Independence of algebraic monodromy groups in compatible systems

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Abstract

In this paper we develop a general method to prove independence of algebraic monodromy groups in compatible systems of representations, and we apply it to deduce independence results for compatible systems both in automorphic and in positive characteristic settings. In the abstract case, we prove an independence result for compatible systems of Lie-irreducible representations, from which we deduce an independence result for compatible systems admitting what we call a Lie-irreducible decomposition. In the case of geometric compatible systems of Galois representations arising from certain classes of automorphic forms, we prove the existence of a Lie-irreducible decomposition. From this we deduce an independence result. We conclude with the case of compatible systems of Galois representations over global function fields, for which we prove the existence of a Lie-irreducible decomposition, and we deduce an independence result. From this we also deduce an independence result for compatible systems of lisse sheaves on normal varieties over finite fields.

Keywords Compatible systems of Galois representations · Independence of algebraic monodromy groups · Automorphic compatible systems · Compatible systems over global function fields

Mathematics Subject Classification 11F80 · 11F70 · 11S37 · 14G17

Introduction

Questions on ℓ -independence of algebraic monodromy groups in compatible systems originate in the literature from a classical result by Serre, see [19], which states that if E is an elliptic curve without complex multiplication over a number field, then the image of the Galois representation on the ℓ -adic Tate module of E is an open subgroup

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of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ for every prime ℓ , and is equal to $\mathrm{GL}_2(\mathbb{Z}_\ell)$ for all but finitely many primes ℓ . The problem of extending this result to more general geometric settings led, for instance, to the formulation of the Mumford-Tate conjecture for abelian varieties, see [16]. Nevertheless, one can naturally study ℓ -independence questions for any abstract rational compatible system of representations. In this context, the foundational work of Larsen and Pink, see [15], provides some breakthroughs.

It is immediate to notice how the main results of [15] can be extended verbatim to compatible systems having coefficients in any number field. In our paper, we use these results to develop a general method to prove λ -independence of the neutral components of the algebraic monodromy groups of compatible systems over a finite extension of the field of coefficients. The main idea is that one can prove a λ -independence result of this form whenever a compatible system admits a *Lie-irreducible decomposition* over a finite extension of its field of coefficients. Let us briefly sketch our approach. First of all, we apply the results of [15] to prove λ -independence of the neutral components of the algebraic monodromy groups of compatible systems of Lie-irreducible (i.e. irreducible after restricting to any open subgroup) representations over a finite extension of the field of coefficients, see Proposition 4.1. We then introduce the notion of Lie-irreducible decomposition, see Definition 4.2, and we deduce from the Lie-irreducible case an analogous λ -independence result for compatible systems admitting a Lie-irreducible decomposition, see Corollary 4.4.

In the case of automorphic compatible systems (and actually for a slightly larger class of geometric compatible systems), we prove the existence of a Lie-irreducible decomposition at a set of primes of positive Dirichlet density, see Theorem 5.4, from which we deduce a λ -independence result, see Corollary 5.5. Assuming a classical irreducibility conjecture for automorphic compatible systems, the set of primes of positive Dirichlet density can be actually replaced by a set of primes of Dirichlet density 1.

In the positive characteristic setting, we prove the existence of a Lie-irreducible decomposition for compatible systems of representations of the absolute Galois group of a global function field, see Theorem 6.2, from which we deduce a λ -independence result, see Corollary 6.3. We also deduce a λ -independence result for compatible systems of lisse sheaves on normal varieties over finite fields, see Corollary 6.5.

The structure of this paper is the following. In Sect. 1 we introduce the basic terminology, while in Sects. 2 and 3 we recall the main tools from [15] which will be useful in Sect. 4. The general method of this paper and the main results in the general setting are presented in Sect. 4. Finally, we see applications of the results of Sect. 4 to the case of geometric compatible systems of Galois representations in Sect. 5, and to compatible systems in the positive characteristic case in Sect. 6.

Notation

For a prime ℓ , we denote by ζ_ℓ a primitive ℓ th-root of 1.

Given a profinite group Γ , an integer $n \geq 1$, and a continuous representation $\rho : \Gamma \rightarrow \mathrm{GL}_n(\mathbb{Q}_\ell)$, we let $\bar{\rho} : \Gamma \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ denote the semisimplification of the reduction of ρ modulo ℓ , which is defined up to conjugacy.

Given a field k , we let $\bar{k}$ be an algebraic closure of k , and k^s be a separable closure of k inside $\bar{k}$, and we denote by $\Gamma_k = \text{Gal}(k^s/k)$ the absolute Galois group of k .

Given a global field F , we denote by $|F|$ the set of finite places of F . If Σ is a set of places of F , and ℓ and p are primes, we set $\Sigma_\ell = \{v \in \Sigma : v \mid \ell\}$ and $\Sigma_{\neq p} = \bigcup_{\ell' \neq p} \Sigma_{\ell'}$. We denote by $\mathbb{A}_F$ the ring of adèles of F .

If K is a non-Archimedean local field, we let I_K be the inertia subgroup of Γ_K , and $\text{Frob}_K \in \Gamma_K/I_K$ be the geometric Frobenius. When $K = F_v$, for F a global field, and v a finite place of F , we write $\text{Frob}_v = \text{Frob}_{F_v}$.

If K is a finite extension of $\mathbb{Q}_p$, given a Weil-Deligne representation (r, N) of the Weil group W_K of K , we let $(r, N)^{\text{F-ss}}$ denote its Frobenius-semisimplification. Given a continuous representation $\rho : \Gamma_K \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_\ell)$, with $\ell \neq p$, we denote by $\text{WD}(\rho)$ the corresponding Weil-Deligne representation of W_K . We denote by rec_K the local Langlands correspondence for GL_n over K of [11]. When $K = F_v$, for F a number field, and v a finite place of F , we write $\text{rec}_v = \text{rec}_{F_v}$.

If K is a finite extension of $\mathbb{Q}_\ell$, if V is a finite dimensional vector space over $\overline{\mathbb{Q}}_\ell$, and $\rho : \Gamma_K \rightarrow \text{GL}(V)$ is a continuous semisimple de Rham representation, we denote by $\text{WD}(\rho)$ the corresponding Weil-Deligne representation of W_K . Also, given an embedding $\tau : K \rightarrow \overline{\mathbb{Q}}_\ell$ we define the multiset $\text{HT}_\tau(\rho)$ of τ -Hodge-Tate weights of ρ to be the multiset of $\dim_{\overline{\mathbb{Q}}_\ell} V$ integers containing i with multiplicity $\dim_{\overline{\mathbb{Q}}_\ell} (V \otimes_{\tau, K} \widehat{K}(i))^{\Gamma_F}$, where $\widehat{K}$ denotes the completion of an algebraic closure of K . If F is a number field, if V is a finite dimensional vector space over $\overline{\mathbb{Q}}_\ell$, and $\rho : \Gamma_F \rightarrow \text{GL}(V)$ is a continuous semisimple representation which is de Rham at all places above ℓ , given an embedding $\tau : F \rightarrow \overline{\mathbb{Q}}_\ell$ we define the multiset of τ -Hodge-Tate weights of ρ to be $\text{HT}_\tau(\rho) = \text{HT}_\tau(\rho|_{\Gamma_{F_v(\tau)}})$, where $v(\tau)$ is the place of F induced by τ .

Given a scheme X of finite type over $\mathbb{F}_q$, where q is a power of p , we denote by $|X|$ its set of closed points. For each $x \in |X|$, let $\bar{x}$ be an algebraic geometric point of X localised at x , and let $\text{Frob}_x \in \Gamma_{k(x)} = \text{Gal}(k(\bar{x})/k(x))$ be the geometric Frobenius. Given a sheaf $\mathcal{F}$ on X , we write $\det(1 - \text{Frob}_x t, \mathcal{F})$ for the characteristic polynomial of the image of Frob_x under the monodromy representation of $\Gamma_{k(x)}$ on $\mathcal{F}_{k(\bar{x})}$.

1 Compatible systems and algebraic monodromy groups

A *group with Frobenii*, or simply an *F-group*, is a pair $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ where Γ is a profinite group, and $\{F_\alpha\}_{\alpha \in A}$ is a collection of elements of Γ , called *Frobenius elements*, which are dense in Γ . Motivating examples for the definition of group with Frobenii are clearly the absolute Galois group of a global field, together with the collection of geometric Frobenii at its finite places, and the étale fundamental group of a scheme of finite type over a finite field, together with the collection of geometric Frobenii at its closed points.

Let $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ be a group with Frobenii, let E be a number field, and let Λ be a set of finite places of E . We give the following definition.¹

¹ Compare [15, Definition 6.5].

Definition 1.1 A *compatible system* of rank n representations of Γ defined over E is a family $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ of continuous representations

$$\rho_\lambda : \Gamma \rightarrow \mathrm{GL}_n(\overline{E}_\lambda)$$

such that there is a subset $\mathcal{X} \subset A \times \Lambda$ satisfying the following conditions.

- (1) For every $\alpha \in A$, $(\alpha, \lambda) \in \mathcal{X}$ for all but finitely many $\lambda \in \Lambda$.
- (2) For all $(\alpha, \lambda) \in \mathcal{X}$, the characteristic polynomial of $\rho_\lambda(F_\alpha)$ has coefficients in E and is independent of λ .
- (3) For any places $\lambda_1, \dots, \lambda_m \in \Lambda$, the set $\{F_\alpha : (\alpha, \lambda_i) \in \mathcal{X} \text{ for all } i = 1, \dots, m\}$ is dense in Γ .

In the last two sections we will see two main (motivating) classes of examples of compatible systems, i.e. geometric compatible systems of Galois representations, see Sect. 5, and compatible systems of lisse sheaves on schemes of finite type over finite fields, see Sect. 6.

We say that a compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ of rank n representations of Γ defined over E has *coefficients* in a finite extension E' of E if for any $\lambda \in \Lambda$ the image of ρ_λ is contained in $\mathrm{GL}_n(E'_\lambda)$ for some place λ' of E' above λ .

Given two sets $\mathcal{L}$ and $\mathcal{L}'$ of rational primes, and a non empty subset $\mathcal{L}'' \subset \mathcal{L} \cap \mathcal{L}'$, we say that two compatible systems $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ and $\mathcal{R}' = \{\rho'_\lambda\}_{\lambda \in \Lambda'}$ defined over a number field E , where $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$ and $\Lambda' = \cup_{\ell \in \mathcal{L}'} |E|_\ell$, are *isomorphic at $\mathcal{L}''$* if $\rho_\lambda \cong \rho'_\lambda$ for all $\ell \in \mathcal{L}''$ and $\lambda \mid \ell$. In this case, we write $\mathcal{R} \cong_{(\mathcal{L}'')} \mathcal{R}'$. When $\mathcal{R}$ and $\mathcal{R}'$ have coefficients in E , we say that they are *isomorphic over E* if the isomorphism $\rho_\lambda \cong \rho'_\lambda$ is defined over E_λ for all $\ell \in \mathcal{L}''$ and $\lambda \mid \ell$.

Let $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ be a group with Frobenii, and let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n representations of Γ with coefficients in a number field E . For any $\lambda \in \Lambda$, denote by G_λ the Zariski closure of $\rho_\lambda(\Gamma)$ in $\mathrm{GL}_n(E_\lambda)$. This is an algebraic subgroup of $\mathrm{GL}_{n, E_\lambda}$, which we call the *algebraic monodromy group* of ρ_λ . We recall the following lemma.

Lemma 1.2 [15, Lemma 6.9] *If ρ_λ is semisimple, then G_λ is reductive. If ρ_λ is absolutely irreducible, then G_λ is reductive, and its natural representation $G_\lambda \rightarrow \mathrm{GL}_{n, E_\lambda}$ is absolutely irreducible.*

Let G_λ° be the connected component of the identity of G_λ . It is immediate to notice that if Γ' is an open subgroup of Γ , and G'_λ denotes the Zariski closure of $\rho_\lambda(\Gamma')$ in $\mathrm{GL}_n(E_\lambda)$, then the connected component of the identity $G_\lambda^{\circ'}$ of G'_λ coincides with G_λ° .

In the next sections, we consider the problem of studying the algebraic monodromy groups G_λ of $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ varying λ . The hope in this context is to find a global algebraic subgroup G of $\mathrm{GL}_{n, E}$ defined over E such that G_λ is conjugate to $G \times_E E_\lambda$ over E_λ for any $\lambda \in \Lambda$. This is the prototype of a λ -independence problem for the compatible system $\mathcal{R}$. Unfortunately, proving a λ -independence result of this form in the abstract setting is actually too optimistic. For this reason, we will focus on weaker λ -independence statements.

2 The formal character and the variety of characteristic polynomials

If G is a reductive group over a field k of characteristic zero, $T \subset G$ is a maximal torus of G over k , and $\rho : G \rightarrow \mathrm{GL}_{n,k}$ is a representation of G over k , we define the *formal character* of ρ to be the restriction of ρ to T .

Let $\mathrm{ch} : \mathrm{GL}_{n,k} \rightarrow \mathbb{G}_{m,k} \times \mathbb{A}_k^{n-1}$ be the morphism which associates to a matrix the coefficients of its characteristic polynomial. Given a connected reductive algebraic subgroup G of $\mathrm{GL}_{n,k}$, and a maximal torus $T \subset G$ over k , we have that $\mathrm{ch}(G) = \mathrm{ch}(T)$ is a variety defined over $\mathbb{Q}$, which determines uniquely the formal character of the natural representation $G \rightarrow \mathrm{GL}_{n,k}$ up to isomorphism, see [15, §4].

Let $\mathbb{G}_{m,k}^n$ be the split maximal torus of $\mathrm{GL}_{n,k}$, and let $T_0 \subset \mathbb{G}_{m,k}^n$ be a subtorus such that $T_0 \times_k \bar{k}$ is conjugate to $T \times_k \bar{k}$ over $\bar{k}$. Since the semisimple part of any point in G can be conjugated into T_0 over $\bar{k}$, we have that $\mathrm{ch}(G) = \mathrm{ch}(T_0)$ pointwise. We have that the Weyl group $\mathrm{Norm}_{\mathrm{GL}_{n,k}}(\mathbb{G}_{m,k}^n) / \mathrm{Cent}_{\mathrm{GL}_{n,k}}(\mathbb{G}_{m,k}^n)$ of $\mathrm{GL}_{n,k}$ with respect to $\mathbb{G}_{m,k}^n$ is the symmetric group $\mathfrak{S}_n$, acting on $\mathbb{G}_{m,k}^n$ by permutation of factors. For every $\sigma \in \mathfrak{S}_n \setminus \mathrm{Cent}_{\mathfrak{S}_n}(T_0)$, we define a proper subgroup $H_\sigma \subset T_0$ in the following way. We let $H_\sigma = \{t \in T : \sigma(t) = t\}$ if $\sigma(T_0) = T_0$, and $H_\sigma = T_0 \cap \sigma(T_0)$ otherwise. We define now Y to be the union of all $\mathrm{ch}(H_\sigma)$. This is a Zariski-closed proper subset of $\mathrm{ch}(T_0) = \mathrm{ch}(G)$. We say that a point $g \in G$ is Γ -regular if $\mathrm{ch}(g) \notin Y$. We have the following result.

Proposition 2.1 [15, Proposition 4.7].

- (1) For every $x \in (\mathrm{ch}(G) \setminus Y)(k)$, there exist a torus $T \subset \mathrm{GL}_{n,k}$, and an element $t \in T(k)$, such that $\mathrm{ch}(T) = \mathrm{ch}(G)$ and $\mathrm{ch}(t) = x$. The pair (t, T) is unique up to conjugation by $\mathrm{GL}_n(k)$.
- (2) Let $g \in G(k)$ be Γ -regular. Then g lies in a unique maximal torus T of G , and the $\mathrm{GL}_n(k)$ -conjugacy class of (g, T) is uniquely determined by $\mathrm{ch}(g)$ and $\mathrm{ch}(G)$.

Let $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ be a group with Frobenii, and let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n semisimple representations of Γ with coefficients in a number field E . Keep notations as in Sect. 1. We have:

Proposition 2.2 [15, Proposition 6.12] *The formal character of the natural representation $G_\lambda^\circ \times_{E_\lambda} \bar{E}_\lambda \rightarrow \mathrm{GL}_{n, \bar{E}_\lambda}$ is independent of λ . More precisely, there exist a torus T_0 over E and a faithful representation $\rho_0 : T_0 \rightarrow \mathrm{GL}_{n, E}$ such that $T_0 \times_E \bar{E}_\lambda$ is isomorphic to a maximal torus in $G_\lambda^\circ \times_{E_\lambda} \bar{E}_\lambda$, and $\rho_0 \otimes_E \bar{E}_\lambda$ is equivalent to the formal character of $G_\lambda^\circ \times_{E_\lambda} \bar{E}_\lambda \rightarrow \mathrm{GL}_{n, \bar{E}_\lambda}$ for all $\lambda \in \Lambda$.*

Proposition 2.3 [15, Proposition 6.14] *There is an open normal subgroup $\Gamma' \subset \Gamma$ such that ρ_λ induces an isomorphism $\Gamma / \Gamma' \xrightarrow{\sim} G_\lambda / G_\lambda^\circ$ for all $\lambda \in \Lambda$.*

Note that Proposition 2.3 implies that the group $G_\lambda / G_\lambda^\circ$ of connected components of G_λ is independent of λ . Also, if we let $A' = \{\alpha \in A : F_\alpha \in \Gamma'\}$, then the pair $(\Gamma', \{F_\alpha\}_{\alpha \in A'})$ is a group with Frobenii, and $\mathcal{R}' = \{\rho'_\lambda = \rho_\lambda|_{\Gamma'}\}_{\lambda \in \Lambda}$ forms a compatible system, with $\mathcal{X}' = \mathcal{X} \cap (A' \times \Lambda)$. Clearly, the algebraic monodromy group of ρ'_λ is G_λ° for every $\lambda \in \Lambda$.

3 Frobenius tori and the splitting field of a compatible system

Let $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ be a group with Frobenii, and let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n semisimple representations of Γ with coefficients in a number field E . Let us assume in this section that for every $\lambda \in \Lambda$ the reductive group G_λ is connected. We have the following result.

Proposition 3.1 [15, Proposition 7.2] *For any $\lambda \in \Lambda$, the set of all $\gamma \in \Gamma$ such that $\rho_\lambda(\gamma)$ is Γ -regular is open and dense in Γ .*

For every $(\alpha, \lambda) \in \mathcal{X}$, the point $\text{ch}(\rho_\lambda(F_\alpha)) \in (\mathbb{G}_{m, E_\lambda} \times \mathbb{A}_{E_\lambda}^{n-1})(E)$ depends only on α , and so the condition on $\rho_\lambda(F_\alpha)$ being Γ -regular depends only on α as well. When this condition holds, we simply say that α is Γ -regular. By Proposition 3.1, if we replace A with the set of Γ -regular $\alpha \in A$ the corresponding set $\mathcal{X} \subset A \times \Lambda$ still satisfies the conditions (1)-(3) in Definition 1.1. From now on, let us assume that every $\alpha \in A$ is Γ -regular.

By Proposition 2.1, for every $\alpha \in A$ we are then given a torus $T_\alpha \subset \text{GL}_{n, E}$, canonical up to conjugacy, associated to $\text{ch}(\rho_\lambda(F_\alpha))$, for $\lambda \in \Lambda$ such that $(\alpha, \lambda) \in \mathcal{X}$, so that the set $\mathcal{X}$ satisfies the following conditions.

- (1) For every $\alpha \in A$, $(\alpha, \lambda) \in \mathcal{X}$ for all but finitely many $\lambda \in \Lambda$.
- (2) For all $(\alpha, \lambda) \in \mathcal{X}$, $T_\alpha \times_E E_\lambda$ is conjugate to a maximal torus of G_λ over E_λ .
- (3) For all pairwise distinct places $\lambda_1, \dots, \lambda_m \in \Lambda$, and all maximal tori T_i of G_{λ_i} , there exists an $\alpha \in A$ such that, for every i , $(\alpha, \lambda_i) \in \mathcal{X}$ and $T_\alpha \times_E E_{\lambda_i}$ is conjugate to T_i over E_{λ_i} .

The tori T_α over E are called *Frobenius tori*. For every $\alpha \in A$, let L_α be the splitting field of T_α over E_α , that is the smallest degree extension L_α/E such that $T_\alpha \times_E L_\alpha$ is split. Let L be the intersection of all L_α . We call this field the *splitting field* of the compatible system $\mathcal{R}$.

Assume now that $\Lambda = \bigcup_{\ell \in \mathcal{L}} |E|_\ell$, where $\mathcal{L}$ is a set of rational primes of Dirichlet density $\sigma > 0$. The role of Frobenius tori and the splitting field of a compatible system is clarified by the following results.

Proposition 3.2 [15, Proposition 8.9] *There exists a subset $\mathcal{L}'$ of $\mathcal{L}$ of Dirichlet density σ such that for each $\ell \in \mathcal{L}'$ and $\lambda \mid \ell$ the connected reductive group G_λ is unramified, and split over LE_λ .*

Corollary 3.3 *Let $\mathcal{L}'$ be a subset of $\mathcal{L}$ such that for each $\ell \in \mathcal{L}'$ and $\lambda \mid \ell$ the group G_λ is unramified. Then, there exist a E -split torus T_0 and a faithful representation $\rho_0 : T_0 \rightarrow \text{GL}_{n, E}$ such that, for each $\ell \in \mathcal{L}'$, each $\lambda \mid \ell$, and each finite place $\mathfrak{l}$ of L lying above λ , the torus $T_0 \times_E L_{\mathfrak{l}}$ is conjugate to a maximal torus in $G_\lambda \times_{E_\lambda} L_{\mathfrak{l}}$ over $L_{\mathfrak{l}}$, and $\rho_0 \otimes_E L_{\mathfrak{l}}$ is equivalent to the formal character of $G_\lambda \times_{E_\lambda} L_{\mathfrak{l}} \rightarrow \text{GL}_{n, L_{\mathfrak{l}}}$ over $L_{\mathfrak{l}}$.*

Theorem 3.4 (part of [15, Theorem 9.4]) *Assume that each ρ_λ is absolutely irreducible, and that $L = E$. Let $\mathcal{L}'$ be a subset of $\mathcal{L}$ such that for each $\ell \in \mathcal{L}'$ and $\lambda \mid \ell$ the group G_λ is unramified. Then, for all $\ell \in \mathcal{L}'$ and $\lambda \mid \ell$ the root datum Ψ_λ of G_λ is independent of λ up to isomorphism.*

4 Independence in compatible systems of Lie-irreducible representations

Let $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ be a group with Frobenii, let E be a number field, let $\mathcal{L}$ be a set of rational primes of Dirichlet density $\sigma > 0$, and let $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$. The following result holds.

Proposition 4.1 *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n absolutely Lie-irreducible² representations of Γ with coefficients in E . Let $\mathcal{L}'$ be a subset of $\mathcal{L}$ such that for each $\ell \in \mathcal{L}'$ and $\lambda \mid \ell$ the group G_λ° is unramified. Then, there exist a finite extension L of E and a split connected reductive algebraic subgroup G of $\mathrm{GL}_{n,L}$ defined over L such that for each $\ell \in \mathcal{L}'$, each $\lambda \mid \ell$, and each finite place $\mathfrak{l}$ of L lying above λ , the group $G_\lambda^\circ \times_{E_\lambda} L_{\mathfrak{l}}$ is conjugate to $G \times_L L_{\mathfrak{l}}$ over $L_{\mathfrak{l}}$.*

Proof Let Γ' be the open subgroup of Γ given by Proposition 2.3. As each ρ_λ is absolutely Lie-irreducible, the restrictions $\rho_\lambda|_{\Gamma'}$ are still absolutely irreducible. Up to restrict to Γ' , we can assume each G_λ to be connected. Let L be the splitting field of $\mathcal{R}$, let $\mathcal{L}' = \cup_{\ell \in \mathcal{L}'} |L|_\ell$, and for each $\mathfrak{l} \in \mathcal{L}'$, denoting by λ its restriction to E , let $\rho_{\mathfrak{l}} = \rho_\lambda \otimes_{E_\lambda} L_{\mathfrak{l}}$. Consider now the compatible system $\{\rho_{\mathfrak{l}}\}_{\mathfrak{l} \in \mathcal{L}'}$. Clearly, the algebraic monodromy group $G_{\mathfrak{l}}$ of $\rho_{\mathfrak{l}}$ is equal to $G_\lambda \times_{E_\lambda} L_{\mathfrak{l}}$ by construction. By Theorem 3.4, the root datum $\Psi_{\mathfrak{l}}$ of $G_{\mathfrak{l}}$ is independent of $\mathfrak{l}$ up to isomorphism. Fix such a root datum Ψ . Then, by [8, Theorem A.4.6], there exists a (unique up to isomorphism) split connected reductive subgroup G of $\mathrm{GL}_{n,L}$ with root datum Ψ , and such that $G \times_L L_{\mathfrak{l}}$ is conjugate to $G_{\mathfrak{l}}$ over $L_{\mathfrak{l}}$ for any $\mathfrak{l} \in \mathcal{L}'$. $\square$

Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n representations of Γ with coefficients in E . We introduce the following definition.

Definition 4.2 We say that $\mathcal{R}$ has a *Lie-irreducible decomposition* over E and at a subset $\mathcal{L}'$ of $\mathcal{L}$ if there exist open subgroups Γ_i of Γ , compatible systems $\mathcal{S}_i = \{\sigma_{i,\lambda}\}_{\lambda \in \Lambda'}$ of rank m_i absolutely Lie-irreducible representations of Γ_i with coefficients in E , where $\Lambda' = \cup_{\ell \in \mathcal{L}'} |E|_\ell$, and rank d_i Artin representations³ ω_i of Γ_i , for $i = 1, \dots, k$, such that

$$\rho_\lambda \cong \oplus_{i=1}^k \mathrm{Ind}_{\Gamma_i}^\Gamma (\sigma_{i,\lambda} \otimes \omega_i)$$

over E_λ for all $\ell \in \mathcal{L}'$ and $\lambda \mid \ell$. In this case, we write

$$\mathcal{R} \cong_{(\mathcal{L}')} \oplus_{i=1}^k \mathrm{Ind}_{\Gamma_i}^\Gamma (\mathcal{S}_i \otimes \omega_i).$$

Remark 4.3 For some choices of the group Γ , one can prove that any λ -adic representation $\rho : \Gamma \rightarrow \mathrm{GL}_n(\overline{E}_\lambda)$ has a Lie-irreducible decomposition $\rho \cong \oplus_{i=1}^k \mathrm{Ind}_{\Gamma_i}^\Gamma (\sigma_i \otimes \omega_i)$,

² We say that a rank n representation ρ of Γ with coefficients in a field k of characteristic zero is *absolutely Lie-irreducible* if $\rho|_{\Gamma'}$ is absolutely irreducible for any open subgroup Γ' of Γ .

³ In this setting, we say that a continuous representation $\omega : \Gamma \rightarrow \mathrm{GL}_d(\overline{E}_\lambda)$ is *Artin* if it factors through a finite quotient of Γ . Any Artin representation ω can be actually realised over $\mathbb{Q}$.

where Γ_i is an open subgroup of Γ , $\sigma_i : \Gamma_i \rightarrow \mathrm{GL}_{m_i}(\overline{E}_\lambda)$ is Lie-irreducible, and $\omega_i : \Gamma_i \rightarrow \mathrm{GL}_{d_i}(\overline{E}_\lambda)$ is Artin, for $i = 1, \dots, k$. This has been proved by Katz, see [12, Proposition 1], when Γ is the étale fundamental group of a smooth connected affine curve over an algebraically closed field of positive characteristic, and by Patrikis, see [17, Proposition 3.4.1], by adapting Katz's argument, when Γ is the absolute Galois group of a number field. To prove a decomposition result for a compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ of representations of Γ , one would then start from a single λ -adic representation ρ_λ in $\mathcal{R}$, decompose it as $\rho_\lambda \cong \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_i}^\Gamma(\sigma_{i,\lambda} \otimes \omega_i)$, and extend each $\sigma_{i,\lambda}$ to a compatible system $\mathcal{S}_i$ of Lie-irreducible representations. The isomorphism $\mathcal{R} \cong_{(\mathcal{L}')} \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_i}^\Gamma(\mathcal{S}_i \otimes \omega_i)$ would then follow by considerations on the traces of Frobenius elements, see the proofs of Theorem 5.4 and Theorem 6.2. However, extending a λ -adic representation to a compatible system is a highly non trivial problem, and can be achieved, for instance, under suitable assumptions in the number field case by means of potential automorphy techniques, see Sect. 5, and in the positive characteristic case by means of the global Langlands correspondence for GL_n of L. Lafforgue, see Sect. 6.

Proposition 4.1 has the following consequence.

Corollary 4.4 *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n representations of Γ with coefficients in E . Assume that $\mathcal{R}$ has a Lie-irreducible decomposition over a finite extension E' of E and at a subset $\mathcal{L}'$ of $\mathcal{L}$. Let $\mathcal{L}''$ be a subset of $\mathcal{L}'$ such that for each $\ell \in \mathcal{L}''$ and $\lambda \mid \ell$ the group G_λ° is unramified. Then, there exist a finite extension L of E' and a split connected reductive algebraic subgroup G of $\mathrm{GL}_{n,L}$ defined over L such that for each $\ell \in \mathcal{L}''$, each $\lambda \mid \ell$, and each finite place $\mathfrak{l}$ of L lying above λ , the group $G_\lambda^\circ \times_{E_\lambda} L_{\mathfrak{l}}$ is conjugate to $G \times_L L_{\mathfrak{l}}$ over $L_{\mathfrak{l}}$.*

Proof By assumption, there exist open subgroups Γ_i of Γ , compatible systems $\mathcal{S}_i = \{\sigma_{i,\lambda'}\}_{\lambda' \in \Lambda'}$ of rank m_i absolutely Lie-irreducible representations of Γ_i with coefficients in E' , where $\Lambda' = \bigcup_{\ell \in \mathcal{L}'} |\mathcal{L}'|_\ell$, and rank d_i Artin representations ω_i of Γ_i , for $i = 1, \dots, k$, such that

$$\rho_\lambda \otimes_{E_\lambda} E'_{\lambda'} \cong \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_i}^\Gamma(\sigma_{i,\lambda'} \otimes \omega_i)$$

over $E'_{\lambda'}$ for each $\ell \in \mathcal{L}'$, each $\lambda \mid \ell$, and each $\lambda' \mid \lambda$. The algebraic monodromy group $G_\lambda \times_{E_\lambda} E'_{\lambda'}$ of $\rho_\lambda \otimes_{E_\lambda} E'_{\lambda'}$ only depends on the algebraic monodromy groups of $\sigma_{i,\lambda'}$ and ω_i , for $i = 1, \dots, k$.

For any i and any $\lambda' \in \Lambda'$, denote by $H_{\sigma_{i,\lambda'}}$ the algebraic monodromy group of $\sigma_{i,\lambda'}$. Note that for each $\ell \in \mathcal{L}''$ and $\lambda' \mid \ell$, the group $H_{\sigma_{i,\lambda'}}^\circ$ is unramified. Then, by Proposition 4.1, for any i there exist a finite extension L_i of E' and a connected reductive algebraic subgroup H_i of GL_{m_i,L_i} defined over L_i such that for each $\ell \in \mathcal{L}''$, each $\lambda' \mid \ell$, and each finite place $\mathfrak{l}$ of L_i lying above λ , the group $H_{\sigma_{i,\lambda'}}^\circ \times_{E'_{\lambda'}} L_{i,\mathfrak{l}}$ is conjugate to $H_i \times_{L_i} L_{i,\mathfrak{l}}$ over $L_{i,\mathfrak{l}}$.

Since for any i the representation ω_i is Artin, it can be defined over $\overline{\mathbb{Q}}$. Its algebraic monodromy group H_{ω_i} is then a finite subgroup of $\mathrm{GL}_{n,\overline{\mathbb{Q}}}$, which can be defined over a number field L_{ω_i} . Let us enlarge L_i so that it contains L_{ω_i} .

The conclusion follows by taking L to be the composite of the extensions L_i for $i = 1, \dots, k$. $\square$

5 Geometric compatible systems of Galois representations

We see now an application of the abstract results of Sect. 4 to the case of geometric compatible systems of Galois representations attached to automorphic representations. In this context, we refer to [1] for the background terminology.

Let F be a number field, let Γ_F denote the absolute Galois group of F , and let S be a finite set of places of F . Let E be a number field, and let Λ be a set of finite places of E . Let $n \geq 1$ be an integer. We recall the following definition.

Definition 5.1 A *geometric*⁴ compatible system of rank n representations of Γ_F defined over E and unramified outside S is a family $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ of continuous semisimple representations

$$\rho_\lambda : \Gamma_F \rightarrow \mathrm{GL}_n(\overline{E}_\lambda)$$

such that:

- (1) If $v \notin S$ is a finite place of F , then for all λ not dividing the residue characteristic of v , the representation ρ_λ is unramified at v , and the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_v)$ has coefficients in E and is independent of λ .
- (2) Each representation ρ_λ is de Rham at all places above the residue characteristic of λ , and in fact crystalline at any place $v \notin S$ above the residue characteristic of λ .
- (3) For each embedding $\tau : F \rightarrow \overline{E}$ the τ -Hodge-Tate weights of ρ_λ are independent of λ .

Clearly, any geometric compatible system $\mathcal{R}$ is a compatible system in the sense of Definition 1.1, with $\mathcal{X} = \{(v, \lambda) \in |F| \times \Lambda : v \notin S, \lambda \nmid \mathrm{char}(k(v))\}$, where $k(v)$ denotes the residue field of F at v . All the definitions introduced in the case of general compatible systems, as well as all the results proved, are therefore valid also for geometric compatible systems of Galois representations.

Let F be a CM field, let F^+ be the maximal totally real subfield of F , and let c be a generator of $\mathrm{Gal}(F/F^+)$. Thanks to the work of many people, e.g. [6,7,11,13,21], geometric compatible systems can be attached to regular algebraic, cuspidal, polarised automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. We recall here the general statement of this construction,⁵ which also involves results of [4,22] and [5]. We have:

Theorem 5.2 *Let (π, χ) be a regular algebraic, cuspidal, polarised automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, unramified outside a finite set S of places of F . Then, there*

⁴ We add “geometric” to the terminology of [1, §5.1] in order to distinguish these compatible systems from the “abstract” compatible systems of Definition 1.1.

⁵ To simplify the exposition, in this paper we decided to restrict to the case of imaginary CM fields. Also, an analogous construction holds when F is totally real.

exist a number field M_π , a compatible system $\mathcal{R}_\pi = \{\rho_{\pi,\lambda}\}_{\lambda \in |M_\pi|}$ of rank n semisimple representations

$$\rho_{\pi,\lambda} : \Gamma_F \rightarrow \mathrm{GL}_n(\overline{M}_{\pi,\lambda})$$

defined over M_π , and an integer w such that:

- (1) $(\rho_{\pi,\lambda}, \epsilon_\lambda^{1-n} \rho_{\chi,\lambda})$ is a totally odd, polarised λ -adic representation.⁶
- (2) If v is a place of F not dividing the residue characteristic of λ , then, given an isomorphism $\iota : \overline{M}_{\pi,\lambda} \xrightarrow{\sim} \mathbb{C}$, we have the local-global compatibility

$$\iota \mathrm{WD}(\rho_{\pi,\lambda} |_{\Gamma_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}_v(\pi_v \otimes |\det|_v^{(1-n)/2}),$$

and these Weil-Deligne representations are pure of weight w .

- (3) Each representation $\rho_{\pi,\lambda}$ is de Rham at all places above the residue characteristic of λ , and for each embedding $\tau : F \rightarrow \overline{M}_\pi$ the τ -Hodge-Tate weights of $\rho_{\pi,\lambda}$ have multiplicity at most one, and are given by

$$\mathrm{HT}_\tau(\rho_{\pi,\lambda}) = \{a_{\tau,1} + n - 1, a_{\tau,2} + n - 2, \dots, a_{\tau,n}\},$$

where $a = (a_{\tau,i})$ is the weight of π . Moreover

$$\mathrm{HT}_{\tau \circ c}(\rho_{\pi,\lambda}) = \{w - h : h \in \mathrm{HT}_\tau(\rho_{\pi,\lambda})\}.$$

- (4) If v is a place of F dividing the residue characteristic of λ , then, given an isomorphism $\iota : \overline{M}_{\pi,\lambda} \xrightarrow{\sim} \mathbb{C}$, we have the local-global compatibility

$$\iota \mathrm{WD}(\rho_{\pi,\lambda} |_{\Gamma_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}_v(\pi_v \otimes |\det|_v^{(1-n)/2}).$$

In particular, $\rho_{\pi,\lambda}$ is semistable at v , and if $v \notin S$ then it is crystalline.

The compatible system $\mathcal{R}_\pi$ is then a geometric compatible system of rank n representations of Γ_F defined over M_π and unramified outside S , in the sense of Definition 5.1. Moreover, it has coefficients in a finite extension E_π of M_π by [6, Proposition 3.2.5], it is totally odd, polarised by the geometric compatible system $\mathcal{M} = \{\epsilon_\lambda^{1-n} \rho_{\chi,\lambda}\}_{\lambda \in |M_\pi|}$ of characters of Γ_{F^+} , and it is strictly pure of weight w , and regular.

The global Langlands conjectures in this case predict all the representations $\rho_{\pi,\lambda}$ to be irreducible. For the purposes of this paper, we consider the following weaker conjecture.

Conjecture 5.3 *Let π be a regular algebraic, polarisable, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then, there exists a set of rational primes $\mathcal{L}$ of Dirichlet density 1 such that for all $\ell \in \mathcal{L}$ and $\lambda \mid \ell$ the representation*

$$\rho_{\pi,\lambda} : \Gamma_F \rightarrow \mathrm{GL}_n(\overline{M}_{\pi,\lambda})$$

⁶ Here ϵ_λ denotes the λ -adic cyclotomic character of Γ_{F^+} , and $\rho_{\chi,\lambda}$ denotes the λ -adic character of Γ_{F^+} attached to χ by global class field theory.

is irreducible.

The proof of this conjecture is a widely open problem in general, and only some partial progress has been obtained so far, see for instance [1, Theorem 5.5.2]. A weaker irreducibility result has been proved in [18]. First of all, let us note that we may assume that the field M_π is a CM full field of definition for $\mathcal{R}_\pi$, see [18, Lemma 1.2, 1.4]. We say that a place λ of a CM field M is *conjugation invariant* if ${}^c\lambda = \lambda$. By [18, Theorem 1.7], there exists a set of rational primes $\mathcal{L}$ of Dirichlet density 1 such that for all $\ell \in \mathcal{L}$ and all conjugation invariant $\lambda \mid \ell$ the representation $\rho_{\pi,\lambda}$ is irreducible. In particular, there exists a set of rational primes $\mathcal{L}'$ of positive Dirichlet density such that if λ divides some $\ell \in \mathcal{L}'$, then $\rho_{\pi,\lambda}$ is irreducible.

Let F be a CM field, let E be a number field, let $\mathcal{L}$ be a set of rational primes of Dirichlet density 1, and let $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$. We prove the following result.

Theorem 5.4 *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a pure, regular, totally odd, polarisable geometric compatible system of rank n representations of Γ_F with coefficients in E . Then, $\mathcal{R}$ has a Lie-irreducible decomposition over a finite extension E' of E and at a subset $\mathcal{L}'$ of $\mathcal{L}$ of positive Dirichlet density. Moreover, assuming Conjecture 5.3, we can choose the set $\mathcal{L}'$ to have Dirichlet density 1.*

Proof By [18, Lemma 1.2, 1.4], there exists a CM full field of definition M for $\mathcal{R}$. Up to enlarging E , we can assume that $M \subset E$. Let $\mathcal{M} = \{\mu_\lambda\}_{\lambda \in \Lambda}$ be a geometric compatible system of characters of Γ_{F^+} such that for each $\lambda \in \Lambda$ the pair $(\rho_{\pi,\lambda}, \mu_\lambda)$ is a totally odd, polarised λ -adic representation. For each $\lambda \in \Lambda$, let

$$\rho_\lambda \otimes_{E_\lambda} \overline{E}_\lambda \cong \bigoplus_{i=1}^{k_\lambda} \rho_{i,\lambda}$$

be an irreducible decomposition. Then, if the restriction of λ to M is conjugation invariant, the pair $(\rho_{i,\lambda}, \mu_\lambda)$ is a totally odd, polarised λ -adic representation for each $i = 1, \dots, k_\lambda$ by [18, Lemma 1.6]. Up to removing finitely many primes from $\mathcal{L}$, we can also assume that if $\ell \in \mathcal{L}$, and $\lambda \mid \ell$, then

- $\ell \geq 2(n+1)$,
- $\zeta_\ell \notin F$,
- ℓ is unramified in F and lies below no element of the set of bad places for $\mathcal{R}$,
- the Hodge-Tate weights of ρ_λ lie in a range of the form $[a, a + \ell - 2]$.

Note that for each $\ell \in \mathcal{L}$ and $\lambda \mid \ell$, and each $i = 1, \dots, k_\lambda$, the set of Hodge-Tate weights of $\rho_{i,\lambda}$ is a subset of the set of Hodge-Tate weights of ρ_λ , and so it is contained in $[a, a + \ell - 2]$. It follows by [1, Lemma 1.4.3] that each $\rho_{i,\lambda}$ is potentially diagonalisable at each place w of F lying above ℓ . By [1, Proposition 5.3.2], there exists a subset $\mathcal{L}^0$ of $\mathcal{L}$ of Dirichlet density 1 such that if $\ell \in \mathcal{L}^0$, then $\overline{\rho}_{i,\lambda} \mid_{\Gamma_{F(\zeta_\ell)}}$ is irreducible for every $\lambda \mid \ell$, and $i = 1, \dots, k_\lambda$. Let $\mathcal{L}'$ be the subset of $\mathcal{L}^0$ consisting of the primes ℓ such that each place of M above ℓ is conjugation invariant. This set has positive Dirichlet density by [18, Lemma 1.5]. Let $\Lambda' = \cup_{\ell \in \mathcal{L}'} |E|_\ell$. For each $\lambda \in \Lambda'$ and $i = 1, \dots, k_\lambda$, by [1, Theorem 5.5.1] the representation $\rho_{i,\lambda}$ is part of a strictly pure compatible system $\mathcal{R}_i = \{\rho_{i,\lambda}\}_{\lambda \in \Lambda'}$, with coefficients in a finite extension E_i of E . By [18, Theorem 1.7], for each $\lambda \in \Lambda'$, each $\rho_{i,\lambda}$ is irreducible, and so k_λ is

independent of $\lambda \in \Lambda'$. Note that assuming Conjecture 5.3, we can find a subset $\mathcal{L}'$ of $\mathcal{L}$ of Dirichlet density 1 such that the same conclusion holds. Set $k = k_\lambda$ for $\lambda \in \Lambda'$.

Fix $i = 1, \dots, k$, and consider the compatible system $\mathcal{R}_i$. For all $\lambda \in \Lambda'$, let $G_{i,\lambda}$ be the algebraic monodromy group⁷ of $\rho_{i,\lambda}$, and let $G_{i,\lambda}^\circ$ be the connected component of the identity of $G_{i,\lambda}$. By Lemma 2.3 there exists a finite Galois extension F'_i/F such that the representation $\rho_{i,\lambda}$ induces an isomorphism $\text{Gal}(F'_i/F) \xrightarrow{\sim} G_{i,\lambda}/G_{i,\lambda}^\circ$ for all $\lambda \in \Lambda'$. Up to removing finitely many places from $\mathcal{L}'$, we may further assume that if $\ell \in \mathcal{L}'$, and $\lambda \mid \ell$, then

- $\zeta_\ell \notin F'_i$,
- ℓ is unramified in F'_i and lies below no element of the set of bad places for $\mathcal{R}_i$,

Let F_i^{cm} denote the maximal CM subfield of F'_i , and $F_i^{\text{cm},+}$ the maximal totally real subfield of F_i^{cm} . Then, by [18, Lemma 1.6], for each irreducible component s of $\rho_{i,\lambda} \mid_{\Gamma_{F_i^{\text{cm}}}}$, the pair $(s, \mu_\lambda \mid_{\Gamma_{F_i^{\text{cm},+}}})$ is a totally odd, polarised λ -adic representation. Again, note that assuming Conjecture 5.3 this holds true above a set $\mathcal{L}'$ of primes of Dirichlet density 1.

By [1, Lemma 5.3.1(2)] each $\rho_{i,\lambda}$ is Lie-multiplicity free.⁸ Therefore, for each $\lambda \in \Lambda'$, by [17, Lemma 3.4.6(1)] there exist an intermediate field $F \subset F_i^\lambda \subset F'_i$, and a rank m_i^λ Lie-irreducible representation σ_i^λ of $\Gamma_{F_i^\lambda}$ such that we can write $\rho_{i,\lambda} \cong \text{Ind}_{\Gamma_{F_i^\lambda}}^{\Gamma_F} \sigma_i^\lambda$.

We now prove that there exists $\lambda_0 \in \Lambda'$ such that $F_i^{\lambda_0}$ is a CM field.⁹ Denote by $F_i^{\lambda,\text{cm}}$ the maximal CM subfield of F_i^λ . For simplicity, let us enlarge E_i to contain the maximal CM subfield of F'_i , so that $F_i^{\lambda,\text{cm}} \subset E_i$ for all $\lambda \in \Lambda'$, and let us take its Galois closure. If $F_i^\lambda \neq F_i^{\lambda,\text{cm}}$ for all $\lambda \in \Lambda'$, then we can find a place $\lambda_0 \in \Lambda'$ with residue characteristic ℓ_0 such that ℓ_0 splits in E_i but not in $F_i^{\lambda_0}$. Let now w_0 be a non-split place of $F_i^{\lambda_0}$ of residue characteristic ℓ_0 . Since $F_{i,w_0}^{\lambda_0}$ does not embed in $E_{i,\lambda_0} = \mathbb{Q}_{\ell_0}$ by assumption, we can deduce by an argument analogous to that of [17, Lemma 3.4.13] that ρ_{i,λ_0} is not regular, which is a contradiction. Therefore, we get that $F_i^{\lambda_0} = F_i^{\lambda_0,\text{cm}}$. Also, $\ell_0 \geq 2(m_i^{\lambda_0} + 1)$ and $\zeta_{\ell_0} \notin F_i^{\lambda_0}$ by the previous assumptions. For simplicity of notation, we set $F_i = F_i^{\lambda_0}$. Also, let F_i^+ be the maximal totally real subfield of F_i .

The representation $\sigma_i^{\lambda_0}$ is an irreducible component of $\rho_{i,\lambda_0} \mid_{\Gamma_{F_i}}$, and so the pair $(\sigma_i^{\lambda_0}, \mu_{\lambda_0} \mid_{\Gamma_{F_i^+}})$ is a totally odd, polarised λ_0 -adic representation, as $F_i \subset F_i^{\text{cm}}$, and the set of Hodge-Tate weights of $\sigma_i^{\lambda_0}$ is a subset of the set of Hodge-Tate weights of ρ_{i,λ_0} . This gives that $\sigma_i^{\lambda_0}$ has Hodge-Tate weights lying in $[a, a + \ell_0 - 2]$. Since ℓ_0 is unramified in F'_i , and so it is in F_i as well, we have that $\sigma_i^{\lambda_0}$ is crystalline at each

⁷ This is a reductive algebraic subgroup of $\text{GL}_n, E_{i,\lambda'}$, for λ' a place of E_i above λ .

⁸ We say that a rank n representation ρ of a profinite group Γ with coefficients in an algebraically closed field k of characteristic zero is *Lie-multiplicity free* if for any open subgroup Γ' of Γ any irreducible Γ' -subrepresentation of ρ has multiplicity 1.

⁹ The following argument is largely adapted on that of [17, Lemma 3.4.13].

place w_0 of F_i lying above ℓ_0 by [17, Lemma 2.2.9]. Therefore, $\sigma_i^{\lambda_0}$ is potentially diagonalisable at each place w_0 of F_i lying above ℓ_0 by [1, Lemma 1.4.3].

By applying the Mackey restriction formula to $\bar{\rho}_{i,\lambda_0}|_{\Gamma_{F(\zeta_{\ell_0})}} \cong \text{Res}_{\Gamma_{F(\zeta_{\ell_0})}}^{\Gamma_F} \text{Ind}_{\Gamma_{F_i}}^{\Gamma_F} \bar{\sigma}_i^{\lambda_0}$, which is irreducible by assumption, we deduce that $\bar{\sigma}_i^{\lambda_0}|_{\Gamma_{F_i(\zeta_{\ell_0})}}$ is irreducible. Therefore, we can apply [1, Theorem 5.5.1] to get that $\sigma_i^{\lambda_0}$ is part of a strictly pure compatible system $\mathcal{S}_i = \{\sigma_{i,\lambda}\}_{\lambda \in \Lambda'}$ of representations of Γ_{F_i} with coefficients in a finite extension E'_i of E_i .

Since $\mathcal{S}_i$ is regular, each representation $\sigma_{i,\lambda}$ is Lie-multiplicity free, again by [1, Lemma 5.3.1(2)]. Also, each $\sigma_{i,\lambda}$ is irreducible by [18, Theorem 1.7]. Since $\sigma_{i,\lambda_0} \cong \sigma_i^{\lambda_0}$ is Lie-irreducible, and Lie-irreducibility in a compatible system of irreducible, Lie-multiplicity free representations is independent of λ by [17, Corollary 3.4.11], we get that $\sigma_{i,\lambda}$ is Lie-irreducible for each $\lambda \in \Lambda'$. As above, let us remark that assuming Conjecture 5.3, we can find a set $\mathcal{L}'$ of primes of Dirichlet density 1 containing ℓ_0 , and such that this conclusion holds.

For every finite place v of F outside the set of bad places for $\mathcal{R}_i$ and every $\lambda \in \Lambda'$ not lying above the residue characteristic of v we have that

$$\begin{aligned} \text{Tr } \rho_{i,\lambda}(\text{Frob}_v) &= \text{Tr } \rho_{i,\lambda_0}(\text{Frob}_v) \\ &= \text{Tr } \text{Ind}_{\Gamma_{F_i}}^{\Gamma_F} \sigma_i^{\lambda_0}(\text{Frob}_v) \\ &= \text{Tr } \text{Ind}_{\Gamma_{F_i}}^{\Gamma_F} \sigma_{i,\lambda}(\text{Frob}_v), \end{aligned}$$

where the first equality follows from the independence of λ of the characteristic polynomials at the Frobenius elements in the compatible system $\mathcal{R}_i$, and the last equality follows from the usual formula for the trace of an induced representation, see [1, §5.5] for instance, and the independence of λ of the characteristic polynomials at the Frobenius elements in the compatible system $\mathcal{S}_i$. Combining the Čebotarev density theorem with [3, §12.1, Proposition 3] we deduce that $\rho_{i,\lambda} \cong \text{Ind}_{\Gamma_{F_i}}^{\Gamma_F} \sigma_{i,\lambda}$ for every $\lambda \in \Lambda'$.

Let E' be a finite extension of E containing all the fields E'_i . Descend each $\sigma_{i,\lambda}$ to a representation $\sigma_{i,\lambda'}$ over $E'_{\lambda'}$ for $\lambda' \mid \lambda$. Since $\sigma_{i,\lambda'}$ is absolutely Lie-irreducible for each $\ell \in \mathcal{L}'$ and $\lambda' \mid \ell$, we have that

$$\mathcal{R} \otimes_E E' \cong_{(\mathcal{L}')} \bigoplus_{i=1}^k \text{Ind}_{\Gamma_{F_i}}^{\Gamma_F} \mathcal{S}_i \otimes_{E'_i} E'$$

is a Lie-irreducible decomposition over E' and at $\mathcal{L}'$. □

Combining this with Corollary 4.4 we get the following result.

Corollary 5.5 *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a pure, regular, totally odd, polarisable geometric compatible system of rank n representations of Γ_F with coefficients in E . Then, there exist a subset $\mathcal{L}'$ of $\mathcal{L}$ of positive Dirichlet density, a finite extension L of E , and a split connected reductive algebraic subgroup G of $\text{GL}_{n,L}$ such that for each $\ell \in \mathcal{L}'$, each $\lambda \mid \ell$, and each finite place $\mathfrak{l}$ of L lying above λ , the group $G_\lambda^\circ \times_{E_\lambda} L_{\mathfrak{l}}$ is conjugate to $G \times_L L_{\mathfrak{l}}$ over $L_{\mathfrak{l}}$. Moreover, assuming Conjecture 5.3, we can choose the set $\mathcal{L}'$ to have Dirichlet density 1.*

Remark 5.6 Note that if the compatible system $\mathcal{R}$ has extremely regular weights, then the sets $\mathcal{L}'$ of Theorem 5.4 and Corollary 5.5 can be chosen to have Dirichlet density 1 unconditionally.

Remark 5.7 Let π be a regular algebraic, polarisable, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and let $\mathcal{R}_\pi = \{\rho_{\pi,\lambda}\}_{\lambda \in |M_\pi|}$ be the corresponding compatible system of representations of Γ_F . Then, the conclusions of Theorem 5.4 and Corollary 5.5 clearly hold for $\mathcal{R}_\pi$. We then have that there exist a finite extension E' of E_π , a set $\mathcal{L}'$ of rational primes of positive Dirichlet density, a finite (possibly trivial) CM extension F_π of F , and a compatible system $\mathcal{S} = \{\sigma_\lambda\}_{\lambda \in \Lambda'}$ of absolutely Lie-irreducible representations of Γ_{F_π} with coefficients in E' , where $\Lambda' = \cup_{\ell \in \mathcal{L}'} |E'|_\ell$, such that $\mathcal{R}_\pi \otimes_{E_\pi} E' \cong_{\mathcal{L}'} \mathrm{Ind}_{\Gamma_{F_\pi}}^{\Gamma_F} \mathcal{S}$. Notice in particular that the compatible system $\mathcal{S}$ is by construction the compatible system attached to a regular algebraic, polarisable, cuspidal automorphic representation τ of $\mathrm{GL}_m(\mathbb{A}_{F_\pi})$, and so π should be thought of as the “automorphic induction” of τ from F_π . In analogy with the case of modular forms of weight $k \geq 2$, we suggest to think of π as having “complex multiplication” by F_π . Also, we have that there exists a connected reductive group G_π defined over a finite extension of E_π which interpolates the groups of connected components of the algebraic monodromy groups of $\rho_{\pi,\lambda}$, for λ above a set of primes of positive Dirichlet density. In analogy with the Mumford-Tate conjecture, the group G_π should be related to a notion of “Mumford-Tate group” for π .

6 Compatible systems in the positive characteristic case

In this last section we see applications of the abstract independence results of Sect. 4 to positive characteristic settings. We start with the case of compatible systems of representations of the absolute Galois group of a global function field, and we then move to compatible systems of lisse sheaves on a variety over a finite field.

Let p be a prime, let q be a power of p , and let C be a smooth projective curve, geometrically connected over $\mathbb{F}_q$. Let F be the field of rational functions on C , and let Γ_F denote the absolute Galois group of F . The following preliminary result is essentially a reformulation in the characteristic p setting of [17, Proposition 3.4.1], which in turn adapts the arguments of [12, Proposition 1].

Lemma 6.1 *Let $\rho : \Gamma_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be an irreducible representation. Then either ρ is induced from a representation of an open proper subgroup of Γ_F , or ρ is Lie-irreducible, or there exist an integer $d \geq 2$ dividing n , a Lie-irreducible representation σ of Γ_F of dimension n/d , and an Artin representation ω of Γ_F of dimension d such that $\rho \cong \sigma \otimes \omega$.*

The proof of this lemma is just the same as the proof of [17, Proposition 3.4.1], where the key ingredient is the vanishing of $H^2(\Gamma_F, \mathbb{Q}/\mathbb{Z})$ given by Tate’s theorem. This holds true also in the positive characteristic case, see [20, Theorem 4].

Let E be a number field, let $\mathcal{L}$ be a set of rational primes of Dirichlet density 1, and let $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$. We prove the following result.

Theorem 6.2 *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of semisimple representations of Γ_F with coefficients in E . Then, $\mathcal{R}$ has a Lie-irreducible decomposition over a finite extension E' of E and at $\mathcal{L} \setminus \{p\}$.*

Proof Fix $\lambda_0 \in \Lambda_{\neq p}$. By Lemma 6.1 there exist finite extensions F_i of F , Lie-irreducible representations $\sigma_i : \Gamma_{F_i} \rightarrow \mathrm{GL}_{m_i}(\overline{E}_{\lambda_0})$, Artin representations $\omega_i : \Gamma_{F_i} \rightarrow \mathrm{GL}_{d_i}(\overline{E}_{\lambda_0})$, for $i = 1, \dots, k$, such that

$$\rho_{\lambda_0} \otimes_{E_{\lambda_0}} \overline{E}_{\lambda_0} \cong \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} (\sigma_i \otimes \omega_i).$$

For each i , choose $\chi_i : \Gamma_{F_i} \rightarrow \overline{E}_{\lambda_0}^\times$ such that the determinant of $\tau_i = \sigma_i \otimes \chi_i$ has finite order. Then, τ_i extends to a compatible system $\{\tau_{i,\lambda}\}_{\lambda \in \Lambda_{\neq p}}$ by the global Langlands correspondence [14, Théorème VI.9]. Also, by global class field theory we can extend χ_i to a compatible system $\{\chi_{i,\lambda}\}_{\lambda \in \Lambda_{\neq p}}$. For each $\lambda \in \Lambda_{\neq p}$, let $\sigma_{i,\lambda} = \tau_{i,\lambda} \otimes \chi_{i,\lambda}^{-1}$. We then have that $\mathcal{S}_i = \{\sigma_{i,\lambda}\}_{\lambda \in \Lambda_{\neq p}}$ is a compatible system extending σ_i .

Let E' be a finite extension of E such that each $\mathcal{S}_i$ has coefficients in E' , and each ω_i can be realised over E' . Then, for any place v of F and any $\lambda \in \Lambda_{\neq p}$ not lying above the residue characteristic of v we have

$$\begin{aligned} \mathrm{Tr} \rho_\lambda(\mathrm{Frob}_v) &= \mathrm{Tr} \rho_{\lambda_0}(\mathrm{Frob}_v) \\ &= \mathrm{Tr} \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} (\sigma_i \otimes \omega_i)(\mathrm{Frob}_v) \\ &= \mathrm{Tr} \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} (\sigma_{i,\lambda} \otimes \omega_i)(\mathrm{Frob}_v), \end{aligned}$$

by independence of λ of the characteristic polynomials at the Frobenius elements, and by the usual formulas for the traces of direct sums, tensor products, and induced representations. Combining the Čebotarev density theorem with [3, §12.1, Proposition 3], it then follows that

$$\mathcal{R} \otimes_E E' \cong_{\mathcal{L} \setminus \{p\}} \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} (\mathcal{S}_i \otimes \omega_i) \otimes_E E'$$

is a Lie-irreducible decomposition for $\mathcal{R}$ over E' . □

By Corollary 4.4 we then have the following immediate consequence.

Corollary 6.3 *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of semisimple representations of Γ_F with coefficients in E . Then, there exist a subset $\mathcal{L}'$ of $\mathcal{L} \setminus \{p\}$ of Dirichlet density 1, a finite extension L of E , and a split connected reductive algebraic subgroup G of $\mathrm{GL}_{n,L}$ such that for each $\ell \in \mathcal{L}'$, each $\lambda \mid \ell$, and each finite place $\mathfrak{l}$ of L lying above λ , the group $G_\lambda^\circ \times_{E_\lambda} L_\mathfrak{l}$ is conjugate to $G \times_L L_\mathfrak{l}$ over $L_\mathfrak{l}$.*

Let us now consider the case of compatible systems of lisse sheaves on a variety over $\mathbb{F}_q$. Refer to [10, §1.1–§1.3] for the basic definitions and results in this setting.

Let X be a geometrically connected variety over $\mathbb{F}_q$. Let $\bar{\eta}$ be a geometric point of X , and let $\pi_1^{\text{ét}}(X, \bar{\eta})$ be the arithmetic fundamental group of X . Let $\ell \neq p$ be a prime.

The functor which assigns to each lisse $\overline{\mathbb{Q}}_\ell$ -sheaf $\mathcal{F}$ on X its fibre $\mathcal{F}_{\overline{\eta}}$ over $\overline{\eta}$ induces an equivalence between the category of lisse $\overline{\mathbb{Q}}_\ell$ -sheaves on X and the category of finite dimensional continuous representations of $\pi_1^{\text{ét}}(X, \overline{\eta})$ with coefficients in $\overline{\mathbb{Q}}_\ell$. Via this equivalence, standard notions about representations (e.g semisimplicity, irreducibility, or Lie-irreducibility) can be translated to the context of lisse sheaves.

Let E be a number field, and let Λ be a set of finite places of E . We recall the following definition.

Definition 6.4 A *compatible system* of lisse E_λ -sheaves on X is a family $\mathcal{R} = \{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ of lisse E_λ -sheaves $\mathcal{F}_\lambda$ on X such that, for any closed point $x \in |X|$ of X and any $\lambda \in \Lambda$, the polynomial $\det(1 - \text{Frob}_x t, \mathcal{F}_\lambda)$ has coefficients in E and is independent of λ .

Any compatible system of lisse E_λ -sheaves $\mathcal{R} = \{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ on X defines a compatible system $\{\rho_\lambda\}_{\lambda \in \Lambda \neq p}$ of representations of $\pi_1^{\text{ét}}(X, \overline{\eta})$ with coefficients in E and indexed by Λ , in the sense of Definition 1.1, where ρ_λ is the representation $\mathcal{F}_{\lambda, \overline{\eta}}$, and $\mathcal{X} = |X| \times \Lambda \neq p$. The algebraic monodromy group G_λ of ρ_λ is called the *arithmetic monodromy group* of $\mathcal{F}_\lambda$. As usual, we denote by G_λ° the connected component of the identity of G_λ . Also, for simplicity of notation we omit the base point $\overline{\eta}$ and write just $\pi_1^{\text{ét}}(X)$ for the étale fundamental group of X .

Assume that $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$, where $\mathcal{L}$ is a set of rational primes of Dirichlet density 1. From Corollary 6.3 we deduce the following λ -independence result.

Corollary 6.5 *Let X be a normal geometrically connected irreducible variety over $\mathbb{F}_q$, and let $\mathcal{R} = \{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of semisimple lisse E_λ -sheaves on X . Then, there exist a subset $\mathcal{L}'$ of $\mathcal{L} \setminus \{p\}$ of Dirichlet density 1, a finite extension L of E , and a split connected reductive algebraic subgroup G of $\text{GL}_{n, L}$ such that for each $\ell \in \mathcal{L}'$, each $\lambda \mid \ell$, and each finite place $\mathfrak{l}$ of L lying above λ , the group $G_\lambda^\circ \times_{E_\lambda} L_{\mathfrak{l}}$ is conjugate to $G \times_L L_{\mathfrak{l}}$ over $L_{\mathfrak{l}}$.*

Proof Let $\{\rho_\lambda\}_{\lambda \in \Lambda \neq p}$ be the compatible system of representations of $\pi_1^{\text{ét}}(X)$ corresponding to $\mathcal{R}$. By [2, Corollary 3.33] there exist a finite étale cover $\varphi : X' \rightarrow X$, and a curve C embedding into X' via some ι such that $\rho_\lambda(\varphi_* \pi_1^{\text{ét}}(X')) = \rho_\lambda(\varphi_* \iota_* \pi_1^{\text{ét}}(C))$, for all $\lambda \in \Lambda \neq p$.

For any $\lambda \in \Lambda \neq p$, let $\rho'_\lambda = \rho_\lambda \circ \varphi_*$, and let G'_λ be its algebraic monodromy group. Since the morphism φ is finite étale, then $\varphi_* \pi_1^{\text{ét}}(X')$ is an open subgroup of $\pi_1^{\text{ét}}(X)$, and so $G'^\circ_\lambda = G^\circ_\lambda$, since restriction to an open subgroup leaves the connected component of the identity of the algebraic monodromy group of a λ -adic representation invariant.

Up to birational equivalence, we can assume that C is smooth and projective. Let F be the field of rational functions on C , and let $\pi_C : \Gamma_F \rightarrow \pi_1^{\text{ét}}(X')$ be the composition of the natural surjection of Γ_F onto $\pi_1^{\text{ét}}(C)$ with ι_* .

For any $\lambda \in \Lambda \neq p$, let $\rho_{C, \lambda} = \rho'_\lambda \circ \pi_C$. Note that $\rho_{C, \lambda}(\Gamma_F) = \rho'_\lambda(\pi_1^{\text{ét}}(X'))$, so that the algebraic monodromy groups of $\rho_{C, \lambda}$ and ρ'_λ coincide. The family $\{\rho_{C, \lambda}\}_{\lambda \in \Lambda \neq p}$ is a compatible system of semisimple representations of Γ_F , and then, by Corollary 6.3 there exist a subset $\mathcal{L}'$ of $\mathcal{L} \setminus \{p\}$ of Dirichlet density 1, a finite extension L of E , and a split connected reductive algebraic subgroup G of $\text{GL}_{n, L}$ such that for each

$\ell \in \mathcal{L}'$, each $\lambda \mid \ell$, and each finite place l of L lying above λ , the group $G_\lambda^\circ \times_{E_\lambda} L_l$ is conjugate to $G \times_L L_l$ over L_l . $\square$

Remark 6.6 A crucial point in the proof of this result is the reduction to the case of curves, which relies on the work of Böckle, Gajda, and Petersen in [2, §3.3]. A similar λ -independence result, for all $\lambda \in \Lambda_{\neq p}$, has been obtained with a different approach by D’Addezio, see [9, Theorem 4.3.2], assuming the variety X to be also smooth.

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References

1. Barnet-Lamb, T., Gee, T., Geraghty, D., Taylor, R.: Potential automorphy and change of weight. *Ann. Math.* **179**, 501–609 (2014)
2. Böckle, G., Gajda, W., Petersen, S.: On the semisimplicity of reductions and adelic openness for E -rational compatible systems over global function fields. *Trans. Am. Math. Soc.* **372**, 5621–91 (2019)
3. Bourbaki, N.: *Algèbre: Modules et Anneaux Semi-simples*, vol. 8. Hermann, Paris (1958)
4. Caraiani, A.: Local-global compatibility and the action of monodromy on nearby cycles. *Duke Math. J.* **161**(12), 2311–2413 (2012)
5. Caraiani, A.: Monodromy and local-global compatibility for $l = p$. *Algebra Number Theory* **8**(7), 1597–1646 (2014)
6. Chenevier, G., Harris, M.: Construction of automorphic Galois representations. II. *Camb. J. Math.* **1**, 53–73 (2013)
7. Clozel, L.: *Motifs et Formes Automorphes: Applications du Principe de Fonctorialité*. In: Clozel, L., Milne, J.S. (eds), *Automorphic Forms, Shimura Varieties, and L -functions—Volume I*, volume 10 of *Perspect. Math.*, chapter 2 (pp. 77–159). Academic Press, San Diego (1990)
8. Conrad, B., Gabber, O., Prasad, G.: *Pseudo-reductive Groups -*. New Math, vol. 26, 2nd edn. Monogr. Cambridge University Press, Cambridge (2015)
9. D’Addezio, M.: The monodromy groups of lisse sheaves and overconvergent F -isocrystals. *Selecta Math.* **26**(3), 1–41 (2020)
10. Deligne, P.: La Conjecture de Weil, II. *Publ. Math. Inst. Hautes Études Sci.* **52**, 137–252 (1980)
11. Harris, M., Taylor, R.: *The Geometry and Cohomology of Some Simple Shimura Varieties*, volume 151 of *Ann. of Math. Stud.* Princeton University Press, Princeton and Oxford (2001)
12. Katz, N.M.: On the monodromy groups attached to certain families of exponential sums. *Duke Math. J.* **54**(1), 41–56 (1987)
13. Kottwitz, R.E.: On the λ -adic representations associated to some simple Shimura varieties. *Invent. Math.* **108**, 653–665 (1992)
14. Lafforgue, L.: Chtoucas de Drinfeld et correspondance de Langlands. *Invent. Math.* **147**(1), 1–241 (2002)
15. Larsen, M., Pink, R.: On ℓ -independence of algebraic monodromy groups in compatible systems of representations. *Invent. Math.* **107**(3), 603–636 (1992)

16. Mumford, D.: Families of abelian varieties. *Proc. Sympos. Pure Math.* **9**, 347–51 (1966)
17. Patrikis, S.: Variations on a theorem of Tate. *Mem. Am. Math. Soc.*, **258**(1238) (2019)
18. Patrikis, S., Taylor, R.: Automorphy and irreducibility of some l -adic representations. *Compos. Math.* **151**(2), 207–229 (2015)
19. Serre, J.-P.: *Abelian l -Adic Representations and Elliptic Curves*. W. A. Benjamin, New York (1968)
20. Serre, J.-P.: Modular forms of weight one and Galois representations. In: *Algebraic number fields: L -functions and Galois properties* (Proc. Sympos., Univ. Durham, Durham, 1975), pp. 193–268, London (1977). Academic Press
21. Shin, S.W.: Galois representations arising from some compact Shimura varieties. *Ann. Math.* **173**(3), 1645–1741 (2011)
22. Taylor, R., Yoshida, T.: Compatibility of local and global Langlands correspondences. *J. Am. Math. Soc.* **20**, 467–493 (2007)

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