Selecta Mathematica (2021) 27:33
https://doi.org/10.1007/s00029-021-00650-5

Selecta Mathematica
New Series

CORRECTION

Correction to: Stable conjugacy and epipelagic L-packets for Brylinski–Deligne covers of Sp(2n)

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Published online: 19 May 2021

Correction to: Selecta Mathematica (2020) 26:12 https://doi.org/10.1007/s00029-020-0537-0

The computations of the toral invariants $f_{(G,S)}(\alpha)$ in Sect. 7.2 of the paper *Stable conjugacy and epipelagic L-packets for Brylinski–Deligne covers of* Sp$(2n)$ in Selecta Mathematica (N.S.), Vol. 26, (2020), no. 1 (ditto for the arXiv version) are flawed when the root α takes the form $\alpha = \epsilon_i \pm \epsilon_j$, in both the symplectic and the odd orthogonal case. The resulting $f_{(G,S)}(\alpha)$ can vary within a stable conjugacy class $\mathcal{E}$ of embeddings $j : S \hookrightarrow G$ of maximal tori. The main impacts of this mistake are listed below.

- The proof of Lemma 7.2.2 no longer works for general tori S. This lemma is used to prove the key Theorem 7.6.3 on the stability of epipelagic L-packets.
- Theorem 8.3.1 is directly built upon Sect. 7.2. Its aim is interpret the *stable system* (a notion of obstruction to stability, see Definition 7.3.3) constructed in Sect. 8.2 for $m \equiv 2 \pmod 4$, as a ratio $\epsilon_j(\gamma)/\epsilon_k(\gamma)$. This is used in Sect. 9.3 to show the compatibility with Θ-lifting when $m = 2$.

The goal here is to show that

- Lemma 7.2.2 still holds for a restricted class of maximal tori, which suffices for our purposes;
- Theorem 8.3.1 can be turned into a definition for the stable system when $m \equiv 2 \pmod 4$, in order to force the results in Sect. 9.3 to be true. The new definition is more transparent.

The impacted results can thus be restored with reasonable modifications. Remarkably, this is a reversion to the strategy in an early draft of this work.

The author is grateful to Chuijia Wang for kindly pointing our these mistakes.

The original article can be found online at https://doi.org/10.1007/s00029-020-0537-0.

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Correction to Lemma 7.2.2

One should assume in Lemma 7.2.2 that S is a maximal torus of type (ER) in $G = \mathrm{Sp}(W)$, as in Definition 6.1.1. This is all what one needs for the epipelagic supercuspidals studied in Sect. 7.3–7.6. The corrected arguments are given below.

Retain the notation in the original proof. The goal is to show $\epsilon_j(\gamma_0)$ depends only on the stable conjugacy class of $j : S \hookrightarrow G$ and $\gamma_0 \in S(F)_{p'}$. Given j, define the sign

$$E_j(S, G) = \prod_{\alpha \in R(G, jS)_{\mathrm{sym}}(\overline{F})/\Gamma_F} f_{G, jS}(\alpha).$$

Set $G_{j\gamma_0} := Z_G(j(\gamma_0))^\circ$. By [1, Lemma 4.12], which applies to S since it is of type (ER),

$$\epsilon_j(\gamma_0) = \prod_{\substack{\alpha \in R(G, jS)_{\mathrm{sym}}(\overline{F})/\Gamma_F \\ \alpha(\gamma_0) \neq 1}} f_{G, jS}(\gamma_0) = \frac{E_j(S, G)}{E_j(S, G_{j\gamma_0})}. \tag{1}$$

Fix any additive character ψ of F. Kottwitz's formula [1, Corollary 4.11] gives

$$E_j(S, G) = \frac{e(G)\epsilon(X^*(S)_{\mathbb{C}} - X^*(T)_{\mathbb{C}}, \psi)}{\prod_{\alpha \in R(G, jS)_{\mathrm{sym}}(\overline{F})/\Gamma_F} \lambda_{F_\alpha/F_{\pm\alpha}}(\psi \circ \mathrm{Tr}_{F_\alpha/F_{\pm\alpha}})}; \tag{2}$$

- $e(G)$ is the Kottwitz sign of G;
- T is a minimal Levi subgroup in the quasisplit inner form G^* of G;
- $\epsilon(\cdots, \psi)$ is the ϵ-factor with Langlands' normalization associated with the virtual Γ_F-representation $X^*(S)_{\mathbb{C}} - X^*(T)_{\mathbb{C}}$ of degree zero;
- $F_\alpha/F_{\pm\alpha}$ are the quadratic extensions associated with symmetric roots α, and $\lambda_{F_\alpha/F_{\pm\alpha}}(\cdots)$ are Langlands' constants.

Formulas (1) and (2) apply to connected reductive F-groups in general. In particular, we have the formula for $E_j(S, G_{\gamma_0})$ by replacing T with a minimal Levi subgroup $T_j \subset G^*_{j\gamma_0}$. The denominator of (2) only depends on the stable conjugacy class of j.

Return to our setting that S is of type (ER) and $G = \mathrm{Sp}(W)$. It remains to show that

$$e(G_{j\gamma_0})\epsilon(X^*(S)_{\mathbb{C}} - X^*(T_j)_{\mathbb{C}}, \psi)$$

depends only on the stable conjugacy class of j.

Using Theorem 6.2.2 we know that $j\gamma_0$ can only have eigenvalues ± 1, with multiplicities $2n_\pm$ respectively. Hence

$$G_{j\gamma_0} = \mathrm{Sp}(W_+) \times \mathrm{Sp}(W_-)$$

where $W_{\pm} \subset W$ are symplectic vector subspaces of dimension $2n_{\pm}$. The stable conjugacy class of j and γ_0 determine the isomorphism class of $G_{j\gamma_0}$, hence determines $e(G_{j\gamma_0})$ and $X^*(T_j)_{\mathbb{C}}$. The proof is now complete.

Correction to Section 8.2 and Theorem 8.3.1

The obscure construction in Sect. 8.2 should be abandoned. Theorem 8.3.1 should instead be turned into a definition for the stable system when $m \equiv 2 \pmod 4$. Specifically, we shall define

$$\theta_j^{\dagger}(\gamma_0) := \frac{\epsilon_k(\gamma_0)}{\epsilon_j(\gamma_0)},$$

where $\gamma_0 \in S(F)_{p'}$. Then we put $\theta_j := \theta_j^{\circ}\theta_j^{\dagger}$ to obtain a stable system as in Theorem 8.2.6.

Let us recall the notation from Sect. 8.2 and 8.3 briefly.

- Fix an additive character ψ of the non-Archimedean local field F. Fix a symplectic F-vector space $(W, \langle\cdot|\cdot\rangle$ to define $G = \mathrm{Sp}(W)$ and its m-fold BD-cover $\tilde{G}$.
- Consider an inducing datum $(\mathcal{E}, S, \theta^{\flat})$ (Definition 7.3.1) and pick any $Y = Y_{\psi} \in \mathfrak{s}(F)$ as in (8.3).
- For each conjugacy class of embeddings $j : S \hookrightarrow G$ in $\mathcal{E}$, we plug Y into the moment map correspondence to produce $H = \mathrm{SO}(V, q)$ and the conjugacy class of embeddings $k : S \hookrightarrow H$.

This will guarantee the compatibility with Θ-lifting of epipelagic representations [3] when $m = 2$. All the statements in Sect. 9.3 remain valid.

To achieve this, we must check the conditions SS.1–SS.3 in Definition 7.3.3, and show that $\theta_j^{\dagger}$ is independent of the choice of Y.

Suppose Y is chosen. We begin with SS.2. It cannot be reduced to $\dim W = 2$ like in the original arguments for Theorem 8.2.6. However, we can still

- parameterize j by a datum $(K, K^{\sharp}, \vec{c})$ with $K = \prod_{i \in I} K_i$ an étale F-algebra, $K^{\sharp} = \prod_{i \in I} K_i^{\sharp}$ the subalgebra fixed by an involution τ, and $\vec{c} = (c_i)_{i \in I} \in K^{\times}$ with $\tau(\vec{c}) = -\vec{c}$;
- parameterize $jY \in j\mathfrak{s}(F)$ by $(K, K^{\sharp}, \vec{c}, \vec{y})$ where $\vec{y} = (y_i)_{i \in I} \in K^{\times}$ with $\tau(\vec{y}) = -\vec{y}$; the part $\vec{y}$ depends only on the stable conjugacy class of j.

This gives $S(F) \simeq \prod_{i \in I} K_i^1$ where K_i^1 is the norm-one subtorus of $K_i^{\times}$. A similar parameterization for conjugacy classes of maximal tori and their elements exists for H and its pure inner forms. See for example [2, Sect. 3.1].

By Theorem 6.2.2, $\gamma_0 = (\gamma_{0,i})_{i \in I} \in \{\pm 1\}^I$. Accordingly,

$$I = I_+ \sqcup I_-, \quad I_{\pm} := \{i \in I : \gamma_{0,i} = \pm 1\}.$$

The analysis of stable conjugacy in $\tilde{G}$ in the proof of Theorem 8.2.6 reduces SS.2 to the equality

$$\theta^{\dagger}_{j'}(\gamma_0) = \theta^{\dagger}_{j}(\gamma_0) \prod_{\substack{i \in I \\ \gamma_{0,i} = -1}} \operatorname{sgn}_{K_i/K_i^{\sharp}} \left(c_i/c'_i\right) \tag{3}$$

for all $j, j' \in \mathcal{E}$, parameterized by $(K, K^{\sharp}, \vec{c})$ and $(K, K^{\sharp}, \vec{c}')$ respectively.

Denote by $k' : S \hookrightarrow H' = \mathrm{SO}(V', q')$ the embedding arising from j' by moment map correspondence. It is stably conjugate to k (possibly across pure inner forms). By definition of the moment map correspondence in Sect. 8.1, k (resp. k') is parameterized by $(K, K^{\sharp}, \vec{c}\vec{y})$ (resp. $(K, K^{\sharp}, \vec{c}'\vec{y})$).

Let $V_{\pm} \subset V$ and $V'_{\pm} \subset V'$ are the quadratic subspaces on which $k\gamma_0$ and $k'\gamma_0$ act by ± 1, respectively. Note that $\dim V_+ = \dim V'_+$ (resp. $\dim V_- = \dim V'_-$) are odd (resp. even). We have

$$H_{k\gamma_0} \simeq \mathrm{SO}(V_+) \times \mathrm{SO}(V_-), \quad H'_{k'\gamma_0} \simeq \mathrm{SO}(V'_+) \times \mathrm{SO}(V'_-).$$

Note that k and k' are related by a pure inner twist from H to H' coming from $H^1(F, S)$. In turn, this restricts to a pure inner twists from $H_{j\gamma_0}$ to $H'_{j'\gamma_0}$, and similarly on their $\pm$-factors. The parameter of kS_+ in $kS = kS_+ \times kS_-$ is carved out from that of kS by $I_+ \subset I$. Ditto for $k'S_+$.

We have shown that $\epsilon_j(\gamma_0) = \epsilon_{j'}(\gamma_0)$, so (3) reduces into

$$\epsilon_{k'}(\gamma_0) = \epsilon_k(\gamma_0) \prod_{\substack{i \in I \\ \gamma_{0,i} = -1}} \operatorname{sgn}_{K_i/K_i^{\sharp}} \left(c_i/c'_i\right). \tag{4}$$

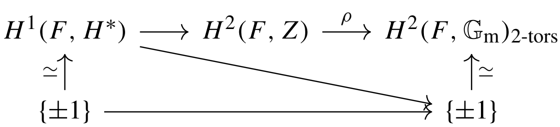
We shall prove it by Kottwitz's formula. The Langlands constants are stably invariant and $T_k \simeq T'_{k'}$ since $(H_{k\gamma_0})^* = (H'_{k'\gamma_0})^*$. Thus (1) and (2) imply

$$\frac{\epsilon_{k'}(\gamma_0)}{\epsilon_k(\gamma_0)} = \frac{E_{k'}(S, H')/E_{k'}(S, H'_{k'\gamma_0})}{E_k(S, H)/E_k(S, H_{k\gamma_0})} = \frac{e(H')e(H_{k'\gamma_0})}{e(H)e(H_{k\gamma_0})}$$

1. First, we claim that

$$\frac{e(H')}{e(H)} = \prod_{i \in I} \operatorname{sgn}_{K_i/K_i^{\sharp}}(c_i/c'_i).$$

Indeed, $e(H)$ is computed by the diagonal arrow below.



where $Z = \ker[H^*_{\mathrm{SC}} \to H^*]$, and ρ is induced by the half-sum of positive roots. The diagonal arrow is non-trivial since there does exist a pure inner form H^{**} of

H^* with $e(H^{**}) = -1$. Hence the bottom arrow is the identity map.
The pure inner twist from H to H' in question comes from

$$\begin{array}{ccccccc} \left(\mathrm{sgn}_{K_i/K_i^\sharp}(c_i/c_i')\right)_{i\in I} & \in & \{\pm 1\}^I & \xrightarrow{\text{mult.}} & \{\pm 1\} & & \\ & & \simeq\uparrow & & \simeq\uparrow & & \\ & H^1(F,S) & \xrightarrow[\text{natural}]{} & H^1(F,H) & \xrightarrow[\text{natural}]{\sim} & H^1(F,H^*). & \end{array}$$

The claim follows.

2. Since $\dim V_+ = \dim V_+'$ is odd, the same arguments yield

$$\frac{e(\mathrm{SO}(V_+'))}{e(\mathrm{SO}(V_+))} = \prod_{i\in I_+} \mathrm{sgn}_{K_i/K_i^\sharp}(c_i/c_i').$$

3. To establish (4), it suffices to show $e(\mathrm{SO}(V_-')) = e(\mathrm{SO}(V_-))$. Apart from the trivial case $\dim V_- = 2$, there are exactly two pure inner forms of $\mathrm{SO}(V_-)$: the other one arises from a quadratic vector space $V_{-,\diamond}$ with the same dimension and discriminant, but opposite Hasse invariant. Denote by V_-^{an} and $V_{-,\diamond}^{\mathrm{an}}$ their anisotropic kernels. It is known that

$$\{\dim V_-^{\mathrm{an}}, \dim V_{-,\diamond}^{\mathrm{an}}\} = \{0,4\} \text{ or } \{2\}.$$

In either case, the split ranks of $\mathrm{SO}(V_-)$ and $\mathrm{SO}(V_{-,\diamond})$ have the same parity, hence $e(\mathrm{SO}(V_-)) = e(\mathrm{SO}(V_{-,\diamond}))$. This establishes SS.2.

For SS.1, one has $\theta_j = \theta_j^\circ \theta_j^\dagger$ by construction. Rescaling ψ or $\langle\cdot|\cdot\rangle$ by some $a \in F^\times$ amounts to a similar rescaling of Y. This amounts to rescale the $q\langle Y\rangle$ in Remark 8.1.2 by a. This also amounts to replacing the parameter $(K, K^\sharp, \vec{c})$ (see Sect. 3.1) of j by $(K, K^\sharp, a\vec{c})$, corresponding to some $j' \in \mathcal{E}$. The required sign-change is already predicated by SS.2.

Now consider SS.3. Following the proof of Theorem 8.2.6, one should show that $\theta_j^\dagger$ depends only on the pro-p part of the datum $\theta^\flat$. Granting this fact, one shows $\theta_j^\dagger = \theta_{\mathrm{Ad}(w)j}^\dagger$ for any $w \in \Omega(G, jS)(F)$ by repeating the original arguments.

To prove this, assume for convenience that the $a(\psi)$ in (8.2) is zero. The only ambiguity is that $Y \in \mathfrak{s}(F)_{-1/e}$ can be replaced by any $Y' \in Y + \mathfrak{s}(F)_0$. Consider the parameter $\vec{y}' = (y_i')_{i\in I}$ of jY'. Then

$$\frac{y_i'}{y_i} - 1 \text{ is topologically nilpotent in } K_i^\sharp$$

for all $i \in I$. Since we assumed $p \neq 2$ from Sect. 7.3 onward, the group of 1 + topological nilpotents is 2-divisible in $(K_i^\sharp)^\times$, hence $\mathrm{sgn}_{K_i/K_i^\sharp} = 1$ on them. The resulting $k : S \hookrightarrow H$, etc. are thus unaltered up to conjugacy.

The above also shows that $\theta_j^\dagger(\gamma_0)$ depends only on the pro-p part of $\theta^\flat$ determined by Y, not on Y itself.

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