



On the K -theory of regular coconnective rings

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Abstract

We show that for a coconnective ring spectrum satisfying regularity and flatness assumptions, its algebraic K -theory agrees with that of its π_0 . We prove this as a consequence of a more general devissage result for stable infinity categories. Applications of our result include giving general conditions under which K -theory preserves pushouts, generalizations of \mathbb{A}^n -invariance of K -theory, and an understanding of the K -theory of categories of unipotent local systems.

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1 Introduction

In this paper we examine the relationship between coconnectivity, regularity, and algebraic K -theory. As a consequence of our investigation we prove that the K -theory of a large collection of coconnective rings agrees with that of their π_0 .

Theorem 1.1 *Given a coconnective \mathbb{E}_1 -algebra R such that*

- (1) $\pi_0 R$ is left regular coherent and
- (2) $\tau_{\leq -1} R$ has Tor amplitude in $[-\infty, -1]$ as a right $\pi_0 R$ -module,

the natural map in connective K -theory

$$K(\pi_0 R) \rightarrow K(R)$$

is an equivalence and both $\pi_0 R$ and R have vanishing K_{-1} .

Although not immediately clear, Theorem 1.1 is a devissage theorem. The core step in the proof is an application of Quillen’s devissage theorem [21, Theorem 4] and condition (1) is exactly what is needed for the canonical t -structure on $\pi_0 R$ -modules to restrict to a bounded t -structure on perfect $\pi_0 R$ -modules with heart finitely presented, discrete $\pi_0 R$ -modules. The essential novelty in Theorem 1.1 comes from condition (2) as a simple condition, easily checked in practice,¹ which guarantees a K -equivalence. As a demonstration we work through the prototypical example of devissage.

Example 1.2 From the localization sequence

$$\text{Perf}(\mathbb{Z})^{p\text{-nil}} \hookrightarrow \text{Perf}(\mathbb{Z}) \twoheadrightarrow \text{Perf}(\mathbb{Z}[1/p])$$

we obtain a cofiber sequence on non-connective K -theory

$$K^{\text{nc}}(\text{Perf}(\mathbb{Z})^{p\text{-nil}}) \rightarrow K^{\text{nc}}(\text{Perf}(\mathbb{Z})) \rightarrow K^{\text{nc}}(\text{Perf}(\mathbb{Z}[1/p])).$$

Identifying \mathbb{F}_p as a generator of $\text{Perf}(\mathbb{Z})^{p\text{-nil}}$ we have an identification

$$\text{Perf}(\mathbb{Z})^{p\text{-nil}} \simeq \text{Perf}(\text{End}_{\mathbb{Z}}(\mathbb{F}_p)).$$

Devissage can then be phrased as the assertion that $K(\text{End}_{\mathbb{Z}}(\mathbb{F}_p)) \simeq K(\mathbb{F}_p)$ and K_{-1} vanishes, from which we obtain a cofiber sequence on connective algebraic K -theory

$$K(\mathbb{F}_p) \rightarrow K(\mathbb{Z}) \rightarrow K(\mathbb{Z}[1/p]).$$

In order to prove this using Theorem 1.1 we compute the homotopy groups of $\text{End}_{\mathbb{Z}}(\mathbb{F}_p)$, which are

¹ Condition (2) is satisfied if $\pi_{-i} R$ has Tor dimension $< i$ as a right $\pi_0 R$ -module.

$$\pi_s \operatorname{End}_{\mathbb{Z}}(\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & s = 0, -1 \\ 0 & \text{otherwise} \end{cases}$$

and observe that conditions (1) and (2) are satisfied. ◁

Proving this localization sequence using Quillen’s devissage theorem directly involves checking that p -torsion perfect \mathbb{Z} -modules admits a bounded t -structure and that every p -torsion abelian group has a filtration whose associated graded consists of \mathbb{F}_p -modules. For a more general \mathbb{E}_1 -algebra R constructing t -structures and filtrations in order to apply devissage involves contemplating the behavior of a generic perfect R -module, which can become unwieldy. By contrast, the Tor amplitude condition in Theorem 1.1 lives at the level of the homotopy groups of R and is therefore quite concrete. This change represents a considerably gain in practical usability.

Before moving on, let us point out that this example also highlights another key feature of devissage: namely, while localization sequences occur at the level of non-commutative motives, devissage does not.² It is rather a property K -theory satisfies in addition to being localizing. Similarly, Theorem 1.1 does not hold in general for localizing (or additive) invariants.

We prove Theorem 1.1 as a corollary of our main result, Theorem 1.3, which is a more general devissage result taking place at the level of stable categories.

Theorem 1.3 *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor between small, stable, idempotent complete categories and let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a bounded t -structure on \mathcal{C} . If we assume that*

- (A) *the image of F generates \mathcal{D} under finite colimits and retracts and*
- (B) *F is fully faithful when restricted to \mathcal{C}^{\heartsuit}*

then there exists a unique bounded t -structure on \mathcal{D} for which F is t -exact. Moreover, the induced maps on connective K -theory

$$\begin{array}{ccc} K(\mathcal{C}^{\heartsuit}) & \longrightarrow & K(\mathcal{D}^{\heartsuit}) \\ \downarrow & & \downarrow \\ K(\mathcal{C}) & \longrightarrow & K(\mathcal{D}) \end{array}$$

are all equivalences and K_{-1} of each term vanishes.

Theorem 1.3 is proved in Sect. 2 and the key step is the construction of the t -structure on \mathcal{D} . This is the most technical point in the proof and it uses all of the conditions of the theorem in an essential way. In fact, as a byproduct of this argument we obtain relatively fine-grained control over the abelian category \mathcal{D}^{\heartsuit} . Specifically, we find that the inclusion $\mathcal{C}^{\heartsuit} \hookrightarrow \mathcal{D}^{\heartsuit}$ satisfies the hypotheses of Quillen’s devissage theorem [21, Theorem 4] from which we obtain the K -equivalence between \mathcal{C}^{\heartsuit} and \mathcal{D}^{\heartsuit} . The proof ends with applying Barwick’s theorem of the heart [2] to identify the

² For example $\operatorname{THH}(\operatorname{End}_{\mathbb{Z}}(\mathbb{F}_p))$ is the fiber of the map $\operatorname{THH}(\mathbb{Z}) \rightarrow \operatorname{THH}(\mathbb{Z}[1/p])$, which does not agree with $\operatorname{THH}(\mathbb{F}_p)$.

K -theories of \mathcal{C} and \mathcal{D} with that of their heart and the main results of [22] and [1] to obtain the vanishing of K_{-1} .

Examining the relation between \mathcal{C}^\heartsuit and \mathcal{D}^\heartsuit we see that conditions (A) and (B) together can be thought of as asking that \mathcal{D} behave like a category of unipotent local systems with coefficients in \mathcal{C} . Indeed, (B) is analogous to the fact that maps between trivial representations can be computed on underlying and (A) is analogous to the fact that unipotent representations are generated from trivial representations under extensions. For this reason we say that a map is *unipotent* if it satisfies these conditions. With this reformulation we can now introduce the key slogan of this paper:

Devissage is the invariance of K -theory under unipotent maps.

In Sect. 3 we deduce Theorem 1.1 from Theorem 1.3 and discuss several points which are complementary to Theorem 1.3. A subtlety worth noting here is that up to this point we have been working entirely with *connective* K -theory as this is the setting where Barwick's theorem of the heart and Quillen's devissage are applicable. In fact, assuming \mathcal{C}^\heartsuit is Noetherian, we show in Lemma 3.12 that \mathcal{D}^\heartsuit is Noetherian as well. This lets us extend Theorem 1.3 to non-connective K -theory in the Noetherian setting for the simple reason that negative K -groups vanish in the Noetherian setting [1]. One might wonder whether Theorem 1.3 holds for negative K -groups in the non-Noetherian setting and since we are not aware of any counter-example we are led to ask³:

Question 1.4 Do the theorem of the heart and devissage hold for negative K -theory?

In Sects. 4 and 5 we turn to applications of our main theorem. Combining our work with the work of Land and Tamme on the K -theory of pullbacks and pushouts [12, 13] we provide general conditions under which K -theory preserves pushouts.

Theorem 1.5 *Suppose $\mathcal{C} \xleftarrow{g} A \xrightarrow{f} B$ is a span of discrete rings where A is left regular coherent and both f and g are right faithfully flat. Then connective K -theory preserves the pushout of this span.*

Theorem 1.5 allows us to generalize Waldhausen's theorems on the K -theory of generalized free products [25] to the setting of non-discrete rings. Using similar techniques, we then obtain an \mathbb{A}^n -invariance result for K -theory.

Theorem 1.6 [\mathbb{A}^n -invariance of algebraic K -theory] *Let \mathcal{C} be a small, stable, idempotent complete category equipped with a bounded t -structure. Then $K_i(\mathcal{C}) \cong K_i(\mathcal{C}[x_1, \dots, x_n])$ for $i \geq n - 1$.*

In the $n = 1$ case, for a regular, Noetherian ring this recovers Quillen's fundamental theorem of algebraic K -theory [21, Theorem 8]. An alternative proof extending the result to regular coherent rings was given by Waldhausen, again in the $n = 1$ case [25]. If \mathcal{C}^\heartsuit is in addition Noetherian, then through a combination of [1, Proposition 3.14], Barwick's theorem of the heart and the extension of Quillen's argument to abelian categories (see [18]) we in fact know that $K^{\text{nc}}(\mathcal{C}) \simeq K^{\text{nc}}(\mathcal{C}[x_1, \dots, x_n])$ and that the negative K -groups vanish. The case of non-Noetherian regular coherent rings and

³ See also [1, Conjecture C], which is part of Question 1.4.

$n > 1$ is more difficult, because $R[x]$ may not even be coherent, so one cannot induct on n in the obvious way. Nevertheless, the $n > 1$ case was already known when R is a discrete ring as a corollary of the Farrell-Jones conjecture for the groups \mathbb{Z}^n and the $n = 1$ case⁴ (see for example [6, Corollary 2]). The degree bounds in the above theorem ultimately come from our use of connective K -theory and a positive answer to Question 1.4 would allow us to remove these restrictions.

In Sect. 5.2 we examine the category of unipotent local systems on a connected space X with coefficients in a category \mathcal{C} with a bounded t -structure. As one might expect, we find that the K -theory of unipotent local systems agrees with the K -theory of \mathcal{C} , generalizing [1, Theorem 4.8]. In the final pair of subsections we work through a collection of examples which demonstrate that the conditions in Theorem 1.1 cannot be weakened.

1.1 Notations and conventions

In order to preserve the brevity of this paper we assume the reader is generally familiar with higher algebra and algebraic K -theory. We also make use of the following notations and conventions throughout.

- The term category will refer to an ∞ -category as developed by Joyal and Lurie.
- $\text{Map}(a, b)$ will denote the space of maps from a to b (in some ambient category).
- In a stable category $\text{map}(a, b)$ will denote the mapping spectrum between a and b .
- For an \mathbb{E}_1 -algebra R , $\text{Mod}(R)$ will refer to its category of left modules and $\text{Perf}(R)$ will refer to the category of perfect left R -modules, i.e. the compact objects in $\text{Mod}(R)$.
- We use \mathcal{C}, \mathcal{D} to denote small, idempotent complete stable categories, and use Cat^{perf} to denote the category of such categories and exact functors.
- Given an exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$ in Cat^{perf} , $F^* : \text{Ind}(\mathcal{C}) \rightarrow \text{Ind}(\mathcal{D})$ denotes $\text{Ind}(F)$, and $F_* : \text{Ind}(\mathcal{D}) \rightarrow \text{Ind}(\mathcal{C})$ denotes the right adjoint of F^* .
- We use \mathcal{U}_{loc} for the noncommutative motive (or just nc motive for short) functor of Blumberg–Gepner–Tabuada [3].
- We use $K(-)$ for connective K -theory and $K^{\text{nc}}(-)$ for non-connective K -theory.
- We use x_n for a polynomial generator in degree n and ϵ_n for an exterior generator in degree n . As an example, $\mathbb{S}[x_n]$ is the free \mathbb{E}_1 -algebra on a class in degree n .
- We use $\mathcal{C}[\epsilon_n]$ as notation for $\mathcal{C} \otimes \text{Perf}(\mathbb{S}[\epsilon_n])$ and similarly for polynomial generators.

2 The main theorem

In this section we prove our main theorem, Theorem 1.3, whose statement we reproduce below.

⁴ While the Farrell-Jones conjecture is about nonconnective K -theory, \mathbb{A}^1 -invariance is only known for connective K -theory, and for this reason this alternative approach arrives at the same degree bound, $i \geq n - 1$, for \mathbb{A}^n -invariance.

Theorem 2.1 [Theorem 1.3] *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor between small, stable, idempotent complete categories and let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a bounded t -structure on \mathcal{C} . If we assume that*

- (A) *the image of F generates \mathcal{D} under finite colimits and retracts and*
- (B) *F is fully faithful when restricted to \mathcal{C}^\heartsuit*

then there exists a unique bounded t -structure on \mathcal{D} for which F is t -exact. Moreover, the induced maps on connective K -theory

$$\begin{array}{ccc} K(\mathcal{C}^\heartsuit) & \longrightarrow & K(\mathcal{D}^\heartsuit) \\ \downarrow & & \downarrow \\ K(\mathcal{C}) & \longrightarrow & K(\mathcal{D}) \end{array}$$

are all equivalences and K_{-1} of each term vanishes.

Before proceeding, we give a sketch of the strategy we follow in proving this theorem. The final step is applying Barwick’s theorem of the heart and Quillen’s devissage theorem to produce K -theory equivalences, along with the vanishing results on K_{-1} due to Schlichting and Antieau–Gepner–Heller. In order to apply these results we need to produce a bounded t -structure on \mathcal{D} which is relatively well behaved. The key idea is that after passing to categories of ind-objects it is in fact quite easy to produce such a t -structure. Condition (B) is then rigged so that we have the control necessary to restrict this t -structure to compact objects in $\text{Ind}(\mathcal{D})$ (i.e. \mathcal{D}). For the remainder of this section the notation from the statement of Theorem 1.3 will remain in place and we assume F satisfies conditions (A) and (B).

Passing to ind-completions gives us an induced commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & & \downarrow \\ \text{Ind}(\mathcal{C}) & \xrightarrow{F^*} & \text{Ind}(\mathcal{D}) \end{array}$$

where F^* is a left adjoint and the vertical arrows are each the inclusion of the full subcategory of compact objects.⁵

As promised, we begin by producing a t -structure on the level of ind-objects. This is rather easy since the category of ind-objects is presentable.

Lemma 2.2 ([15, Proposition 1.4.4.11]) *Let \mathcal{A} be a presentable, stable category. If $\{X_\alpha\}$ is a small collection of objects in \mathcal{A} , then there is an accessible⁶ t -structure, $(\mathcal{A}_{\geq 0}, \mathcal{A}_{\leq 0})$, on \mathcal{A} such that $\mathcal{A}_{\geq 0}$ is the smallest full subcategory of \mathcal{A} containing each X_α and closed under colimits and extensions. The full subcategory of coconnective objects is characterized by the condition $Y \in \mathcal{A}_{\leq 0}$ if and only if $\text{Map}(\Sigma X_\alpha, Y) = 0$ for each X_α .*

We equip $\text{Ind}(\mathcal{C})$ with the t -structure whose connective part is generated by $\mathcal{C}_{\geq 0}$ and we equip $\text{Ind}(\mathcal{D})$ with the t -structure whose connective part is generated by $F(\mathcal{C}_{\geq 0})$. Applying Ind to the split localization sequence

⁵ We will suppress any further mention of these inclusions.

⁶ A t -structure on a presentable category is accessible if the full subcategory of connective objects is accessible.

$$\mathcal{C}_{\geq 0} \rightarrow \mathcal{C} \rightarrow \mathcal{C}_{< 0}$$

we can then read off that $\text{Ind}(\mathcal{C}_{\geq 0}) \subseteq \text{Ind}(\mathcal{C})_{\geq 0}$ and $\text{Ind}(\mathcal{C}_{< 0}) \subseteq \text{Ind}(\mathcal{C})_{< 0}$ which in turn implies that $\text{Ind}(\mathcal{C}_{\geq 0}) \cong \text{Ind}(\mathcal{C})_{\geq 0}$ and $\text{Ind}(\mathcal{C}_{< 0}) \cong \text{Ind}(\mathcal{C})_{< 0}$.

Lemma 2.3 F^* is t -exact.

Proof F^* sends connective objects to connective objects by construction. To show that F^* preserves coconnectivity we need to check that for every $c \in \mathcal{C}_{\geq 1}$ and $x \in \text{Ind}(\mathcal{C})_{\leq 0}$ the mapping space $\text{Map}(F^*(c), F^*(x))$ is contractible. Since the t -structure on $\text{Ind}(\mathcal{C})$ restricts to compact objects we can write x as a filtered colimit of compact, coconnective objects. This implies (since F^* is a left adjoint) that it suffices to prove $\text{Map}(F^*(c), F^*(x)) = 0$ when x is compact. Via the boundedness of the t -structure on \mathcal{C} this follows from condition (B). \square

At this point we are now ready to prove that the t -structure on $\text{Ind}(\mathcal{D})$ restricts to a bounded t -structure on \mathcal{D} . The main idea in proving this is that on the one hand, (A) guarantees that every object in \mathcal{D} is only finitely many steps away from being in the image of F , while on the other hand, the t -structure on \mathcal{C} can be used to produce a rich collection of compact objects in $\text{Ind}(\mathcal{D})^\heartsuit$.

Proposition 2.4 The t -structure on $\text{Ind}(\mathcal{D})$ restricts to a bounded t -structure on \mathcal{D} . Each $d \in \mathcal{D}^\heartsuit$ has a finite filtration with associated graded in the image of $F|_{\mathcal{C}^\heartsuit} : \mathcal{C}^\heartsuit \rightarrow \mathcal{D}^\heartsuit$.

Proof In order to prove the proposition it will suffice to show that every object $d \in \mathcal{D}$ satisfies the following condition:

- (*) d is t -bounded and each $\pi_i^\heartsuit(d)$ has a finite filtration whose associated graded lies in $\mathcal{C}^\heartsuit \subseteq \text{Ind}(\mathcal{D})^\heartsuit \cap \mathcal{D}$.

Note that if d satisfies (*), then the finite filtrations on the homotopy groups implies that each $\pi_i^\heartsuit(d)$ lies in $\mathcal{D} \subseteq \text{Ind}(\mathcal{D})$ and similarly the t -boundedness implies $d \in \mathcal{D}$.

Since F^* is t -exact $F(c)$ satisfies (*) for each $c \in \mathcal{C}^\heartsuit$. Using hypothesis (A) it will now suffice to show that the full subcategory of objects of \mathcal{D} satisfying (*) is thick. As the condition (*) is stated entirely in terms of homotopy groups, it will suffice to show that the corresponding condition (**) on the level of the heart cuts out a subcategory closed under kernels, cokernels and extensions.⁷

- (**) $d \in \text{Ind}(\mathcal{D})^\heartsuit$ has a finite filtration whose associated graded lies in \mathcal{C}^\heartsuit .

Given two objects $A, B \in \mathcal{D}$, each with a finite filtration in \mathcal{D} and a map $r : A \rightarrow B$, we can paste the filtrations to form a filtration on the cofiber $\text{cof } r$ as follows. Let us view our filtration on A (and similarly on B) as a functor $\text{Fil}_* A : \mathbb{Z} \rightarrow \mathcal{D}$ so that $\text{Fil}_i A$ denotes the i^{th} filtered pieces of A , $\text{gr}_i A = \text{cof}(\text{Fil}_{i-1} A \rightarrow \text{Fil}_i A)$. To be a finite filtration on A means that $\lim_i \text{Fil } A = 0$, $\text{colim}_i \text{Fil } A = A$ and $\text{gr}_* A$ is nonzero only finitely many times. Because the filtrations are finite, by shifting we can suppose

⁷ This uses that fact that these operations suffice to describe how homotopy groups change under cofiber sequences and idempotents

that our filtrations on A and B are such that whenever $\text{gr}_i B, \text{gr}_j A$ are nonzero, $j < i$. r then canonically lifts to a map of filtered objects $\text{Fil}_* A \rightarrow \text{Fil}_* B$ because all the maps of $\text{Fil}_* B$ in the range where $\text{Fil}_* A$ is nonzero are isomorphisms. Taking the cofiber of this map of filtered objects, we obtain a finite filtration on the cofiber. The associated graded of this pasted filtration agrees with $\text{gr}_* B$ at first and switches over to being $\Sigma \text{gr}_* A$.

By pasting filtrations, we learn that the collection of objects satisfying $(**)$ is closed under extensions, since extensions of A by B are cofibers of maps of the form $\Sigma^{-1}A \rightarrow B$. We will handle kernels and cokernels simultaneously. Suppose $A, B \in \text{Ind}(\mathcal{D})^\heartsuit$ satisfy $(**)$ and r is a map between them. We paste the filtrations on A and B to form a filtration on the cofiber, $\text{cof}(r)$. In the spectral sequence associated to this filtered object, the E_1 -page is the homotopy groups of the associated graded of the filtration. By the previous paragraph has a copy of the associated graded of B in topological degree 0, and a copy of the associated graded of A in topological degree 1. This spectral sequence converges to the associated graded of a finite filtration on the π_*^\heartsuit of the cofiber. By hypothesis, the E_1 -page of this spectral sequence involves only objects of \mathcal{C}^\heartsuit . Now, since F is t -exact and fully faithful on \mathcal{C}^\heartsuit , kernels and cokernels of maps between objects in the image of $F|_{\mathcal{C}^\heartsuit}$ remain in the image of $F|_{\mathcal{C}^\heartsuit}$. Consequently, as we run the differentials in this spectral sequence the terms remain in the essential image of \mathcal{C}^\heartsuit . Since the spectral sequence has only finitely many pages we learn that it abuts to a filtration of the desired type on the kernel and cokernel of r (which appears as $\pi_1^\heartsuit(\text{cof}(r))$ and $\pi_0^\heartsuit(\text{cof}(r))$ respectively).

The second conclusion follows from knowing that $(*)$ applies to the objects of $\mathcal{D}^\heartsuit \subseteq \mathcal{D}$. □

We now recall Quillen’s devissage and the theorem of the heart, which we use to finish the proof of the main theorem.

Theorem 2.5 [[2] Barwick’s theorem of the heart] *Let \mathcal{C} be a stable category with bounded t -structure. Then the inclusion $\mathcal{C}^\heartsuit \rightarrow \mathcal{C}$ induces an equivalence on connective K -theory.*

Theorem 2.6 [[21, Theorem 4] Quillen’s devissage] *Let $\mathcal{A} \subset \mathcal{B}$ be an exact fully faithful inclusion of abelian categories with \mathcal{A} closed in \mathcal{B} under subobjects, and such that every object of \mathcal{B} has a finite filtration with associated graded in \mathcal{A} . Then the inclusion $\mathcal{A} \rightarrow \mathcal{B}$ induces an equivalence on connective K -theory.*

Proof (of Theorem 1.3). At this point we have already shown that \mathcal{D} admits a bounded t -structure Proposition (2.4) for which F is t -exact (Lemma 2.3). Next we argue that this t -structure on \mathcal{D} is unique. We start by observing that the filtration condition from Proposition 2.4 implies that the extension closure of $F(\mathcal{C}_{\geq 0})$ is $\mathcal{D}_{\geq 0}$ and the extension closure of $F(\mathcal{C}_{< 0})$ is $\mathcal{D}_{< 0}$. Now suppose we have another t -structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{< 0})$ on \mathcal{D} for which F is t -exact. Using the assumption that F is t -exact the above lets us conclude that $\mathcal{D}_{\geq 0} \subseteq \mathcal{D}_{\geq 0}$ and $\mathcal{D}_{< 0} \subseteq \mathcal{D}_{< 0}$. Using the orthogonality of positive and negative parts of t -structures in turn implies that in fact $\mathcal{D}_{\geq 0} = \mathcal{D}_{\geq 0}$ and $\mathcal{D}_{< 0} = \mathcal{D}_{< 0}$ giving the desired uniqueness.

Finally, we examine the square on K -theory,

$$\begin{array}{ccc}
 K(\mathcal{C}^\heartsuit) & \longrightarrow & K(\mathcal{D}^\heartsuit) \\
 \downarrow & & \downarrow \\
 K(\mathcal{C}) & \xrightarrow{K(F)} & K(\mathcal{D}).
 \end{array}$$

Because the t -structures on \mathcal{C} and \mathcal{D} are bounded, we can use Theorem 2.5 to see that the vertical maps are equivalences. Moreover, the results of [22] and [1] show that K_{-1} of \mathcal{C} and \mathcal{D} vanish.

In order to finish the proof it suffices to show that the top horizontal map is an equivalence, which we show by applying Theorem 2.6. The map $f : \mathcal{C}^\heartsuit \rightarrow \mathcal{D}^\heartsuit$ is fully faithful and exact by construction, and we showed that the filtration condition is satisfied in Proposition 2.4.

It remains to check that if $d \in \mathcal{D}^\heartsuit$ is a subobject of $c \in \mathcal{C}^\heartsuit$, then $d \in \mathcal{C}^\heartsuit$. Using the exactness of the inclusion it will suffice to instead show that $\text{coker}(d \rightarrow c) \in \mathcal{C}^\heartsuit$. Using Proposition 2.4 we can equip d with a finite filtration with associated graded in \mathcal{C}^\heartsuit . The cokernel $\text{coker}(d \rightarrow c)$ can be produced by successively quotienting c by the pieces in the associated graded of the filtration on d , thus we only need to know that quotients by subobjects coming from \mathcal{C}^\heartsuit stay in \mathcal{C}^\heartsuit . This last statement follows from the fact that f is fully faithful and exact. □

Remark 2.7 In fact, the assumption that \mathcal{C} and \mathcal{D} be idempotent complete in Theorem 1.1 can be removed. For \mathcal{C} this does not provide additional generality as any stable category with a bounded t -structure is automatically idempotent complete.⁸ If \mathcal{D} is not idempotent complete, then we can instead apply Theorem 1.1 to the map $\mathcal{C} \rightarrow \mathcal{D}^{\text{idem}}$ to learn that the composite

$$K_0(\mathcal{C}) \rightarrow K_0(\mathcal{D}) \rightarrow K_0(\mathcal{D}^{\text{idem}})$$

is an isomorphism. This means that $K_0(\mathcal{D}) \rightarrow K_0(\mathcal{D}^{\text{idem}})$ is surjective, which implies that \mathcal{D} is idempotent complete by [24].

Remark 2.8 We end this section by observing that the following converse to Theorem 1.3 holds: if we have $F : \mathcal{C} \rightarrow \mathcal{D}$ a map which we can use the theorem of the heart and Quillen’s devissage to prove is a K -equivalence, then (A) and (B) must hold.

To see this, first note that to apply the theorem of the heart we need bounded t -structures on \mathcal{C} and \mathcal{D} so that F is t -exact. To apply Quillen’s devissage, the induced functor on hearts should be fully faithful and its image should generate \mathcal{D}^\heartsuit under extensions. The condition on generating \mathcal{D}^\heartsuit implies (A). The t -exactness of F , plus fully faithfulness on the heart implies (B).

Simply put, this is saying that Theorem 1.3 is essentially equivalent to the combination of Quillen’s devissage and the theorem of the heart. ◁

3 Complements

In this section we discuss a couple points which are complementary to Theorem 1.3. We begin by introducing some ideas from noncommutative geometry which provide a

⁸ It suffices to split idempotents on the heart and abelian categories are automatically idempotent complete.

convenient language for thinking about our main theorem. Then, we discuss variants of the main theorem and prove Theorem 1.1 from the introduction. In the third subsection we consider improvements to Theorem 1.3 which are possible when \mathcal{C}^\heartsuit is Noetherian. We end the section by briefly discussing negative K -groups.

3.1 Some nc geometry

For us noncommutative geometry refers to thinking about small idempotent complete stable categories equipped with a “positive half” closed under finite colimits and extensions. This is quite close to established notions of noncommutative geometry such as in [20], with the notable difference being that we work relative to the sphere rather than relative to a discrete base ring k . We explore this setting in some depth in [5] and in this section we build on the groundwork from that paper.⁹ Before proceeding we remind the reader of the main definitions.

Definition 3.1 We use Cat^{perf} to denote the category of small idempotent complete stable categories. Our main objects of study are objects of $\text{Cat}_{\geq 0}^{\text{perf}}$. This is the category of $\mathcal{C} \in \text{Cat}^{\text{perf}}$ equipped with an idempotent complete prestable¹⁰ full subcategory $\mathcal{C}_{\geq 0}$ that generates \mathcal{C} .

Being prestable amounts to asking that $\mathcal{C}_{\geq 0}$ be closed under finite colimits and extensions. Often, we abuse notation by writing $\mathcal{C} \in \text{Cat}_{\geq 0}^{\text{perf}}$, leaving the subcategory of positive objects, $\mathcal{C}_{\geq 0}$, implicit.

Example 3.2 Given an \mathbb{E}_1 -algebra R , the category compact R -modules, $\text{Mod}(R)^\omega$, naturally lives in $\text{Cat}_{\geq 0}^{\text{perf}}$. The positive objects are those built from R via extensions, finite colimits and retracts. ◀

Given $\mathcal{C} \in \text{Cat}_{\geq 0}^{\text{perf}}$, the subcategory $\text{Ind}(\mathcal{C}_{\geq 0}) \subset \text{Ind}(\mathcal{C})$ determines a t -structure on $\text{Ind}(\mathcal{C})$ (see Lemma 2.2). In fact, $\mathcal{C}_{\geq 0}$ can be recovered from the t -structure as $\text{Ind}(\mathcal{C}_{\geq 0}) \cap \mathcal{C}$.

Example 3.3 In the t -structure associated to Example 3.2, a connective object is one built out of copies of R under colimits and extensions, and a coconnective object is one whose underlying spectrum is coconnective. ◀

Definition 3.4 Given $\mathcal{C} \in \text{Cat}_{\geq 0}^{\text{perf}}$,

- \mathcal{C} is *regular* if the t -structure on $\text{Ind}(\mathcal{C})$ restricts to \mathcal{C} (i.e truncations of compact objects are compact),
- \mathcal{C} is *bounded* if each $c \in \mathcal{C}$ is bounded as an object of $\text{Ind}(\mathcal{C})$,
- a functor $\mathcal{C} \rightarrow \mathcal{D}$ is *quasi-affine* if its image generates \mathcal{D} under finite colimits and retracts and

⁹ Even though we cite results in [5], we do not use anything particularly difficult from there, and so the results here can be considered independent of that paper.

¹⁰ As introduced in [16, Appendix C]

- a quasi-affine functor $\mathcal{C} \rightarrow \mathcal{D}$ is *unipotent* if it induces a fully faithful functor on $\text{Ind}(\mathcal{C})^\heartsuit$. ◁

Example 3.5 We say a ring R is regular if $\text{Mod}(R)^\omega$ is regular. If R is a discrete ring then, as a result of well-known arguments, R is regular in this sense iff R is left regular coherent. For a proof stating things this way, see [5, Proposition 2.4]. ◁

Theorem 1.1 gives sufficient conditions for a coconnective ring R to be regular, but they are not necessary. Moreover for a general regular coconnective ring, $K(R)$ may not agree with $K(\pi_0 R)$. See Example 5.14 for an example of such a regular coconnective ring.

Example 3.6 We show in [5, Proposition 2.16] that if \mathcal{C} is regular, and $n \neq 0$, then $\mathcal{C}[x_n]$ is regular. ◁

Remark 3.7 If we think in terms of categories of quasicoherent sheaves, the reasoning behind the term quasi-affine is relatively transparent.

The term unipotent bears more explanation. The key identifying features of a unipotent group are that maps between trivial representations can be computed on underlying and every representation is built out of extensions of trivial reps. Our definition takes these properties as the definition of unipotent.

Note that conditions (A) and (B) of Theorem 1.3 are equivalent to saying that F is unipotent. This lets us reinterpret Theorem 1.3 as saying that bounded regularity can be transferred along unipotent maps, and that such maps induce K -equivalences. This reinterpretation is a precise form of the slogan in the introduction. ◁

3.2 Other forms of the main theorem

In practice unipotence can be difficult to check so we recall an equivalent condition which is often more transparent. The functor $F^* : \text{Ind}(\mathcal{C}) \rightarrow \text{Ind}(\mathcal{D})$ has a right adjoint F_* , which preserves colimits since F^* preserves compact objects.

Lemma 3.8 ([5, Corollary 4.12]) *For a map $F : \mathcal{C} \rightarrow \mathcal{D}$ as in Theorem 1.3 condition (B) is equivalent to:*

(B') *For every $c \in \mathcal{C}^\heartsuit$, the cofiber of the unit map $c \rightarrow F_*F^*(c)$ is in $\text{Ind}(\mathcal{C})_{\leq -1}$.*

Proof sketch Unraveling (B') gives the statement that the cofiber of $\text{map}(d, c) \rightarrow \text{map}(Fd, Fc)$ is coconnected for all $c \in \mathcal{C}^\heartsuit$ and $d \in \mathcal{C}_{\geq 0}$. This visibly implies (B). The key point in proving the reverse implication is using the fact that \mathcal{C}^\heartsuit is closed under extensions in $\text{Ind}(\mathcal{C})$.¹¹ ◻

We now provide a version of Theorem 1.3 for categories of modules over an \mathbb{E}_1 -algebra from which Theorem 1.1 will follow.

Proposition 3.9 *Let $f : A \rightarrow B$ be a map of \mathbb{E}_1 -algebras such that*

1. *$\text{Mod}(A)^\omega$ is bounded and regular and*

¹¹ In this paper we only use that (B') implies (B) and not the reverse implication.

2. $\text{cof}(f)$ has Tor amplitude in $[-\infty, -1]$ as a right A -module,

then $\text{Mod}(B)^\omega$ is bounded and regular; the base-change functor $(-)\otimes_A B$ is a t -exact K -equivalence, and K_{-1} of A and B vanish.

Proof We apply Theorem 1.3 to the base-change functor

$$B \otimes_A - : \text{Mod}(A)^\omega \rightarrow \text{Mod}(B)^\omega.$$

This functor is quasi-affine since A is sent to B which is a generator. Condition (B') of Lemma 3.8 asks that for every $N \in \text{Mod}_A^\heartsuit$ the A -module $\text{cof}(f)\otimes_A N$ be coconnected. This is equivalent to the given Tor amplitude bound on $\text{cof}(f)$. \square

Proof (of Theorem 1.1). We apply Proposition 3.9 to the connective cover map $f : \pi_0 R \rightarrow R$. From Example 3.5 we know that $\pi_0 R$ is left regular coherent iff $\text{Mod}(\pi_0 R)^\omega$ is regular. Boundedness is automatic. Since $\tau_{<0} R \simeq \text{cof}(f)$, the Tor amplitude bounds in Proposition 3.9 and Theorem 1.1 match up. \square

Remark 3.10 Amplifying Remark 2.8, we note that the conditions of Theorem 1.1 are actually equivalent to (A) and (B') for the connective cover map, implying a converse to this theorem. \triangleleft

Remark 3.11 The reader might wonder when the the abelian categories \mathcal{C}^\heartsuit and \mathcal{D}^\heartsuit appearing in Theorem 1.3 are equivalent. We remark that this will be the case as soon as F satisfies:

(C) for every $c \in \mathcal{C}^\heartsuit$, the cofiber of the unit map $c \rightarrow F_* F^*(c)$ is ≤ -2 in the t -structure on $\text{Ind}(\mathcal{C})$.¹²

\triangleleft

3.3 The noetherian case

In situation of Theorem 1.3 if we further assume that the heart of \mathcal{C} is Noetherian, then we can draw stronger conclusions about K -theory and the induced t -structure on \mathcal{D} .

Lemma 3.12 *In situation of Theorem 1.3, if \mathcal{C}^\heartsuit is Noetherian, then the heart of the induced t -structure on \mathcal{D} is Noetherian as well. Moreover, in the square of nonconnective K -theories,*

$$\begin{array}{ccc} K^{\text{nc}}(\mathcal{C}^\heartsuit) & \longrightarrow & K^{\text{nc}}(\mathcal{D}^\heartsuit) \\ \downarrow & & \downarrow \\ K^{\text{nc}}(\mathcal{C}) & \longrightarrow & K^{\text{nc}}(\mathcal{D}) \end{array}$$

all maps are equivalences.

¹² See [5, Proposition 7.2] for details.

Proof We would first like to show that every $d \in \mathcal{D}^\heartsuit$ is Noetherian. By Proposition 2.4, d has a finite filtration with associated graded in $F(\mathcal{C}^\heartsuit)$, so since Noetherian objects are closed under extensions, it suffices to show that $F(c)$ is Noetherian for each $c \in \mathcal{C}^\heartsuit$. As argued in the proof of Theorem 1.3, F is fully faithful with image closed under passing to subobjects. This implies that the lattice of subobjects of $F(c)$ agrees with that of c , so $F(c)$ is Noetherian since c is.

Now by applying the vanishing theorems of [22] and [1], the negative K -groups of categories with a bounded t -structure with Noetherian heart vanish, so the square of equivalences in Theorem 1.3 extends to negative K -theory. \square

Next we examine the interpretation of condition (B) as “unipotence” more closely. Let $f : \mathcal{C}^\heartsuit \rightarrow \mathcal{D}^\heartsuit$ denote the restriction of F to \mathcal{C}^\heartsuit , as well as its Ind-completion. At the moment we know that

- (1) $f : \mathcal{C}^\heartsuit \rightarrow \mathcal{D}^\heartsuit$ is fully faithful.
- (2) Each $d \in \mathcal{D}^\heartsuit$ has a finite filtration with associated graded in the image of f .
- (3) Every subobject of $f(c)$ comes from a subobject of c .

Note that the finite filtrations are not guaranteed to be functorial,¹³ but when \mathcal{D} is Noetherian, functoriality comes for free in the form of the *socle filtration*:

Construction 3.13 At the level of Ind-categories the functor f has right adjoint given by $g := \tau_{\geq 0}G(-)$. Since f is a fully faithful left adjoint it exhibits $\text{Ind}(\mathcal{C}^\heartsuit)$ as a coreflective subcategory of $\text{Ind}(\mathcal{D}^\heartsuit)$. We let $\text{soc}_0(M)$ denote $fg(M)$, which sits as a subobject of M via the counit map.

We define $\text{soc}_n(-)$ inductively via the pullback

$$\begin{array}{ccc}
 \text{soc}_n & \twoheadrightarrow & \text{soc}_0(\text{Id}/\text{soc}_{n-1}) \\
 \downarrow & \lrcorner & \downarrow \\
 \text{soc}_{n-1} & \hookrightarrow \text{Id} & \twoheadrightarrow \text{Id}/\text{soc}_{n-1}.
 \end{array}$$

The key property of the socle filtration is that $\text{soc}_n/\text{soc}_{n-1}$ is in the image of f for every n . Since the maps $\text{Id}/\text{soc}_{n-1} \rightarrow \text{Id}/\text{soc}_n$ become zero after applying g and since $g = \tau_{\geq 0}G$ and G detects coconnectivity we learn that

$$\text{colim}_n \text{soc}_n \rightarrow \text{Id}$$

is an equivalence, i.e. the socle filtration is exhaustive.

Now, using our Noetherian-ness hypothesis we know that arbitrary subobjects of compact objects are compact in $\text{Ind}(\mathcal{D}^\heartsuit)$, therefore the socle filtration restricts to a functorial filtration on \mathcal{D}^\heartsuit that is finite on each object. \triangleleft

Remark 3.14 The construction of a bounded t -structure together with socle filtrations can be interpreted as a generalization of [11, Theorem 8.1] where a similar result is proven for dg-algebras with strong finiteness assumptions. \triangleleft

¹³ Quillen’s devissage, doesn’t require a functorial filtration, but merely an object-wise one.

It is important to note that outside of the Noetherian setting the socle filtration, while still existent at the level of Ind-categories, need not restrict to compact objects.

Example 3.15 Let \mathcal{C} denote the category of pairs (V_0, V_1) where V_0 is a $k[x_1, \dots]$ -module and V_1 is a k -module. Since the infinite dimensional affine space is regular coherent, \mathcal{C} has a bounded t -structure. Let \mathcal{D} denote the category of triples $(V_0, V_1, V_0 \otimes_{k[x_1, \dots]} k \rightarrow V_1)$.

The natural functor $\mathcal{C} \rightarrow \mathcal{D}$ which uses the zero map is fully faithful on the heart, therefore the hypotheses of Theorem 1.3 are satisfied. On the other hand, the socle of

$$(k[x_1, \dots], k, k[x_1, \dots] \twoheadrightarrow k)$$

is (I, k) and the augmentation ideal of $k[x_1, \dots]$ is not compact. ◁

3.4 Negative K -theory

It would be desirable to extend Theorem 1.3 to negative K -theory, however both the theorem of the heart and Quillen’s devissage only apply to connective K -theory in their current form. For that reason we ask the following question (which we hope has a positive answer):

Question 3.16 Do the theorem of the heart and devissage hold for negative K -theory?

As discussed above, if \mathcal{C}^\heartsuit is Noetherian, then \mathcal{D}^\heartsuit is Noetherian and so the negative K -theory vanishes. This might suggest that one should approach this question by proving a vanishing statement for negative K -theory. However, the example from [19] shows that in general regularity does not imply the vanishing of negative K -groups.

In order to probe question of this type more closely we examine the relation between *stable coherence* and the vanishing of negative K -theory in [5, Section 3.2]¹⁴. We reproduce the statements proved therein for the convenience of the reader interested in thinking about Question 3.16.

Definition 3.17 Given a category \mathcal{C} with bounded t -structure we say that \mathcal{C} is \mathbb{A}^n -coherent if the finitely presented $\mathbb{Z}[t_1, \dots, t_n]$ -modules in $\text{Ind}(\mathcal{C}^\heartsuit)$ form an abelian category¹⁵. If \mathcal{C} is \mathbb{A}^n -coherent for all n , then we say it is *stably coherent*. ◁

Example 3.18 In [8, Example 7.3.13], it is shown that an infinite product of copies of the ring $\mathbb{Q}[[x, y]]$ is regular coherent, but this ring is not \mathbb{A}^1 -coherent (demonstrating that this is a non-trivial condition on a regular coherent ring). ◁

The following lemma uses the vanishing results of [1].

Lemma 3.19 ([5, Corollary 3.14, Lemma 3.17]) *If \mathcal{C} is regular and \mathbb{A}^n -coherent, then the first $n + 1$ negative K -groups of \mathcal{C} vanish.*

¹⁴ see also [1, Section 3.5] for a similar discussion.

¹⁵ If \mathcal{C} is the category of perfect R -modules for R a discrete ring, then this is equivalent to asking that $R[t_1, \dots, t_n]$ be left coherent.

Proposition 3.20 ([5, Proposition 3.18]) *If we are given a functor $F : \mathcal{C} \rightarrow \mathcal{D} \in \text{Cat}^{\text{perf}}$ and a bounded \mathbb{A}^n -coherent t -structure on \mathcal{C} such that the conditions of Theorem 1.3 are satisfied, then the induced t -structure on \mathcal{D} is \mathbb{A}^n -coherent as well.*

4 K -theory of pushouts

In this section we prove Theorem 4.11 (a corollary of which is Theorem 1.5 from the introduction) which says that K -theory preserves pushouts of (well-behaved) regular prestable categories. This theorem arises from the examining the interaction of our main theorem with the Land–Tamme \odot -product. In fact, the idea that a result like Theorem 1.1 should be true was suggested to the authors by Markus Land and Georg Tamme with the intention of using such a result to compute the K -theory of pushouts.

4.1 A review on the \odot -product

The Land–Tamme \odot -product is a relatively new operation on \mathbb{E}_1 -algebras (and categories more generally) first introduced in [12], with generalizations appearing in [4] and the forthcoming [13]. Here we roughly follow [13] in our formulation of this operation.

Construction 4.1 Given a pair of categories \mathcal{B} and \mathcal{C} in Cat^{perf} and an arrow $f \in \text{Fun}^L(\text{Ind}(\mathcal{C}), \text{Ind}(\mathcal{B}))$ we can form the oplax limit of f , which we denote $\mathcal{B} \times_f \mathcal{C} \in \text{Cat}^{\text{perf}}$. This is the category of triples (b, c, r) , where $b \in \mathcal{B}, c \in \mathcal{C}$, and $r : b \rightarrow f(c)$ is a map. For our purposes, the key observation about $\mathcal{B} \times_f \mathcal{C}$ is that the forgetful map $\mathcal{B} \times_f \mathcal{C} \rightarrow \mathcal{B} \times \mathcal{C}$ induces an equivalence at the level of nc motives

$$\mathcal{U}_{\text{loc}}(\mathcal{B} \times_f \mathcal{C}) \cong \mathcal{U}_{\text{loc}}(\mathcal{B} \times \mathcal{C}) \cong \mathcal{U}_{\text{loc}}(\mathcal{B}) \oplus \mathcal{U}_{\text{loc}}(\mathcal{C}).$$

Adding another layer, associated to each square

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow f \\ \mathcal{B} & \longrightarrow & \text{Ind}(\mathcal{B}). \end{array}$$

we have an induced map $\mathcal{A} \rightarrow \mathcal{B} \times_f \mathcal{C}$ in Cat^{perf} and we define the Land–Tamme \odot -product $\mathcal{B} \odot_{\mathcal{A}}^f \mathcal{C}$ to be the cofiber of this map. This cofiber sequence in Cat^{perf} provides a pushout in nc motives

$$\begin{array}{ccc} \mathcal{U}_{\text{loc}}(\text{im } \mathcal{A}) & \longrightarrow & \mathcal{U}_{\text{loc}}(\mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{U}_{\text{loc}}(\mathcal{B}) & \longrightarrow & \mathcal{U}_{\text{loc}}(\mathcal{B} \odot_{\mathcal{A}}^f \mathcal{C}) \end{array}$$

where $\text{im } \mathcal{A}$ denotes the image of \mathcal{A} inside $\mathcal{B} \times_f \mathcal{C}$. ◁

Remark 4.2 If we specialize to the case where \mathcal{A}, \mathcal{B} and \mathcal{C} are module categories of \mathbb{E}_1 -algebras A, B and C , then we can move down a categorical level:

- (1) The category $\text{Fun}^L(\text{Mod}(C), \text{Mod}(B))$ can be identified with $\text{Mod}(B \otimes C^{\text{op}})$, meaning the bonding map is just a choice of (B, C) -bimodule M .
- (2) $\text{im } \mathcal{A}$ is generated by the image of A , meaning $\text{im}(\mathcal{A}) \cong \text{Mod}(\text{im } A)^\omega$ where $\text{im } A$ is the endomorphism algebra of the image of A .
- (3) If the functors $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{A} \rightarrow \mathcal{C}$ came from \mathbb{E}_1 -algebra maps $A \rightarrow B$ and $A \rightarrow C$, then $\mathcal{B} \odot_{\mathcal{A}}^f \mathcal{C}$ is generated by the image of B (which is equivalent to the image of C). We let $B \odot_A^M C$ denote the ring of endomorphisms of this object where M is the bimodule used as the bonding map.

◁

Speaking practically, the fundamental difficulty in working with the \odot -product lies in identifying the categories $\text{im}(\mathcal{A})$ and $\mathcal{B} \odot_{\mathcal{A}}^f \mathcal{C}$. A fundamental insight of Land and Tamme is that in many cases of interest these categories are surprisingly computationally accessible.

Example 4.3 In [12], where the \odot -product was introduced, the following example of Construction 4.1 is analyzed. Suppose we are given a pullback square of \mathbb{E}_1 -algebras

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & D. \end{array}$$

Using D as our (B, C) -bimodule, $\text{Mod}(A)$ for \mathcal{A} and the map $B \rightarrow D$ of (B, A) -bimodules for the natural transformation, we can construct a \odot -product $B \odot_A^D C$. In this situation they prove that $\text{im}(A) \cong A$ and the spectrum underlying $B \odot_A^D C$ is equivalent to $B \otimes_A C$ as a (B, C) -bimodule. Furthermore, they show in [12, Proposition 1.13] that the underlying C -bimodule of $B \odot_A^D C$ is the cofiber of the map $I \otimes_A C \rightarrow C$, where I is the fiber of $C \rightarrow D$.

◁

In the forthcoming [13] another, somewhat dual, situation is analyzed.

Theorem 4.4 ([13]) *Given a span $B \xleftarrow{b} A \xrightarrow{c} C$ in Cat^{perf} there is a square*

$$\begin{array}{ccc} A & \xrightarrow{c} & C \\ \downarrow b & \nearrow b^* \eta_c & \downarrow b^* c_* \\ B & \longrightarrow & \text{Ind}(B) \end{array}$$

and an equivalence of the associated \odot -product with the pushout of the span,

$$B \odot_A^{b^* c_*} C \simeq B \coprod_A C.$$

Corollary 4.5 ([13]) *Given a span $B \leftarrow A \rightarrow C$ of \mathbb{E}_1 -algebras we have an equivalence*

$$B \odot_A^{B \otimes_A C} C \simeq B \coprod_A C$$

of the \odot -product with the pushout of the span in \mathbb{E}_1 -algebras.

Remark 4.6 In Theorem 4.5 if we have a span of commutative algebras instead, then the base-change equivalence $B \otimes_A C \otimes_C - \cong B \otimes_A -$ allows us to recognize that we are actually in the situation of Example 4.3 for the cospan $B \rightarrow B \otimes_A C \leftarrow C$. The benefit of making this identification is that we can identify $\text{im}(A)$ as the pullback of this cospan. \triangleleft

Lemma 4.7 *The \odot -product is compatible with base-change, i.e.*

$$(B \odot_A^f C) \otimes D \cong (B \otimes D) \odot_{A \otimes D}^{f \otimes D} (C \otimes D).$$

Proof This follows from the fact that $- \otimes B$ preserves fully faithful maps and localization sequences (see for example [1, Lemma 3.3, Corollary 3.5]) and commutes with pullbacks, lax pullbacks and oplax limits of arrows. \square

The \odot -product allows us to produce examples of equivalences of nc motives which do not arise from equivalences of categories. We end our recollection by working through a pair of examples which illustrate this flexibility phenomenon.

Definition 4.8 Let $\mathbb{S}[x_n]$ denote the polynomial algebra on a generator in degree n . We let N_n denote the cofiber

$$N_n := \text{cof}(\mathcal{U}_{\text{loc}}(\mathbb{S}) \rightarrow \mathcal{U}_{\text{loc}}(\mathbb{S}[x_n]))$$

and use N to mean N_0 . \triangleleft

If we think about N as a homology theory on nc motives it is the NK -theory of Bass which measures the failure of \mathbb{A}^1 -invariance.

Example 4.9 Consider the span of commutative algebras $\mathbb{S} \leftarrow \mathbb{S}[x_n] \rightarrow \mathbb{S}$. Applying Corollary 4.5 we obtain a pullback of nc motives

$$\begin{array}{ccc} \mathcal{U}_{\text{loc}}(\text{im } \mathbb{S}[x_n]) & \longrightarrow & \mathcal{U}_{\text{loc}}(\mathbb{S}) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{U}_{\text{loc}}(\mathbb{S}) & \longrightarrow & \mathcal{U}_{\text{loc}}(\mathbb{S}[x_{n+1}]). \end{array}$$

Using Remark 4.6 we can identify $\text{im}(\mathbb{S}[x_n])$ as $\mathbb{S}[\epsilon_n]$ (the exterior algebra on a class in degree n) since $\mathbb{S} \otimes_{\mathbb{S}[x_n]} \mathbb{S} \simeq \mathbb{S}[\epsilon_{n+1}]$ and the pullback moves the exterior generator down a degree. Since the diagram above is diagonally symmetric we have a splitting

$$\mathcal{U}_{\text{loc}}(\mathbb{S}[\epsilon_n]) \simeq \mathbb{1} \oplus \Sigma^{-1} N_{n+1}.$$

\triangleleft

We learned of next example, which allows us to turn a copy of the coordinate axes in the plane into a polynomial algebra, from Markus Land and Georg Tamme.

Example 4.10 Consider the algebra $R := \mathbb{S}[x_a, x_b]/(x_a x_b)$ which is built from the pullback square on the left below (where x_a is in degree a and x_b is in degree b).

$$\begin{array}{ccc}
 \mathbb{S}[x_a, x_b]/(x_a x_b) & \longrightarrow & \mathbb{S}[x_a] \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbb{S}[x_b] & \longrightarrow & \mathbb{S}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{U}_{\text{loc}}(\mathbb{S}[x_a, x_b]/(x_a x_b)) & \longrightarrow & \mathcal{U}_{\text{loc}}(\mathbb{S}[x_a]) \\
 \downarrow & \lrcorner & \downarrow \\
 \mathcal{U}_{\text{loc}}(\mathbb{S}[x_b]) & \longrightarrow & \mathcal{U}_{\text{loc}}(\mathbb{S}[x_a] \coprod_{\mathbb{S}[x_a, x_b]} \mathbb{S}[x_b])
 \end{array}$$

Applying Corollary 4.5 and Remark 4.6 we then obtain the pullback of nc motives on the right. In fact, we can simplify this by exhibiting an equivalence of \mathbb{E}_1 -algebras

$$\mathbb{S}[x_a] \coprod_{\mathbb{S}[x_a, x_b]} \mathbb{S}[x_b] \simeq \mathbb{S}[x_{a+b+2}].$$

To show this, suppose first that $a, b > 0$. Then using the fact that

$$\mathcal{S}_* \xrightarrow{\Sigma_+^\infty \Omega^-} \text{Alg}(\text{Sp})$$

is a left adjoint and so preserves pushouts, we reduce to the pushout square

$$\begin{array}{ccc}
 \mathcal{S}^{a+1} \times \mathcal{S}^{b+1} & \longrightarrow & \mathcal{S}^{a+1} \\
 \downarrow & \lrcorner & \downarrow \\
 \mathcal{S}^{b+1} & \longrightarrow & \mathcal{S}^{a+b+3}.
 \end{array}$$

We use a trick in order to extend this equivalence to the case where a, b are not strictly positive. First, we lift the pushout above to a pushout in graded rings where x_a and x_b are in grading 1. Since the functor forgetting the grading on an \mathbb{E}_1 -algebra preserves colimits, it will suffice to compute the pushout in the graded setting. Next we use the \mathbb{E}_2 -monoidal shearing functor which suspends by $2n$ in grading n constructed in [14] to reduce to the case where a, b are positive.

In the graded setting the generator x_{a+b+2} is in grading 2 and as a consequence of the fact that $\mathbb{S}[x_a]$ is the free graded algebra on the class x_a in grading 1 we obtain a factorization $\mathbb{S}[x_a] \rightarrow \mathbb{S} \rightarrow \mathbb{S}[x_{a+b+2}]$. With control over the maps in the square above we now obtain an equivalence of nc motives

$$\mathcal{U}_{\text{loc}}(\mathbb{S}[x_a, x_b]/(x_a x_b)) \simeq \mathbb{1} \oplus N_a \oplus N_b \oplus \Sigma^{-1} N_{a+b+2}.$$

◁

4.2 K-theory of pushouts

There is a sharp contrast between the ideas behind the Land–Tamme \odot -product and our main theorem. The \odot -product arises from 2-categorical maneuvers and essentially operates at the level categories and nc motives. Meanwhile our main theorem is specific to K -theory, exploiting additive but non-exact operations (such as truncation) in an essential way. The complementary nature of these approaches allows us to combine them to surprising effect.

Theorem 4.11 *Suppose we are given a span $\mathcal{B} \xleftarrow{b} \mathcal{A} \xrightarrow{c} \mathcal{C}$ in Cat^{perf} where \mathcal{A} is equipped with a bounded t -structure. If we assume that*

(D) *the induced functor $\mathcal{A}^\heartsuit \rightarrow \mathcal{B} \times_{b^*c_*} \mathcal{C}$ is fully faithful,*

then connective K -theory preserves the pushout of the span, i.e the diagram below is a pushout square.

$$\begin{array}{ccc} K(\mathcal{A}) & \longrightarrow & K(\mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ K(\mathcal{B}) & \longrightarrow & K(\mathcal{B} \coprod_{\mathcal{A}} \mathcal{C}) \end{array}$$

Proof From Theorem 4.4, we have a pushout square

$$\begin{array}{ccc} K^{\text{nc}}(\text{im } \mathcal{A}) & \longrightarrow & K^{\text{nc}}(\mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ K^{\text{nc}}(\mathcal{B}) & \longrightarrow & K^{\text{nc}}(\mathcal{B} \coprod_{\mathcal{A}} \mathcal{C}). \end{array}$$

Condition (D) implies that the functor $\mathcal{A} \rightarrow \text{im}(\mathcal{A})$ satisfies the hypotheses of Theorem 1.3, so we have an equivalence $K(\mathcal{A}) \simeq K(\text{im } \mathcal{A})$, and K_{-1} of each vanishes. This implies that the square above remains a pushout when we take connective covers and replace $K(\text{im } \mathcal{A})$ by $K(\mathcal{A})$. □

In order to make this theorem easier to apply we give a simpler condition which implies (D) and is more natural to check in practice.

Lemma 4.12 *In the situation of Theorem 4.11 condition (D) is implied by*

(D') *The functors $\mathcal{A}^\heartsuit \rightarrow \mathcal{B}$ and $\mathcal{A}^\heartsuit \rightarrow \mathcal{C}$ are faithful.*

Proof Let F denote the functor $\mathcal{A} \rightarrow \mathcal{B} \times_{b^*c_*} \mathcal{C}$. Using Lemma 3.8 it suffices to show that $\text{cof}(a \rightarrow F_*F^*(a))$ is ≤ -1 for each $a \in \mathcal{A}^\heartsuit$. In order to proceed we'll give a formula for F_*F^* . From the pullback square

$$\begin{array}{ccc} \text{Map}_{\mathcal{B} \times_{b^*c_*} \mathcal{C}}(F^*x, F^*y) & \longrightarrow & \text{Map}_{\mathcal{B}}(b^*x, b^*y) \\ \downarrow & \lrcorner & \downarrow \\ \text{Map}_{\mathcal{C}}(c^*x, c^*y) & \xrightarrow{b^*c_*} \text{Map}_{\mathcal{B}}(b^*c_*c^*x, b^*c_*c^*y) & \xrightarrow{b^*\eta_{c^*}} \text{Map}_{\mathcal{B}}(b^*x, b^*c_*c^*y) \end{array}$$

natural in both x and y we learn that F_*F^* sits in a pullback square

$$\begin{array}{ccc} F_*F^* & \longrightarrow & b_*b^* \\ \downarrow & \lrcorner & \downarrow b_*b^*\eta_c \\ c_*c^* & \xrightarrow{\eta_{b \circ c^*c^*}} & b_*b^*c_*c^*. \end{array}$$

We can then read off that

$$\text{cof}(\text{Id} \rightarrow F_*F^*) \simeq \Sigma^{-1} \text{cof}(\text{Id} \rightarrow b_*b^*) \circ \text{cof}(\text{Id} \rightarrow c_*c^*)$$

where \circ is the composition monoidal structure on $\text{Fun}^L(\text{Mod}(A), \text{Mod}(A))$. Using [5, Remark 4.9] (which is a variant of Lemma 3.8) and compatibility with colimits we can reformulate the faithfulness hypothesis as saying that $\text{cof}(\text{Id} \rightarrow c_*c^*)$ and $\text{cof}(\text{Id} \rightarrow b_*b^*)$ preserve coconnectivity. Composing and desuspending we obtain the desired coconnectivity bound on $\text{cof}(\text{Id} \rightarrow F_*F^*)$. \square

For discrete rings condition (D') has a simple interpretation: A map $A \rightarrow B$ is faithful on the heart exactly when B is right faithfully flat as an A -module (see [5, Lemma 4.7]). Consequently, we obtain the following corollary, which appeared in the introduction as Theorem 1.5.

Corollary 4.13 *Suppose $B \xleftarrow{f} A \xrightarrow{g} C$ is a span of discrete rings where A is left regular coherent and both f and g are right faithfully flat. Then connective K -theory preserves the pushout of this span.*

Remark 4.14 Note that (D) does not imply (D'). For example if X and Y are (well-behaved) smooth varieties which form a Zariski covering of Z , then (D) is satisfied for the span

$$\text{QCoh}(X) \leftarrow \text{QCoh}(Z) \rightarrow \text{QCoh}(Y)$$

while (D') need not be satisfied. \triangleleft

Remark 4.15 Corollary 4.13 (and in turn Lemma 4.12 and Theorem 4.11) can be viewed as a generalization of [25, Theorems 1 and 4] where the stronger condition that $f : A \rightarrow B$ and $g : A \rightarrow C$ are pure inclusions¹⁶ was imposed. \triangleleft

5 Applications and examples

In this section we work through a collection of applications and examples which use Theorem 1.1. Of particular note are

- (Prop.5.1) which proves \mathbb{A}^1 -invariance for regular categories.
- (Prop.5.2) which proves \mathbb{A}^n -invariance for regular categories in high degrees.
- (Prop.5.10) which analyzes the K -theory of unipotent local systems.
- (Exm.5.13, 5.15 and 5.16) which show that the conditions of Theorem 1.1 are sharp.

5.1 Invariance theorems

We give a short proof of \mathbb{A}^1 -invariance of K -theory for categories with a bounded t -structure. This result was first proven for regular Noetherian rings by Quillen in his foundational paper [21]. Building on this we then prove that $K_j(-)$ is \mathbb{A}^n -invariant once $j \geq n - 1$ (again for categories with a bounded t -structure). Using Theorem 4.11 we then extend \mathbb{A}^1 -invariance to the case of adjoining free variables generalizing the main results of [7].

¹⁶ This asks that B have a splitting $B \cong f(A) \oplus I$ as an A -bimodule where I is a projective right A -module.

Proposition 5.1 [\mathbb{A}^1 -invariance for regular categories] *If $\mathcal{C} \in \text{Cat}^{\text{perf}}$ admits a bounded t -structure, then $K(\mathcal{C}) \simeq K(\mathcal{C}[x_0])$.*

Proof In order to prove this we must show that $K^{\text{nc}}(\mathbb{N} \otimes \mathcal{C})$ vanishes in non-negative degrees. Applying Theorem 1.3 to the map $\mathcal{C} \rightarrow \mathcal{C}[\epsilon_{-1}]$ and applying the equivalence of motives $\mathcal{U}_{\text{loc}}(\mathcal{C}[\epsilon_{-1}]) \cong \mathcal{U}_{\text{loc}}(\mathcal{C}) \oplus \Sigma^{-1}\mathbb{N} \otimes \mathcal{U}_{\text{loc}}(\mathcal{C})$ from Example 4.9 we learn that $K^{\text{nc}}(\mathbb{N} \otimes \mathcal{C})$ vanishes in non-negative degrees as desired.¹⁷ \square

Just as \mathbb{N} controls \mathbb{A}^1 -invariance, \mathbb{A}^n -invariance is controlled by tensor-powers of \mathbb{N} . Using the same ideas we can show that K -theory is \mathbb{A}^n -invariant in sufficiently large degrees as well.

Proposition 5.2 [\mathbb{A}^n - invariance for regular categories] *Suppose $\mathcal{C} \in \text{Cat}^{\text{perf}}$ admits a bounded t -structure. Then $\tau_{\geq n-1}K(\mathcal{C}) \simeq \tau_{\geq n-1}K(\mathcal{C}[x_1, \dots, x_n])$, where $|x_i| = 0$.*

Proof From the equivalence

$$\mathcal{U}_{\text{loc}}(\mathbb{S}[x_1, \dots, x_n]) \simeq \mathcal{U}_{\text{loc}}(\mathbb{S}[x]^{\otimes n}) \simeq (\mathcal{U}_{\text{loc}}(\mathbb{S}[x]))^{\otimes n} \simeq (\mathbb{1} \oplus \mathbb{N})^{\otimes n}$$

we can read off that the obstructions to \mathbb{A}^n -invariance in degree j are $K_j(\mathbb{N}^{\otimes k} \otimes \mathcal{C})$ for $1 \leq k \leq n$. Using Example 4.9 we can find $\Sigma^{-k}\mathbb{N}^{\otimes k} \otimes \mathcal{C}$ as a summand in $\mathcal{U}_{\text{loc}}(\mathcal{C}[\epsilon_1, \dots, \epsilon_k])$ (where each exterior generator is in degree -1). Applying Theorem 1.3 to the map $\mathcal{C} \rightarrow \mathcal{C}[\epsilon_1, \dots, \epsilon_k]$ we learn that $K^{\text{nc}}(\Sigma^{-k}\mathbb{N}^{\otimes k} \otimes \mathcal{C})$ vanishes in degrees ≥ -1 , which lets us conclude. \square

Corollary 5.3 *Let R be a left regular coherent ring. Then $K_i(R) \cong K_i(R[x_1, \dots, x_n])$ for $i \geq n - 1$.*

As mentioned in the introduction, the above corollary, which is more subtle for $n > 1$, was already known when R is a discrete ring, where it follows from the Farrell–Jones conjecture for the groups \mathbb{Z}^n .

Proposition 5.4 [Free generator invariance] *Let $\mathcal{C} \in \text{Cat}^{\text{perf}}$ have a bounded t -structure. Then $K(\mathcal{C}) \simeq K(\mathcal{C}\{x_1, \dots, x_n\})$, where $|x_i| = 0$.*

Proof We proceed by induction on n with base-case given by Proposition 5.1. If we consider the pushout of categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i} & \mathcal{C}\{x_1, \dots, x_{n-1}\} \\ \downarrow j & \lrcorner & \downarrow \\ \mathcal{C}[x] & \longrightarrow & \mathcal{C}\{x_1, \dots, x_n\}. \end{array}$$

then condition (D') holds since the arrows labeled i and j are t -exact at the level of ind-completions and each have a section (sending all the x 's to zero). As a consequence we can apply Theorem 4.11 and conclude. \square

Corollary 5.5 *Let R be a left regular coherent ring, then $K(R) \simeq K(R\{x_1, \dots, x_n\})$.*

¹⁷ In degree zero this uses that K_{-1} of \mathcal{C} and $\mathcal{C}[\epsilon_{-1}]$ both vanish.

In fact, Corollary 5.5 is a special case of the next example, which allows for a more general module in place of the indeterminants x_1, \dots, x_n .

Example 5.6 Applying Corollary 4.5 to the span of rings

$$R \leftarrow R\{\Sigma^{-1}M\} \rightarrow R$$

we get a pullback square

$$\begin{CD} K^{nc}(R \oplus \Sigma^{-1}M) @>>> K^{nc}(R) \\ @VVV @VVV \\ K^{nc}(R) @>>> K^{nc}(R\{M\}). \end{CD}$$

Applying Theorem 1.1 to the section $R \rightarrow R \oplus \Sigma^{-1}M$ we learn that the top horizontal map is an equivalence on -1 -connective covers, so via the pullback square we learn that $K(R) \cong K(R\{M\})$. ◁

There are many more invariance-type results that can be proven using a combination of Theorem 1.3 and the Land-Tamme \odot -product and we end this subsection with a more generic example.

Example 5.7 Suppose we are given a map of discrete rings $R \rightarrow S$ with R left regular coherent and an S -bimodule M which is right flat over R . We can form the pullback of \mathbb{E}_1 -algebras

$$\begin{CD} R \oplus \Sigma^{-1}M @>>> S \\ @VVV @VVV \\ R @>>> S \oplus M \end{CD}$$

where $S \oplus M$ is the trivial square-zero extension of S by M and likewise for $R \oplus \Sigma^{-1}M$. From Theorem 1.1 we know that $K(R \oplus \Sigma^{-1}M) \simeq K(R)$ and therefore

$$K(S) \simeq K\left(R \odot_{R \oplus \Sigma^{-1}M}^{S \oplus M} S\right).$$

We can think of this as a relative version of \mathbb{A}^1 -invariance for the map $R \rightarrow S$ and bi-module M . The underlying (R, S) -bimodule of the \odot -product is given by $R \otimes_{R \oplus \Sigma^{-1}M} S$ which is equivalent to $R\{M\} \otimes_R S$. The free algebra $R\{M\}$ is discrete and right flat as an R -module since M is, so $R\{M\} \otimes_R S$ is discrete as well.

We now will finish completely describing its ring structure. By the previous example, which is the case $R = S$, the \odot -product receives ring maps from both $R\{M\}$ and S . It remains then to determine the left multiplication of an element of S by one of M . This can be read off using Example 4.3, which gives a cofiber sequence of S -bimodules

$$S \rightarrow R \odot_{R \oplus \Sigma^{-1}M}^{S \oplus M} S \rightarrow M \otimes_{R \oplus \Sigma^{-1}M} S,$$

showing that the left multiplication of S on M is the one coming from the left S -module structure. ◁

Note that in Example 5.7 the ring S is not required to be regular! For example, we can let $R = k$ be a field, take $S = k[\epsilon]/\epsilon^2$ and let M be k thought of as an S -bimodule via the augmentation. In this case we obtain an equivalence

$$K(k[\epsilon]/\epsilon^2) \simeq K(k\{\epsilon, y\}/(\epsilon^2, \epsilon y)).$$

5.2 K -theory of unipotent representations

Next we analyze the K -theory of categories of local systems with values in a regular category. The following generalizes the discussion in [1, Section 4.3], in which they analyze the K -theory of cochain algebras of finite, connected spaces with coefficients in commutative Noetherian rings using the Koszul dual description of the module categories in [17, Proposition 7.8] as ind-unipotent representations of the loop space.

Definition 5.8 Given $\mathcal{C} \in \text{Cat}^{\text{perf}}$ and a connected $X \in \mathcal{S}_*$, let $\text{Rep}(\Omega X, \mathcal{C})$ denote the category of local systems on X with values in \mathcal{C} (this is just $\text{Fun}(X, \mathcal{C})$).¹⁸ Pullback along the map $X \rightarrow *$ provides a functor

$$(-)^{\text{triv}} : \mathcal{C} \rightarrow \text{Rep}(\Omega X, \mathcal{C})$$

which associates to $c \in \mathcal{C}$ the constant local system at c . Let $\text{Rep}(\Omega X, \mathcal{C})^{\text{uni}}$ denote $\text{im}((-)^{\text{triv}})$ (where the image is taken as an idempotent complete stable category). We refer to this as the category of unipotent local systems valued in \mathcal{C} . \triangleleft

Remark 5.9 One can make similar definitions for \mathcal{A} a small abelian category. Namely, we let $\text{Rep}(\Omega X, \mathcal{A})$ denote $\text{Fun}(X, \mathcal{A})$, and let $\text{Rep}(\Omega X, \mathcal{A})^{\text{uni}}$ denote unipotent representations, i.e the category generated under extensions, kernels and cokernels by the image of $(-)^{\text{triv}}$. \triangleleft

Proposition 5.10 If $\mathcal{C} \in \text{Cat}_{\geq 0}^{\text{perf}}$ is bounded and regular, and X is connected, then

- (1) Truncation on \mathcal{C} provides $\text{Rep}(\Omega X, \mathcal{C})$ a bounded t -structure with heart $\text{Rep}(\pi_1 X, \mathcal{C}^{\heartsuit})$.
- (2) The t -structure on $\text{Rep}(\Omega X, \mathcal{C})$ restricts to $\text{Rep}(\Omega X, \mathcal{C})^{\text{uni}}$, with heart $\text{Rep}(\pi_1 X, \mathcal{C}^{\heartsuit})^{\text{uni}}$.
- (3) $(-)^{\text{triv}}$ induces an equivalence $K(\mathcal{C}) \simeq K(\text{Rep}(\Omega X, \mathcal{C})^{\text{uni}})$.

Proof For (1), in order to check that $\tau_{\geq 0}$ and $\tau_{< 0}$ determine a t -structure on $\text{Rep}(\Omega X, \mathcal{C})$ we just need to check that the space of maps from $\tau_{\geq 0}c$ to $\tau_{< 0}d$ is contractible. To do this we use the formula

$$\text{Map}_{\text{Rep}(\Omega X, \mathcal{C})}(\tau_{\geq 0}c, \tau_{< 0}d) \simeq \text{Map}_{\mathcal{C}}(\tau_{\geq 0}c, \tau_{< 0}d)^{h\Omega X}$$

where ΩX acts on the space of maps in \mathcal{C} by conjugation. Since $\text{Map}_{\mathcal{C}}(\tau_{\geq 0}c, \tau_{< 0}d)$ is contractible, so is the limit under the action. Boundedness is inherited from \mathcal{C} since the

¹⁸ There is a subtlety here, which is that in general $\text{Rep}(\Omega X, \mathcal{C})$ and $\text{Fun}(X, \text{Ind}(\mathcal{C}))^{\omega}$ differ. It is this which motivated us to use Rep as notation when Fun would appear to suffice.

underlying object functor $\text{Rep}(\Omega X, \mathcal{C}) \rightarrow \mathcal{C}$ is t -exact and conservative. The heart is clearly $\text{Rep}(\Omega X, \mathcal{C}^\heartsuit)$, and since \mathcal{C} is a 1-category, this is the same as $\text{Rep}(\Omega\tau_{\leq 1} X, \mathcal{C}^\heartsuit)$. We conclude since $\tau_{\leq 1} X \cong B\pi_1 X$.

For (2) and (3), we check that the functor $(-)^{\text{triv}}$ is unipotent (see Definition 3.4), so that we can apply Theorem 1.3 to conclude. Quasi-affineness follows from construction, and fully faithfulness on the heart follows from the fact that equivariant maps between objects in \mathcal{C}^\heartsuit with trivial $\pi_1(X)$ -action are just given by the underlying maps in \mathcal{C}^\heartsuit since it is a 1-category. \square

Remark 5.11 If R is an \mathbb{E}_1 -algebra, then the R -module R (with trivial action) is a generator of $\text{Rep}(\Omega X, \text{Mod}(R))^{\text{uni}}$, therefore we may identify this category with $\text{Mod}(C^*(X; R))$, the category of modules over the cochain algebra of X with values in R . If R is a regular coherent discrete ring, Proposition 5.10(3) then provides an equivalence¹⁹

$$K(R) \simeq K(C^*(X; R)).$$

When X is additionally compact and R Noetherian and commutative, the above result combined with Lemma 3.12 and vanishing of negative K -theory coincides with [1, Theorem 4.8]. However, as pointed out to us by Markus Land, their proof is not quite correct, since they claim that the heart of the t -structure on $C^*(X; R)$ agrees with that of R , which is not true if for example $X = S^1$. The hearts will agree exactly when all unipotent representations of $\pi_1 X$ are trivial, which is the same as asking that $\text{cof}(R \rightarrow \text{End}_{R[\pi_1 X]}(R, R))$ has Tor amplitude in $[-\infty, -2]$ as a right R -module. Despite this, the hearts are in general sufficiently similar that Quillen’s devissage provides an equivalence on K -theory. \triangleleft

5.3 Testing the limits of Theorem 1.1

In the next sequence of examples we probe the limits of Theorem 1.1. Summarizing what we find: the conditions of Theorem 1.1 are sharp. To see that regularity of $\pi_0 R$ is necessary we look at an example where \mathbb{A}^1 -invariance fails.

Example 5.12 We consider the exterior algebra $k[\epsilon_0, \epsilon_{-1}]$ over a field k . From Example 4.9 we have an equivalence of non-connective K -theories

$$K^{\text{nc}}(k[\epsilon_0, \epsilon_{-1}]) \simeq K^{\text{nc}}(k[\epsilon_0]) \oplus \Sigma^{-1} K^{\text{nc}}(N_0 \otimes k[\epsilon_0]).$$

Since \mathbb{A}^1 -invariance fails for $k[\epsilon_0]$ (see [10]), both terms in the sum are non-trivial. On the other hand Theorem 1.1 predicts only the first term. \triangleleft

Now we turn to the tor condition of Theorem 1.1. Essentially the simplest example of an algebra which violates it is the trivial square zero extension $S := \mathbb{F}_p[x] \oplus \Sigma^{-1} \mathbb{F}_p$

¹⁹ One could also have deduced the equivalence by a direct application of Theorem 1.1: to do this, one can use the fact that a product of flat right modules over a left coherent ring is flat. In particular, the t -structure on $\text{Mod}(C^*(X; R))$ is the standard one.

where x is in degree zero and acts by zero on \mathbb{F}_p . In a conversation with Markus Land and Georg Tamme we determined that $K_1(S)$ differs from K_1 of $\mathbb{F}_p[x]$ by using the \odot -product to reduce to a connective ring and then using trace methods²⁰.

Example 5.13 The algebra S fits into the pullback square on the left.

$$\begin{array}{ccc}
 S & \longrightarrow & \mathbb{F}_p[x] \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbb{F}_p[x] & \longrightarrow & \mathbb{F}_p[x] \oplus \mathbb{F}_p
 \end{array}
 \qquad
 \begin{array}{ccc}
 K^{\text{nc}}(S) & \longrightarrow & K^{\text{nc}}(\mathbb{F}_p[x]) \\
 \downarrow & \lrcorner & \downarrow \\
 K^{\text{nc}}(\mathbb{F}_p[x]) & \longrightarrow & K^{\text{nc}}(\mathbb{F}_p[x]\{\mathbb{F}_p\})
 \end{array}$$

Writing the $\mathbb{F}_p[x]$ -bimodule $\mathbb{F}_p[x] \oplus \mathbb{F}_p$ as the tensor product²¹ $\mathbb{F}_p[x] \otimes_{\mathbb{F}_p[x]\{\Sigma^{-1}\mathbb{F}_p\}} \mathbb{F}_p[x]$ we can apply Corollary 4.5 to identify $\mathbb{F}_p[x] \odot_{\mathbb{F}_p[x]\{\Sigma^{-1}\mathbb{F}_p\}} \mathbb{F}_p[x]$ with $\mathbb{F}_p[x]\{\mathbb{F}_p\}$. From this we obtain the pullback square of K -theories on the right.

Using \mathbb{A}^1 -invariance we have an isomorphism of relative K -theories

$$\text{cof}(K^{\text{nc}}(\mathbb{F}_p) \rightarrow K^{\text{nc}}(S)) \simeq \Sigma^{-1} \text{cof}(K^{\text{nc}}(\mathbb{F}_p) \rightarrow K^{\text{nc}}(\mathbb{F}_p[x]\{\mathbb{F}_p\}))$$

To conclude that $K_1(S)$ differs from $K_1(\mathbb{F}_p)$ we will argue that $K_2(\mathbb{F}_p[x]\{\mathbb{F}_p\})$ is not even finitely generated.

Let $R := \mathbb{F}_p[x]\{\mathbb{F}_p\}$. We can construct a DGA model for R which is $\mathbb{F}_p[x]\{y, z\}$ with $|y| = 0, |z| = 1, d(z) = xy$. From this we can compute that

- $\pi_0 R \cong \mathbb{F}_p[x, y]/xy$,
- x acts by zero on $\pi_1 R$ and
- $\pi_1 R$ is a free \mathbb{F}_p -vector space on the classes $y^{a_0}[z, y]y^{a_1}$ with $a_0, a_1 \geq 0$.

As a consequence of Waldhausen’s calculation of the first nonzero vanishing homotopy group of the fiber of $K(A) \rightarrow K(\pi_0 A)$ for a connective simplicial ring A ([26, Proposition 1.2]), we learn that the fiber of $K(R) \rightarrow K(\pi_0 R)$ is 1-connected, and has second homotopy group given by

$$\text{HH}_0(\mathbb{F}_p[x, y]/xy; \pi_1 R) \cong \mathbb{F}_p\{y^a[z, y] \mid a \geq 0\}.$$

Since $K_3(\mathbb{F}_p[x, y]/xy)$ is finitely generated (see [9]) we learn that $K_2(R)$ is not finitely generated as promised. ◁

In the example above, although the K -theory differs from that of the connective cover, if we think in terms of Theorem 1.3 it is not immediately clear at what point things broke down. Possibilities include:

- The ring failed to be regular (in the sense of Definition 3.4).
- The base-change functor from the connective cover failed to be t -exact.
- The base-change failed to be fully faithful on the heart.

In view of this we now proceed to give several more geometric examples where we have better control over how things break down.

²⁰ A similar analysis also works for $S = \mathbb{Z} \oplus \Sigma^{-1}\mathbb{F}_p$.

²¹ Here $\{M\}$ denotes the free algebra on a bimodule.

Example 5.14 Consider the quasi-affine variety $X := \mathbb{A}_k^n \setminus \{0\}$ over a field k for $n \geq 2$. Since this scheme is quasi-affine, its category of quasicoherent sheaves is equivalent to the category of modules over the ring of global sections, R . This is a commutative k -algebra whose homotopy groups are the coherent cohomology groups of $\mathbb{A}_k^n \setminus \{0\}$.

In this case we have

$$\pi_s R \cong \begin{cases} k[x_1, \dots, x_n] & s = 0 \\ (\prod_i x_i^{-1})k[x_1^{-1}, \dots, x_n^{-1}] & s = 1 - n \\ 0 & \text{otherwise} \end{cases}$$

The divisible module which shows in degree $1 - n$ has tor dimension n and therefore violates condition (2) in Theorem 1.1. There is a localization sequence

$$\text{Mod}(k[x_1, \dots, x_n]^{x_i\text{-nil}}) \longrightarrow \text{Mod}(k[x_1, \dots, x_n]) \longrightarrow \text{Mod}(R)$$

coming from the fact that $\mathbb{A}_k^n - 0$ is an open subset of \mathbb{A}_k^n . $\text{Mod}(k[x_1, \dots, x_n]^{x_i\text{-nil}})$ is generated by $\otimes_{i=1}^n \text{cof}(x_i)$, whose endomorphism is an exterior algebra over k on n classes in degree -1 . Thus by Theorem 1.1, $K(\text{Mod}(k[x_1, \dots, x_n]^{x_i\text{-nil}})) \cong K(k)$, and so the map $K(\mathbb{A}_k^n) \rightarrow K(R)$ is not an equivalence, as its fiber is $K(k)$. \triangleleft

In Example 5.14, the tor condition fails and the K -theories differ, but R is regular anyway. What happens here is that the map $\text{Mod}(\pi_0 R)^\heartsuit \rightarrow \text{Mod}(R)^\heartsuit$ isn't faithful because the module k supported at the origin is sent to zero. This example also exhibits another more subtle behavior. In [26, Proposition 1.1] (which is extended to general connective ring spectra by [12, Lemma 2.4]), Waldhausen shows that an n -connective map of connective algebras induces an $(n + 1)$ -connective map on K -theory. A similar phenomenon does not occur our setting. In Example 5.14 the first degree where R differs from its $\pi_0 R$ is $1 - n$ while the K -theory first differs in degree 1, which is independent of the parameter n .

Since Example 5.14 isn't tight with respect to the tor condition we now provide another family of examples which, although more geometrically degenerate, do show that the tor condition is tight.

Example 5.15 For $n \geq 1$, consider \mathbb{A}^n with a doubled origin over the same field k , that is to say we look at the pullback below

$$\begin{array}{ccc} R & \longrightarrow & \Gamma(\mathbb{A}^n) \\ \downarrow & \lrcorner & \downarrow \\ \Gamma(\mathbb{A}^n) & \longrightarrow & \Gamma(\mathbb{A}^n - 0). \end{array}$$

From our examination of $\Gamma(\mathbb{A}^n - 0)$ in Example 5.14²² we know that $\pi_{-n} R$ has tor dimension n . Since $\Gamma(\mathbb{A}^n) \rightarrow \Gamma(\mathbb{A}^n - 0)$ is a localization the induced square on K -theory is a pullback (see [23]). From this we can read off that

$$K(R) \cong K(\mathbb{A}_k^n) \oplus K(k)$$

²² the same analysis works for $n = 1$, which isn't covered in Example 5.14.

where again the comparison map $\pi_0 R \rightarrow R$ induces the inclusion of the left summand. <

In the previous two examples the K -theory of R and its connective cover differed, but R was still regular (in the sense of Definition 3.4). Our next example will show that it is possible for $\pi_0 R$ to be regular while R is non-regular. To do this we use an affine nodal cubic curve C over a field k , which has non-vanishing K_{-1} (see [27, III.4.4]). The main result of [1] then implies that the category of perfect coherent sheaves on C is not regular.

Example 5.16 Consider the nodal cubic curve $C := \text{Spec}(k[x, y]/y^2 - x^2(x - 1))$ and let R denote the pullback below

$$\begin{array}{ccc} R & \longrightarrow & k[x^{\pm 1}] \\ \downarrow & \lrcorner & \downarrow \\ C & \longrightarrow & C[x^{-1}]. \end{array}$$

Geometrically, the bottom horizontal arrow corresponds to removing the nodal point from $\text{Spec}(C)$ and the right vertical arrow corresponds to the quotient of C minus the node by the C_2 action that sends y to $-y$. The homotopy groups of R are

$$\pi_s R = \begin{cases} k[x] & s = 0 \\ (yx^{-1})k[x^{-1}] & s = -1 \\ 0 & \text{otherwise} \end{cases} .$$

As in the previous example, since the top horizontal arrow is a localization the induced square on K -theory is a pullback [23]. Since $C[x^{-1}]$ is regular and Noetherian its negative K -groups vanish. This implies that $K_{-1}(R) \cong K_{-1}(C) \neq 0$. On the other hand $\pi_0 R = k[x]$ has vanishing K_{-1} . Since $K_{-1}(R)$ is non-zero, the category of compact R -modules cannot be regular. <

This example demonstrates that when the tor condition is violated, $\text{Mod}(R)^\omega$ can fail to be regular in addition to the K -theories of R and $\pi_0 R$ differing. In essence what we have done in Example 5.16 is taken a singularity and hidden it in degree -1 . The fact that this can be done implies that the tor condition in Theorem 1.1 and the condition that $\pi_0 R$ be left regular coherent cannot be disentangled—an idea we explore further in the next example.

Example 5.17 Suppose that R is a regular, discrete, Noetherian, commutative algebra. Using Theorem 1.3, Example 4.9, Example 4.10 and Proposition 5.1 we obtain K -theory equivalences

$$\begin{aligned} K(R[x_0, x_{-1}]/(x_0x_{-1})) &\simeq K(R) \oplus K(N_0 \otimes R) \oplus K(N_{-1} \otimes R) \oplus \Sigma^{-1} K(N_1 \otimes R) \\ &\simeq K(R[x_0]) \oplus K(R[\epsilon_0]) \simeq K(R[\epsilon_0]). \end{aligned}$$

<

What distinguishes this example is that $R[x_0, x_{-1}]/(x_0x_{-1})$ is coconnective, has Noetherian, regular π_0 , but violates the tor condition, while $R[\epsilon_0]$ is discrete (and therefore satisfies the tor condition), but is non-regular. This suggests that at the level of nc motives regularity of $\pi_0 R$ and the tor condition are not individually particularly meaningful. Instead we should think of the combination of these two conditions, i.e. unipotence, as a meaningful single condition.

5.4 An example we do not cover

We end the paper by giving a simple example of a ring R such that the connective cover map $\pi_0 R \rightarrow R$ induces an equivalence on nc motives, but R does not satisfy the hypotheses of Theorem 1.1.

Example 5.18 Let R be the ring $\text{End}_{\mathbb{F}_p[x,y]}((x,y))^{op}$. As a module over $\mathbb{F}_p[x,y]$, (x,y) has three cells, two in degree 0, and one in degree 1 with attaching maps x and y . From this we can compute the homotopy groups of R

$$\pi_s R \cong \begin{cases} k[x,y] & s = 0 \\ (x,y)/(x,y)^2 & s = -1 \\ 0 & \text{otherwise} \end{cases} .$$

$\mathbb{F}_p[x,y]$ is regular, but $(x,y)/(x,y)^2$ has tor dimension 2 over $\mathbb{F}_p[x,y]$, so R does not satisfy the conditions of Theorem 1.1.

Now we proceed to show that R is regular and the connective cover map induces an equivalence of nc motives. In working with perfect R -modules we identify this category with the thick subcategory of $\text{Mod}(\mathbb{F}_p[x,y])^\omega$ generated by (x,y) . To see that R is regular, first observe that \mathbb{F}_p is connective in the standard t -structure for R since it is a retract of $(x,y) \otimes_{\mathbb{F}_p[x,y]} \mathbb{F}_p$. From the extension $(x,y) \rightarrow \mathbb{F}_p[x,y] \rightarrow \mathbb{F}_p$ we can then conclude that $\mathbb{F}_p[x,y]$ is connective in this t -structure as well. This then implies that $\text{Mod}(R)_{\geq 0}^\omega$ is equivalent to $\text{Mod}(\mathbb{F}_p[x,y])_{\geq 0}^\omega$. Since (x,y) represents the class 1 in $K_0(\mathbb{F}_p[x,y])$, base-change along the map $\pi_0 R = \mathbb{F}_p[x,y] \rightarrow R$, which can be identified with the functor $\otimes_{\mathbb{F}_p[x,y]}(x,y)$, induces multiplication by 1 on the nc motive of $\text{Mod}(\mathbb{F}_p[x,y])^\omega$. ◁

In this example the natural map $K(\pi_0 R) \rightarrow K(R)$ is an equivalence despite the fact that R doesn't satisfy the conditions of Theorem 1.1. The essential issue here is that (x,y) is not flat, i.e. the base-change functor $\pi_0 R \rightarrow R$ is not t -exact. Since, at its core Theorem 1.1 operates using Quillen's devissage theorem it cannot be used for examples of this type. The equivalence of K -theories in this example arises because R is Morita equivalent to its connective cover, which is a different (and less interesting) reason for them to agree. As noted above, this implies that the connective cover map induces an equivalence of nc motives in this case, something which rarely happens for rings to which one can apply Theorem 1.1.

Another point contrasting with Theorem 1.1 is the fact that the equivalence $K(\pi_0 R) \cong K(R)$ is not visible at the level of the homotopy ring of R . Indeed, if

R' is the trivial square zero extension of $\mathbb{F}_p[x, y]$ by $\Sigma^{-1}(x, y)/(x, y)^2$, then its homotopy ring agrees with R , but $K(\pi_0 R') \rightarrow K(R')$ is not an equivalence, because the map $\pi_0 R' \rightarrow R'$ has the map $\mathbb{F}_p[x] \rightarrow \mathbb{F}_p[x] \oplus \Sigma^{-1}\mathbb{F}_p$ as a retract, which was shown in Example 5.13 to not be a K -theory equivalence.

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