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Dyadic approximation in the middle-third Cantor set

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Abstract

In this paper, we study the metric theory of dyadic approximation in the middle-third Cantor set. This theory complements earlier work of Levesley et al. (Math Ann 338(1):97–118, 2007), who investigated the problem of approximation in the Cantor set by triadic rationals. We find that the behaviour when we consider dyadic approximation in the Cantor set is substantially different to considering triadic approximation in the Cantor set. In some sense, this difference in behaviour is a manifestation of Furstenberg's times 2 times 3 phenomenon from dynamical systems, which asserts that the base 2 and base 3 expansions of a number are not both structured.

Keywords Diophantine approximation · Middle-third Cantor set · Hausdorff measures · Fourier analysis · $\times 2 \times 3$ phenomenon.

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Dedicated to Professor Sanju Velani

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1 Introduction

1.1 Background and statement of results

In 1984, Mahler wrote a note entitled "Some Suggestions for Further Research" [30] in which he posed a number of interesting questions which he deemed worthy of attention. One of these questions posed by Mahler, which is of particular interest to us here, was the following:

How close can irrational elements of Cantor's set be approximated by rational numbers
(i) in Cantor's set, and
(ii) by rational numbers not in Cantor's set?

Since the publication of Mahler's note, this question has attracted a huge amount of interest and a wide variety of people have uncovered information about various aspects of this problem. See, for example, [2, 4, 5, 8, 9, 13, 19–21, 26, 27, 29, 33, 38] and references therein.

Arguably, the first step towards addressing Mahler's problem outlined above was the work of Weiss [38], who showed that μ-almost no point in the Cantor set is *very well approximable*. Here μ denotes the natural measure on the middle-third Cantor set as defined in (1.4) below. We shall write Cantor set and middle-third Cantor set interchangeably throughout. Recall that Dirichlet's Approximation Theorem tells us that for any $x \in \mathbb{R}$, we have

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2} \tag{1.1}$$

for infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$. We say that $x \in \mathbb{R}$ is *very well approximable* if the exponent 2 in the denominator of the right-hand side of (1.1) can be improved, i.e. increased. That is, $x \in \mathbb{R}$ is very well approximable if there exists $\varepsilon > 0$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}} \tag{1.2}$$

for infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$. A higher-dimensional generalisation of Weiss's result was subsequently demonstrated in the works of Kleinbock, Lindenstrauss, and Weiss [26], and Pollington and Velani [32]. The framework was later vastly generalised by Das, Fishman, Simmons and Urbański [15, 16].

In spite of the work of Weiss [38], in [29] Levesley, Salp, and Velani were able to prove that there do in fact exist very well approximable numbers (aside from Liouville numbers) in the middle-third Cantor set, thus solving a more precise problem attributed to Mahler which is stated in [10, Problem 35].

This result prompts further study of Mahler's question from the point of view of *irrationality exponents*. The *irrationality exponent* $\xi(x)$ of $x \in \mathbb{R}$ is defined as

$$\xi(x) := \sup\left\{\xi \in \mathbb{R} : \left|x - \frac{p}{q}\right| < \frac{1}{q^{\xi}} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N}\right\}.$$

By Dirichlet's Approximation Theorem, we know that $\xi(x) \geq 2$ for all $x \in \mathbb{R}$. A real number $x \in \mathbb{R}$ is very well approximable if we have strict inequality here, i.e. if $\xi(x) > 2$. The result of Weiss [38] shows us that for μ-almost all points in the middle-third Cantor set this is not the case. Nevertheless, the work of Levesley, Salp, and Velani [29] shows that the set of non-Liouville points in the middle-third Cantor set with irrationality exponent strictly greater than 2 is non-empty. Further results concerning the irrationality exponents of points in the middle-third Cantor set have been obtained in [9, 13].

In this paper, we will be concerned with another aspect of Mahler's problem. In particular, we will be concerned with the problem of how well points in the middle-third Cantor set can be approximated by dyadic rationals; that is, rationals with denominators which are powers of 2. To some extent, our present work is motivated by the work of Levesley, Salp, and Velani [29] who, in resolving [10, Problem 35], considered *triadic* approximation in the middle-third Cantor set. In particular, they proved a 'zero-full' dichotomy for the μ-measure of the points in the middle-third Cantor set which are ψ-well approximable by triadic rationals; that is, rationals with denominators which are powers of 3. In fact, the work of Levesley, Salp, and Velani is far more general and they actually proved such a 'zero-full' dichotomy for the Hausdorff measures of the set in question with respect to general gauge functions.

Throughout, we will let K denote the middle-third Cantor set. Recall that K consists of the points $x \in [0, 1]$ which have a ternary expansion consisting entirely of 0's and 2's. We will also denote by γ the Hausdorff dimension of K, namely

$$\gamma := \dim_{\mathrm{H}} K = \frac{\log 2}{\log 3}. \tag{1.3}$$

Throughout, for a subset $X \subset \mathbb{R}$, we will denote the Hausdorff dimension of X by $\dim_{\mathrm{H}} X$. For a real number $s > 0$, we shall write $\mathcal{H}^s(X)$ to denote the Hausdorff s-measure of X.

We will denote by μ the natural measure on K. More precisely, μ is the Hausdorff γ-measure restricted to the middle-third Cantor set, i.e. for $X \subset \mathbb{R}$ which is Borel,

$$\mu(X) = \frac{\mathcal{H}^{\gamma}(X \cap K)}{\mathcal{H}^{\gamma}(K)} = \mathcal{H}^{\gamma}(X \cap K), \tag{1.4}$$

where we have the last equation because it happens that $\mathcal{H}^{\gamma}(K) = 1$, see [17, Theorem 1.14]. Moreover, $\mu(K) = 1$ and so μ is a probability measure on K. For further information on Hausdorff measure and dimension, we refer the reader to [18].

Given $b \in \mathbb{N}$, we will write

$$\mathcal{A}(b) = \{b^n : n = 0, 1, 2, 3, \ldots\}.$$

Given $\psi : \mathbb{N} \to \mathbb{R}^+$, let

$$W_3(\psi) := \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathcal{A}(3) \right\}.$$

Here $\mathbb{R}^+ := [0, \infty)$.

Regarding triadic approximation in the middle-third Cantor set, the following statement follows immediately from [29, Theorem 1].

Theorem 1.1 (Levesley–Salp–Velani [29]) *For* $\psi : \mathbb{N} \to \mathbb{R}^+$,

$$\mu(W_3(\psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \psi(3^n)^\gamma < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \psi(3^n)^\gamma = \infty. \end{cases}$$

In this paper, we will consider analogous statements for approximating points in the Cantor set by *dyadic* rationals rather than *triadic* rationals. With this in mind, given a function $\psi : \mathbb{N} \to \mathbb{R}^+$, we will be concerned with

$$W_2(\psi) := \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathcal{A}(2) \right\}.$$

Along the same lines as Theorem 1.1, we have the following conjecture[1], which represents the clean-cut dichotomy which we expect to be the truth regarding dyadic approximation in the middle-third Cantor set.

Conjecture 1.2 (Velani) *For monotonic* $\psi : \mathbb{N} \to \mathbb{R}^+$, *we have*

$$\mu(W_2(\psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \psi(2^n) < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \psi(2^n) = \infty. \end{cases}$$

Remark 1.3 It is possible that Conjecture 1.2 may even hold without the monotonicity hypothesis.

To give a heuristic idea for why one might believe the above conjecture to be correct, suppose for a moment that dyadic rationals of denominator 2^n were uniform random variables in $[0, 1]$. Then the expected number of dyadic rationals of denominator 2^n lying in a subinterval $I \subset [0, 1]$ would be $\approx 2^n \times$ (the length of I).

For each $n \in \mathbb{N}$, let $A_n := \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n}, \frac{\psi(2^n)}{2^n}\right)$. Then $W_2(\psi) = \bigcap_{j=0}^{\infty} \bigcup_{n=j}^{\infty} A_n$, and $\bigcup_{n=j}^{\infty} A_n$ is a cover of $W_2(\psi)$ for each $j \in \mathbb{N}$. Now, consider a fixed $n \in \mathbb{N}$ and suppose for this n that $\frac{\psi(2^n)}{2^n} \approx 3^{-N}$. In particular, by our assumption on the

[1] We thank Sanju Velani for suggesting to us that this statement should be true. The present work supports this.

distribution of dyadic rationals, we would expect $\ll \left(\frac{2^N}{3^N}\right) \times 2^n$ dyadic rationals to lie within distance $\frac{\psi(2^n)}{2^n}$ of the Nth level of the construction (this is called K_N later on) of the middle-third Cantor set. Thus, this number represents the maximum number of individual balls in A_n which can possibly intersect the middle-third Cantor set. Moreover, it is known that

$$\mu(B(z,r)) \ll r^{\gamma} \qquad (z \in \mathbb{R}, \quad 0 < r \leq 1), \tag{1.5}$$

see for instance [38] or [36, §2]. So, we have

$$\mu(A_n) \ll \left(\frac{2^N}{3^N}\right) \times 2^n \times \left(\frac{\psi(2^n)}{2^n}\right)^{\gamma} \approx 2^N \times \left(\frac{\psi(2^n)}{2^n}\right) \times 2^n \times (3^{-N})^{\gamma} = \psi(2^n).$$

Combining the above heuristics with the convergence Borel–Cantelli Lemma (Lemma 1.16) gives rise to the convergence part of the above conjecture.

By considering balls of radius $\psi(2^n)/2^n$ centred at triadic rationals within the Cantor set, we can obtain a similar heuristic for the complementary lower bound $\mu(A_n) \gg \psi(2^n)$. If we knew that $\mu(A_n \cap A_m)$ were $O(\mu(A_n)\mu(A_m))$ in some suitably averaged sense, then the divergence Borel–Cantelli Lemma [7, Proposition 2] would complete the proof.

Unfortunately, we cannot prove such bold claims about the interaction of the dyadic rationals with the middle-third Cantor set. Unlike in the case of triadic rationals, where we know exactly how they are distributed with respect to the middle-third Cantor set K, we know very little about how dyadic rationals are distributed with respect to K. For this reason, studying dyadic approximation in the Cantor set is significantly harder than studying triadic approximation in the Cantor set and, as of yet, we have been unable to prove Conjecture 1.2 in full. As a first step towards this conjecture, the following convergence statement is relatively straightforward to establish—we will provide a direct proof in Sect. 2.1.

Proposition 1.4 *For* $\psi : \mathbb{N} \to \mathbb{R}^+$, *if*

$$\sum_{n=1}^{\infty} \psi(2^n)^{\gamma} < \infty, \tag{1.6}$$

then $\mu(W_2(\psi)) = 0$.

Using Fourier analysis, we are able to establish the following improvement upon the above benchmark result.

Theorem 1.5 (Main convergence theorem) *If*

$$\sum_{n=1}^{\infty} (2^{-\log n/(\log\log n \cdot \log\log\log n)} \psi(2^n)^{\gamma} + \psi(2^n)) < \infty, \tag{1.7}$$

then $\mu(W_2(\psi)) = 0$.

Remark 1.6 We may deduce Proposition 1.4 from Theorem 1.5 via two observations. First, by replacing ψ by $\psi_0 : y \mapsto \min\{3/4, \psi(y)\}$, we may assume that $\psi(2^n) < 1$ for all n, since

$$\|2^n\alpha\| < \psi(2^n) \iff \|2^n\alpha\| < \psi_0(2^n).$$

Second, observe that

$$2\psi(2^n)^\gamma > 2^{-\log n/(\log\log n\cdot\log\log\log n)}\psi(2^n)^\gamma + \psi(2^n).$$

Thus, assuming (1.6), it follows that (1.7) also holds and we may apply Theorem 1.5 to deduce Proposition 1.4. The improvement is super-logarithmic assuming $\sum_{n=1}^{\infty}\psi(2^n) < \infty$. For instance, (1.7) holds for $\psi(2^n) = (\log n)^\alpha n^{-1/\gamma}$ whenever $\alpha \in \mathbb{R}$, whereas (1.6) fails even for $\psi(2^n) = n^{-1/\gamma}$.

From Theorem 1.1, we know that for μ-almost every $\alpha \in K$ the inequality

$$\|3^n\alpha\| < n^{-\log 3/\log 2} \tag{1.8}$$

admits infinitely many solutions $n \in \mathbb{N}$, where $\|x\|$ denotes the distance from x to the nearest integer for $x \in \mathbb{R}$. However, we see from Theorem 1.5 that for μ-almost every α the inequality

$$\|2^n\alpha\| < n^{-\log 3/\log 2} \tag{1.9}$$

has at most finitely many solutions. Thus, the behaviour is very different in the case of dyadic approximation in the Cantor set. Observe that it is not possible to obtain this conclusion by only using Proposition 1.4, as has already been indicated in Remark 1.6. In this sense, the additional super-logarithmic decaying factor

$$2^{-\log n/(\log\log n\cdot\log\log\log n)}$$

marks a significant difference.

Remark 1.7 Towards the end of the paper we prove, highly conditionally, the convergence part Conjecture 1.2. The Lang–Waldschmidt Conjecture [28, Conjecture 1 (page 212)] is sufficient for us to improve the exponent in (1.9). We defer the detailed statements and an extended discussion until that point.

The difference in behaviour between dyadic and triadic approximation here is further emphasised by the following theorem due to Bugeaud [11, Theorem 7.17] which, despite the differing metric statements, asserts that for each of the inequalities (1.8) and (1.9) there exist uncountably many real numbers in the middle-third Cantor set for which these inequalities are satisfied only finitely often.

Theorem 1.8 (Bugeaud [11]) *There exists a positive real number c, and uncountably many real numbers $x \in K$ which are badly approximable and, for all integers $b \geq 2$ and $n \geq 1$, satisfy*

$$\|b^n x\| > b^{-cb(\log b)}.$$

Here we ask the following complementary question.

Question 1.9 *Does there exist $\alpha \in K \setminus \mathbb{Q}$ such that (1.9) has infinitely many solutions?*

Bugeaud and Durand [13, Conjecture 1.2] anticipate that the answer should be emphatically positive. Assuming their conjecture:

(i) If $v < \frac{\gamma}{1-\gamma}$ then there exists $\alpha \in K \setminus \mathbb{Q}$ such that

$$\|2^n \alpha\| < 2^{-vn} \tag{1.10}$$

has infinitely many solutions.

(ii) If $v > \frac{\gamma}{1-\gamma}$ and $\alpha \in K \setminus \mathbb{Q}$, then (1.10) has at most finitely many solutions.

A pertinent piece of information was provided by Schleischitz [33, Theorem 3.4], who showed that if τ is sufficiently large, $\alpha \in K$, and $a/q \in \mathbb{Q} \setminus K$, then

$$|q\alpha - a| > \exp(-\tau q^{\gamma}),$$

and in fact this was established in a more general iterated function system setting. Since 0, 1/4, 3/4, and 1 are the only dyadic rationals in K, see [37], the inequality

$$\|q\alpha\| > \exp(-\tau q^{\gamma})$$

holds for $\alpha \in K$ and $q = 2^n$ whenever $n \geq 3$. This is very far from the threshold in (1.9), and also far from the threshold prophesied by Bugeaud and Durand, but is nonetheless an elegant special case of Schleischitz's general result.

Complementing Theorem 1.5, we prove the following statement towards the divergence part of Conjecture 1.2.

Theorem 1.10 (Main Divergence Theorem) *For $\psi(2^n) = 2^{-\log\log n/\log\log\log n}$, we have $\mu(W_2(\psi)) = 1$.*

In Sect. 5, we also provide some additional conditional results and empirical evidence which provide further support in favour of Conjecture 1.2. We will see in particular that a modest refinement of a key estimate, namely (1.11) below, would lead to a sharp convergence theory. Moreover, consider the special case

$$\varphi_a : 2^n \mapsto n^{-a},$$

for $a \geq 0$. It is clear that

$$\mu(W_2(\varphi_0)) = 1,$$

and we know from Theorem 1.5 that whenever $a \geq 1/\gamma$ we have

$$\mu(W_2(\varphi_a)) = 0.$$

We shall see conditionally, in Theorems 5.2 and 5.3(a), that

$$\mu(W_2(\varphi_a)) = \begin{cases} 0, & \text{if } a > 1 \\ 1, & \text{if } a \leq 1. \end{cases}$$

In this article, we are only concerned with μ-measure. For the predicted behaviour of the corresponding Hausdorff dimension problem, there is a conjecture of Bugeaud and Durand [13, Conjecture 1.2].

1.2 Main ideas behind the proofs and some preliminaries

The key to proving results of the flavour we are considering is an understanding of how dyadic rationals are distributed with respect to the middle-third Cantor set. This is not an easy task, and is a variant of the infamous 'times two, times three' phenomenon. One way to view this is that expansions in two multiplicatively independent bases cannot both be structured. However, the story began not in number theory but in dynamical systems, with Furstenberg's influential works [23, 24] from the late 1960s.

Here is a conjecture of Furstenberg's that closely resembles the more arithmetic statements of this flavour that we will use. Any positive integer m defines a map $T_m : x \mapsto mx$ on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ by multiplication. We denote the orbit of a point $x \in \mathbb{T}$ under such a map by $\mathcal{O}_m(x) = \{T_m^r(x) : r \in \mathbb{N}\}$.

Conjecture 1.11 (Furstenberg [24]) *Let p and q be multiplicatively independent positive integers, and let $x \in \mathbb{T} \setminus \mathbb{Q}$. Then*

$$\dim_{\mathrm{H}} \overline{\mathcal{O}_p(x)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_q(x)} \geq 1.$$

This remains open. A weaker conjecture of Furstenberg's [24], but still a famous and deep one, was recently settled independently and concurrently by Shmerkin and Wu.

Theorem 1.12 (Shmerkin [34], Wu [40]) *Let p and q be multiplicatively independent positive integers. Let A_p and B_q be closed subsets of $\mathbb{T}$ that are invariant under T_p and T_q, respectively. Then, for $u, v \in \mathbb{R}$, we have*

$$\dim_{\mathrm{H}}(uA_p + v) \cap B_q \leq \max\{0, \dim_{\mathrm{H}} A_p + \dim_{\mathrm{H}} B_q - 1\}.$$

Without further ado, let us now describe our approach to the specific problems at hand. A fairly straightforward counting argument enables us to deduce enough information to establish Proposition 1.4. Refining the argument using Fourier analysis, we are also able to establish Theorems 1.5 and 1.10. The relevance of counting dyadic rationals near the middle-third Cantor set will be described in greater detail in Sect. 2.

In Sect. 2, we see via Fourier analysis that the problem at hand is intimately connected to the number of binary and ternary 'digit changes' in numbers of the form $2^n m$, where $n, m \in \mathbb{N}$. For $b \in \mathbb{Z}$ such that $b \geq 2$ and $y \in \mathbb{R}$, let $D_b(y)$ denote the number of digit changes in the b-adic expansion of y; that is, $D_b(y)$ denotes the number of consecutive pairs of distinct digits in the b-adic expansion of y. We want to understand the sum $D_2(y) + D_3(y)$. To this end, we will make use of the following inequality, originally due to Stewart [35] and extended by Bugeaud, Cipu, and Mignotte [12]. Integers a and b are *multiplicatively independent* if, for any $m, n \in \mathbb{Z}$, we have that $a^m = b^n$ implies that $m = n = 0$. The statement as written below can be found in [11, Theorem 6.9].

Theorem 1.13 (Stewart [35], Bugeaud–Cipu–Mignotte [12]) *Let $a, b \geq 2$ be multiplicatively independent integers. Then, there exists an effectively computable integer c, which depends only on a and b, such that for every natural number $n \geq 20$, we have*

$$D_a(n) + D_b(n) \geq \frac{\log\log n}{\log\log\log n + c} - 1.$$

This is a deep result which uses Alan Baker's work on linear forms in logarithms [3]. The following more specialised statement follows easily from Theorem 1.13.

Lemma 1.14 *For sufficiently large $n \in \mathbb{N}$, we have*

$$D_2(n) + D_3(n) \gg \frac{\log\log n}{\log\log\log n}, \tag{1.11}$$

where the implicit constant is absolute.

Remark 1.15 In light of Theorem 1.13, the multiplicative independence of 2 and 3 is crucial in this work. By the same methods, we could study the approximation of points in K by rationals whose denominator is a power of k, where k is any positive integer that is not a power of three. More generally, as with other manifestations of the 'times two, times three' phenomenon, we could replace 2 and 3 by multiplicatively independent positive integers k and b, respectively, with essentially the same outcomes. For any $a \in \{0, 1, \ldots, b-1\}$, the appropriate analogue of K is the missing-digit Cantor set $K_{b,a}$ comprising elements of $[0, 1]$ that can be written in base b without the digit a, see [33].

Notation. We use the Bachmann–Landau and Vinogradov notations throughout: for functions f and positive-valued functions g, we write $f \ll g$, or $g \gg f$, or $f = O(g)$, if there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all x.

In addition to our proofs relying heavily on Fourier analysis and the bounds on base 2 and base 3 digit changes discussed above, we will make use of both the standard convergence Borel–Cantelli Lemma and the Chung–Erdős inequality from probability.

Lemma 1.16 (Convergence Borel–Cantelli Lemma) *Let* $(\Omega, \mathcal{A}, m)$ *be a finite measure space and let* $\{E_n\}_{n\geq 1} \subset \mathcal{A}$ *be a sequence of* m*-measurable sets in* Ω*. If*

$$\sum_{n=1}^{\infty} m(E_n) < \infty,$$

then

$$m(\limsup_{n\to\infty} E_n) = 0.$$

For the proof of the divergence result Theorem 1.10, we use the following inequality established by Chung and Erdős in [14, Equation 4]. This is similar in flavour to the divergence counterpart of the Borel–Cantelli Lemma (see e.g. [7, Proposition 2]).

Lemma 1.17 (Chung–Erdős Inequality) *Let* $N \geq 1$ *be an integer. Let* $(\Omega, \mathcal{A}, m)$ *be a probability space, and let* $\{E_n\}_{n=1}^{N} \subset \mathcal{A}$ *be an arbitrary sequence of* m*-measurable sets in* Ω*. Then, if* $m\left(\bigcup_{n=1}^{N} E_n\right) > 0$*, we have*

$$m\left(\bigcup_{n=1}^{N} E_n\right) \geq \frac{\left(\sum_{s=1}^{N} m(E_s)\right)^2}{\sum_{s,t=1}^{N} m(E_s \cap E_t)}.$$

In order to prove the full measure statements given by Theorem 1.10 and later by Theorem 5.3, we first intersect our sets of interest with an arbitrary interval I centred in the Cantor set and show that these intersections have measure proportional to the measure of the interval I. Once this has been achieved, the following lemma yields the full measure statements we ultimately desire since μ is doubling (see Sect. 2.2 for a definition). Lemma 1.18, as stated below, can be found in [7, Proposition 1]. We also refer the reader to [7] for several other applications of Lemma 1.18.

Lemma 1.18 *Let* (Ω, d) *be a metric space, and let* m *be a finite doubling measure on* Ω *such that any open set is measurable. Let* E *be a Borel subset of* Ω*. Assume that there are constants* $r_0, c > 0$ *such that for any ball* B *of radius* $r(B) < r_0$ *and centre in* Ω *we have that*

$$m(E \cap B) \geq cm(B).$$

Then E *has full measure in* Ω*, i.e.* $m(\Omega \setminus E) = 0$*.*

1.3 Subsequent developments

This being a topical subject, there have been notable advances on these and related problems since the initial release of this manuscript. Very recently, Baker [6] has shown unconditionally that

$$\mu(W_2(\varphi_a)) = 1 \qquad (a \leq 0.01).$$

Then the three present authors, together with Baker, showed unconditionally in [1] that

$$\mu(W_2(\varphi_a)) = 0 \qquad \left(a \geq \gamma^{-1} - 0.01\right).$$

Those articles also discuss inhomogeneous and counting refinements.

We now describe some results from the recent works of Khalil and Lüthi [25] and the third named author [41]. Owing to its nice properties, the Cantor measure μ is a 'friendly' measure, in the parlance introduced by Kleinbock, Lindenstrauss and Weiss [26]. This class of measures includes many well-known natural measures supported on fractals and manifolds, and has been a hugely successful medium for systematising the metric theory of Diophantine approximation. In [26, Question 10.1], Kleinbock, Lindenstrauss and Weiss asked if the analogue of Khintchine's theorem holds for friendly measures. Strengthening their question very slightly and restricting ourselves to one-dimensional Euclidean space: if ν is a friendly probability measure on $\mathbb{R}$, and $\psi : \mathbb{N} \to \mathbb{R}^+$ is decreasing, and

$$W(\psi) := \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\},$$

then is it true that

$$\nu(W(\psi)) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \psi(q) < \infty \\ 1, & \text{if } \sum_{n=1}^{\infty} \psi(q) = \infty \end{cases} \quad ?$$

This beautiful problem is still open in the case $\nu = \mu$. By dynamical means, Khalil and Lüthi [25] have recently solved it in the case of the natural probability measure on any missing-digit Cantor set $K_{b,a}$ for which $b \geq 5$ is prime, where $K_{b,a}$ was introduced in Remark 1.15. The third named author provided an independent, Fourier-analytic approach, establishing results of this type in [41].

In [41], progress was also made on the Hausdorff theory of Diophantine approximation on missing-digit Cantor sets. The *irrationality exponent* of an irrational number x, denoted $\omega(x)$, is the supremum of w such that

$$\left| x - \frac{p}{q} \right| < q^{-w}$$

holds for infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$. For $\omega \geq 2$, we write

$$\mathcal{M}(\omega) = \{x \in \mathbb{R} : \omega(x) \geq \omega\}.$$

Conjecture 1.19 (Bugeaud–Durand [13]) *For $\omega \geq 2$, we have*

$$\dim_{\mathrm{H}}(\mathcal{M}(\omega) \cap K) = \max\left\{\frac{2}{\omega} + \gamma - 1, \frac{\gamma}{\omega}\right\}. \tag{1.12}$$

Taking $\omega \to 2^{+}$ in (1.12) yields the prediction [29, Equation (29)] of Levesley, Salp and Velani, which we state explicitly below. Here

$$\mathbf{VWA} = \bigcup_{\omega>2} \mathcal{M}(\omega)$$

is the set of very well approximable numbers, whose μ-measure was shown to vanish in the aforementioned work of Weiss [38].

Conjecture 1.20 (Levesley–Salp–Velani [29]) *We have*

$$\dim_{\mathrm{H}}(\mathbf{VWA} \cap K) = \gamma.$$

Levesley, Salp and Velani [29, Equation 2] were able to prove the lower bound

$$\dim_{\mathrm{H}}(\mathbf{VWA} \cap K) \geq \gamma/2.$$

Denoting

$$\kappa = \dim_{\mathrm{H}} K_{b,a} = \frac{\log(b-1)}{\log b},$$

it was shown in [41] that if

$$b \geq 10^7 + 1 \tag{1.13}$$

then we have

$$\dim_{\mathrm{H}}(\mathcal{M}(\omega) \cap K_{b,a}) = \max\left\{\frac{2}{\omega} + \kappa - 1, \frac{\kappa}{\omega}\right\}$$

in the range $2 \leq \omega \leq 2 + \eta$, where $\eta > 0$ is sufficiently small in terms of b. As an aspect of a future work, we plan to relax the condition (1.13) considerably.

2 Dyadic rationals near the Cantor set

We recall a natural construction of the middle-third Cantor set. Set $K_0 := [0, 1]$ and remove the open middle-third of this interval to obtain $K_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Subsequently, remove the open middle-third of each of these component intervals to obtain K_2 which will consist of 4 subintervals of length 3^{-2}, and repeat this process so that K_N consists of the 2^N intervals of length 3^{-N} which remain after removing the open middle-thirds from each of the component subintervals of K_{N-1}. Then $K = \bigcap_{N=0}^{\infty} K_N$.

For $N \in \mathbb{N}$, let $\mathcal{L}_N$ be the set of rational numbers of the form $\frac{b}{3^N}$ where b is an integer such that $0 \leq b \leq 3^N$ and the ternary expansion of b only contains 0's and 2's. Thus, $\mathcal{L}_N$ corresponds to the left-most endpoints of the intervals comprising K_N. We also define $\mathcal{R}_N := \{x = 1 - y : y \in \mathcal{L}_N\}$ to be the set of right-most endpoints of intervals in K_N. We write $\mathcal{C}_N := \mathcal{L}_N \cup \mathcal{R}_N$ to denote the set of all endpoints of intervals comprising K_N. Furthermore, note that $\mathcal{C}_N$ is the set of rationals in $[0, 1]$ with denominator 3^N which lie in the Cantor set. For $n, M \in \mathbb{N}$, let

$$A_n := \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n}, \frac{\psi(2^n)}{2^n}\right), \quad \text{and} \quad A_n(M) := \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n}, \frac{1}{3^M}\right).$$

Observe that if $N \in \mathbb{N}$ and $x \in \mathcal{C}_N$ then $\mu\left(B\left(x, \frac{3^{-N}}{2}\right)\right) = 2^{-(N+1)}$. Indeed, each component interval in K_N has μ-measure 2^{-N}, and $B\left(x, \frac{3^{-N}}{2}\right)$ contains half of such an interval and intersects no others. Moreover, these balls are disjoint. Similarly, for any $x \in \mathbb{R}$ we have $\mu(B(x, 3^{-N})) \leq 2^{-(N-1)}$, since $B(x, 3^{-N})$ intersects at most two of the component intervals in K_N.

As alluded to earlier, in several of the lemmas below, we will intersect our sets of interest with an interval $I = B(z, r)$, where $z \in K$ and $r > 0$. This generality will be needed in the divergence theory in order to obtain full-measure results, as opposed to just positive-measure results, and is standard in metric Diophantine approximation (for several examples, see [7]). The idea is that dyadic rationals should be evenly distributed in $K = \operatorname{supp}(\mu)$. For such an interval $I = B(z, r)$ or $I = [0, 1]$, for $n \in \mathbb{N}$ and $t \in (0, \infty)$, we write

$$tA_n = \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n}, t\frac{\psi(2^n)}{2^n}\right), \quad tA_n(M) := \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n}, \frac{t}{3^M}\right), \quad \text{and} \quad tI = B(z, tr).$$

We adopt analogous notation for translates of A_n and for $A_n(M)$, i.e. for $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$,

$$A_n + \theta = \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n} + \theta, \frac{\psi(2^n)}{2^n}\right) \quad \text{and} \quad A_n(M) + \theta = \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n} + \theta, \frac{1}{3^M}\right).$$

Finally, we introduce the hybrid notations

$$tA_n + \theta = \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n} + \theta, t\frac{\psi(2^n)}{2^n}\right) \quad \text{and} \quad tA_n(M) + \theta = \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n} + \theta, \frac{t}{3^M}\right),$$

for $n \in \mathbb{N}$, $t \in (0, \infty)$, and $\theta \in \mathbb{R}$.

We will assume throughout this section that $\psi(2^n) < 1$ for all $n \in \mathbb{N}$. This is not at all restrictive for the purposes of proving Proposition 1.4, Theorem 1.5, or Theorem 1.10. Indeed, for the first two of these statements we may replace ψ by $y \mapsto \min\{3/4, \psi(y)\}$ as in Remark 1.6, and in Theorem 1.10 the condition is clearly met.

The μ-measure of a union of balls can be estimated by counting nearby triadic rationals in C_N for a sufficiently large $N \in \mathbb{N}$. This is formalised below.

Lemma 2.1 *Let $I = B(z, r)$, where $z \in K$ and $r > 0$,* ***or*** *$I = [0, 1]$. Let $n_0(I, \psi) \in \mathbb{N}$ be sufficiently large so that if $n \geq n_0(I, \psi)$ then $2^{-n} < r$. Let $n \geq n_0(I, \psi)$ and N be positive integers such that $3^{-N} \leq \frac{\psi(2^n)}{5 \cdot 2^n}$, and let B_n be a translate of A_n. Then*

$$2^{-(N+1)}|\mathcal{C}_N \cap 0.2B_n \cap 0.2I| \leq \mu(B_n \cap I) \leq 2^{-(N-1)}|\mathcal{C}_N \cap 5B_n \cap 5I|.$$

Proof If $x \in \mathcal{C}_N \cap 0.2B_n \cap 0.2I$ then $B(x, 3^{-N}/2) \subseteq B_n \cap I$. That $B(x, 3^{-N}/2) \subseteq B_n$ is clear upon noting our choice of N and recalling that B_n consists of disjoint intervals of length $2 \times \frac{\psi(2^n)}{2^n}$. The inclusion $B(x, 3^{-N}/2) \subseteq I$ follows by the triangle inequality and our choices of N and $n_0(I, \psi)$. Hence, by disjointness (since any two distinct points in $\mathcal{C}_N$ are distance at least 3^{-N} apart), we have

$$\mu(B_n \cap I) \geq \sum_{x \in \mathcal{C}_N \cap 0.2B_n \cap 0.2I} \mu\left(B\left(x, 3^{-N}/2\right)\right) = 2^{-(N+1)}|\mathcal{C}_N \cap 0.2B_n \cap 0.2I|.$$

For the second inequality, we cover $[0, 1]$ by $\{B(x, 3^{-N}) : x \in S_N\}$, where S_N is the set of rationals with denominator 3^N in $[0, 1]$. For each $x \in S_N$ we have

$$\mu\left(B_n \cap I \cap B\left(x, 3^{-N}\right)\right) \leq \mu(B(x, 3^{-N})) \leq 2^{-(N-1)}.$$

If $x \in S_N \setminus \mathcal{C}_N$ then $B(x, 3^{-N})$ does not intersect the Cantor set, so

$$\mu\left(B_n \cap I \cap B\left(x, 3^{-N}\right)\right) \leq \mu(B(x, 3^{-N})) = 0.$$

Finally, if $x \in S_N \setminus (5B_n \cap 5I)$ then $B(x, 3^{-N})$ does not intersect $B_n \cap I$, so

$$\mu\left(B_n \cap I \cap B\left(x, 3^{-N}\right)\right) = 0.$$

To see this, one can argue by contradiction, again employing the triangle inequality and recalling our choices of N and $n_0(I, \psi)$.

Putting everything together yields

$$\mu\left(B_n \cap I\right) \leq \sum_{x \in S_N} \mu\left(B_n \cap I \cap B\left(x, 3^{-N}\right)\right) \leq 2^{-(N-1)}|\mathcal{C}_N \cap 5B_n \cap 5I|,$$

as required. □

We also require the following inequality, which enables us to pass between Cantor levels K_N. Here N is roughly the level at which A_n lives since, up to a multiplicative constant, the component intervals in K_N are the same length as the intervals comprising A_n.

Lemma 2.2 *Let $n, M, N \in \mathbb{N}$ be such that*

$$\frac{\psi(2^n)}{125 \cdot 2^n} \leq 3^{-N} \leq \frac{5\psi(2^n)}{2^n} \leq 3^{-M} \leq 2^{9-n}. \tag{2.1}$$

Then

$$|\mathcal{C}_N \cap 5A_n| \ll |\mathcal{C}_M \cap 5A_n(M)|,$$

where the implied constant is absolute.

Proof Suppose $x \in \mathcal{L}_N \cap 5A_n$; the corresponding bound for $x \in \mathcal{R}_N$ follows by symmetry. Since $x \in \mathcal{L}_N$, we have $x = \frac{a}{3^N}$ for some integer $a \in [0, 3^N)$, where the ternary expansion of a consists only of 0's and 2's. As $x \in 5A_n$, there exists an integer $b \in [0, 2^n]$ such that $\left|x - \frac{b}{2^n}\right| < \frac{5\psi(2^n)}{2^n}$. Thus, $|\mathcal{L}_N \cap 5A_n|$ is bounded above by the number of integer solutions (a, b) to the inequality

$$\left|\frac{a}{3^N} - \frac{b}{2^n}\right| < \frac{5\psi(2^n)}{2^n},$$

where $0 \leq b \leq 2^n$, $0 \leq a < 3^N$, and all of the ternary digits of a are 0 or 2.

Let us now decompose a by writing $a = a_1 a_2$, where a_1 represents the first M ternary digits of a and a_2 represents the remaining $N - M$ digits. From this, we see that $|\mathcal{L}_N \cap 5A_n|$ is bounded above by the number of integer solutions (a_1, a_2, b) to

$$\left|\frac{3^{N-M}a_1 + a_2}{3^N} - \frac{b}{2^n}\right| < \frac{5\psi(2^n)}{2^n}, \tag{2.2}$$

where $0 \leq a_1 < 3^M$, $0 \leq a_2 < 3^{N-M}$, $0 \leq b \leq 2^n$, and the ternary digits of a_1, a_2 are all 0 or 2. Next, we note that

$$\frac{a_1}{3^M} \in 5A_n(M),$$

since

$$\left|\frac{a_1}{3^M} - \frac{b}{2^n}\right| \leq \left|\frac{a_1}{3^M} + \frac{a_2}{3^N} - \frac{b}{2^n}\right| + \frac{a_2}{3^N} < \frac{5\psi(2^n)}{2^n} + 3^{-M} \leq \frac{2}{3^M}. \tag{2.3}$$

Given a_1, we see from (2.3) that $b/2^n$ is forced to lie in an interval of length $\frac{4}{3^M}$ centred at $\frac{a_1}{3^M}$. By hypothesis, $\frac{4}{3^M} \ll 2^{-n}$, so there are $O(1)$ possibilities for b. Next, suppose we are given a_1 and b. Then, by (2.2), a_2 must lie in an interval of length

$\frac{3^N \times 10 \times \psi(2^n)}{2^n}$ centred at $\frac{3^N b}{2^n} - \frac{3^N a_1}{3^M}$. Thus, there are $O(1)$ solutions a_2 to (2.2), by our assumption that $3^{-N} \asymp \frac{\psi(2^n)}{2^n}$. Finally, since $\frac{a_1}{3^M} \in \mathcal{L}_M \cap 5A_n(M)$, we conclude that there are $O(|\mathcal{L}_M \cap 5A_n(M)|)$ solutions to (2.2) in total. By symmetry we have $|\mathcal{R}_N \cap 5A_n| = |\mathcal{L}_N \cap 5A_n|$, so

$$|\mathcal{C}_N \cap 5A_n| = 2|\mathcal{L}_N \cap 5A_n| \ll |\mathcal{C}_M \cap 5A_n(M)|.$$

□

2.1 Proof of the basic result

We presently provide a direct proof of Proposition 1.4. Recall that we assume that $\psi(2^n) < 1$ for all $n \in \mathbb{N}$. If $n \geq 7$ is an integer, and $N \in \mathbb{N}$ is such that

$$3^{-N} \leq \frac{\psi(2^n)}{5 \cdot 2^n} < 3^{-(N-1)}$$

then, by Lemma 2.1 (taking $I = [0, 1]$), we have

$$\mu(A_n) \leq 2^{-(N-1)}|\mathcal{C}_N \cap 5A_n|.$$

Choose $M \in \mathbb{N}$ such that

$$3^{-M} < 2^{5-n} \leq 3^{-(M-1)},$$

and note that $M \leq N$. We must also have

$$|\mathcal{C}_N \cap 5A_n| \ll |\mathcal{C}_M \cap 5A_n(M)|,$$

by Lemma 2.2. Therefore, since $3^{-N} \asymp \frac{\psi(2^n)}{2^n}$, $3^{-M} \asymp 2^{-n}$, and $|\mathcal{C}_M \cap 5A_n(M)|$ is trivially bounded above by $|\mathcal{C}_M| = 2^{M+1}$, we have

$$\mu(A_n) \ll 2^{M-N} = (3^{M-N})^{\gamma} \ll (2^n/3^N)^{\gamma} \ll \psi(2^n)^{\gamma}$$

for $n \geq 7$. Hence

$$\sum_{n=1}^{\infty} \mu(A_n) \ll \sum_{n=1}^{\infty} \psi(2^n)^{\gamma}$$

converges, and the convergence Borel–Cantelli Lemma (Lemma 1.16) completes the proof.

2.2 Fourier analysis

In this section, we use Fourier analysis to count instances of dyadic rationals being close to triadic rationals in the Cantor set. Recall that μ possesses the following two properties:

1. (Positivity) If $z \in K$ and $r > 0$, then $\mu(B(z,r)) > 0$, and
2. (Doubling) If $z \in K$ and $r > 0$, then $\mu(B(z,2r)) \ll \mu(B(z,r))$.

In fact μ is Ahlfors–David regular; that is, for $z \in K$ and $r \in (0,1]$, we have

$$\mu(B(z,r)) \asymp r^{\gamma},$$

see [29, §6.1] or [18]. This property implies the properties (1) and (2) stated above.

Before commencing in earnest, we estimate the measure of an interval in a discrete fashion. This is a simpler analogue of Lemma 2.1.

Lemma 2.3 *Let $I = B(z,r)$, where $z \in K$ and $0 < r < 1$, and let $L \geq L_0(I)$ be a large positive integer. In particular, we require $L_0(I)$ to be sufficiently large that $\frac{3^{-L_0(I)}}{2} < \frac{r}{5}$. Then*

$$\mu(I) \asymp \frac{|\mathcal{C}_L \cap I|}{2^L}.$$

The implicit constants are absolute. Moreover, if $B = B(z,r)$, where $z \in \mathbb{R}$ and $0 < r < 1$, and L is a positive integer for which $3^{1-L} < r$, then

$$\mu(B) \leq 2^{-(L-1)} |\mathcal{C}_L \cap 5B|. \tag{2.4}$$

Proof By the doubling property of μ, it suffices to show that

$$2^{-(L+1)} |\mathcal{C}_L \cap 0.2I| \leq \mu(I) \leq 2^{-(L-1)} |\mathcal{C}_L \cap 5I|. \tag{2.5}$$

To see that this is indeed sufficient, suppose first that we wish to show that $2^L \mu(I) \ll |\mathcal{C}_L \cap I|$. This would follow from $2^L \mu(0.2I) \leq 2|\mathcal{C}_L \cap I|$, since doubling gives

$$\mu(I) \leq \mu(1.6I) \ll \mu(0.8I) \ll \mu(0.4I) \ll \mu(0.2I).$$

That is, it would suffice to show that $2^L \mu(J) \leq 2|\mathcal{C}_L \cap 5J|$ for $J = 0.2I$. By a similar argument, the inequality $2^L \mu(I) \gg |\mathcal{C}_L \cap I|$ would follow if we could show that $2^{L+1} \mu(J') \geq |\mathcal{C}_L \cap 0.2J'|$ for $J' = 5I$, since $\mu(5I) \ll \mu(I)$ by the doubling property. Since I, and therefore also J and J', are arbitrary intervals, it suffices to show (2.5).

If $x \in \mathcal{C}_L \cap 0.2I$ then $B(x, 3^{-L}/2) \subseteq I$, by the triangle inequality. Hence, by disjointness (since distinct points in $\mathcal{C}_L$ are distance at least 3^{-L} apart), we have

$$\mu(I) \geq \sum_{x \in \mathcal{C}_L \cap 0.2I} \mu\left(B\left(x, \frac{3^{-L}}{2}\right)\right) = 2^{-(L+1)} |\mathcal{C}_L \cap 0.2I|.$$

For the upper bound, we cover $[0,1]$ by $\{B(x,3^{-L}) : x \in S_L\}$, where S_L is the set of rationals with denominator 3^L in $[0,1]$. For each $x \in S_L$ we have

$$\mu(I \cap B(x,3^{-L})) \le \mu(B(x,3^{-L})) \le 2^{-(L-1)}.$$

If $x \in S_L \setminus \mathcal{C}_L$ then $B(x,3^{-L})$ does not intersect the Cantor set, so

$$\mu(I \cap B(x,3^{-L})) = 0.$$

Finally, if $x \in S_L \setminus 5I$ then $B(x,3^{-L})$ does not intersect I, so

$$\mu(I \cap B(x,3^{-L})) = 0.$$

Therefore

$$\mu(I) \le \sum_{x \in S_L} \mu\left(I \cap B\left(x,3^{-L}\right)\right) \le 2^{-(L-1)}|\mathcal{C}_L \cap 5I|,$$

as required.

For the final statement, witness that in proving (2.5) we did not use the assumption that $z \in K$. □

2.2.1 Schwartz functions, bump functions, and tempered distributions

We briefly discuss some theory in preparation for the Fourier analysis. Full details can be found in the distribution theory textbooks of Mitrea [31], and Friedlander and Joshi [22]. We restrict ourselves to the one-dimensional setting.

Throughout this paper, we write $e(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$. A *Schwartz function* is a function $f : \mathbb{R} \to \mathbb{C}$ such that if $a, b \in \mathbb{Z}$ are such that $a, b \ge 0$, then

$$\sup_{x \in \mathbb{R}} |x^b f^{(a)}(x)| < \infty,$$

where $f^{(a)}$ is the a^{th} derivative of f. *Schwartz space*, denoted $\mathcal{S}(\mathbb{R})$, is the complex vector space of Schwartz functions, together with a natural topology [31, §14.1]. The *Fourier transform* of a Schwartz function f is

$$\hat{f} : \mathbb{R} \to \mathbb{C}, \qquad \hat{f}(t) = \int_{\mathbb{R}} f(\alpha) e(-t\alpha)\, \mathrm{d}\alpha,$$

and this defines an automorphism of Schwartz space [31, Theorem 3.25], with the inverse operator sending $g \in \mathcal{S}(\mathbb{R})$ to

$$t \mapsto \int_{\mathbb{R}} g(\alpha) e(t\alpha)\, \mathrm{d}\alpha.$$

Note that the normalisation in [31] is slightly different.

A *bump function* is a C^∞, compactly supported function $\phi : \mathbb{R} \to [0, \infty)$. In the sequel, we fix a smooth bump function ϕ supported on $[-2, 2]$ such that $\phi(x) = 1$ for $x \in [-1, 1]$ and $0 \le \phi(x) \le 1$ for all $x \in \mathbb{R}$. Such a function exists, by [22, Theorem 1.4.1]. As explained in [31, §3.1], bump functions are Schwartz, and have the rapid Fourier decay property that for each number $N > 0$,

$$\hat{\phi}(t) \ll_N (1 + |t|)^{-N}, \tag{2.6}$$

where the subscript N denotes that the implicit constant depends on N. We note that the bound in (2.6) may be deduced from [31, Eq. (3.1.12)].

A *tempered distribution* is a continuous linear functional $\mathcal{S}(\mathbb{R}) \to \mathbb{C}$. The archetypal example of a tempered distribution is the Dirac delta function, which sends $g \in \mathcal{S}(\mathbb{R})$ to $g(0)$, but is not a function in the classical sense. Note that a Schwartz function f induces a tempered distribution T_f sending $g \in \mathcal{S}(\mathbb{R})$ to

$$\int_{\mathbb{R}} f(\alpha) g(\alpha) \, \mathrm{d}\alpha.$$

We define *translates* of Schwartz functions f and of tempered distributions u by

$$\tau_a f(x) = f(x - a) \qquad (a, x \in \mathbb{R})$$

and

$$\tau_a u(g) = u(\tau_{-a} g) \qquad (a \in \mathbb{R}, \ g \in \mathcal{S}(\mathbb{R})).$$

For example, denoting by δ the Dirac delta function, the distribution $\tau_a \delta$ sends $g \in \mathcal{S}(\mathbb{R})$ to $g(a)$.

The *Fourier transform* of a tempered distribution $u : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is the tempered distribution

$$\hat{u} : \mathcal{S}(\mathbb{R}) \to \mathbb{C}, \qquad \hat{u}(f) = u(\hat{f}).$$

Writing g as the Fourier transform of $f \in \mathcal{S}(\mathbb{R})$, we obtain the *Parseval formula*

$$u(g) = \hat{u}(g^\vee), \tag{2.7}$$

where $g^\vee$ is the inverse Fourier transform of g; that is, $g^\vee(t) = \hat{g}(-t)$.

The Fourier transform of a translate $\tau_a u$ of a tempered distribution is

$$g \mapsto \hat{u}(ge(-a\cdot)).$$

Indeed, we compute that if u is a tempered distribution, $a \in \mathbb{R}$, and $g \in \mathcal{S}(\mathbb{R})$, then

$$\widehat{\tau_a u}(g) = \tau_a u(\hat{g}) = u(\tau_{-a}\hat{g}) = u(x \mapsto \hat{g}(x + a)) = u(\widehat{ge(-a\cdot)}) = \hat{u}(ge(-a\cdot)).$$

To see the fourth equality, observe that the Fourier transform of $t \mapsto g(t)e(-at)$ sends x to

$$\int_{\mathbb{R}} g(t)e(-at)e(-tx)\,\mathrm{d}t = \int_{\mathbb{R}} g(t)e(-t(x+a))\,\mathrm{d}t = \hat{g}(x+a).$$

A continuous, bounded function F is the *function type* of a distribution u if

$$u(f) = \int_{\mathbb{R}} f(t)F(t)\,\mathrm{d}t \qquad (f \in \mathcal{S}(\mathbb{R})).$$

The *distributional Poisson summation formula* [22, Theorem 8.5.1] asserts that if $g \in \mathcal{S}(\mathbb{R})$ then

$$\sum_{n \in \mathbb{Z}} \tau_n \delta(g) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} g(t)e(nt)\,\mathrm{d}t.$$

This admits the following generalisation.

Lemma 2.4 *Let u be a tempered distribution whose Fourier transform has function type F. Then*

$$\sum_{n \in \mathbb{Z}} \widehat{\tau_n u} = \sum_{n \in \mathbb{Z}} \tau_n \delta(F\cdot).$$

Proof If $g \in \mathcal{S}(\mathbb{R})$ then

$$\begin{aligned}\sum_{n \in \mathbb{Z}} \widehat{\tau_n u}(g) &= \sum_{n \in \mathbb{Z}} \widehat{\tau_{-n} u}(g) \\ &= \sum_{n \in \mathbb{Z}} \hat{u}(ge(n\cdot)) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} F(t)g(t)e(nt)\,\mathrm{d}t = \sum_{n \in \mathbb{Z}} \tau_n \delta(Fg),\end{aligned}$$

where for the final inequality we have applied the distributional Poisson summation formula to the Schwartz function Fg. □

2.2.2 Counting using Fourier analysis

We now proceed with the Fourier analysis.

Lemma 2.5 *Let $y \in \mathbb{R}$, and let $n, k \in \mathbb{N}$. Let I be either* $[0, 1]$ *or a subinterval of* $[0, 1]$ *whose endpoints do not lie in K. Let*

$$f(\alpha) = f_n(\alpha; y) = \sum_{b=0}^{2^n - 1} \phi_r(\alpha + y - b/2^n), \qquad \phi_r(\beta) = \phi(r\beta), \qquad r = 2^{n+k}.$$

If $I = [0, 1]$ *then let* $L_0(I) = 0$. *Otherwise, let* $L_0(I)$ *be a positive integer such that* $3^{-L_0(I)}$ *is less than the distance from the boundary of* I *to the boundary of* K. *Finally, let* $M \geq L \geq L_0(I)$ *be integers. Then, for* $T, N \in \mathbb{N}$ *such that* $N > 1$, *we have*

$$\sum_{x \in \mathcal{L}_M \cap I \bmod 1} f(x)$$
$$= 2^{M-L-k} \sum_{|m| \leq 2^k T} \hat{\phi}(2^{-k} m) e(2^n m y) \sum_{x \in \mathcal{L}_L \cap I} e(2^n m x) \prod_{j=L+1}^{M} \frac{1 + e(2^{n+1} m / 3^j)}{2}$$
$$+ O_N \left(2^{M-L} \frac{|\mathcal{L}_L \cap I|}{T^N} \right),$$

where the subscript N *indicates that the implied constant depends on* N. *Here* $x \in \mathcal{L}_M \cap I \bmod 1$ *means that* $x + m \in \mathcal{L}_M \cap I$ *for some* $m \in \mathbb{Z}$.

Proof Consider the distribution

$$u = \sum_{x \in \mathcal{L}_M \cap I} \delta_x,$$

where δ_x denotes the Dirac delta function at x, namely $\delta_x = \tau_x \delta$. This is the distribution that sends $g \in \mathcal{S}(\mathbb{R})$ to $\sum_{x \in \mathcal{L}_M \cap I} g(x)$. By the Parseval formula (2.7) and Lemma 2.4, we have

$$\sum_{x \in \mathcal{L}_M \cap I \bmod 1} f(x) = \sum_{w \in \mathbb{Z}} \tau_w u(f) = \sum_{w \in \mathbb{Z}} \widehat{\tau_w u}(f^\vee) = \sum_{w \in \mathbb{Z}} \tau_w \delta(F f^\vee), \quad (2.8)$$

where F is the function type of $\hat{u}$ and $f^\vee$ is the inverse Fourier transform of f; more explicitly, we recall that

$$f^\vee(t) = \hat{f}(-t).$$

Via the change of variables $\beta = \alpha + y - \frac{b}{2^n}$ and an application of Fubini's Theorem, we compute that

$$\hat{f}(t) = \int_{\mathbb{R}} f(\alpha) e(-t\alpha) \, \mathrm{d}\alpha$$
$$= e(ty) \sum_{b=0}^{2^n - 1} e\left(-tb/2^n\right) \int_{\mathbb{R}} \phi_r(\beta) e(-t\beta) \, \mathrm{d}\beta.$$

Observe that if $t \in \mathbb{Z}$ then, employing the change of variables $\alpha' = r\beta$, we have

$$\hat{f}(t) = e(ty) 2^n 1_{2^n | t} r^{-1} \hat{\phi}(t/r) = 2^{-k} e(ty) \hat{\phi}(t/r) 1_{2^n | t}. \quad (2.9)$$

The function type of $\hat{\delta_x}$ is $e(-x\cdot)$, since

$$\hat{\delta_x}(g) = \delta_x(\hat{g}) = \hat{g}(x) = \int_{\mathbb{R}} g(t)e(-tx)\,\mathrm{d}t$$

for $g \in \mathcal{S}(\mathbb{R})$. Thus, by the linearity of the Fourier transform, the function type F of $\hat{u}$ is

$$\begin{aligned} F(t) &= \sum_{x \in \mathcal{L}_M \cap I} e(-tx) \\ &= \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_M \in \{0,2\} \\ \sum_{j\le M} \varepsilon_j/3^j \in I}} \prod_{j \le M} e(-t\varepsilon_j/3^j). \end{aligned}$$

As $L \ge L_0(I)$, we see that if $\varepsilon_1, \varepsilon_2, \ldots \in \{0,2\}$ then

$$\sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} \in I \iff \sum_{j \le L} \frac{\varepsilon_j}{3^j} \in I.$$

In particular $\sum_{j\le M} \varepsilon_j/3^j \in I$ if and only if $\sum_{j\le L} \varepsilon_j/3^j \in I$. Therefore

$$\begin{aligned} F(t) = &\sum_{\substack{\varepsilon_1,\ldots,\varepsilon_M \in \{0,2\}: \\ \sum_{j\le M} \varepsilon_j/3^j \in I}} e\left(-t\sum_{j=1}^{M} \varepsilon_j/3^j\right) \\ = &\sum_{\substack{\varepsilon_1,\ldots,\varepsilon_L \in \{0,2\}: \\ \sum_{j\le L} \varepsilon_j/3^j \in I}} \sum_{\varepsilon_{L+1},\ldots,\varepsilon_M \in \{0,2\}} e\left(-t\sum_{j=1}^{L} \frac{\varepsilon_j}{3^j}\right) e\left(-t\sum_{j=L+1}^{M} \frac{\varepsilon_j}{3^j}\right). \end{aligned}$$

Since the elements of $\mathcal{L}_L \cap I$ are precisely the $\sum_{j\le L} \varepsilon_j/3^j \in I$ with $\varepsilon_j \in \{0,2\}$ for all j, note that

$$\sum_{\substack{\varepsilon_1,\ldots,\varepsilon_L \in \{0,2\}: \\ \sum_{j\le L} \varepsilon_j/3^j \in I}} e\left(-t\sum_{j\le L} \frac{\varepsilon_j}{3^j}\right) = \sum_{x \in \mathcal{L}_L \cap I} e(-tx).$$

Next, we observe that

$$\sum_{\varepsilon_{L+1},\ldots,\varepsilon_M \in \{0,2\}} e\left(-t\sum_{j=L+1}^{M} \frac{\varepsilon_j}{3^j}\right) = \sum_{\varepsilon_{L+1},\ldots,\varepsilon_M \in \{0,2\}} \prod_{j=L+1}^{M} e(-t\varepsilon_j/3^j)$$

$$= \prod_{j=L+1}^{M} \sum_{\varepsilon_j \in \{0,2\}} e(-t\varepsilon_j/3^j) = \prod_{j=L+1}^{M} (1 + e(-2t/3^j))$$
$$= 2^{M-L} \prod_{j=L+1}^{M} \frac{1 + e(-2t/3^j)}{2}.$$

Hence, we obtain

$$F(t) = 2^{M-L} \sum_{x \in \mathcal{L}_L \cap I} e(-tx) \prod_{j=L+1}^{M} \frac{1 + e(-2t/3^j)}{2}.$$

By (2.8) and (2.9), we now have

$$\sum_{x \in \mathcal{L}_M \cap I \bmod 1} f(x) = \sum_{w \in \mathbb{Z}} \tau_w \delta(F f^\vee) = \sum_{w \in \mathbb{Z}} F(w) \hat{f}(-w)$$
$$= 2^{M-L-k} \sum_{w \in \mathbb{Z}} e(-wy) \sum_{x \in \mathcal{L}_L \cap I} e(-wx)$$
$$\prod_{j=L+1}^{M} \frac{1 + e(-2w/3^j)}{2} \hat{\phi}(-w/r) 1_{2^n | w}.$$

Substituting $w = -2^n m$ and recalling that $r = 2^{n+k}$, we obtain

$$\sum_{x \in \mathcal{L}_M \cap I \bmod 1} f(x) = 2^{M-L-k} \sum_{m \in \mathbb{Z}} \hat{\phi}(m/2^k) e(2^n m y)$$
$$\sum_{x \in \mathcal{L}_L \cap I} e(2^n m x) \prod_{j=L+1}^{M} \frac{1 + e(2^{n+1} m/3^j)}{2}.$$

Let us now estimate the part of the above sum involving m such that $|m| > 2^k T$. First, we see that

$$\left| \sum_{x \in \mathcal{L}_L \cap I} e(2^n m x) \prod_{j=L+1}^{M} \frac{1 + e(2^{n+1} m/3^j)}{2} \right| \leq |\mathcal{L}_L \cap I|.$$

Next, we use the fact that ϕ is Schwartz, and thus has the rapid Fourier decay property (2.6), to deduce that for each number $N > 1$,

$$\sum_{|m| > 2^k T} |\hat{\phi}(m/2^k)| \ll_N \sum_{m > 2^k T} (m/2^k)^{-N} = 2^{kN} \sum_{m > 2^k T} m^{-N}$$
$$\leq 2^{kN} \int_{2^k T}^{\infty} m^{-N} \, \mathrm{d}m = 2^{kN} \frac{(2^k T)^{1-N}}{N-1} \leq \frac{2^k}{T^{N-1}},$$

where the subscript N indicates that the implied constant depends on N. From here the proof concludes. □

To control the product term in Lemma 2.5, we use digit changes in base 3. Set

$$\rho = |1 + e(1/9)| < 0.94$$

throughout.

Lemma 2.6 *Let $n, L, M \in \mathbb{N}$ with $L \leq M$, and let $m \in \mathbb{Z} \setminus \{0\}$. Then*

$$\left| \prod_{j=L+1}^{M} \frac{1 + e(2^{n+1}m/3^j)}{2} \right| \leq \rho^{D_3(2^{n+1}|m|;L,M)},$$

where $D_3(2^{n+1}|m|; L, M)$ is the number of digit changes in the ternary expansion of $2^{n+1}|m|$ between the L'th and M'th digits, counting from the left. Note that when $L = M$, both sides in the above inequality are equal to 1.

Proof We may assume, without loss of generality, that $m \in \mathbb{N}$. Each term in the product has norm at most 1. For $j \geq 2$, observe that $1+e(2^{n+1}m/3^j)$ has norm being bounded away from 2 if the fractional part of $2^{n+1}m/3^j$ is not too close to 0 or 1. Suppose that the $(j-1)$'st and j'th ternary digits of $2^{n+1}m$ are different. Then the ternary expansion of $2^{n+1}m/3^j$ is

$$2^{n+1}m/3^j = [\text{integer part}].ab\ldots,$$

where $ab \in \{01, 02, 10, 12, 20, 21\}$. This means that the fractional part of $2^{n+1}m/3^j$ is bounded away from 0 and 1. Indeed, it is easy to check that $\|\{2^{n+1}m/3^j\}\| \geq 1/9$ in this scenario. Consequently, we have

$$\left| \frac{1 + e\left(2^{n+1}m/3^j\right)}{2} \right| \leq \rho.$$

The lemma now follows from the definition of $D_3(2^{n+1}|m|; L, M)$. □

Finally, we obtain the following estimate.

Lemma 2.7 *Let $\theta \in \mathbb{R}$, and let I be either* $[0, 1]$ *or a subinterval of* $[0, 1]$ *whose centre lies in K. Let L be the maximum of the values $L_0(I)$ in Lemmas 2.3 and 2.5. Let $n \geq n_0(I)$ be a large positive integer, and let k be an integer in the range*

$$1 \leq k \leq \frac{\varepsilon \log n}{\log \log n},$$

where $\varepsilon > 0$ is a sufficiently small, effectively-computable constant. In particular, n should be sufficiently large that Lemma 1.14 can be applied, and so that

$$L + 6 \leq \frac{c \log n}{4 \log \log n} \quad \text{and} \quad 2^n \geq 3^{L+5},$$

where c is the implicit constant in Lemma 1.14.

(a) If M is the positive integer satisfying

$$3^{-5-M} < 2^{-n-k} \leq 3^{-4-M}, \tag{2.10}$$

then

$$|\mathcal{C}_M \cap 0.2A_n(M) \cap 0.2I| \gg 2^{M-k}\mu(I).$$

(b) If M is the positive integer satisfying

$$3^{5-M} < 2^{-n-k} \leq 3^{6-M}, \tag{2.11}$$

then

$$|\mathcal{C}_M \cap (45A_n(M) + \theta) \cap 5I| \ll 2^{M-k}\mu(I).$$

The values of ε and the implicit constants do not depend on θ, n, k, I.

Proof Recall that μ has the doubling property so, if $I \neq [0, 1]$ then by rescaling I, we may assume that its endpoints do not lie in the Cantor set. To prove part (a), it suffices to prove that

$$|\mathcal{C}_M \cap 0.2A_n(M) \cap I| \gg 2^{M-k}\mu(I).$$

This can be seen via a similar argument to that used at the beginning of the proof of Lemma 2.3.

By the construction of the bump function ϕ, assuming (2.10) we have

$$|\mathcal{C}_M \cap 0.2A_n(M) \cap I| \geq |\mathcal{L}_M \cap 0.2A_n(M) \cap I| \geq \sum_{x \in \mathcal{L}_M \cap I \bmod 1} f_n(x; 0), \tag{2.12}$$

with the notation of Lemma 2.5. Indeed, as ϕ_r is supported on $[-2^{1-n-k}, 2^{1-n-k}]$ and has image in $[0, 1]$, the right-hand side is bounded above by the number of pairs $(b, x) \in \{0, 1, \ldots, 2^n\} \times (\mathcal{L}_M \cap I)$ such that

$$\left| x - \frac{b}{2^n} \right| \leq \frac{2}{3^{M+4}},$$

and this count is in turn is bounded above by $|\mathcal{L}_M \cap 0.2A_n(M) \cap I| \le |\mathcal{C}_M \cap 0.2A_n(M) \cap I|$.

Using Lemma 2.5 with $y = 0$ and a suitably large fixed N, together with (2.12), we see that for some constant $c_N > 0$ we have

$$\begin{aligned}
&|\mathcal{C}_M \cap 0.2A_n(M) \cap I| \\
&\quad \ge 2^{M-L-k}\left(\sum_{|m|\le 2^k T} \hat{\phi}(2^{-k}m) \sum_{x\in\mathcal{L}_L\cap I} e(2^n m x) \prod_{j=L+1}^{M} \frac{1+e(2^{n+1}m/3^j)}{2}\right) \\
&\quad - 2^{M-L} c_N \frac{|\mathcal{L}_L \cap I|}{T^N},
\end{aligned}$$

where $T \ge 1$ and $M \ge L$. The zero frequency, i.e. $m = 0$, contributes

$$2^{M-L-k}\hat{\phi}(0)|\mathcal{L}_L \cap I| = c_1 2^{M-k}\mu(I),$$

where $c_1 > 0$ and the above equality can be seen as the definition of c_1. Moreover, by Lemma 2.3,

$$2^{M-L-k}\hat{\phi}(0)|\mathcal{L}_L \cap I| \asymp 2^{M-k}\mu(I).$$

Thus we see that c_1 is bounded (from above and away from zero) by absolute constants, i.e. $E^{-1} < c_1 < E$ for some absolute constant $E > 1$.

For the remaining terms, we first deal with $2^{M-L}c_N|\mathcal{L}_L \cap I|/T^N$. We choose T to be such that

$$2^{M-L} c_N \frac{|\mathcal{L}_L \cap I|}{T^N} \le 0.5 c_1 2^{M-k}\mu(I).$$

For the above to hold, we need

$$T^N \ge \frac{2c_N}{\hat{\phi}(0)} 2^k.$$

We set the value of $\log T$ to be $\max\left\{0, \frac{k\log 2}{N} + \frac{\log(2c_N/\hat{\phi}(0))}{N}\right\}$. That is, we choose

$$T = \max\{1, (2c_N/\hat{\phi}(0))^{1/N} 2^{k/N}\}.$$

Now for the sum with $0 < |m| \le 2^k T$, the triangle inequality gives

$$\left|\sum_{0<|m|\le 2^k T} \hat{\phi}(2^{-k}m) \sum_{x\in\mathcal{L}_L\cap I} e(2^n m x) \prod_{j=L+1}^{M} \frac{1+e(2^{n+1}m/3^j)}{2}\right|$$

$$\leq \hat{\phi}(0)|\mathcal{L}_L \cap I| \sum_{0<|m|\leq 2^k T} \left| \prod_{j=L+1}^{M} \frac{1+e(2^{n+1}m/3^j)}{2} \right|.$$

In light of Lemma 2.3, this is bounded above by a constant times

$$2^L \mu(I) \sum_{0<|m|\leq 2^k T} \left| \prod_{j=L+1}^{M} \frac{1+e(2^{n+1}m/3^j)}{2} \right|.$$

Next, we will apply Lemma 2.6, and so we need to estimate $D_3(2^n|m|; L, M)$. First recall, by Lemma 1.14, that for some constant $c > 0$ we have

$$D_2(2^n|m|) + D_3(2^n|m|) \geq c \frac{\log\log(2^n|m|)}{\log\log\log(2^n|m|)}, \tag{2.13}$$

and note that

$$D_3(2^n|m|; L, M) \geq D_3(2^n|m|) - (\Delta_3(2^n|m|) - M + L),$$

where $\Delta_3(2^n|m|)$ is the number of ternary digits of $2^n|m|$. From (2.10), we see that $M + 5 = \Delta_3(2^{n+k})$. As $2^n|m| \leq 2^{n+k}T$ (since $|m| \leq 2^k T$), we must therefore have

$$\Delta_3(2^n|m|) \leq \Delta_3(2^{n+k}T) \leq \Delta_3(2^{n+k}) + \Delta_3(T),$$

and consequently $\Delta_3(2^n|m|) - M \leq \Delta_3(T) + 5$. Hence

$$D_3(2^n|m|; L, M) \geq D_3(2^n|m|) - \Delta_3(T) - L - 5.$$

Furthermore

$$D_2(2^n|m|) \leq 1 + D_2(|m|) \leq 1 + \frac{\log|m|}{\log 2}.$$

Using (2.13), we now see that

$$\begin{aligned}
D_3(2^n|m|; L, M) &\geq D_3(2^n|m|) - \Delta_3(T) - L - 5 \\
&\geq \frac{c\log\log(2^n|m|)}{\log\log\log(2^n|m|)} - D_2(2^n|m|) - \Delta_3(T) - L - 5 \\
&\geq \frac{c\log\log(2^n|m|)}{\log\log\log(2^n|m|)} - \frac{\log|m|}{\log 2} - 6 - \Delta_3(T) - L.
\end{aligned} \tag{2.14}$$

As $\log\log(\cdot)/\log\log\log(\cdot)$ is increasing, we may replace its argument by 2^n for a lower bound, to see that the first term on the right-hand side of (2.14) is at least

$\frac{c\log n}{2\log\log n}$. Meanwhile, the upper bound on $|m|$ assures us that $\log|m| \leq k\log 2+\log T$. Therefore

$$D_3(2^n|m|; L, M) \geq \frac{c\log n}{2\log\log n} - k - \frac{\log T}{\log 2} - \Delta_3(T) - L - 6.$$

Since $n \geq n_0(I)$, we may assume that n is arbitrarily large compared to L, so that $L+6 \leq \frac{c\log n}{4\log\log n}$ and, hence

$$D_3(2^n|m|; L, M) \geq \frac{c\log n}{4\log\log n} - k - \frac{\log T}{\log 2} - \Delta_3(T).$$

Moreover,

$$\Delta_3(T) = \lfloor \log T/\log 3\rfloor + 1 \leq \log T + 1 \leq \log T + k.$$

Recalling that $\log T = \max\left\{0, \frac{k\log 2}{N} + \frac{\log(2c_N/\hat{\phi}(0))}{N}\right\}$ as well as $k \geq 1$ and noting that $\hat{\phi}(0) \geq 2$ we now see that

$$\begin{aligned} D_3(2^n|m|; L, M) &\geq \frac{c\log n}{4\log\log n} - 2k - 3\log T \\ &\geq \frac{c\log n}{4\log\log n} - k\left(2 + \frac{3\log 2}{N} + \frac{3\log\left(\frac{2c_N}{\hat{\phi}(0)}\right)}{N}\right) \\ &\geq \frac{c\log n}{4\log\log n} - \left(2 + 3\frac{\log 2}{N} + 3|\log(2c_N)|\right)k. \end{aligned}$$

For convenience, we write

$$\kappa_{n,N,k} = \frac{c\log n}{4\log\log n} - \left(2 + 3\frac{\log 2}{N} + 3|\log(2c_N)|\right)k.$$

Now Lemma 2.6 gives

$$\sum_{0<|m|\leq 2^k T} \left|\prod_{j\leq M} \frac{1+e(2^{n+1}m/3^j)}{2}\right| \ll 2^k T \rho^{\kappa_{n,N,k}}.$$

Recalling that

$$T = \max\left\{1, \left(\frac{2c_N}{\hat{\phi}(0)}\right)^{1/N} 2^{k/N}\right\} \leq 1 + \left(\frac{2c_N}{\hat{\phi}(0)}\right)^{1/N} 2^{k/N},$$

we see that

$$2^{-k} \sum_{0<|m|\le 2^k T} \left| \prod_{j\le M} \frac{1+e(2^{n+1}m/3^j)}{2} \right| \ll \rho^{\kappa_{n,N,k}} + \left(\frac{2c_N}{\hat{\phi}(0)}\right)^{1/N} 2^{k/N} \rho^{\kappa_{n,N,k}}.$$

Observe that

$$2^{k/N}\rho^{\kappa_{n,N,k}} = \rho^{\kappa_{n,N,k}+k\log 2/(N\log\rho)}.$$

Again, for convenience, we write

$$\kappa'_{n,N,k} = \kappa_{n,N,k} + k\log 2/(N\log\rho).$$

Thus, the non-zero frequencies contribute at most

$$C\cdot 2^{M-L}2^L\mu(I)\rho^{\kappa'_{n,N,k}} = C\cdot 2^M\mu(I)\rho^{\kappa'_{n,N,k}},$$

for some constant $C>0$. Recalling that the contribution from the zero frequency is $c_1 2^{M-k}\mu(I)$, for some c_1 bounded below by a positive absolute constant, we glean that

$$\begin{aligned} |\mathcal{C}_M \cap 0.2A_n(M)\cap I| &\ge c_1 2^{M-k}\mu(I) - C\cdot 2^M\mu(I)\rho^{\kappa'_{n,N,k}} - 0.5c_1 2^{M-k}\mu(I) \\ &\ge 2^{M-k}\mu(I)(0.5c_1 - C\rho^{\kappa'_{n,N,k}+k\log 2/\log\rho}). \end{aligned} \tag{2.15}$$

We have

$$\kappa'_{n,N,k} + k\frac{\log 2}{\log\rho} = \frac{c\log n}{4\log\log n} - \left(2+\frac{3\log 2}{N} + 3|\log(2c_N)| - (1+N^{-1})\frac{\log 2}{\log\rho}\right)k.$$

Next, we write

$$c'_N = 2+\frac{3\log 2}{N} + 3|\log(2c_N)| - \left(1+N^{-1}\right)\frac{\log 2}{\log\rho}.$$

Then we choose the value of ε to be

$$\varepsilon = \frac{c}{8c'_N}.$$

As $k\le \varepsilon\log n/\log\log n$, this ensures that

$$\kappa'_{n,N,k} + k\frac{\log 2}{\log\rho} \ge \frac{c\log n}{8\log\log n}.$$

Recalling that $\rho\in(0,1)$ is an absolute constant and that c_1 is bounded away from 0, the result concludes by using (2.15).

The second inequality can be proved similarly: the zero frequency again dominates. As Lemma 2.5 applies to $\mathcal{L}_M$ and not $\mathcal{C}_M$, we note that

$$|\mathcal{C}_M \cap (45A_n(M)+\theta) \cap 5I| = |\mathcal{L}_M \cap (45A_n(M)+\theta) \cap 5I| \\ + |\mathcal{R}_M \cap (45A_n(M)+\theta) \cap 5I|,$$

and by symmetry that

$$|\mathcal{R}_M \cap (45A_n(M)+\theta) \cap 5I| = |\mathcal{L}_M \cap (1-(45A_n(M)+\theta)) \cap (1-5I)|.$$

Moreover, observe that

$$1-(45A_n(M)+\theta) = (1-45A_n(M)) - \theta = 45A_n(M) - \theta.$$

Thus, we may also estimate

$$|\mathcal{R}_M \cap (45A_n(M)+\theta) \cap 5I| = |\mathcal{L}_M \cap (45A_n(M)-\theta) \cap (1-5I)|$$

in the same way using Lemma 2.5, with $1-I$ in place of I, since $1-5I$ is the dilation of the interval $1-I$ by a factor of 5 about its midpoint. □

The following technical lemma enables us to slightly relax the hypotheses (2.10) and (2.11).

Lemma 2.8 *Let $\theta \in \mathbb{R}$, and let I be a real interval. Let $k, M, N, J, t \in \mathbb{N}$. Then, supposing that $0 \le J < M$ and $2^{-n} > 2t/3^{M-J}$, we have*

$$|\mathcal{C}_{M-J} \cap (tA_n(M-J)+\theta) \cap I| + 1 \gg |\mathcal{C}_M \cap (tA_n(M)+\theta) \cap I|,$$

where the implied constant depends only on J. If $I \supseteq [0,1]$, then the $+1$ term on the left-hand side can be removed.

Proof First, assume that $I \supseteq [0,1]$. Elements of $\mathcal{C}_{M-J}$ are endpoints of the intervals of length $3^{-(M-J)}$ forming K_{M-J}. We see from the inequality $2^{-n} > 2t/3^{M-J}$ that $tA_n(M-J)+\theta$ is a disjoint union of intervals of length $2t/3^{M-J}$ centred at shifted dyadic rationals in $[0,1]$ with denominator 2^n. As $2t/3^{M-J} > 1/3^{M-J}$, observe that for each interval forming K_{M-J} and intersecting $tA_n(M-J)+\theta$, at least one (and trivially at most two) of the endpoints must be included in $tA_n(M-J)+\theta$. Writing $\mathcal{N}(M-J)$ for the number of such intervals, and $\mathcal{N}(M)$ for the number of intervals forming K_M and intersecting $tA_n(M-J)+\theta$, and employing the same argument for $\mathcal{C}_M$ as for $\mathcal{C}_{M-J}$, we therefore have

$$1 \le \frac{|\mathcal{C}_{M-j} \cap (tA_n(M-J)+\theta)|}{\mathcal{N}(M-j)} \le 2 \qquad (j = 0, J).$$

Let I_M be such an interval counted by $\mathcal{N}(M)$. Then $I_M \subset I_{M-J}$ for some interval I_{M-J} forming K_{M-J} for which $I_{M-J} \cap (tA_n(M-J)+\theta) \neq \emptyset$. Clearly, each interval forming K_{M-J} contains 2^J many intervals forming K_M. From here we see that

$$\begin{aligned} |\mathcal{C}_M \cap (tA_n(M)+\theta)| &\leq |\mathcal{C}_M \cap (tA_n(M-J)+\theta)| \\ &\leq 2\mathcal{N}(M) \\ &\leq 2^{J+1}\mathcal{N}(M-J) \\ &\leq 2^{J+1}|\mathcal{C}_{M-J} \cap (tA_n(M-J)+\theta)|, \end{aligned}$$

where the first inequality follows because $tA_n(M)+\theta \subseteq tA_n(M-J)+\theta$.

Now, let I be a general subinterval of $\mathbb{R}$. As above, we find that $|\mathcal{C}_M \cap (tA_n(M-J)+\theta) \cap I|$ is at most double the number of intervals forming K_M and intersecting $(tA_n(M-J)+\theta) \cap I$. Let I_M, I_{M-J} be as in above. The additional consideration is that endpoints of I_{M-J} may not be in I. However, this can happen for at most two intervals forming K_{M-J}. Thus, writing $\mathcal{N}(M-J; I)$ for the number of intervals forming K_{M-J} and intersecting $(tA_n(M-J)+\theta) \cap I$, we have

$$\mathcal{N}(M-J; I) \leq |\mathcal{C}_{M-J} \cap (tA_n(M-J)+\theta) \cap I| + 2.$$

Finally, we obtain

$$\begin{aligned} |\mathcal{C}_M \cap (tA_n(M)+\theta) \cap I| &\leq |\mathcal{C}_M \cap (tA_n(M-J)+\theta) \cap I| \\ &\leq 2^{J+1}\mathcal{N}(M-J; I) \\ &\leq 2^{J+1}(|\mathcal{C}_{M-J} \cap (tA_n(M-J)+\theta) \cap I| + 2). \end{aligned}$$

□

3 Convergence theory

Here we establish Theorem 1.5. As $\sum_{n=1}^{\infty} \psi(2^n) < \infty$, we may assume that $\psi(2^n) < 3^{-99}$ for n sufficiently large. Let n be such a large positive integer, such that additionally $\log\log\log n > 3/\varepsilon$, and Lemmas 2.1, 2.2, and 2.7 may be applied, where ε is from Lemma 2.7. Next, let

$$k_n = \min\left(3\left\lfloor \frac{\log n}{\log\log n \cdot \log\log\log n}\right\rfloor, \left\lfloor \frac{-\log\psi(2^n)}{\log 2}\right\rfloor + 1\right),$$

and choose $M, N \in \mathbb{N}$ according to

$$3^{-5-M} < 2^{-n-k_n} \leq 3^{-4-M} \qquad \text{and} \qquad 3^{1-N} < \frac{\psi(2^n)}{5 \cdot 2^n} \leq 3^{2-N}. \tag{3.1}$$

These choices ensure that we have all of the inequalities in (2.1), and also the inequalities

$$1 \le k_n \le \frac{\varepsilon \log n}{\log\log n},$$

which will be needed when we apply Lemma 2.7. Now, fix $I = [0, 1]$ and observe that

$$\mu(A_n) = \mu(A_n \cap I).$$

By Lemma 2.1, we have

$$\mu(A_n \cap I) \le 2^{-(N-1)} |\mathcal{C}_N \cap 5A_n \cap 5I|.$$

Applying Lemma 2.2, we deduce that

$$|\mathcal{C}_N \cap 5A_n \cap 5I| \ll |\mathcal{C}_M \cap 5A_n(M) \cap 5I| \le |\mathcal{C}_M \cap 45A_n(M) \cap 5I|.$$

Next, we apply Lemma 2.7(b) with $\theta = 0$. Our current parameters $M, n, k = k_n$ do not satisfy the conditions of (2.11) required to apply Lemma 2.7(b). However, the parameters $M - 10, n, k_n$ do satisfy (2.11), and so

$$|\mathcal{C}_{M-10} \cap 45A_n(M-10) \cap 5I| \ll 2^{M-10-k_n}\mu(I) \ll 2^{M-k_n}\mu(I).$$

Now we use Lemma 2.8 to deduce that

$$|\mathcal{C}_M \cap 45A_n(M) \cap 5I| \ll |\mathcal{C}_{M-10} \cap 45A_n(M-10) \cap 5I|,$$

where the implicit constant is absolute; here we note from the calculation

$$3^{-5-M} < 2^{-n-k_n} < \frac{\psi(2^n)}{2^n} < \frac{3^{-99}}{2^n}$$

that the required condition $2^{-n} > \frac{90}{3^{M-10}}$ is met.

Therefore

$$\mu(A_n) \ll 2^{-N} |\mathcal{C}_M \cap 45A_n(M) \cap 5I| \ll 2^{M-k_n-N}\mu(I) = 2^{M-k_n-N}.$$

Moreover, since $3^M \ll 2^{n+k_n}$ and $3^{-N} \ll \frac{\psi(2^n)}{2^n}$ by hypothesis, we have

$$2^{M-N-k_n} = (3^M \times 3^{-N})^\gamma \times 2^{-k_n} \ll \left(2^{n+k_n} \times \frac{\psi(2^n)}{2^n}\right)^\gamma \times 2^{-k_n} \ll (2^{k_n}\psi(2^n))^\gamma \times 2^{-k_n},$$

and hence

$$\mu(A_n) \ll (2^{k_n}\psi(2^n))^{\gamma} \times 2^{-k_n}.$$

Observe that if

$$k_n = \left\lfloor \frac{-\log\psi(2^n)}{\log 2} \right\rfloor + 1$$

then

$$(2^{k_n}\psi(2^n))^{\gamma} \times 2^{-k_n} \ll \psi(2^n).$$

Otherwise, we have

$$\begin{aligned}&(2^{k_n}\psi(2^n))^{\gamma} \times 2^{-k_n}\\ &= 2^{-3(1-\gamma)\left\lfloor \frac{\log n}{\log\log n \cdot \log\log\log n}\right\rfloor}\psi(2^n)^{\gamma} \ll 2^{-\log n/(\log\log n \cdot \log\log\log n)}\psi(2^n)^{\gamma}.\end{aligned}$$

From the above arguments we see that

$$\sum_{n=1}^{\infty}\mu(A_n) \ll \sum_{n=1}^{\infty} 2^{-k_n(1-\gamma)}\psi(2^n)^{\gamma} \ll \sum_{n=1}^{\infty}\left(2^{-\log n/(\log\log n\cdot\log\log\log n)}\psi(2^n)^{\gamma} + \psi(2^n)\right),$$

which converges by our assumption (1.7). Thus, by the convergence Borel–Cantelli Lemma (Lemma 1.16), the proof of Theorem 1.5 is complete.

4 Divergence theory

Here we establish Theorem 1.10. Let $I = B(z, r)$, for some $z \in K$ and some $r \in (0, 1)$. Define the 'localised' probability measure μ_I by

$$\mu_I(A) = \frac{\mu(A \cap I)}{\mu(I)},$$

for Borel sets A. By Lemma 1.18, recalling that μ has the necessary properties as stated at the beginning of Sect. 2.2, it suffices to prove that

$$\mu_I(W_2(\psi)) \gg 1, \tag{4.1}$$

with an implicit constant independent of I.

We begin by showing that if $n \geq n_0(I)$, where $n_0(I)$ is a sufficiently large integer such that we can apply Lemmas 2.1 and 2.7, then $\mu_I(A_n) \gg \psi(A_n)$, where the implicit constant does not depend on I. Define $N, k \in \mathbb{N}$ according to

$$3^{-N} \leq \frac{\psi(2^n)}{5\cdot 2^n} < 3^{-(N-1)} \qquad \text{and} \qquad 3^{-5-N} < 2^{-n-k} \leq 2\cdot 3^{-5-N}.$$

Now, since $3^{-N} \leq \frac{\psi(2^n)}{2^n}$, we have

$$|\mathcal{C}_N \cap 0.2A_n \cap 0.2I| \geq |\mathcal{C}_N \cap 0.2A_n(N) \cap 0.2I|.$$

Combining this with Lemma 2.1 yields

$$\mu_I(A_n) \geq \frac{2^{-(N+1)}|\mathcal{C}_N \cap 0.2A_n(N) \cap 0.2I|}{\mu(I)}.$$

Thus, it follows from Lemma 2.7(a) that

$$\mu_I(A_n) \gg \frac{2^{-(N+1)} \times 2^{N-k}\mu(I)}{\mu(I)} \gg 2^{-k}.$$

As

$$2^{-k} \gg 2^n 3^{-N} \gg \psi(2^n) = 2^{-\log\log n/\log\log\log n},$$

we have

$$\mu_I(A_n) \gg \psi(2^n), \tag{4.2}$$

as claimed.

Next, we wish to bound $\mu_I(A_n \cap A_m)$. We will do this analytically when

$$n_0(I) \leq n \leq m \leq n^+,$$

for n^+ to be specified in due course. By the triangle inequality, if $B\left(\frac{a}{2^n}, \frac{\psi(2^n)}{2^n}\right)$ intersects $B\left(\frac{b}{2^m}, \frac{\psi(2^m)}{2^m}\right)$ then

$$\left|\frac{a}{2^n} - \frac{b}{2^m}\right| < \frac{\psi(2^n)}{2^n} + \frac{\psi(2^m)}{2^m},$$

whereupon

$$|b - 2^{m-n}a| < 2^{m-n}\psi(2^n) + \psi(2^m). \tag{4.3}$$

We break the solutions (a, b) to (4.3) into at most

$$2(1 + 2^{m-n}\psi(2^n) + \psi(2^m)) \ll 1 + 2^{m-n}\psi(2^n)$$

groups, according to the value of the integer $h = b - 2^{m-n}a$, the idea being to handle each group analytically.

For a given value of $h = b - 2^{m-n}a$, the corresponding intersections $B\left(\frac{a}{2^n}, \frac{\psi(2^n)}{2^n}\right) \cap B\left(\frac{b}{2^m}, \frac{\psi(2^m)}{2^m}\right)$ are contained in

$$X_h := \bigcup_{a=0}^{2^n} B\Big(\frac{a}{2^n} + \frac{h}{2^m}, \frac{\psi(2^m)}{2^m}\Big).$$

To see this, observe that if

$$x \in \bigcup_{a=0}^{2^n} \bigcup_{b=0}^{2^m} B\left(\frac{a}{2^n}, \frac{\psi(2^n)}{2^n}\right) \cap B\left(\frac{b}{2^m}, \frac{\psi(2^m)}{2^m}\right),$$

then, for some $a \in \{0, 1, 2, \ldots, 2^n\}$ and $b \in \{0, 1, 2, \ldots, 2^m\}$, we have

$$x \in B\left(\frac{a}{2^n}, \frac{\psi(2^n)}{2^n}\right) \cap B\left(\frac{b}{2^m}, \frac{\psi(2^m)}{2^m}\right).$$

Next, write $h = b - 2^{m-n}a$. Then,

$$x \in B\left(\frac{b}{2^m}, \frac{\psi(2^m)}{2^m}\right) = B\Big(\frac{a}{2^n} + \frac{h}{2^m}, \frac{\psi(2^m)}{2^m}\Big) \subseteq X_h.$$

Recalling from (4.3) that $|h| < 2^{m-n}\psi(2^n) + \psi(2^m)$, it follows that

$$\mu_I(A_n \cap A_m) = \mu_I\left(\bigcup_{a=0}^{2^n} \bigcup_{b=0}^{2^m} B\left(\frac{a}{2^n}, \frac{\psi(2^n)}{2^n}\right) \cap B\left(\frac{b}{2^m}, \frac{\psi(2^m)}{2^m}\right)\right)$$
$$\leq \sum_{|h|<2^{m-n}\psi(2^n)+\psi(2^m)} \mu_I(X_h). \tag{4.4}$$

We will use our Fourier-analytic machinery from Sect. 2.2 to bound each $\mu_I(X_h)$. To begin with, the triangle inequality gives

$$\mu_I(X_h) \leq \sum_{a=0}^{2^n} \mu\left(B\left(\frac{a}{2^n} + \frac{h}{2^m}, \frac{\psi(2^m)}{2^m}\right)\right).$$

Next, we apply (2.4) to each of the intervals $B(a/2^n + h/2^m, \psi(2^m)/2^m) \cap I$ with $L = R$ therein, where R is the positive integer satisfying

$$3^{1-R} < \frac{\psi(2^m)}{2^m} \leq 3^{2-R}.$$

This gives

$$\mu(X_h \cap I) \ll 2^{-R}|\mathcal{C}_R \cap 5X_h \cap 5I|,$$

where

$$5X_h = \bigcup_{a=0}^{2^n} B\Big(\frac{a}{2^n} + \frac{h}{2^m}, \frac{5\psi(2^m)}{2^m}\Big),$$

since at most 11 of the balls can contain a given point. We thus have

$$\mu(X_h \cap I) \ll 2^{-R} \left| \mathcal{C}_R \cap \left(45A_n(R) + \frac{h}{2^m} \right) \cap 5I \right|. \tag{4.5}$$

Let $k(m, n)$ be the positive integer for which

$$2^{-k(m,n)-9} < \frac{\psi(2^m)}{2^{m-n}} = 2^{n-m-(\log\log m/\log\log\log m)} \leq 2^{-k(m,n)-8}.$$

We will apply Lemma 2.7(b) with $M \in \mathbb{N}$ given by

$$3^{5-M} < 2^{-n-k(m,n)} \leq 3^{6-M}.$$

In order to meet the condition $k(m, n) \leq \frac{\varepsilon \log n}{\log\log n}$ required for this, it suffices to have

$$m - n + \frac{\log\log m}{\log\log\log m} = o\Big(\frac{\log n}{\log\log n}\Big),$$

and this is assured if we choose

$$n^+ = n + \left\lfloor \frac{\log n}{\log\log n \cdot \log\log\log n} \right\rfloor. \tag{4.6}$$

To see this, we note that $m \mapsto \frac{\log\log m}{\log\log\log m}$ is increasing so, with this choice of n^+, we have

$$\begin{aligned} m - n + \frac{\log\log m}{\log\log\log m} &\leq n + \frac{\log n}{\log\log n \cdot \log\log\log n} - n \\ &+ \frac{\log\log\left(n + \frac{\log n}{\log\log n \cdot \log\log\log n}\right)}{\log\log\log\left(n + \frac{\log n}{\log\log n \cdot \log\log\log n}\right)} \\ &= o\left(\frac{\log n}{\log\log n}\right). \end{aligned}$$

Now Lemma 2.7(b) gives

$$\left| \mathcal{C}_M \cap \left(45A_n(M) + \frac{h}{2^m} \right) \cap 5I \right| \ll 2^{M-k(m,n)} \mu(I). \tag{4.7}$$

We see from our choices of M, R, and $k(m,n)$ that

$$3^{6-M} \geq 2^{-n-k(m,n)} \geq \frac{\psi(2^m)}{2^{m-8}} > 2^8 3^{1-R} > 3^{6-R}$$

and

$$3^{4-M} < \frac{3^{5-M}}{2} < 2^{-n-k(m,n)-1} < 2^8 \frac{\psi(2^m)}{2^m} \leq 2^8 3^{2-R} < 3^{8-R},$$

so $R-4 < M < R$. It follows from the above inequalities that $2^{-n} > \frac{90}{3^M}$ and so we may apply Lemma 2.8 with R and M playing the roles of M and $M-J$ respectively, and this tells us that

$$\left|\mathcal{C}_R \cap \left(45A_n(R) + \frac{h}{2^m}\right) \cap 5I\right| \ll \left|\mathcal{C}_M \cap \left(45A_n(M) + \frac{h}{2^m}\right) \cap 5I\right| + 1.$$

Combining this with (4.5) and (4.7) and the fact that $M < R$, we see that

$$\mu_I(X_h) \ll \frac{2^{M-k(m,n)}\mu(I) + 1}{2^M \mu(I)} \ll 2^{-k(m,n)} + \frac{1}{2^M \mu(I)}. \tag{4.8}$$

Note that

$$3^{-M} \asymp \psi(2^m)/2^m$$

and

$$2^{-k(m,n)} \asymp 2^n/3^M \asymp 2^{n-m}\psi(2^m).$$

Observe also that

$$2^{-M} = (3^{-M})^\gamma \asymp (2^{-k(m,n)-n})^\gamma \ll 2^{-n\gamma}.$$

So, since

$$k(m,n) = o\left(\frac{\log n}{\log\log n}\right),$$

we must therefore have

$$\frac{1}{2^M \mu(I)} \ll 2^{-n\gamma}/\mu(I) \ll 2^{-\log n} \ll 2^{-k(m,n)} \qquad \text{whenever } n \geq n_0(I).$$

Substituting this data into (4.8) gives

$$\mu_I(X_h) \ll 2^{-k(m,n)} \asymp 2^{n-m}\psi(2^m).$$

Substituting this into (4.4), we finally obtain

$$\mu_I(A_n \cap A_m) \ll (1 + 2^{m-n}\psi(2^n)) \times 2^{n-m}\psi(2^m) = 2^{n-m}\psi(2^m) + \psi(2^n)\psi(2^m). \tag{4.9}$$

The Chung–Erdős inequality (Lemma 1.17) gives

$$\mu_I\Bigg(\bigcup_{i=n}^{n^+} A_i\Bigg) \geq \frac{\left(\sum_{i=n}^{n^+} \mu_I(A_i)\right)^2}{\sum_{i,j=n}^{n^+} \mu_I(A_i \cap A_j)}.$$

Estimating the denominator using (4.9) and (4.2), we have

$$\begin{aligned}\sum_{i,j=n}^{n^+} \mu_I(A_i \cap A_j) &\ll \sum_{i,j=n}^{n^+} \mu_I(A_i)\mu_I(A_j) + \sum_{n \leq i \leq j \leq n^+} \psi(2^i)2^{i-j} \\ &\ll \Bigg(\sum_{i=n}^{n^+} \mu_I(A_i)\Bigg)^2 + \sum_{i=n}^{n^+} \mu_I(A_i).\end{aligned}$$

For large n, by (4.2) we have

$$\sum_{t=n}^{n^+} \mu_I(A_t) \gg \sum_{t=n}^{n^+} 2^{-\log\log t / \log\log\log t}.$$

If n is large and $n \leq t \leq n^+$, then $t \leq 2n$ and by an application of the mean value theorem we have

$$\frac{\log\log t}{\log\log\log t} - \frac{\log\log n}{\log\log\log n} = (t-n)\frac{\log\log\log c - 1}{c(\log c)(\log\log\log c)^2} \leq 1,$$

for some $c \in [n, n^+]$. Therefore

$$\sum_{t=n}^{n^+} \mu_I(A_t) \gg \frac{\log n}{\log\log n \cdot \log\log\log n} 2^{-\log\log n / \log\log\log n},$$

and this tends to ∞ as $n \to \infty$. Indeed, to see that $2^{\log\log n / \log\log\log n}$ is sub-logarithmic, observe that

$$2^{\log\log n / \log\log\log n} = (\log n)^{\log 2 / \log\log\log n} = (\log n)^{o(1)}.$$

We thus obtain

$$\mu_I\left(\bigcup_{i=n}^{n^+} A_i\right) \gg \frac{1}{1+\frac{1}{\sum_{t=n}^{n^+}\mu_I(A_t)}} \gg 1.$$

Finally, let

$$B_n = \bigcup_{i=n}^{n^+} A_i.$$

Using the continuity of the measure μ_I, we see that

$$\mu_I\left(\limsup_{n\to\infty} A_n\right) = \mu_I\left(\limsup_{n\to\infty} B_n\right) = \mu_I\left(\bigcap_{n\geq 1}\bigcup_{k\geq n} B_n\right)$$
$$= \lim_{n\to\infty}\mu_I\left(\bigcup_{k\geq n} B_n\right) \geq \lim_{n\to\infty}\mu_I(B_n) \gg 1.$$

This completes the proof that

$$\mu_I(W_2(\psi)) = \mu_I\left(\limsup_{n\to\infty} A_n\right) \gg 1,$$

which is (4.1).

5 Further discussion

5.1 Digit changes to different bases

The 'times two, times three' phenomenon is, roughly speaking, the mantra that digit expansions to two multiplicatively independent bases cannot both be structured. One way to quantify this is to bound from below the maximum—or equivalently the sum—of $D_2(y)$ and $D_3(y)$, since $D_b(y)$ being small means that the base b expansion of y is structured.

Question 5.1 *Is it true that for sufficiently large $y \in \mathbb{N}$, we have*

$$D_2(y) + D_3(y) \gg \log y?$$

Note that for all $y \in \mathbb{N}$ we have $D_2(y) + D_3(y) \leq 10(1 + \log y)$. This bound can be obtained using 'naive' estimates: observing that y has at most $1 + \frac{\log y}{\log 2}$ digits in its base 2 expansion and at most $1 + \frac{\log y}{\log 3}$ digits in base 3. Consequently, the number of digit changes in base 2 and base 3 are also bounded above by these values, respectively. Finally, by estimating $\log 2$ and $\log 3$ quite crudely, e.g. $\frac{1}{2} < \log 2, \log 3 < 2$ will do, we easily obtain the claimed upper bound. For Question 5.1, the extremal example that we have in mind is when y is a large power of 2; if the ternary expansion were roughly uniformly random then we would have $D_2(y) + D_3(y) \approx \frac{2\log y}{3\log 3}$. Reformulating, what Question 5.1 is asking is whether, for all sufficiently large y, we have $D_2(y)+D_3(y) \asymp \log y$. We have some empirical evidence in support of a positive answer; see the appendix.

5.2 Conditional approximation results

As we have seen, the estimate (1.11) given in Lemma 1.14 plays an important role in obtaining Theorems 1.5 and 1.10. Thus, one possible means of improving those Diophantine approximation results would be to obtain a better lower bound for $D_2(y)+D_3(y)$. In this subsection, we discuss the extent to which such an improved bound would lead to better approximation results.

Suppose that one has the estimate

$$D_2(y) + D_3(y) \geq h(y),$$

for an increasing function $h : \mathbb{N} \to (0, \infty)$ which tends to infinity as $y \to \infty$. Then, by repeating the argument in the proof of Lemma 2.7, we find that the conclusions of that lemma are valid for $k \leq \varepsilon h(2^n)$, with a suitable constant $\varepsilon > 0$. Now we make the following replacement of the approximation function:

$$\widetilde{\psi} : 2^n \mapsto \max\{\psi(2^n), 2^{-\varepsilon h(2^n)}\}.$$

Suppose that

$$\sum_{n=1}^{\infty} 2^{-\varepsilon h(2^n)} < \infty.$$

Then $\sum_{n=1}^{\infty} \widetilde{\psi}(2^n)$ converges if and only if $\sum_{n=1}^{\infty} \psi(2^n)$ converges.

For $n \in \mathbb{N}$, let

$$\widetilde{A_n} := \bigcup_{a=0}^{2^n} B\left(\frac{a}{2^n}, \frac{\widetilde{\psi}(2^n)}{2^n}\right).$$

By a similar argument as in the proof of Theorem 1.5, we see that

$$\mu(A_n) \leq \mu(\widetilde{A}_n) \ll \widetilde{\psi}(2^n).$$

Thus, by applying the convergence Borel–Cantelli Lemma (Lemma 1.16), we obtain the following result.

Theorem 5.2 *There exists an effectively computable universal constant* $\varepsilon > 0$ *such that the following holds. If* $D_2(y) + D_3(y) \geq h(y)$ *for all* $y \geq 1$*, where* $h : \mathbb{N} \to (0, \infty)$ *is an increasing function, and*

$$\sum_{n=1}^{\infty}(\psi(2^n) + 2^{-\varepsilon h(2^n)}) < \infty,$$

then

$$\mu(W_2(\psi)) = 0.$$

We thereby obtain the convergence part of Conjecture 1.2 conditionally, for example if we can choose $h(y) = C \log\log y$ for some sufficiently large and effectively computable constant $C > 0$. Recall that we can choose $h(y) \gg \log\log y / \log\log\log y$ for large y unconditionally, by inequality (1.11), so in some sense we are quite close to obtaining the convergence part of Conjecture 1.2.

Similarly, we can also improve Theorem 1.10 if we have a stronger lower bound for $D_2(y) + D_3(y)$.

Theorem 5.3 *Suppose* $D_2(y) + D_3(y) \geq h(y)$ *for all* $y \geq 1$*, where* $h : \mathbb{N} \to (0, \infty)$ *is an increasing function.*

(a) If $h(y) \gg \log y$ *then for* $\psi(2^n) = \frac{1}{n}$ *we have* $\mu(W_2(\psi)) = 1$.
(b) If $h(y) \gg \log\log y$ *then for* $\psi(2^n) = \frac{1}{1+\log n}$ *we have* $\mu(W_2(\psi)) = 1$.

This can be proved by repeating the arguments in Sect. 4. The difference is the choice of n^+ in (4.6). In Case (a), we can choose $n^+ = n + \lfloor \varepsilon n \rfloor$, where $\varepsilon > 0$ is a constant which depends on the implicit constant in $h(y) \gg \log y$. In Case (b), we can choose $n^+ = n + \lfloor \varepsilon \log n \rfloor$ for some $\varepsilon > 0$.

Remark 5.4 In Case (a), which corresponds to a positive answer to Question 5.1, the conclusion is almost the divergence part of Conjecture 1.2. Indeed, as $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^2}$ converges, the function $\psi(2^n) = 1/n$ comes within a log-power factor of the conjectured truth (or doubly-logarithmic in the input). With the same method, one can show that the same conclusion holds for any approximation function ψ satisfying

$$\limsup_{k\to\infty} \sum_{n=2^k}^{2^{k+1}} \psi(2^n) > 0.$$

The reason for introducing Case (b) is that it appears to be closer to our reach; see Theorem 5.6.

5.3 Conditional estimates for $D_2(y) + D_3(y)$

We believe that Question 5.1 should be difficult to answer. In this section, we illustrate how to sharpen inequality (1.11) by assuming the Lang–Waldschmidt Conjecture on Baker's logarithmic sum estimates [28, Conjecture 1 (page 212)].

Conjecture 5.5 (Lang–Waldschmidt Conjecture) *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be non-zero integers with*

$$\Lambda := \sum_i b_i \log a_i \neq 0.$$

Then

$$\log |\Lambda| \geq -C_n(\log A + \log B),$$

where

$$A = \max_i |a_i|, \qquad B = \max_i |b_i|,$$

and $C_n > 0$ is an effectively computable constant which depends only on n.

Theorem 5.6 *Assuming the Lang–Waldschmidt Conjecture, for sufficiently large y, we have*

$$D_2(y) + D_3(y) \gg \log\log y.$$

Remark 5.7 This asserts that, for some constant $c > 0$, we have $D_2(y) + D_3(y) \geq c \log\log y$ for all large enough y. This constant c is effectively computable from Conjecture 5.5. Recalling Theorem 5.2 and the subsequent discussion regarding the constant C, we see that if $c \geq C$ then the convergence part of Conjecture 1.2 would follow.

Proof We follow Stewart's approach in [35]. We may assume that $y \geq 16 > e^e$ so that $\log\log y > 1$. Let

$$y = \sum_{i=0}^{r} a_i 2^i = \sum_{i=0}^{t} b_i 3^i$$

be the binary and ternary expansions of y, and note that $a_r, b_t \geq 1$. Let $m_1 < m_2 < \cdots < m_k$ mark the binary digit changes of y, and let $n_1 < n_2 < \cdots < n_s$ mark the ternary digit changes, i.e. for $j = 1, \ldots, k$ we have $a_{m_j} \neq a_{m_j+1}$ and for $\ell = 1, \ldots, s$ we have $b_{n_\ell} \neq b_{n_\ell+1}$ (and there are no other digit changes). Let

$$2 < \theta_1 < \cdots < \theta_F < t,$$

with these parameters to be decided later. Observe that if each $[\theta_i, \theta_{i+1})$ contains an element of

$$\{r - m_1, \ldots, r - m_k, t - n_1, \ldots, t - n_s\}$$

then $k + s \geq F$.

Otherwise, suppose $[\theta_j, \theta_{j+1})$ fails to intersect this set; we will choose the θ_j in such a way as to contradict this premise. It follows from this assumption that for some $\alpha \in \{0, 1\}$ we have

$$\begin{aligned} y &= \sum_{r-i \notin [\theta_j, \theta_{j+1})} a_i 2^i + \sum_{r-i \in [\theta_j, \theta_{j+1})} \alpha 2^i \\ &= \alpha(2^{r+1} - 1) + \sum_{0 \leq r-i < \theta_j} (a_i - \alpha) 2^i + \sum_{\theta_{j+1} \leq r-i \leq r} (a_i - \alpha) 2^i \\ &= 2^{r-\theta_j} \left(\alpha 2^{\theta_j + 1} + \sum_{u=1}^{\theta_j} (a_{r-\theta_j+u} - \alpha) 2^u \right) - \alpha + \sum_{i=0}^{r-\theta_{j+1}} (a_i - \alpha) 2^i, \end{aligned}$$

and similarly in ternary, for some $\beta \in \{0, 1, 2\}$,

$$y = 3^{t-\theta_j} \left(\frac{\beta}{2} \cdot 3^{\theta_j+1} + \sum_{u=1}^{\theta_j} (b_{t-\theta_j+u} - \beta) 3^u \right) - \frac{\beta}{2} + \sum_{i=0}^{t-\theta_{j+1}} (b_i - \beta) 3^i.$$

Hence

$$2y = A_1 2^{r-\theta_j} + A_2 = B_1 3^{t-\theta_j} + B_2, \tag{5.1}$$

where A_1, A_2, B_1, B_2 are integers satisfying

(i) $2 \cdot 2^{\theta_j} \leq A_1 \leq 4 \cdot 2^{\theta_j}$,
(ii) $|A_2| < 4 \cdot 2^{r-\theta_{j+1}}$,
(iii) $3^{\theta_j} \leq B_1 \leq 6 \cdot 3^{\theta_j}$, and
(iv) $|B_2| < 4 \cdot 3^{t-\theta_{j+1}}$.

Next, observe that by (5.1) we have

$$1 = \frac{A_1 2^{r-\theta_j} + A_2}{B_1 3^{t-\theta_j} + B_2} = R \frac{1 + X}{1 + Y},$$

where

$$R = \frac{A_1 2^{r-\theta_j}}{B_1 3^{t-\theta_j}}, \qquad X = \frac{A_2}{A_1 2^{r-\theta_j}}, \text{ and } \qquad Y = \frac{B_2}{B_1 3^{t-\theta_j}}.$$

Using (i)–(iv) from above, and since necessarily we have $\theta_{j+1} > 2$, we compute that

$$|X| \leq \frac{4 \cdot 2^{r-\theta_{j+1}}}{2 \cdot 2^r} = 2 \cdot 2^{-\theta_{j+1}} < \frac{1}{2}$$

and

$$|Y| \leq \frac{4 \cdot 3^{t-\theta_{j+1}}}{3^t} = 4 \cdot 3^{-\theta_{j+1}} < \frac{1}{2}.$$

In particular, we have $R > 0$ and

$$\max\{R, R^{-1}\} = \max\left\{\frac{1+X}{1+Y}, \frac{1+Y}{1+X}\right\} \leq 1 + 4\max\{|X|, |Y|\} < 1 + 16 \cdot 2^{-\theta_{j+1}}.$$

We now apply the standard inequality

$$\log(1+x) \leq x \qquad (x > -1)$$

with $x = R - 1$ and $R^{-1} - 1$, to obtain

$$|\log R| = \max\{\log R, \log(R^{-1})\} \leq 2^{5-\theta_{j+1}}. \tag{5.2}$$

On the other hand,

$$\log R = \log A_1 + (r - \theta_j)\log 2 - \log B_1 - (t - \theta_j)\log 3.$$

So, assuming Conjecture 5.5 we have

$$\log|\log R| \geq -C_4(\log\max\{|A_1|, |B_1|\} + \log(r - \theta_j)) \tag{5.3}$$

unless $\log R = 0$. We may take $C_4 \geq 1$. The latter, $\log R = 0$, would imply that

$$A_1 2^{r-\theta_j} = B_1 3^{t-\theta_j}.$$

Hence $3^{t-\theta_j} | A_1$, so $3^{t-\theta_j} \leq A_1 \leq 2^{\theta_j+2}$, and therefore

$$2^{r+2} \geq 3^{t-\theta_j} 2^{r-\theta_j}.$$

Observe that $3^{t+1} \geq y \geq 2^r$, and hence $(t+1)\log 3 \geq r\log 2$. Thus,

$$(r+2)\log 2 \geq (t-\theta_j)\log 3 + (r-\theta_j)\log 2 \geq (r\log 2 - \log 3) + r\log 2 - \theta_j \log 6.$$

This is only possible if

$$\theta_j \geq \frac{r\log 2 - \log 12}{\log 6}.$$

We next observe that

$$\theta_{j+1} - 5 \leq -\frac{\log|\log R|}{\log 2} \leq 3C_4(\theta_j + \log\log y + 1).$$

When $\theta_j < \frac{r\log 2 - \log 12}{\log 6}$, the first inequality above follows from (5.2). For the second inequality we apply (5.3) and make use of the upper bounds given in (i) and (iii) for A_1 and B_1 respectively, as well as noting that $2^r \leq y$.

Choosing $\theta_1, \ldots, \theta_F$ so that

$$\theta_{j+1} - 5 > 3C_4(\theta_j + \log\log y + 1), \qquad \theta_j < \frac{r\log 2 - \log 12}{\log 6} \qquad (1 \leq j \leq F)$$

contradicts our premise that $[\theta_j, \theta_{j+1})$ fails to intersect $\{r - m_1, \ldots, r - m_k, t - n_1, \ldots, t - n_s\}$. For example, we can choose $\theta_1 = 3$ and $\theta_{j+1} = 6 + \lfloor 3C_4(\theta_j + \log\log y + 1)\rfloor$ for $j \geq 1$. Let $F \in \mathbb{N}$ be the largest integer such that $\theta_F < \frac{r\log 2 - \log 12}{\log 6}$. Recalling that $\log\log y > 1$, we see that

$$\log\log y \leq \theta_2 \leq 6 + 3C_4(4 + \log\log y) \leq 21C_4\log\log y.$$

Subsequently, we have

$$\theta_{j+1} \leq 21C_4\theta_j$$

for $j \geq 2$. Thus we see that for $j \geq 2$,

$$\theta_j \leq (21C_4)^{j-1}\log\log y.$$

In conclusion, we can find

$$F \geq \frac{\log\left(\frac{r\log 2 - \log 12}{\log\log y \log 6}\right)}{\log(21C_4)} \gg \log\log y$$

many points $\theta_1, \ldots, \theta_F \in (2, t)$ such that each $[\theta_j, \theta_{j+1})$ contains an element of

$$\{r - m_1, \ldots, r - m_k, t - n_1, \ldots, t - n_s\}.$$

This completes the proof. □

Inserting this into Theorem 5.2, we obtain the following further refinement of the benchmark Proposition 1.4.

Corollary 5.8 *Assume the Lang–Waldschmidt Conjecture 5.5. For some effectively computable constant $\varepsilon > 0$, if*

$$\sum_{n=1}^{\infty}(n^{-\varepsilon}\psi(2^n)^{\gamma} + \psi(2^n)) < \infty,$$

then $\mu(W_2(\psi)) = 0$.

Remark 5.9 Corollary 5.8 applies in particular to the special case $\varphi_a : 2^n \mapsto n^{-a}$ discussed in the introduction, for some $a < \gamma^{-1}$.

Corollary 5.10 *Assume the Lang–Waldschmidt Conjecture, and let* ε *be a small positive constant. Then for* μ*-almost every* α*, the inequality*

$$\|2^n \alpha\| < n^{\varepsilon - \log 3/\log 2}$$

has at most finitely many solutions $n \in \mathbb{N}$.

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Appendix

Here we present empirical data suggesting that Question 5.1 has a positive answer (Fig. 1).

This was produced using the software *Mathematica* [39], with the following code:

```
M = 10^7;
X_; c_; d_; X = ConstantArray[0, M];
For[n = 2, n < M + 2, n++,
c = 0; d = 0;
v = IntegerDigits[n, 2];
L = Length[v];
For[i = 1, i < L, i++,
If[Part[v, i] != Part[v, i + 1], c++, null]];
v = IntegerDigits[n, 3];
L = Length[v];
For[i = 1, i < L, i++,
```

```
If[Part[v, i] != Part[v, i + 1], d++, null]];
Part[X, n - 1] = (c + d)/Log[n];
]
ListLinePlot[X]
```

We obtain more data by only considering powers of 2 (Fig. 2).

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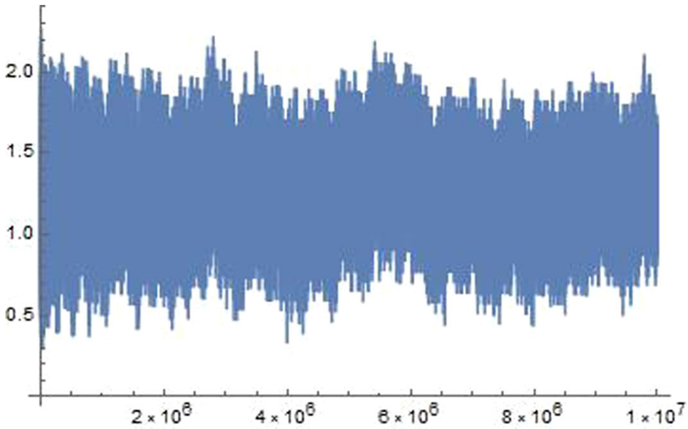

Fig. 1 $(D_2(y) + D_3(y))/\log y$ against y

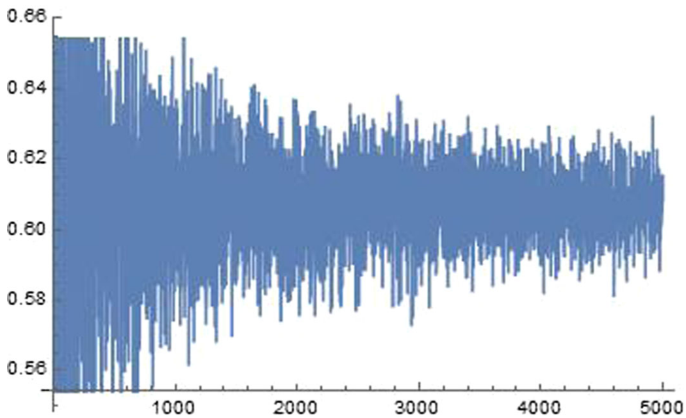

Fig. 2 $D_3(2^y)/(y \log 2)$ against y

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