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Bigrassmannian permutations and Verma modules

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Abstract
We show that bigrassmannian permutations determine the socle of the cokernel of an inclusion of Verma modules in type A. All such socular constituents turn out to be indexed by Weyl group elements from the penultimate two-sided cell. Combinatorially, the socular constituents in the cokernel of the inclusion of a Verma module indexed by $w \in S_n$ into the dominant Verma module are shown to be determined by the essential set of w and their degrees in the graded picture are shown to be computable in terms of the associated rank function. As an application, we compute the first extension from a simple module to a Verma module.

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1 Introduction, motivation and description of results

1.1 Setup

Consider the symmetric group S_n on $\{1, 2, \dots, n\}$ as a Coxeter group with simple reflections s_i, given by the elementary transpositions $(i, i+1)$, where $i = 1, 2, \dots, n-1$. Denote by $\leq$ the corresponding Bruhat order. Given $w \in S_n$, we call i a *left descent* of w if $s_i w < w$, and a *right descent* of w if $ws_i < w$. An element $w \in S_n$ is called *bigrassmannian* provided that it has exactly one left descent, and exactly one right descent. Various aspects of bigrassmannian permutations were studied in [12,13,

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22,23,25,30,31]. The most relevant for this paper is the property that bigrassmannian permutations are exactly the join-irreducible elements of S_n with respect to $\leq$, see [25]. Join-irreducible elements of Weyl groups appear in representation-theoretic context in [18]. Bigrassmannian elements are the first protagonists of the present paper. We denote by $\mathbf{B}_n$ the set of all bigrassmannian permutations in S_n.

Denote by $\mathfrak{g}$ the simple Lie algebra $\mathfrak{sl}_n$ over $\mathbb{C}$ with the standard triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where $\mathfrak{h}$ is the Cartan subalgebra of all (traceless) diagonal matrices and $\mathfrak{n}_+$ is the nilpotent subalgebra of all strictly upper triangular matrices. Consider the BGG category $\mathcal{O}$ associated to this triangular decomposition and its principal block $\mathcal{O}_0$, see [7,16].

The group S_n is the Weyl group of $\mathfrak{g}$, and thus acts naturally on $\mathfrak{h}^*$. We also consider the *dot-action* of S_n on $\mathfrak{h}^*$, that is

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

for $\lambda \in \mathfrak{h}^*$, where ρ is the half of the sum of all positive roots in $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$, we denote by $\Delta(\lambda)$ the Verma module with highest weight λ and by $L(\lambda)$ the unique simple quotient of $\Delta(\lambda)$, see [10,16,38]. The isomorphism classes of simple objects in $\mathcal{O}_0$ are naturally indexed by the elements in S_n as follows: $S_n \ni w \mapsto L(w \cdot 0)$, where 0 denotes the zero element of $\mathfrak{h}^*$. For $w \in S_n$, we set $\Delta_w := \Delta(w \cdot 0)$ and $L_w := L(w \cdot 0)$.

It is well-known, see [10, Chapter 7], that, for $x, y \in S_n$, we have:

- $\dim \mathrm{Hom}_{\mathfrak{g}}(\Delta_x, \Delta_y) \leq 1$,
- a non-zero homomorphism from Δ_x to Δ_y exists if and only if $x \geq y$,
- each non-zero homomorphism from Δ_x to Δ_y is injective.

In particular, all Δ_w, where $w \in S_n$, are uniquely determined submodules of the *dominant Verma module* Δ_e.

The composition multiplicity of L_x in Δ_x can be computed in terms of *Kazhdan–Lusztig (KL) polynomials*, by [2,8,11,19], see Sect. 2.3. The associated *KL combinatorics* divides S_n into subsets, called *two-sided cells*, ordered with respect to the *two-sided order* $\leq_{\mathtt{J}}$. This coincides with the division of S_n given by the Robinson-Schensted correspondence: to each $w \in S_n$, we associate a pair (P_w, Q_w) of standard Young tableaux of the same shape λ which is a partition of n, see [32–34]. The two-sided cells in S_n are indexed by such λ and correspond precisely to the fibers of the map from S_n to the set of all partitions of n induced by the Robinson–Schensted correspondence. Moreover, the two-sided order coincides with the dominance order on partitions, see [15]. The longest element w_0 of S_n forms a two-sided cell which is the maximum with respect to the two-sided order. If we delete this two-sided cell, in what is left there is again a unique maximum two-sided cell, which we denote by $\mathcal{J}$ and call the *penultimate* two-sided cell. The two-sided cell $\mathcal{J}$ is the second protagonist of the present paper. Under the Robinson-Schensted correspondence, it corresponds to the partition

$(2, 1^{n-2})$ of n. Under the involution $x \mapsto xw_0$, the two-sided cell $\mathcal{J}$ corresponds to the two-sided cell which contains all simple reflections, studied in, for example, [20,21]. The Kazhdan–Lusztig cell representation of S_n associated with (any left cell inside) $\mathcal{J}$ is exactly the representation of S_n on $\mathfrak{h}^*$. For each $i, j \in \{1, 2, \ldots, n-1\}$, the cell $\mathcal{J}$ contains a unique element w such that $s_i w > w$ and $ws_j > w$. We denote this w by $w_{i,j}$. The image of $w_{i,j}$ under the involution $x \mapsto xw_0$ is the unique element $\tilde{w}_{i,j}$ of S_n with left decent s_i and right descent $w_0 s_j w_0$. The element $\tilde{w}_{i,j}$ is just the product of simple reflections along the unique path in the Dynkin diagram from the vertex corresponding to s_i to the vertex corresponding to $w_0 s_j w_0$, written from left to right.

1.2 Motivation

The original motivation for this paper comes from a question by Sascha Orlik and Matthias Strauch which is discussed in more detail in the next subsection. A very special case of that question leads to the problem of determining $\mathrm{Ext}^1_{\mathcal{O}}(L_x, \Delta_y)$ in the case $x \neq w_0$. In the case $x = w_0$, we note that $L_{w_0} = \Delta_{w_0}$ and hence the computation of $\mathrm{Ext}^1_{\mathcal{O}}(L_x, \Delta_y)$ can be reduced, using twisting functors, to [28, Theorem 32] (see the proof of Corollary 2). The case $x \neq w_0$ requires new techniques and is completed in Corollary 2.

1.3 Description of the main results

The main result of this paper is the following theorem which, in particular, reveals a completely unexpected connection between $\mathbf{B}_n$ and $\mathcal{J}$.

Theorem 1 (i) *For $w \in S_n$, the module Δ_e/Δ_w has simple socle if and only if $w \in \mathbf{B}_n$.*

(ii) *The map $\mathbf{B}_n \ni w \mapsto \mathrm{soc}(\Delta_e/\Delta_w)$ induces a bijection between $\mathbf{B}_n$ and simple subquotients of Δ_e of the form L_x, where $x \in \mathcal{J}$.*

(iii) *For $w \in S_n$, the simple subquotients of Δ_e/Δ_w of the form L_x, where $x \in \mathcal{J}$, correspond under the bijection from (ii), to $y \in \mathbf{B}_n$ such that $y \leq w$.*

(iv) *For $w \in S_n$, the socle of Δ_e/Δ_w consists of all L_x, where $x \in \mathcal{J}$, which correspond, under the bijection from (ii), to the Bruhat maximal elements in $\{y \in \mathbf{B}_n \,:\, y \leq w\}$.*

The bijection from Theorem 1(ii) is explicitly given in Sect. 4.2. One of the ways to think about this bijection is as follows: each graded simple module $\mathrm{soc}(\Delta_e/\Delta_w)$, where $w \in \mathbf{B}_n$, has multiplicity one in Δ_e (see Proposition 12), moreover, the images of these $\mathrm{soc}(\Delta_e/\Delta_w)$ in the graded Grothendieck group of $\mathcal{O}_0$ are linearly independent.

Theorem 1 provides a categorical, or, alternatively, a representation theoretic interpretation of the poset $\mathbf{B}_n$. We note that this interpretation is completely different from the one in [18]. The crucial step towards the formulation of Theorem 1 was an accidental numerical observation which can be found in Corollary 13.

For $x \in S_n$, denote by $\ell(x)$ the *length* of x (as an element of a Coxeter group), and by $\mathbf{c}(x)$, the *content* of x, that is the number of different simple reflections appearing in a reduced expression of x (this number does not depend on the choice of a reduced expression). Denote by $\Phi : \mathbf{B}_n \to \mathcal{J}$ the map given by $\Phi(w) = w_{i,j}$, if $w \in \mathbf{B}_n$ is such that $s_i w < w$ and $ws_j < w$. For $x \in S_n$, denote by $\mathbf{BM}(x)$ the set of all Bruhat maximal elements in the set $\mathbf{B}(x) := \{y \in \mathbf{B}_n : y \leq x\}$. For $w \in S_n$, we denote by ∇_x the *dual Verma module* obtained from Δ_x by applying the simple preserving duality on $\mathcal{O}_0$, see [16].

Theorem 1, combined with [28, Theorem 32], has the following homological consequence:

Corollary 2 *Let $x, y \in S_n$. Then we have*

$$\dim \operatorname{Ext}^1_{\mathcal{O}}\left(L_x, \Delta_y\right) = \dim \operatorname{Ext}^1_{\mathcal{O}}\left(\nabla_y, L_x\right) = \begin{cases} \mathbf{c}(xy), & x = w_0; \\ 1, & x \in \Phi(\mathbf{BM}(y)); \\ 0, & \textit{otherwise}. \end{cases}$$

Theorem 1 and Corollary 2 are proved in Sect. 2. Both Theorem 1 and Corollary 2 extend to singular blocks of $\mathcal{O}$, which we prove in Theorem 16 and Proposition 15 in Sect. 3.

The starting point of this paper was a question the second author received from Sascha Orlik and Matthias Strauch, namely, whether, for $x, y \in S_n$ such that $x < y$, the space $\operatorname{Ext}^1_{\mathcal{O}}(\Delta_y/\Delta_{w_0}, \Delta_x)$ can be non-zero?

From Corollary 2, it follows that the answer is "yes". For example, for the algebra $\mathfrak{sl}_4$, one can take $y = s_2 w_0$, in which case $\Delta_y/\Delta_{w_0} \cong L_y$, and $x = s_2$. By Corollary 2, we have $\operatorname{Ext}^1_{\mathcal{O}}(L_y, \Delta_x) = \mathbb{C}$.

1.4 Description of additional results

In Sect. 4, we relate the main results with a number of combinatorial tools. In Sect. 4.1 we explain how the combinatorics and the Hasse diagram of the poset $\mathbf{B}_n$ appear naturally in our representation theoretic interpretation of this poset. In Sect. 4.3 we establish representation theoretic significance of the notions of the essential set of a permutation and the associated rank function. In particular, in Corollary 19 we show that the socular constituents in the cokernel of the inclusion of a Verma module indexed by $w \in S_n$ into the dominant Verma module are determined by the essential set of w. Further, in Proposition 22 we show how the associated rank function can be used to compute the degrees of these simple socular constituents in the graded picture.

2 Proofs of the main results

2.1 Category $\mathcal{O}$ tools

Denote by $\underline{\mathcal{O}}_0$ the full subcategory of $\mathcal{O}_0$ which consists of all modules on which the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ acts diagonalizably. Note that all Verma modules and all simple modules in $\mathcal{O}_0$ are objects in $\underline{\mathcal{O}}_0$. By [35], the category $\underline{\mathcal{O}}_0$ has an auto-equivalence, denoted by Θ, which maps L_x to $L_{x^{-1}}$, for $x \in S_n$.

Consequently, $\Theta(\Delta_x) \cong \Delta_{x^{-1}}$, for $x \in S_n$. We note that Θ does **not** extend to the whole of $\mathcal{O}_0$.

For $x \in S_n$, we have the corresponding *twisting functor* T_x on $\mathcal{O}_0$, see [1], and the corresponding *shuffling functor* C_x on $\mathcal{O}_0$, see [9,29]. Both $\mathcal{L}T_x$ and $\mathcal{L}C_x$ are self-equivalences of the bounded derived category $\mathcal{D}^b(\mathcal{O})$. In the proof below we usually use twisting functors and Θ, however, one can, alternatively, use twisting and shuffling functors or shuffling functors and Θ.

The category $\mathcal{O}_0$ is equivalent to the category of finite dimensional modules over a finite dimensional basic algebra, which we denote by A. The algebra A is unique, up to isomorphism. By [36], it is Koszul and hence admits a Koszul $\mathbb{Z}$-grading. We denote by ${}^{\mathbb{Z}}\mathcal{O}_0$ the category of finite dimensional $\mathbb{Z}$-graded A-modules. We denote by $\langle k\rangle$ the grading shift on ${}^{\mathbb{Z}}\mathcal{O}_0$ normalized such that it maps degree k to degree zero, and we fix the standard graded lifts of L_w concentrated in degree zero, and of Δ_w such that its top is concentrated in degree zero.

2.2 Potential socle of the cokernel of an inclusion of Verma modules

Proposition 3 *Let $x, y, z \in S_n$ be such that $x \geq y$ and L_z is in the socle of Δ_y/Δ_x. Then $z \in \mathcal{J}$.*

Proof As Δ_y/Δ_x is a submodule of Δ_e/Δ_x, it is enough to prove the claim in the case $y = e$.

Consider first the case $x = w_0$. Then Δ_{w_0} is the socle of Δ_e and we claim that the socle of Δ_e/Δ_{w_0} consists of all L_x, where $x \in W$ is such that $\ell(x) = \ell(w_0) - 1$. The best way to argue that the socle of Δ_e/Δ_{w_0} is as described above is to recall that Δ_e is rigid, see [3,17]. This means that the socle filtration and the radical filtration of Δ_e coincide. Moreover, these two filtrations also coincide with the graded filtration, see [3, Proposition 2.4.1]. The submodule $\Delta_{w_0} = L_{w_0}$ in Δ_e lives in degree $\ell(w_0)$. By rigidity, the socle of Δ_e/Δ_{w_0} consists of simple modules which live in degree $\ell(w_0) - 1$. Since, by KL combinatorics, a simple module L_x appears in Δ_e only in degrees $\leq \ell(x)$ and L_{w_0} has multiplicity one, the only simples in degree $\ell(w_0) - 1$ are those L_x for which $\ell(x) = \ell(w_0) - 1$. Note also that all such x belong to $\mathcal{J}$. This proves the claim of the proposition in the case $x = w_0$.

For $x \neq w_0$, using the self-equivalence $\mathscr{L}T_{w_0x^{-1}}$, we have

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\left(L_z, \Delta_e/\Delta_x\right) \cong \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\left(\mathscr{L}T_{w_0x^{-1}}(L_z), \Delta_{w_0x^{-1}}/\Delta_{w_0}\right). \tag{1}$$

By the previous two paragraphs, the socle of $\Delta_{w_0x^{-1}}/\Delta_{w_0}$ consist of the simples L_w, where $w \in \mathcal{J}$. Therefore, for the right hand side of (1) to be non-zero, the module $T_{w_0x^{-1}}(L_z)$ must contain some simple subquotient of the form L_w, where $w \in \mathcal{J}$. From [1, Theorem 6.3], it follows that all simple subquotients of $T_{w_0x^{-1}}(L_z)$ are of the form L_u, where $u \leq_{\mathtt{J}} z$. This yields $z \in \mathcal{J}$, completing the proof. □

Proposition 4 *Let $x \in S_n$ and s be a simple reflection.*

(i) *If $sx < x$, then the socle of Δ_e/Δ_x contains some L_y such that $sy > y$.*
(ii) *If $xs < x$, then the socle of Δ_e/Δ_x contains some L_y such that $ys > y$.*

Proof Due to the assumption $sx < x$, we have $\Delta_x \subset \Delta_{sx}$, in particular, the socle of Δ_e/Δ_x contains the socle of Δ_{sx}/Δ_x. The module Δ_{sx}/Δ_x is non-zero and, by construction, s-finite (i.e. the action on this module of the $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ corresponding to s is locally finite). In particular, any L_y in the socle of Δ_{sx}/Δ_x is also s-finite. Therefore $sy > y$ for each y such that L_y is in the socle of Δ_{sx}/Δ_x. Claim (i) follows. Claim (ii) follows from claim (i) using Θ. □

Corollary 5 *Let $w \in S_n$ be such that the socle of Δ_e/Δ_w is simple. Then w is a bigrassmannian permutation.*

Proof Assume that s and t are different simple reflections such that $sw < w$ and $tw < w$. By Proposition 4, the socle of Δ_e/Δ_w contains some L_y such that $sy > y$ and some L_z such that $tz > z$. Both $y, z \in \mathcal{J}$. Note that, for each element u of $\mathcal{J}$, there is at most one simple reflection r such that $ru > u$. This means that $y \neq z$ and hence the socle of Δ_e/Δ_w is not simple. A similar argument works in the case when there are different simple reflections s and t such that $ws < w$ and $wt < w$. The claim follows. □

Proposition 6 *Assume that $x \in S_n$ and $y \in \mathcal{J}$ be such that L_y is in the socle of Δ_e/Δ_x. Assume that s is a simple reflection such that $sy > y$. Then $sx < x$.*

Proof Assume $sx > x$. Applying $\mathscr{L}T_s$ and using [1, Theorem 6.1], similarly to (1), we have

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\left(L_y, \Delta_e/\Delta_x\right) \cong \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\left(L_z[1], \Delta_s/\Delta_{sx}\right).$$

The right hand side of this equality is 0 since Δ_s/Δ_{sx} is a module in homological position 0 while $L_z[1]$ is a module in homological position -1. The claim follows. □

2.3 Combinatorial tools

Consider the Hecke algebra $\mathbb{H}_n$ associated to the Coxeter system (S_n, S), where S is the set $\{s_1, \ldots, s_{n-1}\}$ of simple reflections. $\mathbb{H}_n$ is a $\mathbb{Z}[v, v^{-1}]$-algebra generated by H_i, for $1 \leq i \leq n-1$, which satisfy the braid relations

$$H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1},$$

for all $1 \leq i \leq n-2$, and the quadratic relation

$$(H_i + v)\left(H_i - v^{-1}\right) = 0,$$

for all $1 \leq i \leq n-1$. Given a reduced expression $w = s_i s_j \cdots s_k$ of $w \in S_n$, we let $H_w = H_i H_j \cdots H_k$. The element H_w is independent of the choice of the reduced expression, and $\{H_w : w \in S_n\}$ is a ($\mathbb{Z}[v, v^{-1}]$-)basis of $\mathbb{H}_n$ called the *standard basis*.

Consider the ($\mathbb{Z}$-algebra-)involution on H_n denoted by a bar, such that $\overline{v} = v^{-1}$ and $\overline{H_i} = H_i^{-1}$. Then there is a unique element $\underline{H}_w$ in $\mathbb{H}_n$ such that $\overline{\underline{H}_w} = \underline{H}_w$ and

$$\underline{H}_w = H_w + \sum_y p_{y,w} H_y,$$

for some $p_{y,w} \in v\mathbb{Z}[v]$. The elements $\underline{H}_w$, where $w \in W$, form a basis of $\mathbb{H}_n$ called the *Kazhdan–Lusztig (KL) basis*. Given $w, y \in S_n$, the coefficient of v in $p_{w,y} + p_{y,w}$ is denoted by $\mu(w, y) = \mu(y, w)$, defining the *(Kazhdan–Lusztig)*, *μ-function*. If $s \in S$ then $\underline{H}_s = H_s + v$ and we have

$$\underline{H}_s \underline{H}_y = (v + v^{-1}) \underline{H}_y, \tag{2}$$

for $sy < y$, and

$$\underline{H}_s \underline{H}_y = \underline{H}_{sy} + \sum_{sx<x, x<y} \mu(x, y) \underline{H}_x, \tag{3}$$

for $sy > y$. Another basic fact is that

$$p_{x,w_0} = v^{\ell(w_0)-\ell(x)}. \tag{4}$$

For more details about KL basis we refer to [19].

A consequence of the Kazhdan–Lusztig conjecture (which is now a theorem, see [2,8,11,19]), is that the polynomial $p_{x,y}$ above describes the composition multiplicities of the Verma modules in ${}^{\mathbb{Z}}\mathcal{O}_0$ in the following sense:

$$\sum_{i\in\mathbb{Z}} [\Delta_w : L_y\langle -i\rangle] v^i = p_{w,y}.$$

From the Kazhdan–Lusztig conjecture/theorem, combined with the Koszul self-duality of $\mathcal{O}_0$ from [3,36], it follows that the integer $\mu(w, y)$ equals the dimension of the first extension space between L_w and L_y (and hence determines the Gabriel quiver of the algebra A).

We denote by $\mathbf{a}\colon W \to \mathbb{Z}_{\geq 0}$ Lusztig's $\mathbf{a}$-function, see [27]. The value $\mathbf{a}(w)$ does not change when w varies over a two-sided cell. We write $\mathbf{a}(\mathcal{J}) = \mathbf{a}(w)$ for $w \in \mathcal{J}$. By definition, the value $\mathbf{a}(w)$ for $w \in \mathcal{J}$ describes the minimal possible degree shift for simple subquotients L_u, with $u \in \mathcal{J}$, of Δ_e (i.e. the degree shift for the top layer of the tetrahedron on Fig. 2). The minimal shift is achieved exactly when w is a Duflo element. It follows that

$$p_{e,w} \in \mathbb{Z}\{v^a \mid \mathbf{a}(w) \leq a \leq \ell(w)\} \tag{5}$$

where the first equality holds if and only if w is a Duflo element. Since $\mathcal{J}$ contains $w_{1,1}$, which is the longest element of some parabolic subgroup of S_n, we have $\mathbf{a}(\mathcal{J}) = \ell(w_{1,1}) = \frac{(n-1)(n-2)}{2}$.

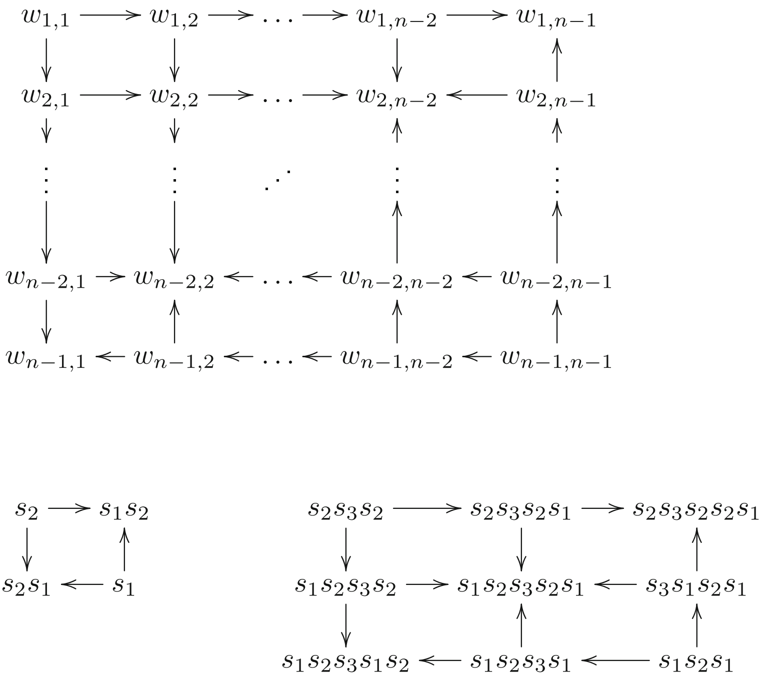

Fig. 1 Bruhat graph of the penultimate cell (the rows are right, and the columns are left cells) with the explicit special cases $n = 3, 4$

Proposition 7 *Let $x, y \in \mathcal{J}$. Then $\mu(x, y) \neq 0$ if and only if x, y are adjacent in the Bruhat graph. In the latter case, we have $\mu(x, y) = 1$.*

Proof By the Kazhdan–Lusztig inversion formula, see [19], (or, equivalently, by Koszul duality, see [3]), we have $\mu(x, y) = \mu(w_0 x^{-1}, w_0 y^{-1})$. Now, the claim follows from the similar fact about the small two-sided cell (see [26, Proposition 3.8 (d)] or [20,21]). □

For convenience of the reader, on Fig. 1 we give the Bruhat graph of $\mathcal{J}$ which is also illustrated by the explicit special for $n = 2, 3, 4$.

Lemma 8 *Suppose $y = w_{i,j} \in \mathcal{J}$, with $i \neq j$. Then*

$$p_{s,y} = v^{-1} p_{e,y}, \quad \text{for all } s \in S. \tag{6}$$

Proof Suppose $s \neq s_i$. Then, comparing the H_e-coefficients in (2), gives (6). If $s = s_i$, then we have $s \neq s_j$, so comparing the H_e-coefficients in the equation $\underline{H}_y \underline{H}_s = (v + v^{-1}) \underline{H}_y$ gives (6). □

Lemma 9 *Suppose $y = w_{i,j} \in \mathcal{J}$, with $i \neq j$, $s \in S$ and $sy > y$, we have*

$$(v + v^{-1})p_{e,y} = p_{e,sy} + \sum_{\substack{u \in S \\ uy \sim_{\mathrm{L}} y \\ uy < y}} p_{e,uy}. \tag{7}$$

Proof Consider Equation (3), for $y \in \mathcal{J}$. Since all x appearing in this equation with non-zero coefficient satisfy, at the same time, $x < y$ and $x \geq_{\mathrm{L}} y$, and, since $y \in \mathcal{J}$, we have $x \sim_{\mathrm{L}} y$. Now, comparing the H_e-coefficients on both sides in Equation (3), we get

$$(v + v^{-1})p_{e,y} = p_{s,y} + vp_{e,y} = p_{e,sy} + \sum_{\substack{x \sim_{\mathrm{L}} y \\ x < y}} \mu(x, y)p_{e,x}, \tag{8}$$

where we used (6) for the first equality. Then Proposition 7 reduces (8) to (7). □

Proposition 10 *Let $y = w_{1,1}$ or $y = w_{n-1,n-1}$. Then $p_{e,y} = v^{\ell(y)}$.*

Proof The element $w_{n-1,n-1}$ is the longest element for the parabolic subgroup of S_n generated by $\{1, \dots, n-2\}$. The element $w_{1,1}$ is the longest element for the parabolic subgroup of S_n generated by $\{2, \dots, n-1\}$. Since the KL basis can be computed in the parabolic subgroups, the claim of the lemma follows from (4) applied to the Coxeter group S_{n-1} with the corresponding identification of simple reflections. □

For $i, j \in \{0, \dots, n\}$, put $p_{i,j} := p_{e,w_{i,j}}$ if $1 \leq i, j \leq n-1$, and $p_{i,j} := 0$ otherwise.

Lemma 11 *Suppose $w_{i,j} \in \mathcal{J}$, with $i \neq j$. Then*

$$\left(v + v^{-1}\right) p_{i,j} = p_{i-1,j} + p_{i+1,j} + \delta_{i,n-j} \cdot v^{\ell(w_0)}, \tag{9}$$

where δ denotes the Kronecker symbol.

Proof This follows from (7) with $s = s_i$, (4) and Fig. 1. □

Proposition 12 *Let $w = w_{i,j} \in \mathcal{J}$. We have*

$$p_{e,w} = v^{\ell(w)} + v^{\ell(w)-2} + \dots + v^{\ell(w)-2d(w)}, \tag{10}$$

where

$$d(w) = \min\{i-1, j-1, n-1-i, n-1-j\}.$$

Proof We proceed by induction on $d = d(w)$.

Let $d = 0$. By symmetry (flipping the Dynkin diagram on one hand, and taking inverses of permutations on the other hand), it is enough to consider $w = w_{i,1}$. If $i = 1$, then Proposition 10 gives (10). If $i = 2$, then (9) gives

$$(v + v^{-1})p_{2,1} = v^a + p_{3,1},$$

where $a := \mathbf{a}(\mathcal{J})$. Since $\ell(w_{2,1}) = a + 1$, by (5), we have $p_{2,1} = cv^{a+1}$ for some $c \in \mathbb{Z}$. For the same reason, $p_{3,1}$ does not have term proportional to v^a, so we get $p_{2,1} = v^{a+1}$ and $p_{3,1} = v^{a+2}$. From this we can obtain (10) for any $w_{i,1}$, using (9) and a two-step induction.

Let $d = d(w) > 0$. By symmetry, we may assume $w = w_{d+1,j}$, with $d < j < n - d$. We assume that (10) is true for $w_{d,1}$ and (if exists) for $w_{d-1,1}$. Equation (9) applied to $w = w_{d,j}$ gives

$$\left(v + v^{-1}\right) p_{d,j} = p_{d-1,j} + p_{d+1,j}.$$

From this, it easily follows that (10) is also true for $w_{d+1,j}$. □

For $i, j \in \{1, 2, \ldots, n-1\}$, we denote by $\mathbf{B}_n^{(i,j)}$ the set of all $w \in \mathbf{B}_n$ such that $s_i w < w$ and $w s_j < w$.

Corollary 13 *We have*

$$|\mathbf{B}_n| = \sum_{w \in \mathcal{J}} p_{e,w}(1), \tag{11}$$

moreover, for each $i, j \in \{1, 2, \ldots, n-1\}$, we have $|\mathbf{B}_n^{(i,j)}| = p_{e,w_{i,j}}(1)$.

Proof By Proposition 12, the right hand side of (11) evaluates to the tetrahedral number $\frac{(n-1)n(n+1)}{6}$. That the left hand side of (11) is given by the same number follows from the main result of [23]. The refined version, for each $i, j \in \{1, 2, \ldots, n-1\}$, follows from the main result of [12]. Both statements are best seen by comparing the main result of [12] to the picture described in Sect. 4.1, where the right hand side (11) is realized as a tetrahedron in $\mathbb{R}^3$ and the elements corresponding to $\mathbf{B}_n^{(i,j)}$ consist of all elements of this tetrahedron aligned along a vertical line. □

Proposition 14 *Let $i, j \in \{1, 2, \ldots, n-1\}$. The restriction of the Bruhat order to each $\mathbf{B}_n^{(i,j)}$ is a chain, moreover, for $x, y \in \mathbf{B}_n^{(i,j)}$ such that $x < y$, there exist $w \in \mathbf{B}_n \setminus \mathbf{B}_n^{(i,j)}$ such that $x < w < y$.*

Proof This follows directly from the main result of [12]. The poset $\mathbf{B}_n$ can be realized as a tetrahedron in $\mathbb{R}^3$, see Sect. 4.1. The subset $\mathbf{B}_n^{(i,j)}$ consist of all elements of this tetrahedron aligned along a vertical line. If such a line contains more than one element of the tetrahedron, we have a diamond shaped subset of $\mathbf{B}_n$ and the elements of this diamond outside the original line provide the necessary w, see for example the points (2, 2, 5) and (2, 2, 3), in the case $n = 4$, where w can be chosen as any of the elements (2, 1, 4), (1, 2, 4), (3, 2, 4) or (2, 3, 4). □

2.4 Proof of Theorem 1(i), (ii)

Corollary 5 gives one direction of (i). For the other direction, we need that, for any $w \in \mathbf{B}_n$, the module Δ_e/Δ_w has simple socle. Assume that s_i and s_j are such that $s_i w < w$ and $ws_j < w$. Then, by Proposition 6, $L_{w_{i,j}}$ is the only possible simple subquotient in the socle of Δ_e/Δ_w. We need to prove that the multiplicity of $L_{w_{i,j}}$ in the socle of Δ_e/Δ_w equals 1.

By Proposition 14, there is a chain

$$w_1 < u_1 < w_2 < u_2 < \cdots < w_{k-1} < u_{k-1} < w_k,$$

where $\mathbf{B}_n^{(i,j)} = \{w_1, w_2, \ldots, w_k\}$ and all $u_l \in \mathbf{B}_n \backslash \mathbf{B}_n^{(i,j)}$. This chain gives an a sequence of inclusions

$$\Delta_e \supsetneq \Delta_{w_1} \supsetneq \Delta_{u_1} \supsetneq \Delta_{w_2} \supsetneq \Delta_{u_2} \supsetneq \cdots \supsetneq \Delta_{w_k}$$

which, in turn, gives rise to a sequence of projections

$$\Delta_e/\Delta_{w_1} \twoheadleftarrow \Delta_e/\Delta_{u_1} \twoheadleftarrow \Delta_e/\Delta_{w_2} \twoheadleftarrow \Delta_e/\Delta_{u_2} \twoheadleftarrow \cdots \twoheadleftarrow \Delta_e/\Delta_{w_k}.$$

This implies that the socle of each Δ_e/Δ_{w_l} contains a summand which is not a summand of the socle of Δ_e/Δ_{w_m}, for any $m < l$. From Corollary 13 we obtain that $k = p_{e,w_{i,j}}(1)$ and hence the socle of each Δ_e/Δ_{w_l} contains a unique summand which is not in the socle of any Δ_e/Δ_{w_m}, for $m < l$.

Therefore, for both claims (i) and (ii), it is enough to prove that the socle of each Δ_e/Δ_{w_l} is simple. We argue that the socle constituents of any Δ_e/Δ_{w_m}, where $m < l$, cannot contribute to the socle of Δ_e/Δ_{w_l}. Assume that this is not the case and some socle constituent of some Δ_e/Δ_{w_m} contributes to the socle of Δ_e/Δ_{w_l}. Then it also must contribute to the socle of $\Delta_e/\Delta_{u_{l-1}}$, since $w_m < u_{l-1} < w_l$. But this contradicts Proposition 6 and completes the proof.

2.5 Proof of Theorem 1(iii) and (iv)

Both statements follow from the bijection given in Theorem 1(ii) and the fact that $\Delta_x \subset \Delta_y$ is equivalent to $x \geq y$.

2.6 Proof of Corollary 2

First of all, we note that the equality

$$\dim \mathrm{Ext}^1_{\mathcal{O}}\left(L_x, \Delta_y\right) = \dim \mathrm{Ext}^1_{\mathcal{O}}\left(\nabla_y, L_x\right)$$

follows by applying the simple preserving duality on $\mathcal{O}_0$. Therefore we only need to prove the following:

$$\dim \operatorname{Ext}^1_{\mathcal{O}}\left(L_x, \Delta_y\right) = \begin{cases} \mathbf{c}(xy), & x = w_0; \\ 1, & x \in \Phi(\mathbf{BM}(y)); \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Consider first the case $x = w_0$. In this case $L_{w_0} = \Delta_{w_0}$. Applying the twisting functor T_y and using the fact that it is acyclic on Verma modules, see [1, Theorem 2.2], we observe that

$$\operatorname{Ext}^1_{\mathcal{O}}\left(\Delta_{y^{-1}w_0}, \Delta_e\right) = \operatorname{Ext}^1_{\mathcal{O}}\left(\Delta_{w_0}, \Delta_y\right).$$

By [28, Theorem 32], the dimension of $\operatorname{Ext}^1_{\mathcal{O}}(\Delta_{y^{-1}w_0}, \Delta_e)$ equals $\mathbf{c}(y^{-1}w_0) = \mathbf{c}(xy)$. This establishes (12) in the case $x = w_0$.

Assume now that $x \neq w_0$. Let

$$0 \to \Delta_y \to M \to L_x \to 0 \tag{13}$$

be a non-split short exact sequence. Since L_x is a simple object and (13) is non-split, the socle of M coincides with the socle of Δ_y and hence is isomorphic to L_{w_0}. In particular, M is a submodule of the injective envelope I_{w_0} of L_{w_0}. As the multiplicity of L_{w_0} in M is one, all nilpotent endomorphisms of I_{w_0} send M to 0. Since Δ_e, being a projective object in $\mathcal{O}$, is copresented by projective-injective objects in $\mathcal{O}$ (see [24, § 3.1]) and since I_{w_0} is the only indecomposable projective-injective object in $\mathcal{O}_0$, it follows that M is a submodule of Δ_e. This means that $L_x \cong M/\Delta_y$ corresponds to a socle constituent of Δ_e/Δ_y. Recall that, for any module M and any simple module L, the dimension of $\operatorname{Ext}^1(L, M)$ coincides with the multiplicity of L in the socle of I/M, where I is a minimal injective envelope of M. Combining all these observations together, we have that, for $x \neq w_0$, formula (12) follows from Theorem 1(iv).

3 Singular blocks

In this subsection we extend our results on $\mathcal{O}_0$ to an arbitrary block $\mathcal{O}_\mu$ (that is, the Serre subcategory of $\mathcal{O}$ with the simple objects $L(w \cdot \mu)$, for w in the μ-integral subgroup $S_n^{(\mu)}$ of S_n) thus to the entire category $\mathcal{O}$. Let $\mu \in \mathfrak{h}^*$ be an $S_n^{(\mu)}$-dominant weight. The first, standard, remark is that, up to equivalence with a block for some other n, it is enough to consider the case when μ is integral and hence $S_n^{(\mu)} = S_n$, see Soergel's combinatorial description of $\mathcal{O}$ in [36]. Therefore, from now on we assume μ to be integral.

Note that, if $I = \{i \in \{1, \dots, n-1\} \mid s_i \cdot \mu = \mu\}$ and S_I is the parabolic subgroup of S_n generated by I, then $S_I \cong S_\mu := \operatorname{Stab}_W(\mu)$. So, we have $L(w \cdot \mu) \cong L(wz \cdot \mu)$ and $\Delta(w \cdot \mu) \cong \Delta(wz \cdot \mu)$, for any $z \in S_\mu$. Given $w \in S_n$, we denote by $\underline{w}$ and $\overline{w}$ the unique shortest and the unique longest coset representatives for the coset wS_μ in

S_n/S_μ. We have the indecomposable projective functors

$$\theta_\mu^0 : \mathcal{O}_0 \to \mathcal{O}_\mu \qquad \text{and} \qquad \theta_0^\mu : \mathcal{O}_\mu \to \mathcal{O}_0$$

called *translation to the* μ*-wall* and *translation out of the* μ*-wall*, respectively. (These functors are sometimes denoted by T_0^μ and T_μ^0, respectively, e.g., in [16].) They are biadjoint, exact and determined uniquely, up to isomorphism, by $\theta_\mu^0 \Delta_e = \Delta_\mu$ and $\theta_0^\mu \Delta_\mu = P_{\overline{e}}$, respectively, see [5]. In particular, from the biadjointness it follows that the action of the functor θ_μ^0 on simple modules is given, for $w \in S_n$, by

$$\theta_\mu^0 L_w \cong \begin{cases} L(w \cdot \mu), & w = \overline{w}; \\ 0, & \text{otherwise.} \end{cases}$$

On the level of the Grothendieck group, the functor $\theta_0^\mu \theta_\mu^0$ corresponds to the right multiplication with the sum of all elements in S_μ, see [5, § 3.4].

Proposition 15 *Let* $x, y \in S_n$ *be such that* $x < y$*. Then we have*

$$\operatorname{soc}(\Delta(x \cdot \mu)/\Delta(y \cdot \mu)) \cong \theta_\mu^0 \left(\operatorname{soc} \Delta_{\underline{x}}/\Delta_{\underline{y}} \right).$$

Proof We may assume $x = \underline{x}$ and $y = \underline{y}$.

Suppose L_w is a socle component of Δ_x/Δ_y. Then, by exactness, the module $\theta_\mu^0 L_w$, whenever it is nonzero, that is, whenever $w = \overline{w}$, is a socle component of

$$\theta_\mu^0 \left(\Delta_x/\Delta_y\right) \cong \left(\theta_\mu^0 \Delta_x\right) / \left(\theta_\mu^0 \Delta_y\right) \cong \Delta(x \cdot \mu)/\Delta(y \cdot \mu).$$

Now, assume that $L(w \cdot \mu)$ is a socle component of $\Delta(x \cdot \mu)/\Delta(y \cdot \mu)$. Then we have

$$\operatorname{Ext}^1_{\mathcal{O}}(L(w \cdot \mu), \Delta(y \cdot \mu)) \neq 0.$$

By adjunction, we also have

$$\operatorname{Ext}^1_{\mathcal{O}}(L(w \cdot \mu), \Delta(y \cdot \mu)) \cong \operatorname{Ext}^1_{\mathcal{O}} \left(\theta_\mu^0 L_{\overline{w}}, \Delta(y \cdot \mu)\right) \cong \operatorname{Ext}^1_{\mathcal{O}} \left(L_{\overline{w}}, \theta_0^\mu \Delta(y \cdot \mu)\right).$$

As θ_0^μ is a projective functor, $\theta_0^\mu \Delta(y \cdot \mu)$ has a Verma flag. As, on the level of the Grothendieck group, $\theta_0^\mu \theta_\mu^0$ corresponds to the right multiplication with the sum of all elements in S_μ, the Verma flag of $\theta_0^\mu \Delta(y \cdot \mu)$ has subquotients Δ_{yz}, where $z \in S_\mu$, each appearing with multiplicity one. It follows that $\operatorname{Ext}^1_{\mathcal{O}}(L_{\overline{w}}, \Delta_{yz}) \neq 0$, for some $z \in S_\mu$, which means $\overline{w} \in \mathcal{J}$ by Corollary 2. Moreover, we obtain that $L_{\overline{w}}$ appears in the socle of Δ_e/Δ_{yz}. Since $\theta_\mu^0 L_{\overline{w}} \neq 0$ while $\theta_\mu^0(\Delta_y/\Delta_{yz}) = 0$, the subquotient $L_{\overline{w}}$ is also contained in the socle of Δ_e/Δ_y. To see that $L_{\overline{w}}$ is in the socle of Δ_x/Δ_y, we note that Δ_e/Δ_x cannot contain this subquotient in the socle, for in that case

$\theta_\mu^0(\Delta_e/\Delta_x) \cong \Delta(\mu)/\Delta(x\cdot\mu)$ would contain $L(w\cdot\mu)$ in the socle, contradicting our assumption (note that the graded multiplicities of simples from $\mathcal{J}$ in Δ_e, as well as of the corresponding translated simples in $\Delta(\mu)=\theta_\mu^0\Delta_e$ are one or zero, so we can distinguish all such simple subquotients). The proof is complete. □

Theorem 16 *Let $x, y \in S_n$ and let μ be an integral, dominant weight. Then we have*

$$\dim \operatorname{Ext}^1_{\mathcal{O}}(L(x\cdot\mu), \Delta(y\cdot\mu)) = \dim \operatorname{Ext}^1_{\mathcal{O}}(\nabla(y\cdot\mu), L(x\cdot\mu)) = \begin{cases} \mathbf{c}(\overline{x}\underline{y}) - |I|, & \overline{x} = w_0; \\ 1, & \overline{x} \in \Phi(\mathbf{BM}(\underline{y})); \\ 0, & \textit{otherwise}. \end{cases}$$

Proof Similarly to the proof of Corollary 2, the proof of Proposition 15 takes care of the case $\overline{x} \neq w_0$.

For the case $\overline{x} = w_0$, we need to generalize [28, Theorem 32], which is proved in the regular setup, to singular blocks. So, let $x = w_0$. We may assume $y = \underline{y}$. For the proof, we work in the graded category ${}^{\mathbb{Z}}\mathcal{O}$, where hom- and ext-functors are, as usual, denoted by hom and $\operatorname{ext}^i_{\mathcal{O}}$. The precise claim in this case is

$$\dim \operatorname{ext}^1_{\mathcal{O}}\left(L\left(w_0\cdot\mu\right), \Delta(y\cdot\mu)\langle i\rangle\right) = \begin{cases} \mathbf{c}\left(w_0 y\right) - |I|, & \text{if } i = \ell(w_0 y) - 2; \\ 0, & \text{otherwise}. \end{cases} \tag{14}$$

If $\mu = 0$, then, similarly to the proof of Corollary 2, Formula (14) reduces to [28, Theorem 32] using twisting functors.

In the general case, we first note that $L(w_0\cdot\mu) \cong \theta_\mu^0 L_{w_0}$ where we now use the graded translation functor $\theta_\mu^0 : {}^{\mathbb{Z}}\mathcal{O}_0 \to {}^{\mathbb{Z}}\mathcal{O}_\mu$, see [37]. By adjunction, this gives

$$\operatorname{ext}^1_{\mathcal{O}}\left(L\left(w_0\cdot\mu\right), \Delta(y\cdot\mu)\langle i\rangle\right) \cong \operatorname{ext}^1_{\mathcal{O}}\left(L_{w_0}, \theta_0^\mu \Delta(y\cdot\mu)\langle i\rangle\right).$$

The graded module $\theta_0^\mu\Delta(y\cdot\mu)$ has a filtration

$$0 = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_{\ell(\overline{e})} = \theta_\mu^0\Delta(y\cdot\mu), \tag{15}$$

with subquotients

$$M_i/M_{i-1} \cong \bigoplus_{z\in S_\mu:\ell(z)=i} \Delta_{yz}\langle\ell(z)\rangle, \qquad \text{for } i = 0, 1, 2, \ldots, \ell(\overline{e}).$$

In particular, we have the short exact sequence

$$0 \to M_0 \to \theta_0^\mu\Delta(y\cdot\mu) \to N \to 0, \tag{16}$$

where $M_0 \cong \Delta_y$ and N has an Verma flag induced from (15). Consider the long exact sequence obtained by applying $\hom(L_{w_0}, {}_-\langle i\rangle)$ to (16), for $i \in \mathbb{Z}$. By (14) applied to

$\mu = 0$ and $i = \ell(w_0 y) - 2$, each of the spaces $\mathrm{ext}^1_{\mathcal{O}}(L_{w_0}, \Delta_{yz}\langle \ell(w_0 y) - 2 + \ell(z)\rangle)$ is zero and thus $\mathrm{ext}^1_{\mathcal{O}}(L_{w_0}, N\langle \ell(w_0 y) - 2\rangle) = 0$. Since $\theta_0^\mu \Delta(y \cdot \mu)$ has simple socle and this socle lives in degree $\ell(w_0 y)$, we also have $\hom(L_{w_0}, \theta_0^\mu \Delta(y \cdot \mu)\langle \ell(w_0 y) - 2\rangle) = 0$. Therefore, our long exact sequence gives the short exact sequence

$$\begin{aligned} 0 &\to \hom\left(L_{w_0}, N\langle \ell\left(w_0 y\right) - 2\rangle\right) \to \mathrm{ext}^1_{\mathcal{O}}\left(L_{w_0}, \Delta_y\langle \ell\left(w_0 y\right) - 2\rangle\right) \\ &\to \mathrm{ext}^1_{\mathcal{O}}\left(L_{w_0}, \theta_0^\mu \Delta(y \cdot \mu)\langle \ell(w_0 y) - 2\rangle\right) \to 0. \end{aligned}$$

Using (14), for $\mu = 0$, again, the dimension of the middle space in this short exact sequence is given by $\mathbf{c}(w_0 y)$. Since the socle of Δ_{yz} in ${}^{\mathbb{Z}}\mathcal{O}_0$ is $L_{w_0}\langle -\ell(w_0 yz)\rangle$, we have

$$\begin{aligned} \hom\left(L_{w_0}, N\langle \ell(w_0 y) - 2\rangle\right) &= \hom\left(L_{w_0}, \bigoplus_{s \in I} \Delta_{ys}\langle \ell(s) + \ell\left(w_0 y\right) - 2\rangle\right) \\ &= \hom\left(L_{w_0}, \bigoplus_{s \in I} \Delta_{ys}\langle \ell(w_0 ys)\rangle\right) = \mathbb{C}^{|I|}. \end{aligned}$$

It follows that

$$\dim \mathrm{ext}^1_{\mathcal{O}}\left(L_{w_0}, \theta_0^\mu \Delta(y \cdot \mu)\langle \ell\left(w_0 y\right) - 2\rangle\right) = \mathbf{c}\left(w_0 y\right) - |I|.$$

It remains to show that, for $i \neq \ell(w_0 y) - 2$, we have

$$\mathrm{ext}^1_{\mathcal{O}}\left(L_{w_0}, \theta_0^\mu \Delta(y \cdot \mu)\langle i\rangle\right) = 0.$$

For this, we consider the short exact sequence

$$0 \to \theta_0^\mu \Delta(y \cdot \mu)\langle \ell(w_0 y)\rangle \to I_{w_0} \to M \to 0$$

which induces

$$\begin{aligned} &\hom\left(L_{w_0}, I_{w_0}\langle -\ell(w_0 y) + i\rangle\right) \to \hom\left(L_{w_0}, M\langle -\ell(w_0 y) + i\rangle\right) \\ &\to \mathrm{ext}^1_{\mathcal{O}}\left(L_{w_0}, \theta_0^\mu \Delta(y \cdot \mu)\langle i\rangle\right) \to 0. \end{aligned}$$

Thus, it remains to show that $\hom(L_{w_0}, M\langle j\rangle) = 0$, for $j = -\ell(w_0 y) + i \neq -2$. We denote by K the cokernel of the inclusion $\theta_0^\mu \Delta(y \cdot \mu)\langle \ell(w_0 y)\rangle \subset \theta_0^\mu \Delta(\mu)\langle \ell(w_0)\rangle$. In particular, the multiplicity of any shift of L_{w_0} in K is zero. Then we have a canonical map $K \hookrightarrow M$ by definition of K and M. This defines a short exact sequence

$$0 \to K \to M \to Q \to 0,$$

where Q has a filtration by $\Delta_w\langle \ell(w_0) + \ell(w)\rangle$, for each $w \in S_n \backslash S_I$.

We claim that the socle of Q comes from the socle of $\Delta_s\langle \ell(w_0) + 1\rangle$, for $s \in S\backslash I$. Indeed, The module I_{w_0} is Koszul dual to the projective resolution of L_e. The latter can be obtained from the BGG resolution of L_e, given in terms of Verma modules, by gluing projective resolutions of Verma constituents of the BGG resolution into a projective resolution of L_e. In this way, the Verma modules in the BGG resolution give rise to the subquotients of the dual Verma filtration of I_{w_0}. Note that any Δ_x appears in the BGG resolution once and, moreover, for any y such that $\ell(y) = \ell(x) + 1$ and $y > x$, the map $\Delta_y\langle 1\rangle \to \Delta_x$ in the BGG resolution is non-zero, see [6]. The Koszul dual of this property is that the corresponding subquotient of the dual Verma flag of I_{w_0} is given by a nonzero morphism in the derived category and thus by a non-split short exact sequence

$$0 \to \nabla_{w_0x^{-1}}\langle -1\rangle \to R \to \nabla_{w_0y^{-1}} \to 0.$$

Applying the simple preserving duality $\star$ on $\mathcal{O}$, we consider the dual short exact sequence

$$0 \to \Delta_{w_0y^{-1}} \to R^\star \to \Delta_{w_0x^{-1}}\langle 1\rangle \to 0. \tag{17}$$

Write $w_0x^{-1} = u_1tu_2$ reduced, where t is a simple reflection and $w_0y^{-1} = u_1u_2$. Then we can consider (17) as the image, under the u_2-shuffling, of a non-split short exact sequence

$$0 \to \Delta_{u_1} \to R_1 \to \Delta_{u_1t}\langle 1\rangle \to 0. \tag{18}$$

If $w_0 = u_3u_1t$ is reduced, then, twisting (18) by u_3, we obtain a non-split short exact sequence

$$0 \to \Delta_{w_0t} \to R_2 \to \Delta_{w_0}\langle 1\rangle \to 0. \tag{19}$$

Here $\Delta_{w_0} = L_{w_0}$ and hence the fact that (19) is non-split means that R_2 has simple socle. In particular, the action of the center of $\mathcal{O}_0$ on R_2 is not semi-simple. This implies that the action of the center of $\mathcal{O}_0$ on both R_1 and $R^\star$ is not semi-simple either and hence they both have simple socle. Since I_{w_0} is self-dual with respect to $\star$, this implies that the socle of Q comes from its Bruhat minimal subquotients $\Delta_s\langle \ell(w_0)+1\rangle$, where $s \in S\backslash I$.

Applying $\hom(L_{w_0}, -)$, we obtain

$$\hom\left(L_{w_0}, M\langle j\rangle\right) \hookrightarrow \hom\left(L_{w_0}, Q\langle j\rangle\right).$$

However,

$$\hom\left(L_{w_0}, Q\langle j\rangle\right) = \hom\left(L_{w_0}, \bigoplus_{s\in S\backslash I} \Delta_s\langle \ell(w_0) + 1 + j\rangle\right) = 0,$$

for $j \neq -2$. This completes the proof. □

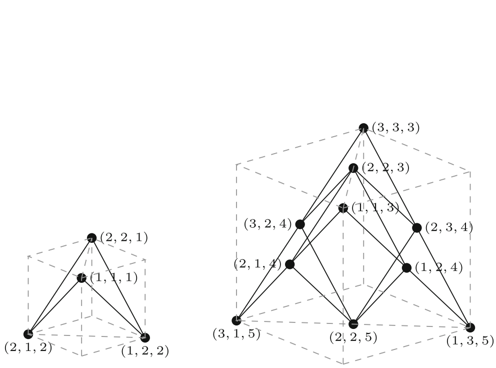

Fig. 2 The tetrahedron for $n = 3, 4, 5$, respectively

4 Representation theory versus combinatorics

4.1 Tetrahedron

Our computation for KL polynomials for $\mathcal{J}$ in Proposition 12 gives a nice geometric diagram for simple subquotients of Δ_e of the form $L_w\langle -i\rangle$, where $w \in \mathcal{J}$ and $i \in \mathbb{Z}$. By representing each simple subquotient $L_{w_{i,j}}\langle -k\rangle$ of Δ_e as the point (i, j, k), we get a tetrahedron in $\mathbb{Z}^3$. In this picture, we join two points if the corresponding subquotients extend in Δ_e (the existence of the extension follows from the arguments in Sect. 2.4). In Fig. 2 we show explicit examples for $n = 3, 4, 5$ (note that, as usual in depicting positively graded algebras, the z-axis is reversed in the pictures, so that the bottom of the picture consists of all elements of the form sw_0, for s a simple reflection).

Via the bijection given by Theorem 1(ii), this gives the Hasse diagram for $(\mathbf{B}_n, \leq)$, cf. [12, Figure 1 and 2].

In Figs. 3 and 4, we give some examples of composition factors of the form L_x, where $x \in \mathcal{J}$, in the modules Δ_e/Δ_w, for $n = 4$ and 5. The composition factors of this form in Δ_e/Δ_w which are not in the socle are given by the black points in the figures and the composition factors in the socle are marked by their coordinates in the figures. The white points correspond to the composition factors in Δ_w (and hence outside of Δ_e/Δ_w).

For convenience of the reader, on Fig. 5 we present the penultimate cell for $n = 5$. In the figure, an element on the position (i, j) has i as the unique left ascent, and j as the unique right ascent.

4.2 Explicit formulae

The set $\mathbf{B}_n$ can be explicitly parameterized by triples of integers

$$(i, j, k) \quad \text{such that} \quad 1 \leq i, j \leq n-1 \quad \text{and} \quad 0 \leq k \leq \min\{i-1, j-1, n-1-i, n-1-j\}.$$

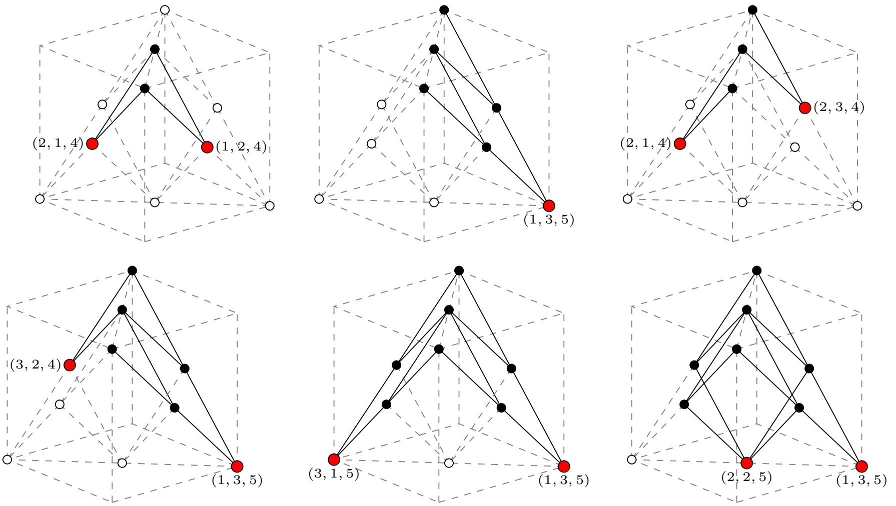

Fig. 3 Socles of the quotients Δ_e/Δ_w for $n = 4$, for $w = s_1s_2s_1, s_1s_2s_3, s_2s_3s_1$, respectively, in the first row, and $w = s_1s_2s_3s_2, s_1s_2s_3s_2s_1, s_1s_2s_3s_1s_2$, respectively, in the second row

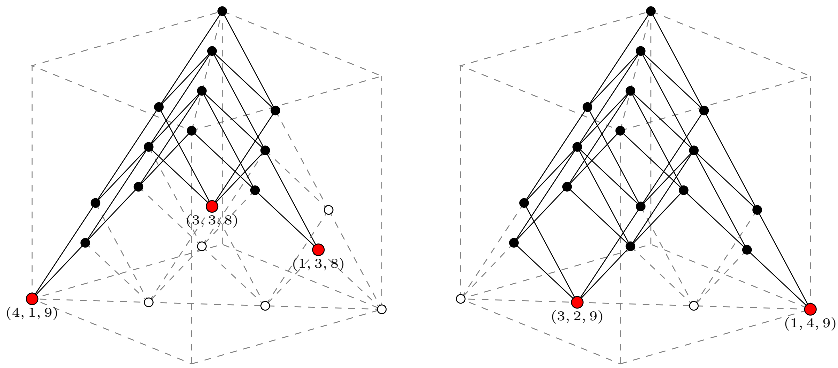

Fig. 4 Socles of the quotients Δ_e/Δ_w for $n = 5$, for $w = s_3s_4s_1s_2s_3s_2s_1$ and $w = s_1s_2s_3s_4s_2s_3s_1s_2$, respectively

	1	2	3	4
1	$s_2s_3s_4s_2s_3s_2$	$s_2s_3s_4s_2s_3s_2s_1$	$s_2s_3s_4s_2s_3s_1s_2s_1$	$s_2s_3s_4s_1s_2s_3s_1s_2s_1$
2	$s_1s_2s_3s_4s_2s_3s_2$	$s_1s_2s_3s_4s_2s_3s_2s_1$	$s_1s_2s_3s_4s_2s_3s_1s_2s_1$	$s_3s_4s_1s_2s_3s_1s_2s_1$
3	$s_1s_2s_3s_4s_1s_2s_3s_2$	$s_1s_2s_3s_4s_1s_2s_3s_2s_1$	$s_1s_2s_3s_4s_3s_1s_2s_1$	$s_4s_1s_2s_3s_1s_2s_1$
4	$s_1s_2s_3s_4s_1s_2s_3s_1s_2$	$s_1s_2s_3s_4s_1s_2s_3s_1$	$s_1s_2s_3s_4s_1s_2s_1$	$s_1s_2s_3s_1s_2s_1$

Fig. 5 The penultimate cell for $n = 5$

For $i \leq j$, put:

$$b(i, j, k) := \left(s_is_{i+1}\dots s_{j+k}\right)\left(s_{i-1}s_i\dots s_{j+k-1}\right)\cdots\left(s_{i-k}s_{i-k+1}\dots s_j\right),$$

and if $j < i$, set $b(i, j, k) := b(j, i, k)^{-1}$.

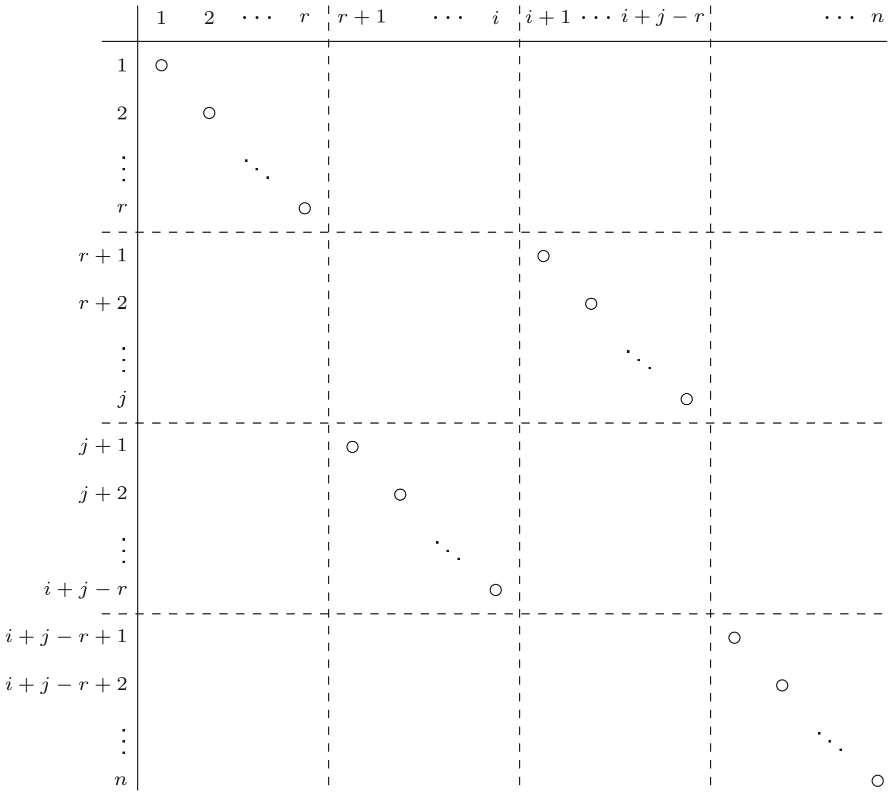

Fig. 6 The bigrassmannian $b(i, j, k)$. Here $r = \min\{i-1, j-1\} - k$

We can draw $w \in S_n$ as a picture on a two-dimensional grid, in the following way: View $w \in S_n$ as a permutation on $\{1, \dots, n\}$ acting on the left, and put$\circ$ on positions $(i, w(i))$, $i = 1, \dots, n$, in matrix coordinates. This way, we can visualize bigrassmannian permutations as on Fig. 6.

Proposition 17 *The bijection from Theorem 1(ii) is given by*

$$b(i, j, k) \mapsto L_{w_{i,j}}\left\langle -\frac{(n-1)(n-2)}{2} - |i-j| - 2k \right\rangle$$

Proof The assignment in the claim does define a bijection by Proposition 12, so we need to check that it agrees with the bijection in Theorem 1. From the definition, we see that $b(i, j, k) < b(i, j, k')$ (in the Bruhat order), for $k < k'$. This property, since the relevant subquotients appear multiplicity-freely, determines the bijection. Since the inclusion of the Verma modules Δ_x agree with (the opposite of) the Bruhat order, the bijection in Theorem 1 also has this property and thus agrees with the one given in our claim. □

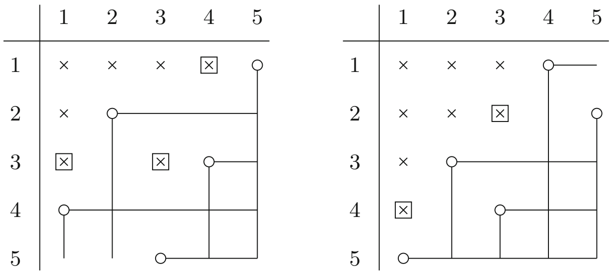

Fig. 7 Essential sets attached to $s_3s_4s_1s_2s_3s_2s_1$ and $s_1s_2s_3s_4s_2s_3s_1s_2$ (for $n = 5$), respectively

4.3 Essential set and rank of a permutation

For $w \in S_n$, the so-called *essential set* attached to w (defined in [14]) is given by

$$\mathrm{Ess}(w) := \left\{(i, j)\colon 1 \le i, j \le n-1,\ i < w^{-1}(j),\ j < w(i),\ w(i+1) \le j,\ w^{-1}(j+1) \le i\right\}.$$

In Fig. 7 we give examples of essential sets for $w \in S_5$. To easily find $\mathrm{Ess}(w)$, following [22], we put $\circ$ on positions $(i, w(i))$, $i = 1, \ldots, n$, in matrix coordinates, and kill all cells to the right or below of these. The surviving cells are denoted by $\times$, and are sometimes called the *diagram* of w. The south-east corners of the diagram (denoted by $\boxtimes$) constitute the essential set attached to w.

In [22] it is shown that there is a bijection between $\mathbf{BM}(w)$ and $\mathrm{Ess}(w)$. More precisely, an element $(i, j) \in \mathrm{Ess}(w)$ corresponds to a certain monotone triangle (see the cited article for definition), denoted by $J_{i,k,j+1}$, for some k, which is then identified with a bigrassmannian permutation. From the description of monotone triangles in [30, Section 8], it follows that $J_{i,k,j+1}$ correspond to a bigrassmannian permutation with left descent j and right descent i. From Proposition 14 we know that such a bigrassmannian element in $\mathbf{BM}(w)$ is uniquely determined. We conclude:

Corollary 18 (A reformulation of [22, Theorem]) *For $w \in S_n$, the map*

$$x \mapsto (\text{right descent of } x, \text{left descent of } x)$$

is a bijection $\mathbf{BM}(w) \to \mathrm{Ess}(w)$.

The following result allows us to determine the socle of Δ_e/Δ_w via the essential set of w which is very easy and efficient to compute.

Corollary 19 *The (ungraded) socle of Δ_e/Δ_w is given by* $\bigoplus_{(i,j)\in \mathrm{Ess}(w)} L_{w_{j,i}}$.

Proof The claim follows from Proposition 17 and Corollary 18. □

To illustrate how this corollary works, one can compare Figs. 7 with Fig. 4 using Fig. 5. Observe that the essential set alone does not provide information on the degrees of the composition factors in the socle of Δ_e/Δ_w. To get these degrees, we need another combinatorial tool called the *rank function*.

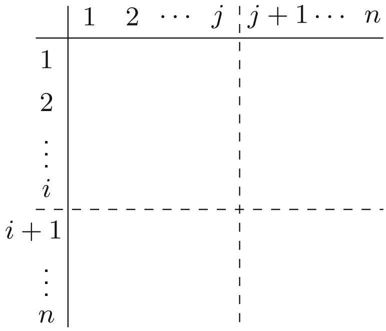

Fig. 8 Areas which determine $r_w(i, j)$ and $t_w(i, j)$

	1	2	3	4	5
1	1	1	1	$\boxed{1}$	0
2	1	1	1	1	0
3	$\boxed{1}$	1	$\boxed{2}$	1	0
4	0	0	1	1	0
5	0	0	0	0	0

	1	2	3	4	5
1	1	1	1	0	0
2	1	2	$\boxed{2}$	1	0
3	1	1	2	1	0
4	$\boxed{1}$	1	1	1	0
5	0	0	0	0	0

Fig. 9 The values of $t_w(i, j)$ for $w = s_3s_4s_1s_2s_3s_2s_1$ and $w = s_1s_2s_3s_4s_2s_3s_1s_2$ (for $n = 5$), respectively. The values at $\mathrm{Ess}(w)$ are boxed

The rest of the subsection provides an upgrade of the description in Corollary 19 to the graded setup. For $w \in S_n$, the so-called *rank function* r_w (defined in [14]) is given by:

$$r_w(i, j) := |\{k \leq i : w(k) \leq j\}|, \qquad 1 \leq i, j \leq n.$$

More useful for us is the function t_w, which we call the *co-rank function*, given by

$$t_w(i, j) := \min\{i, j\} - r_w(i, j).$$

If we again consider a permutation as a picture on a two-dimensional grid as before, then $r_w(i, j)$ is equal to the number of $\circ$ in the north-west area in Fig. 8. If $i \leq j$, then $t_w(i, j)$ is equal to the number of $\circ$ in the north-east area in Fig. 8, and otherwise to the number of $\circ$ in the south-west area.

The co-rank functions for the examples in Fig. 7 are given in Fig. 9.

Lemma 20 *For $w, x \in S_n$, we have $t_w \leq t_x$ if and only if $w \leq x$.*

Proof See [4, Theorem 2.1.5]. □

The following lemma is clear from Fig. 6:

Lemma 21 *For each $b(i, j, k) \in \mathbf{B}_n$, we have $\mathrm{Ess}(b(i, j, k)) = \{(j, i)\}$ and $t_{b(i,j,k)}(j, i) = k + 1$.*

Finally, we can describe the graded shifts of the socle constituents in $\Delta_e/(\Delta_w\langle -\ell(w)\rangle)$.

Proposition 22 *Let $w \in S_n$. The socle of $N_w := \Delta_e/(\Delta_w\langle -\ell(w)\rangle)$ in ${}^{\mathbb{Z}}\mathcal{O}$ is given by*

$$\bigoplus_{(i,j)\in \mathrm{Ess}(w)} L_{w_{j,i}}\left\langle -\frac{(n-1)(n-2)}{2} - |i-j| - 2(t_w(i,j)-1)\right\rangle.$$

Proof Note first that, for $w = b(j,i,k) \in \mathbf{B}_n$, we have

$$\mathrm{soc}(N_w) = L_{w_{j,i}}\langle -\mathbf{a}(\mathcal{J}) - |i-j| - 2(t_w(i,j)-1)\rangle \tag{20}$$

by Lemma 21 and Proposition 17. Recall that $\mathbf{a}(\mathcal{J}) = \frac{(n-1)(n-2)}{2}$.

Now let $w \in S_n$ be arbitrary and let $(i,j) \in \mathrm{Ess}(w)$. By Corollary 19 we only need to determine the degree shift, say m, of $L_{w_{j,i}}$ in the socle of N_w. Let $b(j,i,k)$ be the element in $\mathbf{BM}(w)$ which corresponds to (i,j) under the bijection in Corollary 18. Since $b(j,i,k) \in \mathbf{BM}(w)$, by Lemma 20 we have $b(j,i,k) \leq w$ while $b(j,i,k+1) \not\leq w$, and thus $N_w \not\twoheadrightarrow N_{b(j,i,k+1)}$ while $N_w \twoheadrightarrow N_{b(j,i,k)}$. So $m = \mathbf{a}(\mathcal{J}) + |i-j| + 2(t_w(i,j)-1)$ follows from (20) and the parity of m. □

5 Further remarks

5.1 Inclusions between arbitrary Verma modules

An immediate consequence of Theorem 1 is:

Corollary 23 *Let $v, w \in S_n$ be such that $v < w$.*

(i) *The bijection from Theorem 1(ii) induces a bijection between simple subquotients of Δ_v/Δ_w of the form L_x, where $x \in \mathcal{J}$, and $y \in \mathbf{B}_n$ such that $y \leq w$ and $y \not\leq v$.*
(ii) *The socle of Δ_v/Δ_w consists of all L_x, where x corresponds to an element in $\mathbf{BM}(w)\backslash\mathbf{BM}(v)$.*

A more detailed description of the socle of $\Delta_v\langle -\ell(v)\rangle/(\Delta_w\langle -\ell(w)\rangle)$ as an object in ${}^{\mathbb{Z}}\mathcal{O}_0$ follows from Proposition 22.

5.2 No such clean result in other types

Unfortunately, Theorem 1 is not true, in general, in other types. One of the reasons is that Corollary 13 fails already in types B_3 and D_4. We note that $\mathbf{B}_n$ agrees in type A with the set of join-irreducible elements in W, while, in general, there are bigrassmannian elements that are not join-irreducible. To generalize Theorem 1, we need, to start with, replace $\mathbf{B}_n$ by the set of join-irreducible elements in W. But even then, most of our crucial arguments fail outside type A. For example, in non-simply laced types, for some pairs of simple reflections s and t there will be more than one element $w \in \mathcal{J}$ such that $sw > w$ and $wt > w$. Another problem is that neither bigrassmannian elements nor join-irreducible elements with fixed left and right descents form a chain with respect to the Bruhat order.

Rank two case is, however, special. In this case $\mathcal{J}$ is the set of bigrassmannian elements and all KL polynomials are trivial. So, Theorem 1 is true. Notably, an appropriate analogue of the map Φ in this case is not the identity map.

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References

1. Andersen, H., Stroppel, C.: Twisting functors on $\mathcal{O}$. Represent. Theory **7**, 681–699 (2003)
2. Beilinson, A., Bernstein, J.: Localisation de g-modules. C. R. Acad. Sci. Paris Ser. I Math. **292**(1), 15–18 (1981)
3. Beilinson, A., Ginzburg, V., Soergel, W.: Koszul duality patterns in representation theory. J. Am. Math. Soc. **9**(2), 473–527 (1996)
4. Björner, A., Brenti, F.: Combinatorics of Coxeter Groups, Graduate Texts in Mathematics, vol. 231. Springer, Berlin (2006)
5. Bernstein, I.N., Gelfand, S.I.: Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. Compositio Math. **41**(2), 245–285 (1980)
6. Bernstein, I. N., Gelfand, I. M., Gelfand, S. I.: Differential operators on the base affine space and a study of g-modules. Lie groups and their representations (Proc. Summer School, Bolyai Janos Math. Soc., Budapest, 1971), pp. 21–64. Halsted, New York, (1975)
7. Bernstein, I.N., Gelfand, I.M., Gelfand, S.I.: A certain category of g-modules. (Russian) Funkcional. Anal. i Prilozen. **10**(2), 1–8 (1976)
8. Brylinski, J.-L., Kashiwara, M.: Kazhdan–Lusztig conjecture and holonomic systems. Invent. Math. **64**(3), 387–410 (1981)
9. Carlin, K.: Extensions of Verma modules. Trans. Am. Math. Soc. **294**(1), 29–43 (1986)
10. Dixmier, J.: Enveloping algebras. Revised reprint of the 1977 translation. Graduate Studies in Mathematics, **11**. American Mathematical Society, Providence, RI. xx+379 pp (1996)
11. Elias, B., Williamson, G.: The Hodge theory of Soergel bimodules. Ann. Math. (2) **180**(3), 1089–1136 (2014)
12. Engbers, J., Hammett, A.: On comparability of bigrassmannian permutations. Australas. J. Combin. **71**, 121–152 (2018)
13. Eriksson, K., Linusson, S.: Combinatorics of Fulton's essential set. Duke Math. J. **85**(1), 61–76 (1996)
14. Fulton, W.: Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. Duke Math. J. **65**(3), 381–420 (1992)
15. Geck, M.: Kazhdan–Lusztig cells and the Murphy basis. Proc. Lond. Math. Soc. (3) **93**(3), 635–665 (2006)
16. Humphreys, J. E.: Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$. Grad. Stud. Math., **94**. American Mathematical Society, Providence, RI, xvi+289 pp (2008)
17. Irving, R.: The socle filtration of a Verma module. Ann. Sci. École Norm. Sup. (4) **21**(1), 47–65 (1988)

18. Iyama, O., Reading, N., Reiten, I., Thomas, H.: Lattice structure of Weyl groups via representation theory of preprojective algebras. Compos. Math. **154**(6), 1269–1305 (2018)
19. Kazhdan, D., Lusztig, G.: Representations of Coxeter groups and Hecke algebras. Invent. Math. **53**(2), 165–184 (1979)
20. Kildetoft, T., Mackaay, M., Mazorchuk, V., Zimmermann, J.: Simple transitive 2-representations of small quotients of Soergel bimodules. Trans. Am. Math. Soc. **371**(8), 5551–5590 (2019)
21. Ko, H.: Mazorchuk, V. 2-representations of small quotients of Soergel bimodules in infinite types. Preprint arXiv:1809.11113, to appear in Proc. AMS
22. Kobayashi, M.: Bijection between bigrassmannian permutations maximal below a permutation and its essential set. Electron. J. Combin. **17**(1), Note 27, 8 pp (2010)
23. Kobayashi, M.: Enumeration of bigrassmannian permutations below a permutation in Bruhat order. Order **28**(1), 131–137 (2011)
24. König, S., Slungård, I., Xi, C.: Double centralizer properties, dominant dimension, and tilting modules. J. Algebra **240**(1), 393–412 (2001)
25. Lascoux, A., Schützenberger, M.-P.: Treillis et bases des groupes de Coxeter. Electron. J. Combin. **3**, R27 (1996)
26. Lusztig, G.: Some examples of square integrable representations of semisimple p-adic groups. Trans. Am. Math. Soc. **277**(2), 623–653 (1983)
27. Lusztig, G.: Cells in affine Weyl groups. II. J. Algebra **109**(2), 536–548 (1987)
28. Mazorchuk, V.: Some homological properties of the category $\mathcal{O}$. Pac. J. Math. **232**(2), 313–341 (2007)
29. Mazorchuk, V., Stroppel, C.: Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module. Trans. Am. Math. Soc. **357**(7), 2939–2973 (2005)
30. Reading, N.: Order dimension, strong Bruhat order and lattice propeties for posets. Order **19**(1), 73–100 (2002)
31. Reiner, V., Woo, A., Yong, A.: Presenting the cohomology of a Schubert Variety. Trans. Am. Math. Soc. **363**(1), 521–543 (2011)
32. Robinson, G.D.B.: On the representations of the symmetric group. Am. J. Math. **60**(3), 745–760 (1938)
33. Sagan, B.: The symmetric group. Representations, combinatorial algorithms, and symmetric functions. Second edition. Grad. Texts Math. **203**. Springer-Verlag, New York, xvi+238 pp (2001)
34. Schensted, C.: Longest increasing and decreasing subsequences. Can. J. Math. **13**, 179–191 (1961)
35. Soergel, W.: Équivalences de certaines catégories de g-modules. C. R. Acad. Sci. Paris Sér. I Math. **303**(15), 725–728 (1986)
36. Soergel, W.: Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Am. Math. Soc. **3**(2), 421–445 (1990)
37. Stroppel, C.: Category $\mathcal{O}$: gradings and translation functors. J. Algebra **268**(1), 301–326 (2003)
38. Verma, D.-N.: Structure of certain induced representations of complex semisimple Lie algebras. Bull. Am. Math. Soc. **74**, 160–166 (1968)