q}}}$ to get

$$\begin{aligned}
 \text{Diagram 1} &= \text{Diagram 2} - \left(\frac{1}{\mathfrak{g}_{\mathbb{q}}(a_0-2)} \cdot \text{Diagram 3} - \frac{\mathfrak{f}_{\mathbb{q}}(a_0-1)}{\mathfrak{g}_{\mathbb{q}}(a_0-2)} \cdot \text{Diagram 4} \right) \\
 &= \mathfrak{g}_{\mathbb{q}}(a_0 - 1) \cdot \text{Diagram 5} + \frac{\mathfrak{f}_{\mathbb{q}}(a_0-1)}{\mathfrak{g}_{\mathbb{q}}(a_0-2)} \cdot \text{Diagram 6}, \tag{4.8}
 \end{aligned}$$

and it remains to compute the final term. To this end, we will use the fusion rule for $v - 2a_0 + 2$ on the mini box (using induction), which corresponds to E_{v-2a_0+1} . The last digit of its relevant \mathfrak{p} -adic expansion is an element of $\{3, \dots, \ell - 1\}$, namely $\ell - a_0 + 2$. We claim that fusion results in:

$$\begin{aligned}
 \text{Diagram 7} &= \text{Diagram 8} + \frac{1}{\mathfrak{g}_{\mathbb{q}}(\ell - a_0 + 1)} L_{v-2}^0 + \text{lower order terms} = \frac{1}{\mathfrak{g}_{\mathbb{q}}(\ell - a_0 + 1)} L_{v-2}^0. \tag{4.9}
 \end{aligned}$$

Here, we have three things to check. To start, the first term in the middle is zero because $\mathbb{T}(v - 2)$ and $\mathbb{T}(v - 2a_0 + 2)$ do not share any common Weyl factors. Second, the fusion rule includes a term of the form $B_{v-2a_0+2}^0$, which is typically a sum of two diagrams (although the second may not appear in some cases). The first diagram combines with the present caps and cups to form $\frac{1}{\mathfrak{g}_{\mathbb{q}}(\ell - a_0 + 1)} L_{v-2}^0$. The second diagram (if it is present at all) vanishes when it is sandwiched because of the containment relation $U_0 U_0 = 0$. Third, one checks that all possible terms of even lower order arising from the fusion rule become zero when sandwiched. Such terms only arise if $\text{tl}(v - 2a_0 + 2) > 0$, i.e. if $a_0 = 3$. This finishes the verification of (4.9), which we now use to rewrite (4.8). The coefficient of L_{v-2}^0 is

$$\frac{\mathfrak{f}_{\mathbb{q}}(a_0-1)}{\mathfrak{g}_{\mathbb{q}}(a_0-2)} \cdot \frac{1}{\mathfrak{g}_{\mathbb{q}}(\ell - a_0 + 1)} = \frac{\mathfrak{f}_{\mathbb{q}}(a_0-1)}{\mathfrak{g}_{\mathbb{q}}(a_0-2)} \cdot \mathfrak{g}_{\mathbb{q}}(a_0 - 2) = \mathfrak{f}_{\mathbb{q}}(a_0 - 1),$$

where we have simplified $\mathfrak{g}_{\mathbb{q}}(\ell - a_0 + 1)^{-1} = \mathfrak{g}_{\mathbb{q}}(a_0 - 2)$ using $[\ell]_{\mathbb{q}} = 0$ (this does not hold in the semisimple case). Thus, we have verified the partial trace claim (4.7).

Finally, we consider the case $0 < i \leq \text{tl}(v)$, which, surprisingly, is much easier to prove. Recall that we assume $a_i \neq 0$ and aim to prove:

$$\boxed{v-w} \cup^w = g_q(a_i - 1) \cdot \boxed{v-2w} + f_q(a_i - 1) \cdot L_{v-2w}^i.$$

To verify this claim, we calculate that $v - w + 1 = [a_j, \dots, a_i, 0, \dots, 0]_{p,\ell}$. In particular, we can use (a slight generalization of) Proposition 3.28 to trace off $w - 1 = p^{(i)} - 1$ strands and get

$$\boxed{v-w} \cup^{w-1} = g_q(a_i - 1) \cdot \begin{array}{c} \boxed{x-1} \\ | \\ \boxed{x-1} \end{array} + f_q(a_i - 1) \cdot \begin{array}{c} \boxed{x-1} \\ \cup \\ \boxed{x-1} \end{array},$$

where $x = [a_j, \dots, a_i - 1, 0, \dots, 1]_{p,\ell}$ (if $i = j$, the f -term vanishes but the g -term does not because we assume $a_j > 1$). Now we use shortening to compute the full partial trace as:

$$\begin{aligned} \boxed{v-w} \cup^w &= g_q(a_i - 1) \cdot \begin{array}{c} \boxed{x-1} \\ | \\ \boxed{x-1} \end{array} + f_q(a_i - 1) \cdot \begin{array}{c} \boxed{x-1} \\ \cup \\ \boxed{x-1} \end{array} = g_q(a_i - 1) \cdot \begin{array}{c} \boxed{} \\ | \\ \boxed{} \end{array} \\ &+ f_q(a_i - 1) \cdot \begin{array}{c} \boxed{} \\ \cup \\ \boxed{} \end{array}, \end{aligned}$$

which we pull straight to get the claimed partial trace formula. □

Lemma 4.14 *We have $\text{PT}(v) \implies \mathbf{F}(v)$ for all $v \geq 2$.*

Proof We start with a few observations. First, Proposition 4.4 ensures that we know how many orthogonal, primitive idempotents to expect in the categorified fusion rule. Second, by the same arguments as in the proof of Theorem 3.18 (however, it is easier in this case since we only need to tensor with $\mathbb{T}(1)$) the idempotents projecting onto isotypic components are uniquely determined by the property of absorbing $E_{v-1} \otimes \text{id}_1$. These isotypic idempotents are automatically orthogonal because a straightforward computation, using Proposition 3.3 and Lemma 4.1, shows that the isotypic components share no Weyl factors, which implies that there are no non-zero morphisms between them by Lemma 3.25.

Combining these observations, it remains to show that the morphisms $B_v^i, A_v^i, (A_v^i)^\dagger$ from Definition 4.7 satisfy the absorption property and are indeed idempotents whenever they appear in (4.5). Finally, we also check that A_v^i and $(A_v^i)^\dagger$ are orthogonal.

Absorption. Let us first check that all of the candidate idempotents appearing in Theorem 4.8 absorb $E_{v-1} \otimes \text{id}_1$. For E_v, B_v^0, A_v^0 , and $(A_v^0)^\dagger$, this follows immediately

from the absorption properties of the $p\ell JW$ projectors – see Proposition 3.19.(a). For the cases with $i > 0$, we use the shortening property from Proposition 3.19.(b).

We will now verify the idempotency of the candidate expressions on a case-by-case basis.

Idempotency (and orthogonality) of A_v^i and $(A_v^i)^\dagger$. We start with the term A_v^i (which also covers the symmetric case $(A_v^i)^\dagger$), which is defined as the sum of the two diagrams

$$A_v^i = X + Y, \quad \text{where } X = \begin{array}{c} \text{v-w} \\ \downarrow \\ \text{v-2w} \\ \downarrow \\ \text{v-w} \end{array} \quad \text{and } Y = \begin{array}{c} \text{v-w} \\ \downarrow \\ \text{v-2w} \\ \downarrow \\ \text{v-w} \end{array},$$

in which $w = p^{(i)}$. Next, we compute the pairwise products of X, X^\dagger , and Y . First, we use shortening and absorption of mixed JW projectors to compute $X^2 = X$ and $XY = Y$.

Symmetrically, we also have $(X^\dagger)^2 = X^\dagger$ and $Y(X^\dagger) = Y$. Now we claim that all other products are zero, namely $(X^\dagger)X = X(X^\dagger) = YX = (X^\dagger)Y = Y^2 = 0$. This can be seen as follows. Up to symmetry, these statements all follow from

$$\begin{array}{c} \text{v-2w} \\ \downarrow \\ \text{v-w} \end{array} = 0 \iff \begin{array}{c} \text{v-2w} \\ \downarrow \\ \text{v-w} \end{array} = \begin{array}{c} \text{v-w} \\ \downarrow \\ \text{v-w} \end{array} = E_{v-w} U_i D_i U_i = 0.$$

The equivalence is given by bending, as illustrated. On the right-hand side we undid shortening and translated the caps and cups into morphisms U_i and D_i respectively (this is possible since $a_i = 1$), and then applied (3.11). Taking all of these together shows that

$$\begin{aligned} (X + Y)^2 &= X + Y, & (X^\dagger + Y)^\dagger &= X^\dagger + Y, \\ (X + Y)(X^\dagger + Y) &= (X^\dagger + Y)(X + Y) = 0, \end{aligned}$$

which expresses A_v^i and $(A_v^i)^\dagger$ as orthogonal idempotents.

Idempotency of B_v^i . Next we check that the terms B_v^i are idempotents. Recall that B_v^i for $i \neq j$ is defined as a linear combination of the following two morphisms

$$X = \begin{array}{c} \text{v-w} \\ \downarrow \\ \text{v-w} \end{array} \quad \text{and} \quad Y = \begin{array}{c} \text{v-w} \\ \downarrow \\ \text{v-w} \end{array},$$

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