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Central limit theorems for generic lattice point counting

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Abstract

We consider the problem of counting lattice points contained in domains in $\mathbb{R}^d$ defined by products of linear forms. For $d \geq 9$ we show that the normalized discrepancies in these counting problems satisfy non-degenerate Central Limit Theorems with respect to the unique $\mathrm{SL}_d(\mathbb{R})$-invariant probability measure on the space of unimodular lattices in $\mathbb{R}^d$. We also study more refined versions pertaining to "spiraling of approximations". Our techniques are dynamical in nature and exploit effective exponential mixing of all orders for actions of diagonalizable subgroups on spaces of unimodular lattices.

Keywords Counting problems · Central limit theorems · Exponential mixing of all orders

Mathematics Subject Classification Primary 11K60 · 11N45; Secondary 37A25 · 60F05

1 Introduction

Let Λ be a lattice in $\mathbb{R}^d$, and let (Ω_T) be an increasing family of Borel subsets of $\mathbb{R}^d$ with finite volumes tending to infinity as $T \to \infty$. A fundamental problem in the Geometry of Numbers is to estimate the number of points in Λ which are contained in Ω_T. Under mild regularity conditions, one can usually show that

$$|\Lambda \cap \Omega_T| = \frac{\mathrm{Vol}(\Omega_T)}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} + o\Big(\mathrm{Vol}(\Omega_T)\Big) \qquad \text{as } T \to \infty.$$

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In this paper we study the corresponding *discrepancy function* defined by

$$\mathcal{D}_T(\Lambda) := |\Lambda \cap \Omega_T| - \frac{\mathrm{Vol}(\Omega_T)}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)}. \tag{1.1}$$

When the domain Ω_T is a T-dilation of a region $\Omega \subset \mathbb{R}^d$ with piecewise smooth boundary, one can easily prove that

$$\mathcal{D}_T(\Lambda) = O_\Lambda\Big(\mathrm{Vol}(\Omega_T)^{1-1/d}\Big), \tag{1.2}$$

(see the book [21] for many results of this form), and this estimate is the *best possible* in this generality. However, the estimate has been improved for certain particular classes of domains. A well-studied setting is when the domain Ω has non-vanishing curvature. In this case, Hlawka [16] has shown that

$$\mathcal{D}_T(\Lambda) = O_\Lambda\Big(\mathrm{Vol}(\Omega_T)^{1-2/(d+1)}\Big) \tag{1.3}$$

These bounds have been subsequently improved by a number of people (see, for instance, [17] for a survey).

In this paper we shall be interested in asymptotic behaviour ($T \to \infty$) of the discrepancy function $\mathcal{D}_T(\Lambda)$ for "generic" lattices Λ. The following two questions naturally arise in this setting:

(i) what is the asymptotic "generic" growth of $\mathcal{D}_T(\Lambda)$?
(ii) do suitably normalized discrepancy functions converge in distribution?

Concerning Question (i): it turns out that the estimate (1.2) can be improved for generic lattices. The first striking result in this direction was established by W. Schmidt [25]. He proved that for a *every* increasing family of Borel sets Ω_T as above and almost every lattice Λ,

$$\mathcal{D}_T(\Lambda) = O_{\Lambda,\varepsilon}\Big(\mathrm{Vol}(\Omega_T)^{1/2+\varepsilon}\Big) \quad \text{for all } \varepsilon > 0.$$

However, the exact asymptotic behavior of $\mathcal{D}_T(\Lambda)$ for generic lattices is still quite mysterious, and it turns out that the answer depends very sensitively on the shape of the domains. For instance, Hardy, Littlewood [16] and Khinchin [20] discovered that when Ω_T is a T-dilation of a generic compact polygon in $\mathbb{R}^2$, then

$$\mathcal{D}_T(\mathbb{Z}^2) = O_\varepsilon\Big(\big(\log \mathrm{Vol}(\Omega_T)\big)^{1+\varepsilon}\Big) \quad \text{for all } \varepsilon > 0.$$

This exhibits a striking difference with the estimate (1.3) for strictly convex domains. Skriganov [28] established a far-reaching generalization of this estimate. He showed that when Ω_T is a dilation by a factor T of a compact polyhedron in $\mathbb{R}^d$, then for almost every unimodular lattice Λ,

$$\mathcal{D}_T(\Lambda) = O_{\Lambda,\varepsilon}\Big(\big(\log \mathrm{Vol}(\Omega_T)\big)^{d-1+\varepsilon}\Big) \quad \text{for all } \varepsilon > 0.$$

It is not known whether the above bound is optimal. Another well-studied example is the case when the domains Ω_T are the Euclidean balls in $\mathbb{R}^d$. In this case, it was shown by Kelmer [18] that for any exponentially growing sequence $T_i \to \infty$ and almost all lattices Λ,

$$\mathcal{D}_{T_i}(\Lambda) = O_{\Lambda,\varepsilon}\Big(\operatorname{Vol}(\Omega_{T_i})^{1-(d+1)/(2d)+\varepsilon}\Big) \quad \text{for all } \varepsilon > 0.$$

Concerning Question(ii) above: several results have been proved for certain particular families of lattices. For instance, it was discovered by Beck that the distributions of suitably normalized discrepancy functions are asymptotically Gaussian. We refer to a survey [2] and a monograph [3] for a comprehensive exposition of these results. Beck considered the domains

$$\Omega_T := \Big\{(x, y) \in \mathbb{R}^2 \,:\, x^2 - 2y^2 \in (a, b),\ 0 < x < T,\ y > 0\Big\}$$

and translated lattices $\Lambda_\omega := \mathbb{Z}^2 + (\omega, 0)$ with $0 < \omega < 1$ and showed that there exists an explicit $\sigma > 0$ such that

$$\operatorname{Leb}\Big(\big\{\omega \in (0,1) : \operatorname{Vol}(\Omega_T)^{-1/2}\mathcal{D}_T(\Lambda_\omega) < \xi\big\}\Big) \longrightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\xi} e^{-t^2/2\sigma^2}\,dt \quad \text{as } T \to \infty. \tag{1.4}$$

While this approach seems to work for domains defined by more general indefinite integral binary quadratic forms, it was not clear whether this result could hold in higher dimensions since its proof was based on properties of continued fraction expansions for quadratic irrationals. Furthermore, Beck points out that there are essential difficulties in extending his work to higher dimensions related to the long-standing Littlewood Conjecture.

Levin [22] investigated the discrepancy function of the family of lattices of the form

$$\Lambda_{\underline{a}} := \operatorname{diag}(a_1, \ldots, a_d)^{-1}\mathcal{O}, \quad \underline{a} = (a_1, \ldots, a_d) \in (0, 1)^d,$$

where $\mathcal{O}$ is a fixed lattice in $\mathbb{R}^d$ arising from an order in a totally real number field. He showed that for the boxes $\Omega_{\underline{N}} := [-N_1, N_1] \times \cdots \times [-N_d, N_d]$, suitably normalized discrepancy functions $\mathcal{D}_{\underline{N}}(\Lambda_{\underline{a}})$ are asymptotically Gaussian as $N_1 \cdots N_d \to \infty$, with $\underline{a} \in (0, 1)^d$ considered random. Since the results [2, 3, 22] treat only very particular lattices arising from orders in number fields, one may wonder whether this behavior occurs for truly generic lattices. We will address this question in the present paper.

One should also mention the ground-breaking works of Dolgopyat, Fayad [8, 10] (see also the survey [9]), generalizing Kersten [19], about the discrepancy of distribution for toral translations. Using our terminology, these results can be interpreted in terms of discrepancy functions for the family of lattices given by

$$\Lambda_{\underline{u}} := \big\{(x_1 + u_1 y, \ldots, x_{d-1} + u_{d-1} y, y) : (x_1, \ldots, x_{d-1}, y) \in \mathbb{Z}^d\big\} \quad \text{with } 0 \le u_1, \ldots, u_{d-1} < 1.$$

and certain families of domains $\Omega_T(\underline{\theta})$ depending on additional parameters $\underline{\theta}$. It is shown in [8, 10] that the corresponding discrepancy for $|\Lambda_{\underline{u}} \cap \Omega_T(\underline{\theta})|$ after a suitable normalization converges in distribution as $T \to \infty$, with $(\underline{u}, \underline{\theta})$ considered random. It should be noted that the obtained limit distributions in [8, 10] are different from the Normal Law. Further related results about distribution of Diophantine approximants were proved in [11] and [7].

1.1 Main results

Let $L_1, \dots, L_d : \mathbb{R}^d \to \mathbb{R}$ be linearly independent linear forms and $N(x) := L_1(x) \cdots L_d(x)$. For a bounded interval $I \subset \mathbb{R}^+$ and $T > 0$, we consider the domains

$$\Omega_T(I) := \left\{x \in \mathbb{R}^d : N(x) \in I \;\text{ and }\; 0 < L_1(x), \dots, L_d(x) < T\right\}.$$

We write X for the space of unimodular lattices in $\mathbb{R}^d$ equipped with the unique $\mathrm{SL}_d(\mathbb{R})$-invariant probability measure μ. The following result provides an analogue of (1.4) for μ-generic unimodular lattices:

Theorem 1.1 *Let $\mathcal{D}_T$ denote the discrepancy function for $\Omega_T(I)$. If $d \geq 9$, then*

$$\mu\Big(\big\{\Lambda \in X : \mathrm{Vol}(\Omega_T)^{-1/2}\mathcal{D}_T(\Lambda) < \xi\big\}\Big) \longrightarrow \frac{1}{\sigma(I)\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-t^2/2\sigma(I)^2}\, dt \quad \text{as } T \to \infty,$$

for all $\xi \in \mathbb{R}$, where

$$\sigma(I)^2 := \frac{1}{\zeta(d)} \sum_{p,q=1}^{\infty} \frac{\mathrm{Leb}\left(p^d I \cap q^d I\right)}{p^d q^d\, \mathrm{Leb}(I)}.$$

Remark 1.2 We explain in Sect. 4.1 that one may, without loss of generality, prove Theorem 1.1 in the special case when $L_i(x) = x_i$ for $i = 1, \dots, d$.

Athreya, Ghosh and Tseng [1] studied the related problem of "spiraling" of Diophantine approximants which involves counting the lattice points in the domains

$$\left\{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : \; \|x\| \cdot |y| \in I, \; \frac{x}{\|x\|} \in B, \; 0 < \|x\| < T, \; 0 < y < T\right\},$$

defined for an interval $I \subset \mathbb{R}^+$ and a Borel subset $B \subset S^{d-1}$. Our method allows us to analyze the distribution of the error term for this counting problem.

Theorem 1.1 is a special case of Theorem 1.3 below, which deals with the following general setting. For $k \geq 2$ and positive integers $d_1, \dots, d_k$, we set

$$\underline{d} = (d_1, \dots, d_k) \quad \text{and} \quad d = d_1 + \cdots + d_k,$$

and define $\mathbb{S}_{\underline{d}} := \prod_{j=1}^{k} S^{d_j - 1}$, where S^{d_j-1} denotes the unit sphere in $\mathbb{R}^{d_j}$, endowed with the standard Euclidean inner product, with the convention that $S^0 = \{-1, 1\}$.

The corresponding norm on $\mathbb{R}^{d_j}$ will be denoted by $\|\cdot\|$, the spherical measure on S^{d_j-1} will be denoted by κ_j, and we set

$$\kappa := \kappa_1 \otimes \cdots \otimes \kappa_k. \tag{1.5}$$

Let us also fix rotation-invariant smooth metrics on each S^{d_j-1} with $d_j \geq 2$. If $d_j = 1$, we endow $S^0 = \{-1, 1\}$ with the discrete distance. If $B \subset \mathbb{S}_{\underline{d}}$ is a Borel set and $\varepsilon > 0$, we denote by B_ε the ε-thickening of B with respect to the products of the chosen metrics. We say that a Borel set $B \subset \mathbb{S}_{\underline{d}}$ has a *smooth boundary* if

$$\kappa(B_\varepsilon) - \kappa(B) \ll \varepsilon, \quad \text{for all small enough} \varepsilon > 0,$$

where the implicit constants are independent of ε.

Let now $L_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$, $j = 1, \ldots, k$, be linear maps such that $(L_1, \ldots, L_k)$ is a bijection of $\mathbb{R}^d$. We define

$$N(z) := \prod_{j=1}^{k} \|L_j(z)\|^{d_j} \quad \text{and} \quad \xi(z) := \Big(\frac{L_1(z)}{\|L_1(z)\|}, \ldots, \frac{L_k(z)}{\|L_k(z)\|}\Big). \tag{1.6}$$

Given a bounded interval $I \subset (0, \infty)$, a Borel set $B \subset \mathbb{S}_{\underline{d}}$ and $T > 0$, we consider the domains

$$\Omega_T(I, B) := \big\{z \in \mathbb{R}^d \ : \ N(z) \in I, \ \xi(z) \in B \ \text{and} \ 0 < \|L_1(z)\|, \ldots, \|L_k(z)\| < T\big\}. \tag{1.7}$$

Our main result is the following:

Theorem 1.3 *When $k \geq 2$ and $d \geq 9$, the discrepancy functions for the sets $\Omega_T(I, B)$ satisfy,*

$$\mu\Big(\big\{\Lambda \in X : \operatorname{Vol}(\Omega_T)^{-1/2}\mathcal{D}_T(\Lambda) < \xi\big\}\Big) \longrightarrow \frac{1}{\sigma(I, B)\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-t^2/2\sigma(I,B)^2}\, dt \quad \text{as } T \to \infty,$$

for all $\xi \in \mathbb{R}$, where

$$\sigma(I, B)^2 := \frac{1}{\zeta(d)} \left(\sum_{p,q=1}^{\infty} \frac{\operatorname{Leb}\left(p^d I \cap q^d I\right)}{p^d q^d \operatorname{Leb}(I)}\right) \Big(1 + \frac{\kappa(B \cap -B)}{\kappa(B)}\Big).$$

Theorems 1.1 and 1.3 have been announced in [5] for $d \geq 4$. However, it turned out that the technical part of our argument works only for $d \geq 9$.

In the next section, we summarize the main steps of the proof of Theorem 1.3. Our argument can be roughly divided into two parts that involve:

- a construction of a suitable approximation for the counting function (Sect. 4),
- analysis of such approximations (Sect. 3).

1.2 Concerning novelty

We want to stress that the approximation of the counting function in this paper is very different from, and much more involved than, the approximation employed in our previous paper [7]. In the latter paper, the domains in which the lattice points were counted could be perfectly tiled by a *fixed* subgroup of diagonal matrices, thus essentially reducing the question whether a Central Limit Theorem holds, to (an unbounded version of) the setting in [6].

In this paper however, the relevant domains in which we wish to count, are foliated by lower-dimensional subsets, which all admit nice tilings by (higher rank) diagonal subgroups of matrices, but these subgroups depend in a non-trivial way on the leaf in the foliation. We can approximate the counting function on each of these leaves, and bunch the resulting approximations together into a *functional tiling* (see Sect. 2 for more details). This functional tiling is an integral of averages of a parameterized family of smooth functions over yet another parameterized family of subgroups of diagonal matrices. Each of these parameterized averages can in principle be analyzed using the techniques from [6, 7], but that is *not* enough.

The issue is that the parameterized family of smooth functions in the averages is not bounded in the relevant parameter (even after the cuspidal cut-offs), which causes serious problems in our cumulant machinery, more specifically in our analysis of "clustered tuples". To circumvent this, we need to make use of some special features of the geometry at hand (see Sect. 3.5.2).

2 Outline of the proof

Our argument will involve analysis on the space X of unimodular lattices in $\mathbb{R}^d$, which can be considered as a homogeneous space $X \simeq \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. The space X supports a unique $\mathrm{SL}_d(\mathbb{R})$-invariant probability measure, which we shall denote by μ throughout the paper.

Given a bounded Borel measurable function $f : \mathbb{R}^d \to \mathbb{R}$ with bounded support, its *Siegel transform* $\widehat{f} : X \to \mathbb{R}$ is defined by

$$\widehat{f}(\Lambda) := \sum_{z \in \Lambda \setminus \{0\}} f(z), \quad \text{for } \Lambda \in X.$$

According to Siegel's Mean Value Theorem [27], if f is Riemann integrable, then

$$\int_X \widehat{f}\, d\mu = \int_{\mathbb{R}^d} f(z)\, dz, \tag{2.1}$$

where we normalise the Lebesgue measure dz on $\mathbb{R}^d$ so that the unit cube is assigned volume one.

Suppose that Ω_T is a bounded Borel set in $\mathbb{R}^d$, which does not contain the origin. Then, with the above notations,

$$|\Omega_T \cap \Lambda| = \widehat{\chi}_{\Omega_T}(\Lambda) \quad \text{and} \quad \mathrm{Vol}(\Omega_T) = \int_X \widehat{\chi}_{\Omega_T}\, d\mu, \tag{2.2}$$

so that

$$\mathcal{D}_T(\Lambda) = \widehat{\chi}_{\Omega_T}(\Lambda) - \int_X \widehat{\chi}_{\Omega_T}\, d\mu.$$

In the setting of Theorem 1.3, these formulas can be rewritten further. In what follows, we retain the notation used there. In particular, we have fixed $k \geq 2$ and $d \geq 3$, as well as a k-tuple $\underline{d} = (d_1, \ldots, d_k)$ of positive integers with $d = d_1 + \cdots + d_k$. We have chosen a bounded interval $I \subset (0, \infty)$ and a Borel set $B \subset \mathbb{S}_{\underline{d}}$ with a smooth boundary. We denote by $\Omega_T = \Omega_T(I, B)$ the sets defined in (1.7). There is no loss of generality in assuming that the maps L_j are the standard coordinate projections (see Sect. 4.1). Then the domains Ω_T can be conveniently foliated by the level sets

$$\mathcal{L}_{s,\xi} := \left\{ z \in \mathbb{R}^d \;:\; N(z) = s \text{ and } \xi(z) = \xi \right\}, \quad \text{for} s \in I \text{and} \xi \in B,$$

which are invariant under the subgroup $A < \mathrm{SL}_d(\mathbb{R})$ of diagonal matrices of the form

$$a(u) := \mathrm{Diag}\left(e^{u_1} I_{d_1}, e^{u_2} I_{d_2}, \ldots, e^{u_{k-1}} I_{d_{k-1}}, e^{-\frac{1}{d_k}\sum_{j=1}^{k-1} d_j u_j} I_{d_k}\right), \quad \text{for } u \in \mathbb{R}^{k-1}. \tag{2.3}$$

We note that $A \simeq \mathbb{R}^{k-1}$ since

$$a(u)a(v) = a(u+v) \quad \text{for all} u, v \in \mathbb{R}^{k-1}.$$

The initial idea of our approach is that the level sets $\mathcal{L}_{s,\xi}$ can be tessellated, using the action of a discrete subgroup of A on $\mathbb{R}^d$. Unfortunately, the domains Ω_T themselves do not possess such simple tilings. However, it turns out that each of the intersections $\Omega_T \cap \mathcal{L}_{s,\xi}$ has a tiling where tiles and the discrete subgroup depends on the parameters s and T (but not on the parameter ξ). We will show that the indicator functions χ_{Ω_T} can be approximated by suitable integrals of varying functional averages. These "functional tilings" stem from the above tilings for different values of s and ξ and are constructed using the following data:

- A collection of finite measure spaces $(Y_{T,i}, \kappa_{T,i})$ indexed by $T > 0$ and i in a finite set $\mathcal{I}$,
- A collection of bounded Borel functions $f_{T,i} : \mathbb{R}^d \times Y_{T,i} \to [0, \infty)$ with $T > 0$ and $i \in \mathcal{I}$,
- A collection of finite subsets $Q(y_i)$ of A with $y_i \in Y_{T,i}$.

The corresponding "functional tiling" is given by

$$F_T(z) := \sum_{i\in\mathcal{I}} \int_{Y_{T,i}} \left(\sum_{a\in Q_{T,i}(y_i)} f_{T,i}(az, y_i) \right) d\kappa_{T,i}(y_i), \quad \text{for } z \in \mathbb{R}^d. \tag{2.4}$$

We shall show that for a suitable choice of the data, F_T provides an approximation for the characteristic function χ_{Ω_T} in the sense that

$$\left\| \chi_{\Omega_T} - F_T \right\|_1 = o\big(\mathrm{Vol}(\Omega_T)^{1/2} \big) \quad \text{as } T \to \infty.$$

Assuming this, we can then write

$$\frac{\widehat{\chi}_{\Omega_T} - \mathrm{vol}(\Omega_T)}{\mathrm{Vol}(\Omega_T)^{1/2}} = \frac{\widehat{\chi}_{\Omega_T} - \widehat{F}_T}{\mathrm{Vol}(\Omega_T)^{1/2}} + \frac{\widehat{F}_T - \int_X \widehat{F}_T \, d\mu}{\mathrm{Vol}(\Omega_T)^{1/2}} + \frac{\int_X \big(\widehat{F}_T - \widehat{\chi}_{\Omega_T}\big)\, d\mu}{\mathrm{Vol}(\Omega_T)^{1/2}},$$

where the first and third term on the right hand side tend to zero in the $L^1(\mu)$-norm. Thus, the distributional limit of $\mathcal{D}_T(\Lambda)$ is the same as the distributional limit of the sequence of functions

$$\Upsilon_T(\Lambda) := \mathrm{Vol}(\Omega_T)^{-1/2} \left(\widehat{F}_T(\Lambda) - \int_X \widehat{F}_T \, d\mu \right).$$

The significance of this observation is that Siegel transforms of functional tilings like F_T can be investigated using homogeneous dynamics techniques.

Since averages of this form also arise in other arithmetic problems, we will analyze their behavior in an abstract axiomatic setting (cf. assumptions (I.a)–(I.c) and (II.a)–(II.c) below). This analysis will be carried out in Sect. 3. Our main result here is Theorem 3.19. Notably, it shows that when certain basic norm estimates for functions $f_{T,i}$ hold, the distributional convergence of $\Upsilon_T(\Lambda)$ holds provided that the variance $\|\Upsilon_T\|_{L^2(X)}$ converges. Next, in Sect. 4 we construct an approximation for χ_{Ω_T} of the form (2.4) satisfying our assumptions (I.a)–(I.c) and (II.a)–(II.c). Once such an approximation is available, our main result will be a corollary of Theorem 3.19.

3 Analysis of general functional tilings

In this section we consider a family of functions F_T on $\mathbb{R}^d$ defined by a "functional tiling" as in (2.4). Our goal is to analyze the asymptotic behavior of the sums $\widehat{F}_T(\Lambda) = \sum_{z\in\Lambda\setminus\{0\}} F_T(z)$ for lattices Λ in $\mathbb{R}^d$. We will formulate several assumptions on the objects defining F_T and then in the next section demonstrate that the developed framework does apply to our setting.

We hope that the axiomatic approach outlined in this paper can be used in other counting problems as well. Our main result here is Theorem 3.19, which establishes the Central Limit Theorem for $(\widehat{F}_T)$, with respect to the measure μ.

3.1 Some remarks about the axioms

The goal of this section is to describe a general approach for proving Central Limit Theorems for Siegel transforms of functional tilings (F_T) of the form (2.4). Our approach is based on two sets of assumptions on the data

$$(\mathcal{I}, (Y_{T,i}, \kappa_{T,i}), f_{T,i}, Q(y_i)).$$

The first set of assumptions are labelled I.a,I.b,I.c and are described in Sect. 3.2, while the second set of assumptions are labelled II.a, II.b, II.c and are described in Sect. 3.4.

The first set of assumptions simply describes the objects in the data that make up the functional tiling. The key point here is that the functions $f_{T,i}$ are smooth in the first variable and supported in a fixed compact subset of $\mathbb{R}^d$ (in particular, the Siegel transform of $f_{T,i}(\cdot, y_i)$ is well-defined and smooth for every $y_i \in Y_{T,i}$ and for all $i \in \mathcal{I}$.

The second set of assumptions is deeper. The first two assumptions (II.a and II.b) are concerned with the finite subsets $Q_{T,i}(y_i)$. Roughly speaking, II.a requires that $Q_{T,i}$ are well-separated subsets of the group A described in the previous subsection, while II.b takes this assumption a bit further, namely that there is a sequence $(\widetilde{Q}_{T,i})$ (independent of $y_i \in Y_{T,i}$) and a family of quasi-isometric embeddings $\beta_{T,i}(\cdot, y_i)$ of $\mathbb{R}^{k-1}$ into itself such that $Q_{T,i}(y_i) = \beta_{T,i}(\widetilde{Q}_{T,i}, y_i)$. These two assumptions will be useful when we estimate the contribution to cumulants of (truncations of) $\widehat{F}_{T,i}(\cdot, y_i)$ coming from *separated* tuples (Sect. 3.5.1)

The remaining assumption II.c is the most technical one. It is used to control the contribution to cumulants of (truncations of) $\widehat{F}_T$ coming from *clustered* tuples (Sect. 3.5.2). Roughly speaking, the idea behind this assumption can be explained as follows. By II.b, the y_i-dependence of the map $\beta_{T,i}$ is rather mild, and, up to bounded error, $\beta_{T,i}$ is close to a map $\widetilde{\beta}_{T,i}$ which is independent of y_i. The essence of the assumption II.c is that the sums in functional tilings like (2.4) can be estimated from above by sums over subsets which are *independent* of y_i. Although this assumption probably can be weakened, it holds in the setting that we are interested in, and it simplifies a lot of the upper estimates of integrals involving products of the $f_{T,i}$'s.

3.2 Functional averages and their truncations

Let $\mathcal{I}$ be a finite set. For $T > 0$ and $i \in \mathcal{I}$, we consider:

- (I.a) finite measure spaces $(Y_{T,i}, \kappa_{T,i})$ satisfying $\sup_{T,i} \kappa_{T,i}(Y_{T,i}) < \infty$,
- (I.b) bounded Borel functions $f_{T,i} : \mathbb{R}^d \times Y_{T,i} \to [0, \infty)$ such that for $y_i \in Y_{T,i}$, the map $x \mapsto f_{T,i}(x, y_i)$ is smooth, and supported in a compact set $\mathcal{K} \subset \mathbb{R}^d$, independent of T, i, and y_i,
- (I.c) a set-valued map $y_i \mapsto Q_{T,i}(y_i)$ from $Y_{T,i}$ into the set of finite subsets of the subgroup $A < \mathrm{SL}_d(\mathbb{R})$ of diagonalizable matrices of the form $a(u)$ defined in (2.3) such that

$$\sup_{i,y_i} |Q_{T,i}(y_i)| \ll V_T$$

with a parameter V_T satisfying $V_T \to \infty$ as $T \to \infty$.

For $f \in C_c^\infty(\mathbb{R}^d)$, let $\partial_k f$ denote the partial derivative of f with respect to the k-th coordinate for $k = 1, \ldots, d$. If $\beta = (\beta_1, \ldots, \beta_d)$ is a multi-index, we set $\partial_\beta f = \partial_1^{\beta_1} \cdots \partial_d^{\beta_d} f$, and define

$$\|f\|_{C^p} = \max_{|\beta| \le p} \|\partial_\beta f\|_\infty, \quad \text{for } p \ge 1, \tag{3.1}$$

where $|\beta| = \beta_1 + \cdots + \beta_d$.
We use the notations

$$M_T := \max_{i \in \mathcal{I}} \int_{Y_{T,i}} \left\| f_{T,i}(\cdot, y_i) \right\|_\infty d\kappa_{T,i}(y_i), \tag{3.2}$$

$$M_{T,q} := \max_{i \in \mathcal{I}} \sup_{y_i \in Y_{T,i}} \left\| f_{T,i}(\cdot, y_i) \right\|_{C^q}. \tag{3.3}$$

Given the data in (I.a)–(I.c), we consider the family of functions given by

$$F_T(z) := \sum_{i \in \mathcal{I}} \int_{Y_{T,i}} \left(\sum_{a \in Q_{T,i}(y_i)} f_{T,i}(az, y_i) \right) d\kappa_{T,i}(y_i), \quad \text{for } z \in \mathbb{R}^d, \tag{3.4}$$

and their Siegel transforms

$$\widehat{F}_T(\Lambda) = \sum_{i \in \mathcal{I}} \int_{Y_{T,i}} \left(\sum_{a \in Q_{T,i}(y_i)} \widehat{f}_{T,i}(a\Lambda, y_i) \right) d\kappa_{T,i}(y_i), \quad \text{for } \Lambda \in X. \tag{3.5}$$

Our goal is to show that under suitable assumptions the functions

$$\Upsilon_T(\Lambda) := V_T^{-1/2} \left(\widehat{F}_T(\Lambda) - \int_X \widehat{F}_T \, d\mu \right)$$

converge in distribution. One of the difficulties here is that Siegel transforms (even for bounded Borel functions with bounded support) are *not* bounded. Nonetheless, they are typically only large on sets of very small μ-measure and belong to $L^p(X)$ for $p < d$ (see Lemmas 3.2 and 3.3 below). Here and later in the paper we always assume that $d \ge 3$ so that the Siegel transforms are L^2-integrable. This makes it possible to efficiently approximate a Siegel transform by bounded functions on X whose L^p-distance from the original Siegel transform is small. To make this approximation precise, we shall use a family of compactly supported cutoff functions $\eta_L : X \to [0, 1]$ with $L > 0$, constructed in [7, Lemma 4.11] such that for every compact set $K \subset \mathbb{R}^d$ and $f \in C(K)$, we have

$$\left\| \widehat{f} \eta_L \right\|_{L^\infty(X)} \ll_K L \|f\|_\infty. \tag{3.6}$$

Furthermore, for every $\varepsilon > 0$,

$$\|\widehat{f}(1-\eta_L)\|_{L^1(X)} \ll_{K,\varepsilon} L^{-d+1+\varepsilon}\|f\|_\infty \quad \text{and} \quad \|\widehat{f}(1-\eta_L)\|_{L^2(X)} \ll_{K,\varepsilon} L^{-d/2+1+\varepsilon}\|f\|_\infty, \tag{3.7}$$

where the implicit constants are independent of L.

We introduce a parameter $L_T \to \infty$, which will be specified later, and introduce the functions $\varphi_{T,i} : X \times Y_{T,i} \to [0, \infty)$ defined by

$$\varphi_{T,i}(\Lambda, y_i) := \widehat{f}_{T,i}(\Lambda, y_i)\eta_{L_T}(\Lambda), \quad \text{for } \Lambda \in X \text{ and } y_i \in Y_{T,i},$$

which provide compactly supported truncations of the functions $\widehat{f}_{T,i}(\cdot, y_i)$. We then consider

$$\Phi_T(\Lambda) := \sum_{i\in\mathcal{I}} \int_{Y_{T,i}} \left(\sum_{a\in Q_{T,i}(y_i)} \varphi_{T,i}(a\Lambda, y_i) \right) d\kappa_{T,i}(y_i), \quad \text{for } \Lambda \in X.$$

The following lemma shows that this function approximates the Siegel transform $\widehat{F}_T$ if the parameter L_T grows fast enough.

Lemma 3.1 *If for some $\varepsilon > 0$,*

$$L_T^{-d/2+1+\varepsilon} V_T^{1/2} M_T \to 0 \quad \textit{as } T \to \infty, \tag{3.8}$$

then

$$\|\widehat{F}_T - \Phi_T\|_{L^2(X)} = o\left(V_T^{1/2}\right) \quad \textit{as } T \to \infty.$$

Similarly, if

$$L_T^{-d+1+\varepsilon} V_T^{1/2} M_T \to 0 \quad \textit{as } T \to \infty, \tag{3.9}$$

then

$$\|\widehat{F}_T - \Phi_T\|_{L^1(X)} = o\left(V_T^{1/2}\right) \quad \textit{as } T \to \infty.$$

Before we proceed to the proof of this lemma, we discuss its relevance to our arguments so far. We wish to prove convergence in distribution for the functions

$$\Upsilon_T = V_T^{-1/2}\left(\widehat{F}_T - \Phi_T\right) + V_T^{-1/2}\left(\Phi_T - \int_X \Phi_T\, d\mu\right) + V_T^{-1/2}\int_X \left(\Phi_T - \widehat{F}_T\right) d\mu.$$

If L_T is chosen as in (3.9), then the first and third term of the right hand side tend to zero in the L^1-norm, whence Υ_T converges in distribution to a continuous measure if and only if the functions

$$\Psi_T := V_T^{-1/2}\left(\Phi_T - \int_X \Phi_T\, d\mu\right) \tag{3.10}$$

do. In the upcoming subsections, we will analyse this type of sequences.

Proof of Lemma 3.1 By construction, we have

$$\|\widehat{F}_T - \Phi_T\|_{L^2(X)} \le \sum_{i\in\mathcal{I}} \int_{Y_{T,i}} \sum_{a\in Q_{T,i}(y_i)} \|(\widehat{f}_{T,i}(\cdot, y_i)\circ a)(1-\eta_{L_T}\circ a)\|_{L^2(X)}\, d\kappa_{T,i}(y_i).$$

Since the measure μ is A-invariant, the inner terms are independent of $a \in Q_{T,i}(y_i)$, whence

$$\|\widehat{F}_T - \Phi_T\|_{L^2(X)} \le \sum_{i\in\mathcal{I}} \int_{Y_{T,i}} |Q_{T,i}(y_i)|\ \|\widehat{f}_{T,i}(\cdot, y_i)(1-\eta_{L_T})\|_{L^2(X)}\, d\kappa_{T,i}(y_i).$$

By the assumption (I.b), the supports of the functions $x \mapsto f_{T,i}(x, y_i)$ are all contained in a fixed compact set $\mathcal{K} \subset \mathbb{R}^d$, independent of T, i and y_i. Hence, by (3.7),

$$\|\widehat{f}_{T,i}(\cdot, y_i)(1-\eta_{L_T}(\cdot))\|_{L^2(X)} \ll_{\mathcal{K},\varepsilon} L_T^{-d/2+1+\varepsilon} \|f_{T,i}(\cdot, y_i)\|_\infty, \quad \text{for all } y_i \in Y_{T,i}.$$

Furthermore, by the assumption (I.c), we have $|Q_{T,i}(y_i)| \le V_T$, so that we conclude that

$$\|\widehat{F}_T - \Phi_T\|_{L^2(X)} \ll_{\mathcal{K},\varepsilon} L_T^{-d/2+1+\varepsilon} V_T \left(\sum_{i\in\mathcal{I}} \int_{Y_{T,i}} \|f_{T,i}(\cdot, y_i)\|_\infty\, d\kappa_{T,i}(y_i)\right).$$

This implies the first part of the lemma, and the proof of the second part is similar. □

3.3 Sobolev norms and mixing estimates

In order to obtain quantitative estimates on correlations, we need to control the smoothness of the functions. Our main tool for this purpose are Sobolev norms, which we now introduce. First note that every Y in the Lie algebra $\mathfrak{sl}_d(\mathbb{R})$ of $\mathrm{SL}_d(\mathbb{R})$ induces a differential operator D_Y on $C^\infty(X)$ by

$$(D_Y\varphi)(\Lambda) = \frac{d}{dt}\varphi(e^{tY}\Lambda)\ |_{t=0} \quad \text{for smooth functions } \varphi \text{on } X.$$

More generally, if we fix a basis $Y_1, \ldots, Y_m$ of $\mathfrak{sl}_d(\mathbb{R})$ with $m = d^2 - 1$, and if Y is a monomial in the universal enveloping algebra of $\mathfrak{sl}_d(\mathbb{R})$ with respect to this basis, say $Y = Y_1^{\eta_1} \cdots Y_m^{\eta_m}$ for non-negative integers $\eta_1, \ldots, \eta_m$, then we define $D_\eta := D_{Y_1}^{\eta_1} \cdots D_{Y_m}^{\eta_m}$, and refer to the integer $|\eta| := \eta_1 + \cdots + \eta_m$ as the *order* of D_η, where $\eta = (\eta_1, \ldots, \eta_m)$. We write $C_c^\infty(X)$ for the space of compactly supported functions φ such that all the derivatives $D_\eta\varphi$ exist.

Let $\Lambda \in X$. We say that a linear subspace $V < \mathbb{R}^d$ is Λ-*rational* if $V \cap \Lambda$ is a lattice in V. If V is Λ-rational, we denote by $d_\Lambda(V)$ the volume of $V/V \cap \Lambda$. We define

$$\alpha(\Lambda) = \sup\left\{d_\Lambda(V)^{-1} \,:\, V < \mathbb{R}^d \text{ is} \Lambda - \text{ rational}\right\}.$$

It can readily be checked that α is a *proper* function on X, and that for every compact set $\mathcal{C} \subset \mathrm{SL}_d(\mathbb{R})$, there is a constant $A_\mathcal{C} > 0$ such that

$$A_\mathcal{C}^{-1}\alpha(\Lambda) \leq \alpha(g\Lambda) \leq A_\mathcal{C}\alpha(\Lambda), \quad \text{for all } g \in \mathcal{C} \text{ and} \Lambda \in X. \tag{3.11}$$

Before we introduce Sobolev norm, we mention important properties of the α-function in relation with Siegel transforms.

Lemma 3.2 ([26], Lemma 2) *If $f : \mathbb{R}^d \to \mathbb{R}$ is a bounded function with bounded support, then*

$$\left|\widehat{f}(\Lambda)\right| \ll_{\mathrm{supp}(f)} \alpha(\Lambda)\|f\|_\infty, \quad \textit{for all } \Lambda \in X.$$

The following estimate is also well-known (see e.g. [13, Lemma 3.10]):

Lemma 3.3 $\int_X \alpha^p \, d\mu < \infty$ *for every $p < d$.*

The following norms were introduced and studied by Einsiedler, Margulis and Venkatesh [12].

Definition 3.4 (*Sobolev norms*) Let q be a positive integer. For $\varphi \in C_c^\infty(X)$, its *Sobolev norm $S_q(\varphi)$ of order q* is defined as

$$S_q(\varphi) := \sum_{|\eta| \leq q} \left(\int_X |\alpha^d D_\eta \varphi|^2 \, d\mu\right)^{1/2}.$$

The explicit expression of the norm S_q will not be important in our paper. Instead we shall use as black boxes, the following properties of the norms, established in [12] and in our previous paper [7].

Proposition 3.5 ([12], Subsect. 3.7) *For all sufficiently large q,*

(i) $S_q(\varphi) \ll_q S_{q+1}(\varphi)$ *and* $\|\varphi\|_{L^\infty(X)} \ll_q S_q(\varphi)$ *for all* $\varphi \in C_c^\infty(X)$.
(ii) for some $p \geq 1$, *we have* $S_q(\varphi_1\varphi_2) \ll_q S_{q+p}(\varphi_1)S_{q+p}(\varphi_2)$, *for all* $\varphi_1, \varphi_2 \in C_c^\infty(X)$.
(iii) there exists $\sigma_q > 0$ *such that* $S_q\big(\varphi \circ a(u)\big) \ll_q e^{\sigma_q\|u\|} S_q(\varphi)$, *for all* $u \in \mathbb{R}^{k-1}$, *where* $a(u)$ *is defined in* (2.3) *and* $\|\cdot\|$ *is the* ℓ^∞*-norm on* $\mathbb{R}^{k-1}$.

For our next proposition, we need some notation and preliminary results. First, we recall some further properties of the cut-off functions η_L constructed in [7]:

Proposition 3.6 ([7], Lemma 4.11) *There exists a constant $c > 0$ such that*

$$\operatorname{supp} \eta_L \subset \{\alpha \leq cL\}, \quad \textit{for all } L > 0,$$

and for all $q \geq 1$, $f \in C^\infty(\mathbb{R}^d)$, and $L > 0$,

$$\sup_{|\eta| \leq q} \left\| D_\eta\big(\widehat{f}\, \eta_L\big) \right\|_{L^\infty(X)} \ll_{\operatorname{supp}(f),q} L \, \|f\|_{C^q}.$$

Remark 3.7 The second inequality in Proposition 3.6 is not explicitly stated in [7]. However, Lemma 4.11 in [7] tells us that $\|D_\eta \eta_L\|_{L^\infty(X)} \ll_\eta 1$, so the inequality in the proposition above follows after iterated use of the product rule for derivatives, in combination with Lemma 3.2 and the fact that the supports of the functions $D_\eta \eta_L$ are still contained in $\{\alpha \ll L\}$ for every η (where the implicit constants are independent of L and η).

The following corollary concerning Sobolev norms of truncated Siegel transforms is now immediate.

Corollary 3.8 *For all $q \geq 1$, $f \in C^\infty(\mathbb{R}^d)$, and $L > 0$,*

$$S_q\big(\widehat{f}\, \eta_L\big) \ll_{\operatorname{supp}(f),q} L^{d+1} \|f\|_{C^q}.$$

We also record the following corollary for future references. It is immediate from the inequalities in (3.11) and the first part of Proposition 3.6.

Corollary 3.9 *For every compact set $\mathcal{C} \subset \operatorname{SL}_d(\mathbb{R})$, there is a constant $B_{\mathcal{C}} > 0$ such that*

$$\eta_L \circ g \leq \chi_{\{\alpha \leq B_{\mathcal{C}} L\}} \quad \textit{for all} g \in \mathcal{C} \textit{and} L > 0.$$

Recall that $A \simeq \mathbb{R}^{k-1}$ via the map $u \mapsto a(u)$ defined in (2.3). Let us throughout the rest of the section denote by $\|\cdot\|$ the ℓ^∞-norm on $\mathbb{R}^{k-1}$. The following theorem is a special case of [4, Theorem 1.1]. Roughly speaking, this theorem asserts that if $\varphi \in C_c^\infty(X)$, then the family $u \mapsto \varphi(a(u)\cdot)$ consists of "almost independent" random variables, at least if the u's are far apart.

Theorem 3.10 (Theorem 1.1 in [4]) *For every $r \geq 2$, there exist $q_r \geq 1$ and $\delta_r > 0$ such that for all $q \geq q_r$, $\varphi_1, \ldots, \varphi_r \in C_c^\infty(X)$, and $u^{(1)}, \ldots, u^{(r)} \in \mathbb{R}^{k-1}$,*

$$\left| \int_X \left(\prod_{m=1}^r \varphi_m \circ a(u^{(m)}) \right) d\mu - \prod_{m=1}^r \int_X \varphi_m \, d\mu \right| \ll_{r,q} e^{-\delta_r \min_{j \neq k} \|u^{(j)} - u^{(k)}\|} \prod_{m=1}^r S_q(\varphi_m).$$

Theorem 1.1 in [4] is formulated for general r-tuples of elements in $G = \operatorname{SL}_d(\mathbb{R})$, and not just for r-tuples in A. Furthermore, in the version in [4], the $\min_{i \neq j}$-expression is applied to differences with respect to an invariant Riemannian metric on G. The restriction of any such metric to A is quasi-isometric to the ℓ^∞-distance on $\mathbb{R}^{k-1}$, and the resulting constants are assumed to have been absorbed in δ_r and by the $\ll$-sign.

3.4 Cumulants

We review the notion of cumulants, and a classical CLT-criterion due to Frechet and Shohat. In this subsection (X, μ) can be a general probability measure space.

Definition 3.11 (*Cumulants*) Fix $r \geq 2$. Given $\varphi_1, \ldots, \varphi_r \in L^\infty(X)$, we define their *cumulant* $\mathrm{cum}_r(\varphi_1, \ldots, \varphi_r)$ *of order* r by

$$\mathrm{cum}_{[r]}(\varphi_1, \ldots, \varphi_r) := \sum_{\mathcal{P} \in \mathfrak{P}_{[r]}} (-1)^{|\mathcal{P}|-1} \prod_{I \in \mathcal{P}} \left(\int_X \prod_{i \in I} \varphi_i \, d\mu \right),$$

where $\mathfrak{P}_{[r]}$ denotes the set of partitions of the set $[r] = \{1, \ldots, r\}$. Given $\Phi \in L^\infty(X)$, we define its r-*cumulant* $\mathrm{cum}_r(\Phi)$ by

$$\mathrm{cum}_r(\Phi) := \mathrm{cum}_{[r]}(\Phi, \ldots, \Phi).$$

Remark 3.12 It is clear that $\mathrm{cum}_{[r]}$ is multi-linear in the functions $\varphi_1, \ldots, \varphi_r$, and if one of them is a constant function, then $\mathrm{cum}_{[r]}(\varphi_1, \ldots, \varphi_r) = 0$ (see e.g. [23, Subsect. 3.1]). In particular,

$$\mathrm{cum}_{[r]}(\varphi_1, \ldots, \varphi_r) = \mathrm{cum}_{[r]} \left(\varphi_1 - \int_X \varphi_1 \, d\mu, \ldots, \varphi_r - \int_X \varphi_r \, d\mu \right)$$

and

$$\mathrm{cum}_r \left(\Phi - \int_X \Phi \, d\mu \right) = \mathrm{cum}_r(\Phi).$$

Furthermore, the 2-cumulant of Φ is just the μ-variance of Φ.

The main property of cumulants that makes them valuable to us in this paper is summarized in the following CLT-criterion by Frechet and Shohat, which can be deduced from their results in [14]. It is essentially the classical method of moments tailored for (distributional) convergence to the normal distribution.

Proposition 3.13 (Frechet–Shohat's cumulant criterion) *Let* (Ψ_T) *be a sequence of real-valued, bounded and measurable functions on* X *such that*

- $\int_X \Psi_T \, d\mu = 0$,
- *the limit* $\sigma^2 := \lim_T \|\Psi_T\|^2_{L^2(X)}$ *exists and is finite,*
- $\mathrm{cum}_r(\Psi_T) \to 0$ *for all* $r \geq 3$.

Then the μ*-distributions of* Ψ_T *converge in the sense of distribution to the Normal Law with mean zero and variance* σ^2 *(the case* $\sigma = 0$ *is interpreted as convergence in the sense of distributions to the Dirac measure at* 0*).*

Remark 3.14 There are no explicit mentioning of cumulants in the paper of Frechet and Shohat, so in particular Proposition 3.13 is not directly featured there. A more modern (and explicit) exposition of cumulants can be found in [23], although our formulation of Proposition 3.13 is not explicit there either. However, it is noted in [23, Subsect. 3.2] that cumulants of random variables can be expressed in terms of moments (and vice versa). By the classical method of moments, to prove that the μ-distributions of Ψ_T converges in the sense of distributions to the centered Normal Law with variance σ^2 it suffices to check that all moments (or cumulants) of Ψ_T with respect to μ converge (as real numbers) to the moments (or cumulants) of the centered Normal Law with variance σ^2. Since cumulants of a random variable can be expressed as logarithmic derivatives of the Fourier transform of the corresponding probability distribution (see e.g. [23, Subsect. 3.1]), it follows after some straightforward computations that Normal Laws are characterized as those probability distributions whose cumulants of order $r \geq 3$ all vanish (at least within the class of distributions that are uniquely determined by their moments).

In order to apply this proposition, we have to analyze the cumulants $\operatorname{cum}_r(\Psi_T)$. This task will be carried out in the next section.

3.5 Estimating cumulants of order $r \geq 3$

Let Ψ_T be defined by (3.10). Our goal is to show that under suitable additional conditions,

$$\operatorname{cum}_r(\Psi_T) \to 0 \qquad \text{as } T \to \infty$$

for all $r \geq 3$. Since

$$\operatorname{cum}_r(\Psi_T) = V_T^{-r/2} \operatorname{cum}_r\left(\Phi_T - \int_X \Phi_T \, d\mu\right) = V_T^{-r/2} \operatorname{cum}_r(\Phi_T),$$

this is equivalent to

$$\operatorname{cum}_r(\Phi_T) = o\left(V_T^{r/2}\right) \quad \text{as } T \to \infty. \tag{3.12}$$

Let us from now on fix $r \geq 3$. For each r-tuples $\underline{i} = (i_1, \dots, i_r) \in \mathcal{I}^r$, we set

$$Y_{T,\underline{i}} := Y_{T,i_1} \times \cdots \times Y_{T,i_r} \quad \text{and} \quad \kappa_{T,\underline{i}} := \kappa_{T,i_1} \otimes \cdots \otimes \kappa_{T,i_r},$$

and for $y = (y_1, \dots, y_r) \in Y_{T,\underline{i}}$, we set

$$Q_{T,\underline{i}}(y) := Q_{T,i_1}(y_1) \times \cdots \times Q_{T,i_r}(y_r).$$

We write elements of $Q_{T,\underline{i}}(y)$ as $\underline{u} = (u^{(1)}, \ldots, u^{(r)})$. Using the multi-linearity of the cumulants, we see that $\mathrm{cum}_r(\Phi_T)$ can be written as

$$\sum_{\underline{i}\in\mathcal{I}^r} \int_{Y_{T,\underline{i}}} \sum_{\underline{u}\in Q_{T,\underline{i}}(y)} \mathrm{cum}_{[r]} \left(\varphi_{T,i_1}(\cdot, y_1) \circ a(u^{(1)}), \cdots, \varphi_{T,i_r}(\cdot, y_r) \circ a(u^{(r)})\right) d\kappa_{T,\underline{i}}(y).$$

We shall make the following additional assumptions regarding the data defining the function Φ_T. Throughout this section, $\|\cdot\|$ denotes the ℓ^∞-norm on $\mathbb{R}^{k-1}$ and $B(x, \gamma)$ the ball with respect to this norm.

(II.a) There exist finite sets $\widetilde{Q}_{T,i} \subset \mathbb{R}^{k-1}$ satisfying:

- for all $\gamma \geq 1$

$$\left|\widetilde{Q}_{T,i} \cap B(u, \gamma)\right| \ll \gamma^{k-1}, \tag{3.13}$$

  where the implicit constants are independent of u, T, and i.
- $\max_i |\tilde{Q}_{T,i}| \ll V_T$ with a parameter V_T satisfying $V_T \to \infty$ as $T \to \infty$.

(II.b) There exist Borel maps $\beta_{T,i} : \mathbb{R}^{k-1} \times Y_{T,i} \to \mathbb{R}^{k-1}$ such that

$$Q_{T,i}(y_i) = \beta_{T,i}\left(\widetilde{Q}_{T,i}, y_i\right)$$

satisfying:

- there exist $c_1, c_2 > 0$, independent of T, such that for all $u, v \in \widetilde{Q}_{T,i}$,

$$\min_{i,j} \inf_{y_i\in Y_{T,i}} \inf_{y_j\in Y_{T,j}} \left\|\beta_{T,i}(u, y_i) - \beta_{T,j}(v, y_j)\right\| \geq c_1\|u - v\| - c_2, \tag{3.14}$$

- there exist maps $\widetilde{\beta}_{T,i} : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ such that for all $u \in \widetilde{Q}_{T,i}$,

$$\sup_T \sup_{y_i\in Y_{T,i}} \left\|\beta_{T,i}(u, y_i) - \widetilde{\beta}_{T,i}(u)\right\| < \infty. \tag{3.15}$$

(II.c) For the functions $f_{T,i}$ from (I.b), there exist Borel functions $h_{T,i} : \mathbb{R}^d \times Y_{T,i} \to [0, \infty)$ such that

$$f_{T,i}\left(a(\beta_{T,i}(u, y_i))z, y_i\right) \leq h_{T,i}\left(a(\widetilde{\beta}_{T,i}(u))z, y_i\right)$$

for all $u \in \widetilde{Q}_{T,i}$, $y_i \in Y_{T,i}$, and $z \in \mathbb{R}^d$. We further assume that the family of the functions

$$H_{T,i}(z) := \int_{Y_{T,i}} h_{T,i}(z, y_i)\, d\kappa_{T,i}(y_i)$$

is uniformly bounded, and there exists a fixed compact set $\mathcal{K}' \subset \mathbb{R}^d$ such that

$$\mathrm{supp}(H_{T,i}) \subset \mathcal{K}'$$

for all T and i.

Remark 3.15 We note that the condition (I.c) from Sect. 3.2 follows immediately from condition (II.a) and the first part of condition (II.b).

With this new notation, we set

$$\Xi_{r,T,\underline{i}}(y) := \sum_{\underline{u}\in\widetilde{Q}_{T,\underline{i}}} \mathrm{cum}_{[r]}\left(\varphi_{T,i_1}(\cdot, y_{i_1})\circ a\big(\beta_{T,i_1}(u^{(1)}, y_{i_1})\big), \cdots, \varphi_{T,i_r}(\cdot, y_{i_r})\circ a\big(\beta_{T,i_r}(u^{(r)}, y_{i_r})\big)\right),$$

where $\widetilde{Q}_{T,\underline{i}} := \widetilde{Q}_{T,i_1}\times\cdots\times\widetilde{Q}_{T,i_r}$. Then

$$\mathrm{cum}_r(\Phi_T) = \sum_{\underline{i}\in\mathcal{I}^r}\int_{Y_{T,\underline{i}}} \Xi_{r,T,\underline{i}}(y)\, d\kappa_{T,\underline{i}}(y). \tag{3.16}$$

For $\gamma > 0$, we define the *r-diagonal γ-neighborhood* $\Delta_r(\gamma)$ by

$$\Delta_r(\gamma) := \left\{(u^{(1)},\ldots,u^{(r)})\in(\mathbb{R}^{k-1})^r : \|u^{(j)}-u^{(k)}\|\le\gamma \text{ for all } j,k\right\}.$$

We split the sum defining $\Xi_{r,T,\underline{i}}$ into two subsums subdivided with respect to the set $\Delta_r(\gamma)$. Namely, we choose a parameter $\gamma_{T,r}\to\infty$, which will be specified later, and write

$$\Xi_{r,T,\underline{i}} = \Xi^{(1)}_{r,T,\underline{i}} + \Xi^{(2)}_{r,T,\underline{i}},$$

where $\Xi^{(1)}_{r,T,\underline{i}}(y)$ denotes the sum over *clustered* r-tuples

$$\sum_{\underline{u}\in\widetilde{Q}_{T,\underline{i}}\cap\Delta_r(\gamma_{T,r})} \mathrm{cum}_{[r]}\left(\varphi_{T,i_1}(\cdot, y_{i_1})\circ a\big(\beta_{T,i_1}(u^{(1)}, y_{i_1})\big), \cdots, \varphi_{T,i_r}(\cdot, y_{i_r})\circ a\big(\beta_{T,i_r}(u^{(r)}, y_{i_r})\big)\right), \tag{3.17}$$

and $\Xi^{(2)}_{r,T,\underline{i}}(y)$ denotes the sum over *separated* r-tuples:

$$\sum_{\underline{u}\in\widetilde{Q}_{T,\underline{i}}\cap\Delta_r(\gamma_{T,r})^c} \mathrm{cum}_{[r]}\left(\varphi_{T,i_1}(\cdot, y_{i_1})\circ a\big(\beta_{T,i_1}(u^{(1)}, y_{i_1})\big), \cdots, \varphi_{T,i_r}(\cdot, y_{i_r})\circ a\big(\beta_{T,i_r}(u^{(r)}, y_{i_r})\big)\right). \tag{3.18}$$

The aim in the upcoming subsections is to find conditions on the parameters $\gamma_{T,r}$ and L_T such that for every $\underline{i} = (i_1,\ldots,i_r)\in\mathcal{I}^r$,

$$\int_{Y_{T,\underline{i}}} \left|\Xi^{(1)}_{r,T,\underline{i}}(y)\right| d\kappa_{T,\underline{i}}(y) = o\left(V_T^{r/2}\right) \quad \text{as } T\to\infty, \tag{3.19}$$

and

$$\sup_{y\in Y_{T,\underline{i}}} \left|\Xi^{(2)}_{r,T,\underline{i}}(y)\right| = o\left(V_T^{r/2}\right) \quad \text{as } T\to\infty. \tag{3.20}$$

Together with the assumption (I.a) in Sect. 3.2, these estimates imply (3.12).

3.5.1 Analysis of the separated tuples

Now we prove the estimate (3.20) involving separated tuples. The crucial ingredient here is the estimates on higher-order correlations (Theorem 3.10), which allows us to establish an estimate on cumulants following our approach from [6].

We recall the estimate from Proposition 3.5(iii) that for every $q \geq 1$, there exists $\sigma_q > 0$ such that

$$S_q\big(\varphi \circ a(u)\big) \ll_q e^{\sigma_q \|u\|} S_q(\varphi) \qquad \text{for all } \varphi \in C_c^\infty(X) \text{ and} u \in \mathbb{R}^{k-1}.$$

We may without loss of generality assume that the map $q \mapsto \sigma_q$ is increasing. Furthermore, we may also assume that the map $r \mapsto \delta_r$ in Theorem 3.10 is decreasing. In particular, without loss of generality we can assume that

$$\delta_r < r\sigma_q, \quad \text{for all } q, r \geq 1. \tag{3.21}$$

The following lemma is a corollary of the main technical results from our work [6].

Lemma 3.16 *There is an integer $q_r \geq 1$, such that for every integer $q > q_r$, there exists a constant $c_{r,q} > 0$, with the property that for every $\gamma > 0$ and for all $\varphi_1, \ldots, \varphi_r \in C_c^\infty(X)$ and $u^{(1)}, \ldots, u^{(r)} \in \mathbb{R}^{k-1}$ with*

$$\max_{j \neq k} \|u^{(j)} - u^{(k)}\| > c_{r,q}\gamma,$$

we have

$$\left|\mathrm{cum}_{[r]}\left(\varphi_1 \circ a(u^{(1)}), \ldots, \varphi_r \circ a(u^{(r)})\right)\right| \ll_{r,q} e^{-\gamma} \prod_{j=1}^{r} S_q(\varphi_j).$$

Proof The proof follows the argument in [6, Sec. 6.4]. Let us fix $r \geq 2$, $\gamma > 0$ and an integer $q \geq 1$. We define parameters $\beta_0 = 0$, $\beta_1, \ldots, \beta_r$ recursively by $\beta_{j+1}\delta_r - 3r\sigma_q\beta_j = \gamma$. Then because of (3.21),

$$0 < \beta_1 < 3\beta_1 < \beta_2 < \cdots < \beta_{r-1} < 3\beta_{r-1} < \beta_r.$$

It is also clear from the recursive definition that

$$\beta_r \leq c_{r,q}\,\gamma \tag{3.22}$$

for a constant $c_{r,q} > 0$. Combining [6, Prop. 6.1] and [6, Prop. 6.2], we conclude that there is an integer q_r such that if $q > q_r$, then

$$\left|\mathrm{cum}_{[r]}\left(\varphi_1 \circ a(u^{(1)}), \ldots, \varphi_r \circ a(u^{(r)})\right)\right| \ll_{r,q} e^{-\gamma} \prod_{j=1}^{r} S_q(\varphi_j),$$

for all $u^{(1)}, \dots, u^{(r)} \in \mathbb{R}^{k-1}$ such that $\max_{j \neq k} \|u^{(j)} - u^{(k)}\| > \beta_r$. Together with (3.22), this proves the lemma. ☐

Now we apply Lemma 3.16 to estimate $\Xi^{(2)}_{r,T,\underline{i}}$, and deduce a criterion for (3.20). From now on q_r denotes the integer from Lemma 3.16.

Proposition 3.17 *Let q_r be as in Lemma 3.16 and suppose that the parameters L_T and $\gamma_{T,r}$ are chosen so that for some $q > q_r$,*

$$L_T^{r(d+1)}\, V_T^{r/2}\, e^{-c_1 \gamma_{T,r}/c_{r,q}}\, M_{T,q}^r \to 0, \quad \text{as } T \to \infty, \tag{3.23}$$

where c_1 is the positive constant in condition (II.b), and $c_{r,q}$ is given by Lemma 3.16. Then, for every $\underline{i} = (i_1, \dots, i_r) \in \mathcal{I}^r$,

$$\sup_{y \in Y_{T,\underline{i}}} \left|\Xi^{(2)}_{r,T,\underline{i}}(y)\right| = o\left(V_T^{r/2}\right), \quad \text{as } T \to \infty.$$

Proof We first note that if $(u^{(1)}, \dots, u^{(r)})$ belongs to $\widetilde{Q}_{T,\underline{i}} \cap \Delta_r(\gamma_{T,r})^c$, then by condition (II.b), for all $i_m, i_n \in \mathcal{I}$,

$$\left\|\beta_{T,i_m}(u^{(m)}, y_{i_m}) - \beta_{T,i_n}(u^{(n)}, y_{i_n})\right\| > c_1 \gamma_{T,r} - c_2,$$

for all $y_{i_m} \in Y_{T,i_m}$ and $y_{i_n} \in Y_{T,i_n}$. Applying Lemma 3.16 with γ defined by $c_1\, \gamma_{T,r} - c_2 = c_{r,q}\gamma$, we deduce that

$$\left|\mathrm{cum}_{[r]}\left(\varphi_{T,i_1} \circ a\big(\beta_{T,i_1}(u^{(1)}, y_{i_1})\big), \cdots, \varphi_{T,i_r} \circ a\big(\beta_{T,i_r}(u^{(r)}, y_{i_r})\big)\right)\right|$$

is estimated by

$$\ll_{r,q} e^{-c_1 \gamma_{T,r}/c_{r,q}} \prod_{m=1}^{r} S_q(\varphi_{T,i_m}),$$

where we in the last $\ll$-sign have absorbed the $e^{-c_2/c_{r,q}}$-factor. We recall that

$$\varphi_{T,i}(\Lambda, y_i) = \widehat{f}_{T,i}(\Lambda, y_i)\eta_{L_T}(\Lambda).$$

By Corollary 3.8,

$$S_q\big(\varphi_{T,i}(\cdot, y_i)\big) \ll_{\mathcal{K},q} L_T^{d+1} \left\|f_{T,i}(\cdot, y_i)\right\|_{C^q},$$

where $\mathcal{K} \subset \mathbb{R}^d$ is a fixed compact set which contains all of the supports of the functions $x \mapsto f_{T,i}(x, y_i)$ as y_i ranges over $Y_{T,i}$. We conclude that

$$\sup_{y \in Y_{T,\underline{i}}} \left|\Xi^{(2)}_{r,T,\underline{i}}(y)\right| \ll_{r,\mathcal{K},q} \left(\prod_{m=1}^{r} |\widetilde{Q}_{T,i_m}|\right) e^{-c_1\gamma_{T,r}/c_{r,q}}\, L_T^{r(d+1)} M_{T,q}^r$$
$$= V_T^r\, e^{-c_1\gamma_{T,r}/c_{r,q}}\, L_T^{r(d+1)} M_{T,q}^r.$$

This implies the proposition. □

3.5.2 Analysis of the clustered tuples

Next, we deal with the clustered tuples. Our analysis here is one of the main novelties of this paper. We stress that we do *not* assume that the maps $T \mapsto \|f_{T,i}\|_\infty$ are bounded (otherwise, our analysis could have been carried out as in [6]). This is also where the assumption (II.c) becomes crucial. This condition says roughly that the $\kappa_{T,i}$-integrals of $f_{T,i}$ are bounded functions. The main purpose of this subsection is to explain how this "bounded on average"-condition can be used to derive (3.19).

Proposition 3.18 *Suppose that the parameters L_T and $\gamma_{T,r}$ satisfy for some $\varepsilon > 0$,*

$$L_T^{r-d+\varepsilon}\, V_T^{1-r/2}\, \gamma_{T,r}^{(r-1)(k-1)} \to 0, \quad \text{as } T \to \infty. \tag{3.24}$$

Then,

$$\int_{Y_{T,\underline{i}}} \left|\Xi^{(1)}_{r,T,\underline{i}}(y)\right| d\kappa_{T,\underline{i}}(y) = o\left(V_T^{r/2}\right), \quad \text{as } T \to \infty.$$

Proof Expanding the definition of the cumulant in (3.17), we deduce that

$$\left|\Xi^{(1)}_{r,T,\underline{i}}(y)\right| \ll_r \max_{\mathcal{P}} \sum_{\underline{u} \in \widetilde{Q}_{T,\underline{i}} \cap \Delta_r(\gamma_{T,r})} \prod_{I \in \mathcal{P}} \int_X \left(\prod_{k \in I} \varphi_{T,i_k}\left(a\left(\beta_{T,i_k}(u^{(k)}, y_{i_k})\right)\Lambda, y_{i_k}\right)\right) d\mu(\Lambda). \tag{3.25}$$

We recall that

$$\varphi_{T,i}(\Lambda, y_i) = \widehat{f}_{T,i}(\Lambda, y_i)\eta_{L_T}(\Lambda).$$

By condition (II.c), there exist Borel functions $h_{T,i} : \mathbb{R}^d \times Y_{T,i} \to [0, \infty)$ such that

$$f_{T,i}\left(a\left(\beta_{T,i}(u, y_i)\right)z, y_i\right) \le h_{T,i}\left(a\left(\widetilde{\beta}_{T,i}(u)\right)z, y_i\right),$$

for all $u \in \widetilde{Q}_{T,i}$, $z \in \mathbb{R}^d$, and $y_i \in Y_{T,i}$. Hence, setting

$$h(z) := \sup_{T,i} \int_{Y_{T,i}} h_{T,i}(z, y_i)\, d\kappa_{T,i}(y_i),$$

we deduce that

$$\int_{Y_{T,i}} \widehat{f}_{T,i}\big(a\big(\beta_{T,i}(u,y_i)\big)\Lambda, y_i\big)\, d\kappa_{T,i}(y_i) = \sum_{z\in\Lambda\setminus\{0\}} \int_{Y_{T,i}} f_{T,i}\big(a\big(\beta_{T,i}(u,y_i)\big)z, y_i\big)\, d\kappa_{T,i}(y_i)$$
$$\leq \widehat{h}\big(a\big(\widetilde{\beta}_{T,i}(u)\big)\Lambda\big).$$

We recall that according condition (II.c), the function h is uniformly bounded and its support is contained in a fixed compact set. In particular, it follows from Lemma 3.2 that

$$\widehat{h}(\Lambda) \ll \alpha(\Lambda), \quad \text{for all } \Lambda \in X. \tag{3.26}$$

By condition (II.b), there is a fixed compact set $\mathcal{C} \subset A$ such that

$$a\left(\beta_{T,i}(u,y_i) - \widetilde{\beta}_{T,i}(u)\right) \in \mathcal{C}, \quad \text{for all } u \in \widetilde{Q}_{T,i},\, y_i \in Y_{T,i}, \text{ and } T > 0.$$

By Corollary 3.9, there is a constant $B = B(\mathcal{C}) > 0$ such that

$$\eta_{L_T} \circ g \leq \chi_{\{\alpha \leq B\, L_T\}} \quad \text{for all} T \text{and} g \in \mathcal{C},$$

whence

$$\eta_{L_T}\big(a\big(\beta_{T,i}(u,y_i)\big)\Lambda\big) = \eta_{L_T}\left(a\big(\beta_{T,i}(u,y_i) - \widetilde{\beta}_{T,i}(u)\big)a\big(\widetilde{\beta}_{T,i}(u)\big)\Lambda\right)$$
$$\leq \chi_{\{\alpha \leq B\, L_T\}}\left(a\big(\widetilde{\beta}_{T,i}(u)\big)\Lambda\right).$$

Combining the above estimates, we conclude that

$$\int_{Y_{T,i}} \varphi_{T,i}\left(a\big(\beta_{T,i}(u,y_i)\big)\Lambda, y_i\right)\, d\kappa_{T,i}(y_i) \leq \psi_T\left(a\big(\widetilde{\beta}_{T,i}(u)\big)\Lambda\right),$$

where ψ_T is defined by

$$\psi_T(\Lambda) := \widehat{h}(\Lambda)\, \chi_{\{\alpha \leq B\, L_T\}}(\Lambda), \quad \text{for } \Lambda \in X. \tag{3.27}$$

Therefore, we deduce from (3.25) that

$$\int_{Y_{T,\underline{i}}} |\Xi^{(1)}_{r,T,\underline{i}}(y)|\, d\kappa_{T,\underline{i}}(y) \ll_r \max_{\mathcal{P}} \sum_{\underline{u}\in\widetilde{Q}_{T,\underline{i}}\cap\Delta_r(\gamma_{T,r})} \prod_{I\in\mathcal{P}} \int_X \left(\prod_{k\in I} \psi_T\left(a\big(\tilde{\beta}_{T,i_k}(u^{(k)})\big)\Lambda\right)\right) d\mu(\Lambda). \tag{3.28}$$

We observe that it follows from (3.26), (3.27), and Lemma 3.3 that

$$\sup|\psi_T| = O(L_T) \quad \text{and} \quad \|\psi_T\|_{L^p(X)} = O_p(1) \;\; \text{for } p < d. \tag{3.29}$$

In particular, it also follows that for $p \geq d$,

$$\|\psi_T\|_{L^p(X)} = O_{p,q}\left(L_T^{1-q/p}\right) \quad \text{for all } q < d.$$

According to the general Hölder inequality, for exponents $p_k \in (1, \infty]$ satisfying $\sum_k 1/p_k = 1$,

$$\int_X \left(\prod_{k\in I} \psi_T\left(a\big(\tilde{\beta}_{T,i_k}(u^{(k)})\big)\Lambda\right)\right) d\mu(\Lambda) \leq \prod_{k\in I} \left\|\psi_T \circ a\big(\tilde{\beta}_{T,i_k}(u^{(k)})\big)\right\|_{L^{p_k}(X)} = \prod_{k\in I} \|\psi_T\|_{L^{p_k}(X)}.$$

Therefore, when $|I| < d$,

$$\int_X \left(\prod_{k\in I} \psi_T\left(a\big(\tilde{\beta}_{T,i_k}(u^{(k)})\big)\Lambda\right)\right) d\mu(\Lambda) = O(1),$$

and when $|I| \geq d$,

$$\int_X \left(\prod_{k\in I} \psi_T\left(a\big(\tilde{\beta}_{T,i_k}(u^{(k)})\big)\Lambda\right)\right) d\mu(\Lambda) = O_\varepsilon\left(L_T^{|I|-d+\varepsilon}\right) \quad \text{for all } \varepsilon > 0.$$

We conclude that for every partition $\mathcal{P}$,

$$\prod_{I\in\mathcal{P}} \int_X \left(\prod_{k\in I} \psi_T\left(a\big(\tilde{\beta}_{T,i_k}(u^{(k)})\big)\Lambda\right)\right) d\mu(\Lambda) = O_\varepsilon\left(L_T^{r-d+\varepsilon}\right),$$

and from (3.25),

$$\int_{Y_{T,\underline{i}}} \left|\Xi^{(1)}_{r,T,\underline{i}}(y)\right| d\kappa_{T,\underline{i}}(y) \ll_{r,\varepsilon} \left|\widetilde{Q}_{T,\underline{i}} \cap \Delta_r(\gamma_{T,r})\right| L_T^{r-d+\varepsilon}.$$

Since

$$\left|\widetilde{Q}_{T,\underline{i}} \cap \Delta_r(\gamma_{T,r})\right| \leq \sum_{u\in\widetilde{Q}_{T,i_1}} \prod_{k=2}^{r} \left|\widetilde{Q}_{T,i_k} \cap \{v : \|v-u\| \leq \gamma_{T,r}\}\right|,$$

it follows from condition (II.a) that

$$\left|\widetilde{Q}_{T,\underline{i}} \cap \Delta_r(\gamma_{T,r})\right| \ll V_T \gamma_{T,r}^{(k-1)(r-1)},$$

whence

$$\int_{Y_{T,\underline{i}}} \left|\Xi^{(1)}_{r,T}(y)\right| d\kappa_{T,i}(y) \ll_{r,\varepsilon} V_T \gamma_{T,r}^{(k-1)(r-1)} L_T^{r-d+\varepsilon},$$

for all $\varepsilon > 0$, which implies the assertion of the proposition. □

3.6 Main result

In this section, we finally prove convergence in distribution of the functions

$$\Upsilon_T(\Lambda) = V_T^{-1/2}\left(\widehat{F}_T(\Lambda) - \int_X \widehat{F}_T\, d\mu\right),$$

where

$$\widehat{F}_T(\Lambda) = \sum_{i\in\mathcal{I}} \int_{Y_{T,i}} \left(\sum_{a\in Q_{T,i}(y_i)} \widehat{f}_{T,i}(a\Lambda, y_i)\right) d\kappa_{T,i}(y_i).$$

We recall that the data in this formula satisfy the conditions (I.a)–(I.c) and (II.a)–(II.c). We further put an additional condition on the norms of the functions $f_{T,i}$, using the notation introduced in (3.2)–(3.3).

The main result of Sect. 3 is the following theorem:

Theorem 3.19 *Suppose that*

- *There exists* $\theta_0 > 0$ *such that*

$$M_T = O\left(V_T^{\theta_0}\right).$$

- *For* $q \geq 1$, *there exists* $\theta_q > 0$ *such that*

$$M_{T,q} = O\left(V_T^{\theta_q}\right).$$

- *The limit*

$$\sigma := \lim_{T\to\infty} \|\Upsilon_T\|_{L^2(X)}$$

  exists and is finite.

If $d > 4(1+\theta_0)$, *then the functions* Υ_T *on* (X, μ) *converge in distribution to the Normal Law with variance* σ.

Proof We shall use Proposition 3.13. We recall that by Lemma 3.1, the functions $\widehat{F}_T$ can be approximated by functions

$$\Phi_T(\Lambda) := \sum_{i\in\mathcal{I}} \int_{Y_{T,i}} \left(\sum_{a\in Q_{T,i}(y_i)} \varphi_{T,i}(a\Lambda, y_i)\right) d\kappa_{T,i}(y_i),$$

so that

$$\left\| \widehat{F}_T - \Phi_T \right\|_{L^2(X)} = o\left(V_T^{1/2}\right).$$

This implies that the functions

$$\Psi_T(\Lambda) = V_T^{-1/2}\left(\Phi_T(\Lambda) - \int_X \Phi_T \, d\mu\right)$$

satisfy

$$\left\| \Upsilon_T - \Psi_T \right\|_{L^2(X)} \to 0.$$

Then, in particular, $\lim_{T\to\infty} \|\Psi_T\|_{L^2(X)} = \sigma$. It also follows that if Ψ_T converges in distribution to the Normal Law, so does Υ_T. Hence, it remains to verify that the conditions of Proposition 3.13 hold for the functions Ψ_T, namely, that

$$\mathrm{cum}_r(\Psi_T) = V_T^{-r/2}\,\mathrm{cum}_r(\Phi_T) \to 0 \quad \text{for all } r \geq 3.$$

Since the later cumulant can be expressed as (3.16), this will follow from Propositions 3.17 and 3.18.

Now it remains to choose the parameters L_T and $\gamma_{T,r}$ so that the conditions in Lemma 3.1, Propositions 3.17, and 3.18 are satisfied. To do this, we shall take

$$L_T = V_T^{\rho} \quad \text{and} \quad \gamma_{T,r} = M_r \log V_T, \tag{3.30}$$

where ρ and M_r are positive real numbers, which will be chosen later. The condition (3.8) in Lemma 3.1 is satisfied if ρ is chosen so that for some $\varepsilon > 0$

$$V_T^{\rho(-d/2+1+\varepsilon)+1/2+\theta_0} \to 0,$$

or equivalently, if

$$\rho > \frac{1+2\theta_0}{d-2-2\varepsilon}. \tag{3.31}$$

We write q_r for the index introduced in Lemma 3.16 and fix an integer $q > q_r$. The condition (3.23) in Proposition 3.17 is satisfied if

$$V_T^{\rho r(d+1)+r/2-\frac{c_1 M_r}{c_{r,q}}+r\theta_q} \to 0,$$

which can always be arranged by choosing M_r large enough, depending on r, ρ, d. Finally, the condition (3.24) in Proposition 3.18 is satisfied if we choose the constants ρ and M_r such that for some $\varepsilon > 0$,

$$V_T^{\rho(r-d+\varepsilon)+1-r/2}(M_r \log V_T)^{(r-1)(k-1)} \to 0.$$

This holds provided that

$$\rho(r-d+\varepsilon) < r/2 - 1. \tag{3.32}$$

Hence, it is sufficient to choose ρ so that both (3.31) and (3.32) hold for all $r \geq 3$. This is possible provided that

$$\frac{1+2\theta_0}{d-2-2\varepsilon} < \rho \leq \frac{1}{2}.$$

Since $\varepsilon > 0$ is arbitrary, this argument works provided that $d > 4 + 4\theta_0$. □

Remark 3.20 In order to proceed with the proof above it is sufficient to have that

$$\left\| \Upsilon_T - \Psi_T \right\|_{L^1(X)} \to 0 \tag{3.33}$$

and

$$\lim_{T\to\infty} \left\| \Psi_T \right\|_{L^2(X)} = \sigma. \tag{3.34}$$

According to Lemma 3.1, condition (3.33) holds under assumption (3.9). This assumption is weaker than (3.8), so that we can replace (3.31) by the assumption

$$\rho > \frac{1+2\theta_0}{2d-2-2\varepsilon}. \tag{3.35}$$

Then the argument can be carried out when $d > 2(1+\theta_0)$, provided that we can establish (3.34) independently.

4 Proof of the main theorem

In this section, we prove our main theorem (Theorem 1.3). We recall that our goal is to analyze the lattice counting function for the domains

$$\Omega_T = \Omega_T(I, B) = \left\{ z \in \mathbb{R}^d : N(z) \in I,\ \xi_L(z) \in B \text{ and } 0 < \|L_1(z)\|, \ldots, \|L_k(z)\| < T \right\}. \tag{4.1}$$

Ultimately, we will construct an approximation of the characteristic function χ_{Ω_T} by functional averages of the form (2.4) and show that these functional averages satisfy the assumptions of Theorem 3.19, so that Theorem 1.3 will be a consequence of Theorem 3.19. This is a tedious and rather technical task, so it might be beneficial for the reader to first take a look in Sect. 4.7, where the main objects of the section are summarized, and the most important verifications are indexed.

4.1 A basic reduction

Let $L_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ with $j = 1, \ldots, k$, $I \subset (0, \infty)$, and $B \subset \mathbb{S}_{\underline{d}}$ be the objects defining the sets Ω_T. We also consider the basic domains

$$\Omega_T^0(I, B) := \{z \in \mathbb{R}^d \mid N(z) \in I,\ \xi(z) \in B \text{ and } 0 < \|z_1\|, \ldots, \|z_k\| < T\}, \tag{4.2}$$

where

$$N(z) := \prod_{j=1}^{k} \|z_j\|^{d_j} \quad \text{and} \quad \xi(z) := \left(\frac{z_1}{\|z_1\|}, \ldots, \frac{z_k}{\|z_k\|}\right). \tag{4.3}$$

Then $\Omega_T = L^{-1}(\Omega_T^0)$ for the invertible linear map $L = (L_1, \ldots, L_k)$. Let us write $L = cL_0$ with $c \in \mathbb{R}^\times$ and $\det(L_0) = 1$. Then

$$\Omega_T = L_0^{-1}\big(\operatorname{sgn}(c)|c|^{-1/d}\Omega_T^0(I, B)\big) = L_0^{-1}\big(\Omega_T^0(|c|^{-1}I, \operatorname{sgn}(c)B)\big).$$

Therefore, for any lattice Λ,

$$\big|\Lambda \cap \Omega_T\big| = \big|L_0(\Lambda) \cap \Omega_T^0(|c|^{-1}I, \operatorname{sgn}(c)B)\big|,$$

and

$$\operatorname{Vol}\big(\Omega_T^0(|c|^{-1}I, \operatorname{sgn}(c)B)\big) = \operatorname{Vol}\big(\Omega_T^0(I, B)\big).$$

Since the measure on the space of lattices is invariant under L_0, it is sufficient to analyze the distribution of the function $\Lambda \mapsto |\Lambda \cap \Omega_T^0| - \operatorname{Vol}(\Omega_T^0)$.

From now on we assume that the sets $\Omega_T = \Omega_T(I, B)$ are defined by (4.2), where I is a non-empty bounded interval in $(0, \infty)$, and B is a Borel subset of $\mathbb{S}_{\underline{d}}$ with positive measure.

4.2 A coodinate system

The sets Ω_T are more conveniently studied in a different coordinate system which we now introduce. We use notations

$$\mathbb{R}_*^{\underline{d}} := \prod\nolimits_{j=1}^{k} \mathbb{R}^{d_j}\backslash\{0\} \qquad \text{and} \qquad \mathbb{S}_{\underline{d}} := \prod\nolimits_{j=1}^{k} S^{d_j-1}.$$

Let

$$\pi : \mathbb{R}_*^{\underline{d}} \longrightarrow \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{S}_{\underline{d}} :\ z \mapsto \big(u(z), s(z), \xi(z)\big), \tag{4.4}$$

where

$$u(z) := (\log \|z_1\|, \dots, \log \|z_{k-1}\|),$$
$$s(z) := \log N(z) = \sum_{j=1}^{k} d_j \log \|z_j\|,$$
$$\xi(z) := \left(\frac{z_1}{\|z_1\|}, \dots, \frac{z_k}{\|z_k\|} \right).$$

It is readily checked that the map π is equivariant with respect to the group A defined in (2.3) in the following sense:

$$\pi(a(u)z) = (u(z) + u, s(z), \xi(z)), \quad \text{for all} u \in \mathbb{R}^{k-1} \text{and} x \in \mathbb{R}^{\underline{d}}_*, \tag{4.5}$$

and that the inverse map π^{-1} is given by

$$\pi^{-1}(u, s, \xi) = \left(e^{u_1}\xi_1, \dots, e^{u_{k-1}}\xi_{k-1}, e^{\left(s - \sum_{j=1}^{k-1} d_j u_j\right)/d_k} \xi_k \right). \tag{4.6}$$

If one computes the Jacobian of this inverse map, the following lemma emerges:

Lemma 4.1 *For every bounded Borel function $f : \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{S}_{\underline{d}} \to \mathbb{R}$ with bounded support,*

$$\int_{\mathbb{R}^{\underline{d}}_*} f(\pi(z))\, dz = \frac{1}{d_k} \int_{\mathbb{S}_{\underline{d}}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{k-1}} f(u, s, \xi)\, du \right) e^s\, ds\, d\kappa(\xi).$$

Here dz denote the volume element on $\mathbb{R}^d$ which assigns volume one to the unit cube, du is the volume element on $\mathbb{R}^{k-1}$ such that the unit cube in $\mathbb{R}^{k-1}$ has volume one, and the measure κ is defined in (1.5).

Let us now write out the set Ω_T in (u, s, ξ)-coordinates. We define

$$\Delta_T := \pi(\Omega_T) \subset \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{S}_{\underline{d}},$$

and given a point z in $\mathbb{R}^{\underline{d}}_*$, we set

$$u = u(z) = (u_1, \dots, u_{k-1}), \quad s = s(z), \quad \xi = \xi(z).$$

Then $z \in \Omega_T$ if and only if

$$s \in \log I, \quad \xi \in B, \quad u_1 < \log T, \dots, u_{k-1} < \log T, \quad s - \sum_{j=1}^{k-1} d_j u_j < d_k \log T.$$

We now set $v_j = u_j - \log T$ for $j = 1, \ldots, k-1$. Then, the above conditions on u are equivalent to

$$v_1, \ldots, v_{k-1} < 0 \quad \text{and} \quad \sum_{j=1}^{k-1} d_j v_j > -(d \log T - s). \tag{4.7}$$

For $s < d \log T$, we let $\delta_T(s)$ denote the diagonal $(k-1) \times (k-1)$-matrix whose diagonal elements $\delta_{T,j}(s)$ are given by

$$\delta_{T,j}(s) := \frac{d \log T - s}{d_j}, \quad \text{for } j = 1, \ldots, k-1.$$

We note that since the interval I is bounded, the inequality $s < d \log T$ is satisfied for all $x \in \Omega_T(I, B)$ when $T > e^{\sup(I)/d}$. Then (4.7) can be re-written as

$$\min_j \delta_{T,j}(s)^{-1} v_j < 0 \quad \text{and} \quad \sum_{j=1}^{k-1} \delta_{T,j}(s)^{-1} v_j > -1,$$

Let

$$\mathcal{S}_1 := \left\{ (w_1, \ldots, w_{k-1}) \in \mathbb{R}^{k-1} \, : \, w_1, \ldots, w_{k-1} < 0 \text{ and } \textstyle\sum_{j=1}^{k-1} w_j > -1 \right\} \tag{4.8}$$

and

$$v_T := (\log T, \ldots, \log T).$$

We conclude that

$$\Delta_T = \pi(\Omega_T) = \left\{ (u, s, \xi) \, : \, s \in \log I, \quad \xi \in B, \quad u \in \delta_T(s)\mathcal{S}_1 + v_T \right\} \tag{4.9}$$

when $T > e^{\sup(I)/d}$.

4.3 Volume and variance computations

The above parametrization of Ω_T leads, in particular, to an an easy computation of its volume, and the mean and the variance of the Siegel transforms $\widehat{\chi}_{\Omega_T}$.

Lemma 4.2 *There exists a polynomial $P_{I,B}$ such that*

$$P_{I,B}(t) = c_{k-1}(I, B) t^{k-1} + O(t^{k-2}),$$

where

$$c_{k-1}(I, B) = \frac{d^{k-1}}{d_1 \cdots d_k} \operatorname{Leb}(I) \operatorname{Vol}_{k-1}(\mathcal{S}_1) \, \kappa(B),$$

such that

$$\mathrm{Vol}\left(\Omega_T(I,B)\right) = P_{I,B}(\log T),$$

for all $T > e^{\sup(I)/d}$.

Proof It follows from (4.9) and Lemma 4.1 that

$$\begin{aligned}\mathrm{Vol}(\Omega_T) &= \frac{\kappa(B)}{d_k}\int_{\log I}\mathrm{Vol}_{k-1}\left(\delta_T(s)\mathcal{S}_1 + v_T\right)e^s\,ds\\ &= \frac{\kappa(B)}{d_k}\,\mathrm{Vol}_{k-1}(\mathcal{S}_1)\int_{\log I}\frac{(d\log T - s)^{k-1}}{d_1\cdots d_{k-1}}\,e^s\,ds.\end{aligned}$$

If we expand the inner parenthesis and integrating term-wise, we deduce that $\mathrm{Vol}(\Omega_T) = P_{I,B}(\log T)$ for the polynomial

$$P_{I,B}(t) = \frac{\kappa(B)}{d_k}\,\mathrm{Vol}_{k-1}(S_1)\int_{\log I}\frac{(dt-s)^{k-1}}{d_1\cdots d_{k-1}}\,e^s\,ds.$$

The leading term of this polynomial is $c_{k-1}(I,B)t^{k-1}$ with

$$c_{k-1}(I,B) = \frac{d^{k-1}}{d_1\cdots d_k}\kappa(B)\,\mathrm{Vol}_{k-1}(\mathcal{S}_1)\int_{\log I}e^s\,ds = \frac{d^{k-1}}{d_1\cdots d_k}\kappa(B)\,\mathrm{Vol}_{k-1}(\mathcal{S}_1)\,\mathrm{Leb}(I),$$

which finishes the proof of the lemma. □

From (2.2), we also obtain that

$$\int_X \widehat{\chi}_{\Omega_T}\,d\mu = P_{I,B}(\log T) = c_{k-1}(I,B)(\log T)^{k-1} + O\left((\log T)^{k-2}\right).$$

To compute the variance of the Siegel transform, we need the following

Theorem 4.3 (Rogers' mean-square value theorem, [24]) *Let* $d \geq 3$ *and let* $f:\mathbb{R}^d \to \mathbb{R}$ *be a bounded and non-negative Borel measurable function with bounded support. Then* $\widehat{f} \in L^2(X)$ *and*

$$\int_X\left(\widehat{f} - \int_X \widehat{f}\,d\mu\right)^2 d\mu = \frac{1}{\zeta(d)}\sum_{p,q\geq 1}\left(\int_{\mathbb{R}^d} f(pz)f(qz)\,dz + \int_{\mathbb{R}^d} f(pz)f(-qz)\,dz\right),$$

where ζ *denotes the Riemann zeta-function.*

For a future reference, we also note that a straightforward application of the Cauchy-Schwarz inequality to the expression in Theorem 4.3 yields the following corollary:

Corollary 4.4 *If $d \geq 3$ and $f : \mathbb{R}^d \to \mathbb{R}$ is a bounded and non-negative Borel measurable function with bounded support, then*

$$\|\widehat{f}\|^2_{L^2(X)} \leq \|f\|_1^2 + 2\frac{\zeta(d/2)^2}{\zeta(d)}\|f\|_2^2.$$

Now using Theorem 4.3, we compute the variance:

Corollary 4.5

$$\sigma^2 := \lim_{T\to\infty} \frac{\int_X \left(\widehat{\chi}_{\Omega_T} - \int_X \widehat{\chi}_{\Omega_T}\right)^2 d\mu}{\mathrm{Vol}(\Omega_T)} = \frac{1}{\zeta(d)} \left(\sum_{p,q=1}^{\infty} \frac{\mathrm{Leb}\left(p^d I \cap q^d I\right)}{p^d q^d\, \mathrm{Leb}(I)} \right) \times \left(1 + \frac{\kappa(B \cap -B)}{\kappa(B)}\right).$$

Proof By Theorem 4.3,

$$\int_X \left(\widehat{\chi}_{\Omega_T} - \int_X \widehat{\chi}_{\Omega_T}\right)^2 d\mu = \frac{1}{\zeta(d)} \sum_{p,q\geq 1} \left(\mathrm{Vol}(p^{-1}\Omega_T \cap q^{-1}\Omega_T) + \mathrm{Vol}(p^{-1}\Omega_T \cap -q^{-1}\Omega_T)\right).$$

If we split this sum into sums over $\{p = q\}$ and $\{p \neq q\}$ and use the symmetry of p and q and the formula $\mathrm{Vol}(q^{-1}\Omega_T) = q^{-d}\,\mathrm{Vol}(\Omega_T)$ for every $q \geq 1$, we see that this sum can be written as

$$\mathrm{Vol}(\Omega_T) + \mathrm{Vol}(\Omega_T \cap -\Omega_T) + \frac{2}{\zeta(d)} \sum_{q=1}^{\infty} \frac{1}{q^d} \sum_{p=1}^{q-1} \left(\mathrm{Vol}\left(\Omega_T \cap (q/p)\Omega_T\right) + \mathrm{Vol}\left(\Omega_T \cap -(q/p)\Omega_T\right)\right).$$

We observe that for $c, T > 0$, $I \subset (0, \infty)$, and $B \subset \mathbb{S}_{\underline{d}}$,

$$\pm c\Omega_T(I, B) = \Omega_{cT}(c^d I, \pm B),$$

and for $T_1, T_2 > 0$, $I_1, I_2 \subset (0, \infty)$, and $B_1, B_2 \subset \mathbb{S}_{\underline{d}}$,

$$\Omega_{T_1}(I_1, B_1) \cap \Omega_{T_2}(I_2, B_2) = \Omega_{\min(T_1,T_2)}(I_1 \cap I_2, B_1 \cap B_2).$$

Hence, we deduce from Lemma 4.2 that for every $c \geq 1$,

$$\kappa_{\pm}(c) := \lim_{T\to\infty} \frac{\mathrm{Vol}(\Omega_T \cap \pm c\Omega_T)}{\mathrm{Vol}(\Omega_T)} = \lim_{T\to\infty} \frac{\mathrm{Vol}\left(\Omega_T(I \cap c^d I, B \cap \pm B)\right)}{\mathrm{Vol}(\Omega_T)} = \frac{\mathrm{Leb}(I \cap c^d I)}{\mathrm{Leb}(I)} \frac{\kappa(B \cap \pm B)}{\kappa(B)}.$$

Then since we are assuming that $d \geq 3$, we can apply the Dominated Convergence Theorem to conclude that the limit σ^2 exists and

$$\begin{aligned}\sigma^2 &= 1 + \frac{\kappa(B \cap -B)}{\kappa(B)} + \frac{2}{\zeta(d)} \sum_{q=1}^{\infty} \frac{1}{q^d} \sum_{p=1}^{q-1} \big(\kappa_+(q/p) + \kappa_-(q/p)\big) \\ &= \left(1 + \frac{2}{\zeta(d)} \sum_{q=1}^{\infty} \frac{1}{q^d} \sum_{p=1}^{q-1} \frac{\mathrm{Leb}\left(I \cap (q/p)^d I\right)}{\mathrm{Leb}(I)}\right)\left(1 + \frac{\kappa(B \cap -B)}{\kappa(B)}\right).\end{aligned}$$

This implies the stated formula. □

4.4 Tessellations of the sets $\Omega_T(I, B)$

In this subsection, we construct, for all large enough T, a functional tiling of the indicator function χ_{Ω_T} using the coordinate system introduced in the previous section. This tiling will be the basis for our smooth approximation scheme later. Before we can state our main observation (Corollary 4.10) of this subsection, we need some preliminaries. For a positive integer N, we define

$$\mathcal{S}(N) := \left\{(u_1, \ldots, u_{k-1}) \in \mathbb{R}^{k-1} \;:\; u_1, \ldots, u_{k-1} < 0 \text{ and } \sum_{j=1}^{k-1} u_j > -N\right\},$$

and set

$$\mathcal{S}_1 := \mathcal{S}(1) \quad \text{and} \quad \mathcal{S}_2 := [-1, 0)^{k-1} \backslash \mathcal{S}(1). \tag{4.10}$$

We note that this definition of $\mathcal{S}_1$ coincides with the one given in (4.8) above. Geometrically, $\mathcal{S}_1$ and $\mathcal{S}_2$ are the lower and upper pieces of the unit cube $(-1, 0]^k$ in $\mathbb{R}^{k-1}$ cut in half by the hyperplane $u_1 + \cdots + u_{k-1} = -1$. Furthermore, we define

$$P_{N,i} := \left\{n \in [0, N]^{k-1} \cap \mathbb{Z}^{k-1} \;:\; \mathcal{S}_i - n \subset \mathcal{S}(N)\right\}, \quad \text{for } i = 1, 2.$$

The next lemma tells us that $\mathcal{S}(N)$ can be tesselated by translates of $\mathcal{S}_1$ by vectors in $P_{N,1}$ and by translates of $\mathcal{S}_2$ by elements of $P_{N,2}$. We stress that while the sets of integer vectors $P_{N,1}$ and $P_{N,2}$ are not disjoint, the translates of $\mathcal{S}_1$ and $\mathcal{S}_2$ by vectors in the respective sets are disjoint.

Lemma 4.6 *For every positive integer N,*

$$\mathcal{S}(N) = \left(\bigsqcup_{n \in P_{N,1}} (\mathcal{S}_1 - n)\right) \bigsqcup \left(\bigsqcup_{n \in P_{N,2}} (\mathcal{S}_2 - n)\right).$$

In particular,

$$\max_{n\in P_{N,i}} \|n\|_\infty \ll N \quad \textit{and} \quad |P_{N,i}| \ll \mathrm{Vol}_{k-1}(\mathcal{S}(N)) \ll N^{k-1}.$$

Proof Fix $u \in \mathcal{S}(N)$, and note that since $-N \le u_j \le 0$ for all j, there are *unique* integers $0 \le n_j \le N$ such that

$$w := u + n \in [-1,0)^{k-1}, \quad \text{where } n = (n_1, \dots, n_{k-1}),$$

and thus either $w \in \mathcal{S}_1$ or $w \in \mathcal{S}_2$, whence $u \in \mathcal{S}_i - n$ for either $i = 1, 2$. Clearly these are disjoint events, so in particular,

$$\mathcal{S}(N) = \left(\bigsqcup_{n\in P_{N,1}} (S_1 - n) \right) \bigsqcup \left(\bigsqcup_{n\in P_{N,2}} (S_2 - n) \right),$$

which finishes the proof. □

We observe that in view of (4.9) the sets Δ_T are related to suitable dilations of the sets $\mathcal{S}(N)$. Indeed, for T and s with $s < d \log T$, we let

$$\tau_T(s) := \mathrm{Diag}\left(\tau_{T,1}(s), \dots, \tau_{T,k-1}(s)\right)$$

denote the diagonal $(k-1) \times (k-1)$-matrix with the positive diagonal entries

$$\tau_{T,j}(s) := \frac{d \log T - s}{d_j \lfloor \log T \rfloor}, \quad \text{for } j = 1, \dots, k-1, \tag{4.11}$$

then

$$\Delta_T = \left\{ (u, s, \xi) \;:\; s \in I, \quad \xi \in B, \quad u \in \tau_T(s)\mathcal{S}(\lfloor \log T \rfloor) + v_T \right\}.$$

Therefore, applying Lemma 4.6 to $\mathcal{S}(\lfloor \log T \rfloor)$, we get the following "functional tiling" for the characteristic function χ_{Δ_T}.

Lemma 4.7 *For all* $(u, s, \xi) \in \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{S}_{\underline{d}}$ *with* $s < d \log T$,

$$\begin{aligned}
\chi_{\Delta_T}(u, s, \xi) = &\sum_{n \in P_{\lfloor \log T \rfloor, 1}} \chi_{\mathcal{S}_1}\left(\tau_T(s)^{-1}(u + \tau_T(s)n - v_T)\right) \chi_{\log I}(s)\, \chi_B(\xi) \\
&+ \sum_{n \in P_{\lfloor \log T \rfloor, 2}} \chi_{\mathcal{S}_2}\left(\tau_T(s)^{-1}(u + \tau_T(s)n - v_T)\right) \chi_{\log I}(s)\, \chi_B(\xi).
\end{aligned}$$

In particular, for all $T > e^{\sup(I)/d}$, *this identity holds everywhere.*

4.5 Construction of a functional tiling

Now we construct our functional tiling, namely, the objects satisfying conditions (I.a)–(I.c) and (II.a)–(II.c) with $V_T := \mathrm{Vol}(\Omega_T)$.

4.5.1 Construction of the sets $\widetilde{Q}_{T,i}$, $Q_{T,i}(y)$ and maps $\beta_{T,i}$, $\widetilde{\beta}_{T,i}$ (assumptions (II.a)–(II.b))

Let us now rewrite the assertion of Lemma 4.7, so that it fits the decomposition (2.4). We note that

$$\tau_T(s) = \tau_\infty + O\big(1/(\log T)\big) \quad \text{as } T \to \infty \tag{4.12}$$

uniformly on s in compact sets, where

$$\tau_\infty := \mathrm{Diag}(d/d_1, \dots, d/d_{k-1}).$$

We define

$$\beta_T : \mathbb{R}^{k-1} \times \mathbb{R} \to \mathbb{R}^{k-1} \quad \text{and} \quad \widetilde{\beta}_T : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$$

by

$$\beta_T(u, s) := \tau_T(s)u - v_T \quad \text{and} \quad \widetilde{\beta}_T(u) := \tau_\infty u - v_T \tag{4.13}$$

for $u \in \mathbb{R}^{k-1}$ and $s \in \mathbb{R}$. Let

$$\widetilde{Q}_{T,i} := P_{\lfloor \log T \rfloor, i} \subset \mathbb{R}^{k-1}, \qquad \text{for } i = 1, 2. \tag{4.14}$$

From Lemmas 4.2 and 4.6, we see that $|\widetilde{Q}_{T,i}| \ll \mathrm{Vol}(\Omega_T)$. The condition (3.13) in (II.a) can be also checked easily. The following lemma verifies condition (II.b). We recall that $\|\cdot\|$ denotes the ℓ^∞-norm on $\mathbb{R}^{k-1}$.

Lemma 4.8 *Let $J \subset \mathbb{R}$ be a bounded interval.*

(i) There exist $c_1, c_2 > 0$ such that for all $T \geq T_0(J)$, $s_1, s_2 \in J$, and $u, v \in \widetilde{Q}_{T,i}$,

$$\big\|\beta_T(u, s_1) - \beta_T(v, s_2)\big\| \geq c_1\|u - v\| - c_2.$$

(ii) There exists $c_3 > 0$ such that for all $T \geq T_0(J)$, $s \in J$, and $u \in \widetilde{Q}_{T,i}$,

$$\big\|\beta_T(u, s) - \widetilde{\beta}_T(u)\big\| \leq c_3.$$

Proof Since $\|u\| \ll \log T$ for all $u \in \widetilde{Q}_{T,i}$, this lemma follows immediately from (4.12) and the definitions of the maps β_T and $\tilde{\beta}_T$. □

Remark 4.9 While in Sect. 2 we have allowed β_T and $\widetilde{\beta}_T$ to also depend on i, it is not necessary at this point. However, to properly work with these functions in our setting, we also need to define the finite measure spaces $(Y_{T,i}, \kappa_{T,i})$, for $i = 1, 2$. This will be done in the next section.

Let us now rewrite the decomposition in Lemma 4.7 using the standard coordinates. We set

$$Q_{T,i}(s) := \beta_T\left(\widetilde{Q}_{T,i}, s\right) \tag{4.15}$$

and

$$\widetilde{h}_{T,i}(u, s, \xi) := \chi_{\mathcal{S}_i}\left(\tau_T(s)^{-1}u\right)\chi_{\log I}(s)\,\chi_B(\xi),$$

for $i = 1, 2$, and note that the assertion in the lemma above can be written as

$$\begin{aligned}\chi_{\Delta_T}(u, s, \xi) &= \sum_{w\in\tilde{Q}_{T,1}} \widetilde{h}_{T,1}(u + \beta_T(w, s), s, \xi) + \sum_{w\in\tilde{Q}_{T,2}} \widetilde{h}_{T,2}(u + \beta_T(w, s), s, \xi) \\ &= \sum_{v\in Q_{T,1}(s)} \widetilde{h}_{T,1}(u + v, s, \xi) + \sum_{v\in Q_{T,2}(s)} \widetilde{h}_{T,2}(u + v, s, \xi) \end{aligned} \tag{4.16}$$

for all large enough T. Let us now set

$$h_{T,i} := \widetilde{h}_{T,i} \circ \pi, \quad \text{for } i = 1, 2.$$

Since $\chi_{\Omega_T} = \chi_{\Delta_T} \circ \pi$, the equivariance (4.5) of π yields the following corollary of Lemma 4.7:

Corollary 4.10 *For all large enough T,*

$$\chi_{\Omega_T}(z) = \sum_{v\in Q_{T,1}(s(z))} h_{T,1}(a(v)z) + \sum_{v\in Q_{T,2}(s(z))} h_{T,2}(a(v)z), \quad \textit{for } z \in \mathbb{R}^d_*.$$

We stress that the summation range in the above formula depend on the point z, albeit in a weak way via $s(z)$. In the next subsection, we will get rid of this z-dependence upon introducing an additional average. The price we have to pay for this is that the functions $h_{T,i}$ will be replaced with more complicated functions $f_{T,i}$, which depend on the an extra variable, coming from the average.

4.5.2 Construction of the spaces $(Y_{T,i}, \kappa_{T,i})$ and functions $f_{T,i}$ (assumptions (I.a)–(I.b))

If $\mathcal{T} \subset \mathbb{R}^{k-1}$ is a subset and $r \geq 0$, we denote by $\mathcal{T}_r$ the r-thickening of $\mathcal{T}$ with respect to this norm. Similarly, for a subset B of $\mathbb{S}_{\underline{d}}$, we denote by B_r the r-thickening of B with respect to the rotation-invariant metric on $\mathbb{S}_{\underline{d}}$.

Since $|v| \ll \log T$ for every $v \in \widetilde{Q}_{T,i}$, it follows from (4.12) that for any bounded interval $J \subset \mathbb{R}$, there exist $c(J) > 0$ such that for all $s, t \in J$, $T \geq T_0(J)$, and $v \in \widetilde{Q}_{T,i}$,

$$\left\| \tau_T(s)^{-1}(\beta_T(v,s) - \beta_T(v,t)) \right\| = \left\| \tau_T(s)^{-1}(\tau_T(s)v - \tau_T(t)v) \right\| \leq c(J)\,|s-t|.$$

Hence, we deduce that for all $s, t \in J$ satisfying $|s-t| \leq r$, $T \geq T_0(J)$, $u \in \mathbb{R}^{k-1}$, and $v \in \widetilde{Q}_{T,i}$,

$$\chi_{S_i}\big(\tau_T(s)^{-1}(u + \beta_T(v,s))\big) \leq \chi_{(S_i)_{c(J)r}}\big(\tau_T(s)^{-1}(u + \beta_T(v,t))\big). \tag{4.17}$$

Let us now introduce a parameter $\varepsilon \in (0,1)$ and a non-negative real smooth function ρ_ε on $\mathbb{R}$ with

$$\operatorname{supp}(\rho_\varepsilon) \subset [-\varepsilon/2, \varepsilon/2] \quad \text{and} \quad \int_{\mathbb{R}} \rho_\varepsilon(t)\,dt = 1. \tag{4.18}$$

For future reference, we also note that ρ_ε can be chosen, so that

$$\|\rho_{\varepsilon_T}\|_{C^q} \ll \varepsilon^{-1-q}. \tag{4.19}$$

By the standard properties of convolutions,

$$\chi_{\log I} \leq \rho_\varepsilon * \chi_{(\log I)_\varepsilon} \leq \chi_{(\log I)_{2\varepsilon}}.$$

Then, using (4.17), we deduce that for every $u \in \mathbb{R}^{k-1}$ and $v \in \widetilde{Q}_{T,i}$,

$$\begin{aligned}
\widetilde{h}_{T,i}\big(u + \beta_T(v,s), s, \xi\big) &= \chi_{S_i}\big(\tau_T(s)^{-1}(u + \beta_T(v,s))\big)\chi_{\log I}(s)\,\chi_B(\xi) \\
&\leq \int_{(\log I)_\varepsilon} \chi_{S_i}\big(\tau_T(s)^{-1}(u + \beta_T(v,s))\big)\rho_\varepsilon(s-t)\,\chi_B(\xi)\,dt \\
&\leq \int_{(\log I)_\varepsilon} \chi_{(S_i)_{c\varepsilon}}\big(\tau_T(s)^{-1}(u + \beta_T(v,t))\big)\,\rho_\varepsilon(s-t)\,\chi_B(\xi)\,dt,
\end{aligned}$$

where $c = c(J) > 0$ for a fixed bounded interval J which contains $(\log I)_\varepsilon$ for all $0 < \varepsilon < 1$. Let $\psi_{i,\varepsilon}$ be a smooth function on $\mathbb{R}^{k-1}$ such that

$$\chi_{(S_i)_{c\varepsilon}} \leq \psi_{i,\varepsilon} \leq \chi_{(S_i)_{2c\varepsilon}}, \qquad \text{for } i = 1, 2, \tag{4.20}$$

and let ϑ_ε be a smooth function on $\mathbb{S}_{\underline{d}}$ such that

$$\chi_B \leq \vartheta_\varepsilon \leq \chi_{B_\varepsilon}. \tag{4.21}$$

For future reference, we note that these functions can be constructed, so that

$$\|\psi_{i,\varepsilon}\|_{C^q} \ll \varepsilon^{-1-q} \quad \text{and} \quad \|\vartheta_\varepsilon\|_{C^q} \ll \varepsilon^{-\theta_q} \quad \text{for some } \theta_q > 0. \tag{4.22}$$

From the above estimate, we deduce that for every $u \in \mathbb{R}^{k-1}$ and $v \in \widetilde{Q}_{T,i}$,

$$\widetilde{h}_{T,i}\big(u+\beta_T(v,s),s,\xi\big) \le \int_{(\log I)_\varepsilon} \psi_{i,\varepsilon}\big(\tau_T(s)^{-1}(u+\beta_T(v,t))\big)\,\rho_\varepsilon(s-t)\,\vartheta_\varepsilon(\xi)\,dt. \tag{4.23}$$

By the same argument as in (4.17), we also have for all $s, t \in J$ satisfying $|s-t| \le \varepsilon$, $T \ge T_0(J)$, $u \in \mathbb{R}^{k-1}$, and $v \in \widetilde{Q}_{T,i}$,

$$\chi_{(S_i)_{2c\varepsilon}}\big(\tau_T(s)^{-1}(u+\beta_T(v,t)\big) \le \chi_{(S_i)_{3c\varepsilon}}\big(\tau_T(s)^{-1}(u+\beta_T(v,s))\big).$$

Then it follows from (4.20) and (4.21) that

$$\widetilde{h}_{T,i}\big(u+\beta_T(v,s),s,\xi\big) \le \int_{(\log I)_\varepsilon} \psi_{i,\varepsilon}\big(\tau_T(s)^{-1}(u+\beta_T(v,t))\big)\,\rho_\varepsilon(s-t)\,\vartheta_\varepsilon(\xi)\,dt, \tag{4.24}$$

and

$$\begin{aligned}
&\int_{(\log I)_\varepsilon} \psi_{i,\varepsilon}\big(\tau_T(s)^{-1}(u+\beta_T(v,t))\big)\,\rho_\varepsilon(s-t)\,\vartheta_\varepsilon(\xi)\,dt\\
&\quad\le \int_{(\log I)_\varepsilon} \chi_{(\mathcal{S}_i)_{2c\varepsilon}}\big(\tau_T(s)^{-1}(u+\beta_T(v,t))\big)\,\rho_\varepsilon(s-t)\,\vartheta_\varepsilon(\xi)\,dt\\
&\quad\le \int_{(\log I)_\varepsilon} \chi_{(\mathcal{S}_i)_{3c\varepsilon}}\big(\tau_T(s)^{-1}(u+\beta_T(v,s))\big)\,\rho_\varepsilon(s-t)\,\vartheta_\varepsilon(\xi)\,dt\\
&\quad\le \chi_{(\mathcal{S}_i)_{3c\varepsilon}}\big(\tau_T(s)^{-1}(u+\beta_T(v,s))\big)\,\chi_{(\log I)_{2\varepsilon}}(s)\,\chi_{B_\varepsilon}(\xi).
\end{aligned} \tag{4.25}$$

We introduce a parameter $\varepsilon_T \in (0,1)$, to be specified later, and define

$$Y_T := (\log I)_{\varepsilon_T} \quad\text{and}\quad \kappa_T := \mathrm{Leb}\,|_{Y_T}\,. \tag{4.26}$$

For $y \in Y_T$, we set

$$\widetilde{f}_{T,i}\big((u,s,\xi),y\big) := \psi_{i,\varepsilon_T}(\tau_T(s)^{-1}u)\,\rho_{\varepsilon_T}(s-y)\,\vartheta_{\varepsilon_T}(\xi) \tag{4.27}$$

and consider

$$\begin{aligned}
\widetilde{F}_T(u,s,\xi) := &\int_{Y_T}\Big(\sum_{w\in Q_{T,1}(y)} \widetilde{f}_{T,1}\big((u+w,s,\xi),y\big)\Big)\,d\kappa_T(y)\\
&+\int_{Y_T}\Big(\sum_{w\in Q_{T,2}(y)} \widetilde{f}_{T,2}\big((u+w,s,\xi),y\big)\Big)\,d\kappa_T(y).
\end{aligned}$$

It follows from (4.23) that for every $u \in \mathbb{R}^{k-1}$ and $v \in \widetilde{Q}_{T,i}$,

$$\widetilde{h}_{T,i}\big(u+\beta_T(v,s),s,\xi\big) \le \int_{Y_T} \widetilde{f}_{T,i}\big((u+\beta_T(v,y),s,\xi),y\big)\,d\kappa_T(y).$$

Hence, by (4.16) and (4.24),

$$\begin{aligned}\chi_{\Delta_T}(u,s,\xi) &\le \int_{Y_T}\Big(\sum_{v\in\widetilde{Q}_{T,1}} \widetilde{f}_{T,1}\big((u+\beta_T(v,y),s,\xi),y\big)\Big)\,d\kappa_T(y)\\ &\quad+\int_{Y_T}\Big(\sum_{v\in\widetilde{Q}_{T,2}} \widetilde{f}_{T,2}\big((u+\beta_T(v,y),s,\xi),y\big)\Big)\,d\kappa_T(y)\\ &=\widetilde{F}_T(u,s,\xi).\end{aligned}$$

Let

$$\begin{aligned}\chi^{+}_{\Delta_T}(u,s,\xi) := &\sum_{v\in\widetilde{Q}_{T,1}} \chi_{(\mathcal{S}_1)_{3c\varepsilon_T}}\big(\tau_T(s)^{-1}(u+\beta_T(v,s))\big)\,\chi_{(\log I)_{2\varepsilon_T}}(s)\,\chi_{B_{\varepsilon_T}}(\xi)\\ &+\sum_{v\in\widetilde{Q}_{T,2}} \chi_{(\mathcal{S}_2)_{3c\varepsilon_T}}\big(\tau_T(s)^{-1}(u+\beta_T(v,s))\big)\,\chi_{(\log I)_{2\varepsilon_T}}(s)\,\chi_{B_{\varepsilon_T}}(\xi).\end{aligned} \tag{4.28}$$

Then it follows from (4.25) that

$$\widetilde{F}_T(u,s,\xi) \le \chi^{+}_{\Delta_T}(u,s,\xi).$$

We conclude that

$$\chi_{\Delta_T} \le \widetilde{F}_T \le \chi^{+}_{\Delta_T}. \tag{4.29}$$

The estimate indicates that $\widetilde{F}_T$ provides an approximation for the characteristic function χ_{Δ_T}. Let us now define $f_{T,i}:\mathbb{R}^d\times\mathbb{R}\to[0,\infty)$ by

$$f_{T,i}(z,y)=\widetilde{f}_{T,i}(\pi(z),y) \quad \text{for } z\in\mathbb{R}^d_* \text{ and } \ y\in Y_T, \tag{4.30}$$

and $f_{T,i}(z,y):=0$ for all $z\in\mathbb{R}^d\backslash\mathbb{R}^d_*$. Then $f_{T,i}$ is smooth in the z-coordinate. We also set

$$F_T := \widetilde{F}_T\circ\pi. \tag{4.31}$$

From (4.5) we see that the function F_T can be written as

$$\begin{aligned}F_T(z) &= \int_{Y_T}\Big(\sum_{v\in Q_{T,1}(y)} f_{T,1}\big(a(v)z,y\big)\Big)\,d\kappa_T(y)\\ &\quad+\int_{Y_T}\Big(\sum_{v\in Q_{T,2}(y)} f_{T,2}\big(a(v)z,y\big)\Big)\,d\kappa_T(y),\end{aligned} \tag{4.32}$$

which is exactly the form of functional tiling analyzed in Sect. 3.

The following lemma demonstrates that the function F_T proves a good approximation for the characteristic function $\chi_{\Omega_T} = \chi_{\Delta_T} \circ \pi$.

Lemma 4.11 *Let $p \geq 1$. For $\varepsilon_T = \mathrm{Vol}(\Omega_T)^{-\eta}$ with $\eta > p/2$, then*

$$\left\| \chi_{\Omega_T} - F_T \right\|_{L^p} = o\big(\mathrm{Vol}(\Omega_T)^{1/2}\big) \quad \textit{as } T \to \infty.$$

Proof We shall use the integral formula from Lemma 4.2. From (4.29),

$$\begin{aligned}\|\chi_{\Delta_T} - \tilde{F}_T\|_{L^p} &\leq \|\chi_{\Delta_T}^+ - \chi_{\Delta_T}\|_{L^p} \\ &\ll \left(\int_{\mathbb{S}_{\underline{d}}} \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}} \left|\chi_{\Delta_T}^+(u,s,\xi) - \chi_{\Delta_T}(u,s,\xi)\right|^p e^s \, ds \, du \, d\kappa(\xi) \right)^{1/p}.\end{aligned}$$

We recall that $\chi_{\Delta_T}^+$ and χ_{Δ_T} are given by (4.28) and (4.16) respectively. By successive use of the triangle-inequality, this expression is less than $A_1 + A_2$, where

$$A_i := \sum_{v \in \widetilde{Q}_{T,i}} (A_{i,1}(v) + A_{i,2}(v) + A_{i,3}(v))$$

with

$$A_{i,1}(v) := \left(\int_{\mathbb{S}_{\underline{d}}} \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}} \chi_{(S_i)_{3c\varepsilon_T} \setminus S_i}\big(\tau_T(s)^{-1}(u + \beta_T(v,s))\big) \chi_J(s)\, \chi_{B_{\varepsilon_T}}(\xi)\, e^s \, ds \, du \, d\kappa(\xi) \right)^{1/p},$$

$$A_{i,2}(v) := \left(\int_{\mathbb{S}_{\underline{d}}} \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}} \chi_{(S_i)_{3c\varepsilon_T}}\big(\tau_T(s)^{-1}(u + \beta_T(v,s))\big) \chi_{(\log I)_{2\varepsilon_T} \setminus (\log I)_{\varepsilon_T}}(s)\, \chi_{B_{\varepsilon_T}}(\xi)\, e^s \, ds \, du \, d\kappa(\xi) \right)^{1/p},$$

$$A_{i,3}(v) := \left(\int_{\mathbb{S}_{\underline{d}}} \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}} \chi_{(S_i)_{3c\varepsilon_T}}\big(\tau_T(s)^{-1}(u + \beta_T(v,s))\big) \chi_J(s)\, \chi_{B_{\varepsilon_T} \setminus B}(\xi)\, e^s \, ds \, du \, d\kappa(\xi) \right)^{1/p}.$$

Since

$$\mathrm{Leb}_{k-1}\left(\tau_T(s)((S_i)_{3c\varepsilon} \setminus S_i)\right) \ll \varepsilon$$

uniformly over $s \in J$ and sufficiently large T, we conclude that $A_{i,1}(v) \ll \varepsilon_T^{1/p}$ uniformly over v. Also since

$$\mathrm{Leb}_{k-1}\left(\tau_T(s)(S_i)_{3c\varepsilon}\right) \ll 1$$

uniformly over $s \in J$ and sufficiently large T, and

$$\mathrm{Leb}_1\left((\log I)_{2\varepsilon} \setminus (\log I)_{\varepsilon}\right) \ll \varepsilon \quad \text{and} \quad \kappa(B_\varepsilon \setminus B) \ll \varepsilon,$$

we deduce that $A_{i,2}(v) + A_{i,3}(v) \ll \varepsilon_T^{1/p}$ uniformly on v. Therefore,

$$\left\| \chi_{\Delta_T} - \tilde{F}_T \right\|_{L^p} \ll \left(|\widetilde{Q}_{T,1}| + |\widetilde{Q}_{T,2}|\right) \varepsilon_T^{1/p} \ll \mathrm{Vol}(\Omega_T)\, \varepsilon_T^{1/p}.$$

Hence, when $\varepsilon_T = \mathrm{Vol}(\Omega_T)^{-\eta}$ with $\eta > p/2$, we have $\|\chi_{\Omega_T} - F_T\|_{L^p} = o\big(\mathrm{Vol}(\Omega_T)^{1/2}\big)$. □

4.5.3 Construction of the maps $h_{T,i}$ (assumption (II.c))

Let us now turn to the construction of the maps $h_{T,i}$ satisfying the condition (II.c). We recall that $h_{T,i}$ should be non-negative Borel functions on $\mathbb{R}^d \times Y_{T,i}$ satisfying

$$f_{T,i}\big(a(\beta_T(v,y))z, y\big) \le h_{T,i}\big(a(\widetilde{\beta}_T(v))z, y\big)$$

for all $v \in \widetilde{Q}_{T,i}$, $z \in \mathbb{R}^d$, and $y \in Y_{T,i}$. Moreover, we arrange that the supports the functions $x \mapsto h_{T,i}(x, y)$ lie in a fixed compact set, independent of $y \in Y_T$, and

$$\sup_{z,T} \int_{Y_T} h_{T,i}(z, y)\, d\kappa_T(y) < \infty.$$

We shall use the coordinate system (4.4). Then in view of (4.30), it is sufficient to construct non-negative Borel functions $\widetilde{g}_{T,i}$ on $(\mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{S}_{\underline{d}}) \times Y_T$ such that

$$\widetilde{f}_{T,i}\big((u + \beta_T(v,y), s, \xi), y\big) \le \widetilde{g}_{T,i}\big((u + \widetilde{\beta}_T(v), s, \xi), y\big), \tag{4.33}$$

for all $v \in \widetilde{Q}_{T,i}$ and $(u, s, \xi) \in \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{S}_{\underline{d}}$ and $y \in Y_T$, whose supports lie in a set $\mathcal{K} \times Y_{T,i}$, with a fixed compact $\mathcal{K} \subset \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{S}_{\underline{d}}$, and such that

$$\sup_{(u,s,\xi),T} \int_{Y_T} \widetilde{g}_{T,i}\big((u, s, \xi), y\big)\, d\kappa_T(y) < \infty.$$

Indeed, if such maps have been constructed, we can simply set $h_{T,i} = \widetilde{g}_{T,i} \circ \pi$. We recall from (4.27) that

$$\widetilde{f}_{T,i}\big((u, s, \xi), y\big) = \psi_{i,\varepsilon_T}(\tau_T(s)^{-1}u)\, \rho_{\varepsilon_T}(s - y)\, \vartheta_{\varepsilon_T}(\xi),$$

where ψ_{i,ε_T} satisfies

$$\chi_{(\mathcal{S}_i)_{c\varepsilon_T}} \le \psi_{i,\varepsilon_T} \le \chi_{(\mathcal{S}_i)_{2c\varepsilon_T}}.$$

By Lemma 4.8(ii), there is a compact set $\mathcal{D}_0 \subset \mathbb{R}^{k-1}$ such that

$$\beta_T(v, y) - \widetilde{\beta}_T(v) \in \mathcal{D}_0, \quad \text{for all } v \in \widetilde{Q}_{T,i},\, y \in Y_T, \text{ and} T \ge T_0(J).$$

Furthermore, by the construction of the map τ_T in (4.11), there exists a compact set $\mathcal{D} \subset \mathbb{R}^{k-1}$ such that

$$\tau_T(s)\Big((S_i)_{2c\varepsilon_T} - \mathcal{D}_0\Big) \subset \mathcal{D}, \quad \text{for all} s \in J \text{and sufficently large} T.$$

Hence, for all $s \in J$, $u \in \mathbb{R}^{k-1}$, $v \in \widetilde{Q}_{T,i}$, $y \in Y_T$, and sufficiently large T,

$$\begin{aligned}\psi_{i,\varepsilon_T}\left(\tau_T(s)^{-1}(u+\beta_T(v,y))\right) &\le \chi_{(S_i)_{2c\varepsilon_T}}\left(\tau_T(s)^{-1}(u+\widetilde{\beta}_T(v)+\beta_T(v,y)-\widetilde{\beta}_T(v))\right)\\ &\le \chi_{(S_i)_{2c\varepsilon_T}-\mathcal{D}_0}\left(\tau_T(s)^{-1}(u+\widetilde{\beta}_T(v))\right)\\ &\le \chi_{\mathcal{D}}(u+\widetilde{\beta}_T(v)).\end{aligned}$$

Let us now define

$$\widetilde{g}_{T,i}\big((u,s,\xi),y\big) := \chi_{\mathcal{D}}(u)\,\rho_{\varepsilon_T}(s-y)\,\vartheta_{\varepsilon_T}(\xi).$$

Then the estimate (4.33) clearly holds. Furthermore,

$$\int_{Y_{T,i}} \widetilde{g}_{T,i}((u,s,\xi),y)\,d\kappa_T(y) \le \int_J \chi_{\mathcal{D}}(u)\rho_{\varepsilon_T}(s-y)\,\vartheta_{\varepsilon_T}(\xi)\,dy \le \chi_{\mathcal{D}}(u)\chi_{J_{\varepsilon_T}}(s)\,\chi_{B_{\varepsilon_T}}(\xi),$$

which is clearly compactly supported and bounded, uniformly in T.

4.6 Estimation of the function norms

In order to apply our general result from the previous section (Theorem 3.19), We have to estimate the norms of the functions $f_{T,i}$, specifically, the quantities M_T and $M_{T,q}$ defined in (3.2)–(3.3).

Lemma 4.12 *For the functions $f_{T,i}$ defined in (4.30),*

$$M_T \ll \varepsilon_T^{-1} \quad \text{and} \quad M_{T,q} \ll \varepsilon_T^{-r_q}$$

with $r_q > 0$.

Proof We use that $f_{T,i}(\cdot,y) = \widetilde{f}_{T,i}(\cdot,y)\circ\pi$, and the maps $\widetilde{f}_{T,i}(\cdot,y)$ are supported in a fixed compact subset of $\mathbb{R}^{k-1}\times\mathbb{R}\times\mathbb{S}_{\underline{d}}$, which is independent of $y \in Y_{T,i}$. Then the restrictions to this compact set of all partial derivatives of the map π are uniformly bounded. Therefore, it is sufficient to estimate

$$\widetilde{M}_T := \max_i \int_{Y_T} \big\|\widetilde{f}_{T,i}(\cdot,y)\big\|_{C^0}\,d\kappa_{T,i}(y) \quad\text{and}\quad \widetilde{M}_{T,q} := \max_i \sup_{y_i\in Y_T} \big\|\widetilde{f}_{T,i}(\cdot,y)\big\|_{C^q}.$$

We recall from (4.27) that

$$\widetilde{f}_{T,i}\big((u,s,\xi),y\big) = \psi_{i,\varepsilon_T}\big(\tau_T(s)^{-1}u\big)\,\rho_{\varepsilon_T}(s-y)\,\vartheta_{\varepsilon_T}(\xi).$$

According to (4.20), (4.19), and (4.21),

$$\|\psi_{i,\varepsilon_T}\|_{C^0} \le 1, \qquad \|\rho_{\varepsilon_T}\|_{C^0} \ll \varepsilon_T^{-1}, \qquad \|\vartheta_{\varepsilon_T}\|_{C^0} \le 1.$$

Hence, we conclude that $\|\widetilde{f}_{T,i}(\cdot, y)\|_{C^0} \ll \varepsilon_T^{-1}$. This proves the first estimate. Using additionally (4.22), we conclude that also $\|\widetilde{f}_{T,i}(\cdot, y)\|_{C^q} \ll \varepsilon_T^{-r_q}$ for some $r_q > 0$, which implies the second estimate. □

4.7 Proof of Theorem 1.3

Let us now summarize what we have done in this technical section. The aim has been to produce smooth approximations F_T for the indicator functions χ_{Ω_T} to which the arguments of Sect. 3 apply. These approximations are given explicitly in (4.32). They are integrals of varying averages which are fibered over the finite measure spaces

$$(Y_T, \kappa_T) = \left((\log I)_{\varepsilon_T}, \mathrm{Leb}\,|_{(\log I)_{\varepsilon_T}}\right).$$

These averages are constructed using finite subsets $\widetilde{Q}_{T,i}$ and $Q_{T,i}(y)$ of $\mathbb{R}^{k-1}$, defined in (4.14) and (4.15), and Borel maps $\beta_T : \mathbb{R}^{k-1} \times Y_T \to \mathbb{R}^{k-1}$ and $\widetilde{\beta}_T : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$, defined in (4.13). The approximations F_T depend on a choice of a parameter ε_T, which we take $\varepsilon_T = \mathrm{Vol}(\Omega_T)^{-\eta}$ for some $\eta > 0$. In order for these approximations to be useful for us, we arrange that

$$\|\chi_{\Omega_T} - F_T\|_{L^p(X)} = o\left(\mathrm{Vol}(\Omega_T)^{1/2}\right) \quad \text{as } T \to \infty, \quad \text{for } p = 1, 2. \tag{4.34}$$

According to Lemma 4.11, one can take $\eta > 1$. Then (4.34) holds. The averages are further made up by Borel functions $f_{T,i} : \mathbb{R}^d \times Y_T \to [0, \infty)$, which are defined in (4.27) and (4.30). These functions are smooth in the first variable, but unbounded as $T \to \infty$. They are however "bounded on average", in the sense that there are Borel functions $h_{T,i} : \mathbb{R}^d \times Y_T \to [0, \infty)$ defined in Sect. 4.5.3. Ultimately, this provides the framework outlined in (I.a)–(I.c) and (II.a)–(II.c) from Sect. 3, so that we can apply Theorem 3.19 with $V_T = \mathrm{Vol}(\Omega_T)$. The conditions on the norms M_T and $M_{T,q}$ have been verified in Lemma 4.12 with $\theta_0 = \eta > 1$. We recall that the limit

$$\sigma := \lim_{T\to\infty} V_T^{-1/2} \left\| \widehat{\chi}_{\Omega_T} - \int_X \widehat{\chi}_{\Omega_T} d\mu \right\|_{L^2(X)}.$$

has been computed in Corollary 4.5. In view of (4.34), it follows from Corollary 4.4 that

$$\|\widehat{\chi}_{\Omega_T} - \widehat{F}_T\|_{L^p(X)} = o\big(V_T^{1/2}\big) \quad \text{as } T \to \infty, \quad \text{for } p = 1, 2. \tag{4.35}$$

Hence, we conclude that also

$$\lim_{T\to\infty} V_T^{-1/2} \left\| \widehat{F}_T - \int_X \widehat{F}_T d\mu \right\|_{L^2(X)} = \sigma.$$

Now we have verified all the assumptions of Theorem 3.19.

We conclude that the functions $V_T^{-1/2}\big(\widehat{F}_T - \int_X \widehat{F}_T d\mu\big)$ converge in distribution to the Normal Law with variance σ when $d > 4(1+\eta)$ with some $\eta > 1$, namely, when $d \geq 9$. Because of (4.35), the functions

$$V_T^{-1/2}\left(\widehat{\chi}_{\Omega_T}(\Lambda) - \int_X \widehat{\chi}_{\Omega_T} d\mu\right) = V_T^{-1/2}\Big(|\Lambda \cap \Omega_T| - \mathrm{Vol}(\Omega_T)\Big)$$

also converge in distribution to the same limit.

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