On the toric ideals of matroids of a fixed rank

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Accepted: 24 February 2021 / Published online: 23 March 2021
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Abstract

In 1980 White conjectured that every element of the toric ideal of a matroid is generated by quadratic binomials corresponding to symmetric exchanges. We prove White's conjecture for high degrees with respect to the rank. This extends our result (Lason and Michałek in Adv Math 259:1–12, 2014) confirming White's conjecture 'up to saturation'. Furthermore, we study degrees of Gröbner bases and Betti tables of the toric ideals of matroids of a fixed rank.

Keywords Matroid · Toric ideal · Base exchange · Generating set · Gröbner basis · Betti table

Mathematics Subject Classification 05B35 · 13P10 · 14M25 · 52B40

1 Introduction

1.1 Problem description

Let M be a matroid on the ground set E with the set of bases $\mathfrak{B}$ and the rank function $r : \mathcal{P}(E) \rightarrow \mathbb{N}$. We denote the rank of M , that is $r(E)$, simply by r .

For a fixed field $\mathbb{K}$ consider a $\mathbb{K}$ -homomorphism φ_M between polynomial rings:

$$\varphi_M : S_M = \mathbb{K}[y_B : B \in \mathfrak{B}] \ni y_B \rightarrow \prod_{e \in B} x_e \in \mathbb{K}[x_e : e \in E].$$

Research supported by Polish National Science Centre grant no. 2015/19/D/ST1/01180. The paper was initiated during author's stay at Freie Universität Berlin in the frame of Polish Ministry "Mobilność Plus" program.

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The *toric ideal* of a matroid M , denoted by I_M , is the kernel of the map φ_M . For a representable matroid M the toric variety associated with the toric ideal I_M has a very nice embedding as a subvariety of a Grassmannian [12]. It is the closure of the torus orbit of the point in the Grassmannian corresponding to the matroid M . Furthermore, any closure of the torus orbit of a point in the Grassmannian is of this form for some representable matroid M .

When an ideal is defined only by combinatorial means, one expects to have a combinatorial description of its set of generators. An attempt to achieve this description often leads to surprisingly deep combinatorial questions. White's conjecture is an example. In 1980 Neil White stated in fact three conjectures that describe generators of the toric ideal of a matroid with increasing accuracy.

Conjecture 1.1 (White [31]) *The toric ideal I_M of a matroid M is generated by quadratic binomials.*

The family $\mathfrak{B}$ of bases of the matroid M satisfies *symmetric exchange property* (the reader is referred to [26] for background of matroid theory, and to [20] for other exchange properties). That is, for every bases B_1, B_2 and an element $e \in B_1 \setminus B_2$ there exists an element $f \in B_2 \setminus B_1$, such that both sets $B'_1 = (B_1 \setminus e) \cup f$ and $B'_2 = (B_2 \setminus f) \cup e$ are bases. In such a case we say that the quadratic binomial $y_{B_1} y_{B_2} - y_{B'_1} y_{B'_2}$ corresponds to *symmetric exchange*. It is clear that this binomials belong to the ideal I_M .

Conjecture 1.2 (White [31]) *The toric ideal I_M of a matroid M is generated by quadratic binomials corresponding to symmetric exchanges.*

Conjecture 1.3 (White [31]) *The toric ideal I_M of a matroid M considered in the noncommutative polynomial ring $\mathbb{K}\langle y_B : B \in \mathfrak{B} \rangle$ is generated by quadratic binomials corresponding to symmetric exchanges.*

Conjecture 1.3, the strongest among White's conjectures describing generators of the ideal I_M , turned out to be equivalent to Conjecture 1.2 when considered for all matroids (see the discussion in section 4 of [19]).

Since every toric ideal is generated by binomials, it is not hard to rephrase the above conjectures in the combinatorial language. Conjecture 1.1 asserts that if two multisets of bases of a matroid have equal union (as a multiset), then one can pass between them by a sequence of steps, in each step exchanging two bases for another two bases with the same union (as a multiset). In Conjecture 1.2 additionally each step corresponds to a symmetric exchange. In Conjecture 1.3 one takes sequences of bases instead of multisets, and similarly each step corresponds to a symmetric exchange between consecutive bases. Actually, these are the original formulations due to White. It is immediate that the conjectures do not depend on the field $\mathbb{K}$.

In 2002 Jürgen Herzog and Takayuki Hibi, as they wrote, could not escape from the temptation to ask the following questions strengthening Conjecture 1.1:

Question 1.4 (Herzog, Hibi, [15]) *Let M be a matroid.*

(1) *Is the base ring S_M/I_M a Koszul algebra?*

(2) Does the toric ideal I_M possess a quadratic Gröbner basis?

Recall general implications (see ex. [6,7]). If the quotient ring S/I (where S is the polynomial ring) is a Koszul algebra, then the ideal I is generated by quadrics. If the ideal I has a quadratic Gröbner basis, then S/I is a Koszul algebra.

1.2 State of the art

White's conjectures are known to be true for many special classes of matroids: graphic matroids [1], linear frame matroids [24], strongly base orderable matroids [19] (in particular for transversal matroids), sparse paving matroids [3], and for matroids of low rank—at most 3 [17] (see also other related papers [15,16,21,28]).

Questions of Herzog and Hibi are settled for a fewer number of matroid classes. Base-sortable matroids (called also sort-closed matroids) possess quadratic Gröbner basis [2,18] (in particular uniform matroids [29] and lattice path matroids [27] do), as their base polytope is alcoved hence possesses a quadratic regular unimodular triangulation. Moreover, the answer to Question 1.4 (1) is 'yes' for transversal polymatroids [5].

The first and to our knowledge the only known 'general result', i.e. a result valid for arbitrary matroids, confirmed White's Conjecture 1.2 'up to saturation'. To explain what this means, denote by $\mathfrak{m}$ the so-called *irrelevant ideal*, i.e. the ideal generated by all variables in the polynomial ring S_M . Recall that the ideal $I : \mathfrak{m}^\infty = \{a \in S_M : a\mathfrak{m}^n \subset I \text{ for some } n \in \mathbb{N}\}$ is called the *saturation* of the ideal I with respect to $\mathfrak{m}$. Notice that the ideal I_M , as a prime ideal, is saturated. Let J_M be the ideal generated by quadratic binomials corresponding to symmetric exchanges. Clearly, $J_M \subset I_M$. Conjecture 1.2 asserts that the ideals J_M and I_M are equal, which translates geometrically as 'they define the same affine scheme'.

In [19] we prove that the saturations of I_M and J_M with respect to $\mathfrak{m}$ are equal. That is, in algebraic geometry language, 'they define the same projective scheme'. In particular, ideals I_M and J_M have the same affine set of zeros, so Conjecture 1.2 holds on 'set-theoretic level'. Recall that homogeneous ideals have equal saturations with respect to $\mathfrak{m}$ if and only if their homogeneous parts are equal starting from some degree. Thus, we can rephrase the result of [19] in the following way:

Theorem 1.5 (Lasoń, Michałek, [19]) *For every matroid M there exists a constant $c(M)$ such that the homogeneous parts of degree at least $c(M)$ of the toric ideal I_M are generated by quadratic binomials corresponding to symmetric exchanges.*

Moreover, from the proof of the theorem follows that $c(M) = d + (r - 1)d|\mathfrak{B}|$, where r is the rank of M , $|\mathfrak{B}|$ is the size of the set of bases of M , and d is the degree in which the ideal I_M is generated.

1.3 Our results

This paper studies toric ideals of matroids of a fixed rank. We obtain several finiteness results. Namely, for an infinite class of matroids of a fixed rank we obtain finite bounds

for various parameters—degree of a Gröbner basis, complexity of verifying White’s conjecture, shape of the Betti table, and degree starting from which White’s conjecture holds.

Firstly, towards the answer to the Question 1.4 of Herzog and Hibi, we make an initial step in the study of Gröbner bases of the toric ideal of a general matroid:

Theorem 3.2 *Let M be a matroid of rank r . The toric ideal I_M has a Gröbner basis of degree at most $(r + 3)!$. In particular, it is generated in degree at most $(r + 3)!$.*

As a result, White’s conjectures for matroids of a fixed rank are finite problems:

Corollary 3.3 *Checking if Conjecture 1.1, 1.2 or 1.3 is true for matroids of a fixed rank is decidable (it reduces to connectivity of a finite number of graphs).*

Moreover, we get a bound for the shape of the nonzero part of the Betti table:

Corollary 3.5 *Let M be a matroid of rank r . The i -th graded Betti number of the toric ideal I_M vanishes for degrees greater than $(r + 3)!(i + 1)$.*

Finally, the main result of the paper strengthens Theorem 1.5:

Theorem 5.1 *For every r there exists a constant $c(r)$ such that for every matroid M of rank r the homogeneous parts of degree at least $c(r)$ of the toric ideal I_M are generated by quadratic binomials corresponding to symmetric exchanges.*

By the discussion after Remark 15 in [19] we can deduce the following:

Remark 1.6 Theorems 3.2, 5.1 and Corollaries 3.3, 3.5 are true for discrete polymatroids.

1.4 Outline of the paper

In the next section we discuss how White’s conjectures translate into problems on graphs on bases of a matroid. We use it later in the proof of Corollary 3.3.

In Sect. 3 we prove Theorem 3.1, a structural result on bases coverings. This is a main tool for Theorem 3.2 which bounds the degree of a Gröbner basis of the toric ideal of a matroid only in terms of its rank. Theorem 3.2 leads to Corollaries 3.3, 3.5, and later serves as an ingredient for the proof of Theorem 5.1.

The main result of the paper strengthens Theorem 1.5. Recall that the key part in the proof of Theorem 1.5 is [19, Claim 4]. It asserts that if $b \in I_M$ is a binomial of degree n , then for every variable y_B we have $y_B^{(r-1)n} b \in J_M$. Suppose the ideal I_M is generated in degree d . If $b \in I_M$ is a binomial of degree at least $d + (r - 1)d|\mathfrak{B}|$, then $b = a_1 b_1 + \dots + a_k b_k$ where b_i are generators of I_M of degree d , and a_i are monomials of degree at least $(r - 1)d|\mathfrak{B}|$. From the pigeon hole principle, every monomial a_i contains some variable y_B in degree at least $(r - 1)d$, hence by [19, Claim 4] $a_i b_i \in J_M$, and finally $b \in J_M$. Therefore, the constant $c(M)$ from Theorem 1.5 can be bounded by $d + (r - 1)d|\mathfrak{B}|$.

By Theorem 3.2 we have a bound $d \leq (r + 3)!$. But, the size of the set of bases $|\mathfrak{B}|$ can not be bounded for matroids of rank r . Indeed, we have to be able to generate by quadratic binomials corresponding to symmetric exchanges binomials $y_{B_1} \dots y_{B_n} -$

$y_{B'_1} \cdots y_{B'_n} \in I_M$ of high degree with respect to the rank ($n \gg r$) for which bases $B_1, \dots, B_n$ are pairwise disjoint. For them there is no hope for a single variable in high degree, as every variable can appear in degree at most one.

To overcome this difficulty, in Sect. 4, we introduce a Ramsey-type result for blow-ups of bases. It asserts that if a matroid contains sufficiently many disjoint bases, then it contains an arbitrarily large k -th blow-up of a basis—a matroid obtained by replacing every element of a basis by k parallel elements. Moreover, if we modify this bases by only symmetric exchanges, then we can guarantee that this k -th blow-up agrees with some k bases. This allows us to ‘reveal’ a single variable in high degree in any monomial $y_{B_1} \cdots y_{B_n}$ of sufficiently large degree.

Having these three ingredients—[19, Claim 4], Theorem 3.2, and a Ramsey-type result for blow-ups of bases, we finally prove Theorem 5.1 in the last section.

2 Graphs on bases of a matroid

This section contains preliminaries, in particular notions used in Sect. 3. We discuss here how White’s conjectures translate into problems on graphs on bases of a matroid.

We say that two bases of a matroid are *neighboring* if one is obtained from the other by a symmetric exchange. That is, if their symmetric difference has exactly two elements. A graph on bases of a matroid M with edges between neighboring bases is called the *basis graph* of M , and denoted by $\mathfrak{B}(M)$. Basis graphs have been studied in 1960s and 1970s, and they are well understood. In particular, basis graphs are Hamiltonian (with only two trivial exceptions), even a characterization is known (see [4,22,23] and references within).

For $k \geq 1$, a k -matroid is a matroid whose ground set can be partitioned into k pairwise disjoint bases. We call a basis of a k -matroid *complementary* if its complement can be partitioned into $k - 1$ pairwise disjoint bases. That is, when it is an element of some partition of the ground set into bases. When $\mathfrak{B}$ is the set of bases of a k -matroid, then we denote the set of complementary bases by $\mathfrak{B}^c$.

We recall one of the versions of the matroid union theorem, which will be used several times in this paper. It characterizes k -matroids in terms of rank function.

Theorem 2.1 (Nash-Williams [25], Edmonds [9]) *A matroid M is a k -matroid, if and only if for every $A \subset E$ the inequality $k \cdot r(A) \geq |A|$ holds, and $k \cdot r(E) = |E|$.*

Blasiak [1] proposed a very nice and simple translation of the problem of generating the toric ideal of a matroid to the problem of connectivity of some graphs naturally associated to k -matroids. We are going to use this approach for the proof of Corollary 3.3. Following Blasiak, for $k \geq 3$ the k -base graph of a k -matroid M , denoted by $\mathfrak{B}_k(M)$, is a graph on sets of k pairwise disjoint bases of M (partitions of the ground set into bases), where edges join vertices with nonempty intersection. That is, sets of bases $\{B_1, \dots, B_k\}$ and $\{B'_1, \dots, B'_k\}$ are connected in $\mathfrak{B}_k(M)$ if for some i, j the equality $B_i = B'_j$ holds. Recall that if $\{e, f\}$ is a circuit in a matroid M , then elements e and f are said to be *parallel*. In this case B is a basis of M containing e if and only if $(B \cup f) \setminus e$ is a basis of M containing f . Via this property one can add to a matroid elements parallel to a fixed element (enlarging its ground set), or remove

them. Notice that the reflexive closure of being parallel is an equivalence relation. A simple corollary of the proof of Proposition 2.1 from [1] gives the following.

Proposition 2.2 (Blasiak, [1]) *Let $\mathfrak{C}$ be a class of matroids that is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Then the following conditions are equivalent:*

- (1) *for every $k > d$ and for every k -matroid M in $\mathfrak{C}$ the k -base graph $\mathfrak{B}_k(M)$ is connected,*
- (2) *for every matroid M in $\mathfrak{C}$ the ideal I_M is generated in degree at most d .*

In particular, in order to prove Conjecture 1.1 it is enough to show that for every $k > 2$ and for every k -matroid M the k -base graph $\mathfrak{B}_k(M)$ is connected.

Here we propose another approach to White's conjecture. Consider other graphs that can be naturally associated to k -matroids. The *complementary basis graph* of a k -matroid, denoted by $\mathfrak{B}^c(M)$, is a graph on complementary bases of M with edges between neighboring bases. That is, the complementary basis graph of a k -matroid is the restriction of its basis graph to complementary bases $\mathfrak{B}^c(M) = \mathfrak{B}(M)|_{\mathfrak{B}^c}$.

Graphs $\mathfrak{B}^c(M)$ have been already studied for 2-matroids M . In 1985 Farber, Richter and Shank [11] proved that for a graphic 2-matroid M the graph $\mathfrak{B}^c(M)$ is connected, they also conjectured connectivity for arbitrary 2-matroids. In [1] after the proof of Proposition 2.1 Blasiak observes the following easy equivalence:

Proposition 2.3 (Blasiak, [1]) *Let $\mathfrak{C}$ be a class of matroids that is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Then the following conditions are equivalent:*

- (1) *for every 2-matroid M in $\mathfrak{C}$ the complementary basis graph $\mathfrak{B}^c(M)$ is connected,*
- (2) *for every matroid M in $\mathfrak{C}$, elements of degree 2 in I_M considered in the noncommutative polynomial ring $\mathbb{K}\langle y_B : B \in \mathfrak{B} \rangle$ are generated by quadratic binomials corresponding to symmetric exchanges.*

We state the following two conjectures strongly related to White's conjectures:

Conjecture 2.4 *Complementary basis graph of a k -matroid is connected.*

Conjecture 2.5 *Let $k \geq 2$, and let M be a matroid of rank r on the ground set E of size $kr + 1$. Suppose $x, y \in E$ are two elements such that both sets $E \setminus x$ and $E \setminus y$ can be partitioned into k pairwise disjoint bases. Then there exist partitions of $E \setminus x$ and $E \setminus y$ into k pairwise disjoint bases which share a common basis.*

We learned from Joseph Bonin that Conjecture 2.5 for $k = 2$ was studied in 1980s by Paul Seymour and Neil White, but it was not resolved.

Proposition 2.6 *Let $\mathfrak{C}$ be a class of matroids that is closed under deletions that do not lower the rank of a matroid, and adding parallel elements. Then, considered for all matroids in $\mathfrak{C}$, the following implications between conjectures hold:*

- (1) *the strongest White's Conjecture 1.3 implies complementary basis graph Conjecture 2.4,*

- (2) *conjunction of complementary basis graph Conjecture 2.4 and Conjecture 2.5 implies the strongest White's Conjecture 1.3,*
in general, for every $d \geq 2$ conjunction of Conjecture 2.4 for k -matroids for $k > d$ and Conjecture 2.5 for $k \geq d$ implies that the toric ideal of a matroid, considered in the noncommutative polynomial ring, is generated in degree at most d .

Proof We begin with implication (1). Let M be a k -matroid in $\mathfrak{C}$, and let B_1, B'_1 be complementary bases in M . So, there exist bases $B_2, \dots, B_k, B'_2, \dots, B'_k$ such that entries of the sequences $\mathcal{A} = (B_1, \dots, B_k)$ and $\mathcal{A}' = (B'_1, \dots, B'_k)$ form partitions of the ground set E . Then $b = y_{B_1} \cdots y_{B_k} - y_{B'_1} \cdots y_{B'_k} \in I_M$, or equivalently sequences of bases $\mathcal{A}$ and $\mathcal{A}'$ have equal union (as a multiset). By the assumption, we can generate b using quadratic binomials corresponding to symmetric exchanges, or equivalently we can pass between $\mathcal{A}$ and $\mathcal{A}'$ by a sequence of steps, in each step making a symmetric exchange. Notice that all bases appearing during this process are complementary bases of M . Observe that the first bases of two sequences joined by a single step are either the same or neighboring. Thus we get a path in $\mathfrak{B}^c(M)$ between B_1 and B'_1 .

For the implication (2), by Propositions 2.2 and 2.3 it is enough to show that for every $k \geq 3$ and for every k -matroid M in $\mathfrak{C}$ the k -base graph $\mathfrak{B}_k(M)$ is connected, and for every 2-matroid M in $\mathfrak{C}$ the complementary basis graph $\mathfrak{B}^c(M)$ is connected. The second part we get directly from complementary basis graph Conjecture 2.4. For the first part, let M be a k -matroid in $\mathfrak{C}$ (for $k \geq 3$) and let $\{B_1, \dots, B_k\}, \{B'_1, \dots, B'_k\}$ be two vertices in $\mathfrak{B}_k(M)$. Since $\mathfrak{B}^c(M)$ is connected, it is enough to show that vertices of $\mathfrak{B}_k(M)$ containing neighboring bases are connected in $\mathfrak{B}_k(M)$. If the symmetric difference of B_1 and B'_1 is $\{x, y\}$, then consider the restriction of M to the set $E \setminus (B_1 \cap B'_1)$. This matroid satisfies assumptions of Conjecture 2.5 with points x, y . Thus in $\mathfrak{B}_k(M)$ there are vertices $\{B_1, B''_2, \dots, B''_k\}, \{B'_1, B'''_2, \dots, B'''_k\}$ connected by an edge. The first one is connected by an edge with $\{B_1, \dots, B_k\}$, while the second with $\{B'_1, \dots, B'_k\}$. We get that any two vertices $\{B_1, \dots, B_k\}, \{B'_1, \dots, B'_k\}$ in $\mathfrak{B}_k(M)$ containing neighboring bases are connected by a path. $\square$

Proposition 2.7 *If $k \geq 2^{r-1} + 1$, then Conjecture 2.5 holds.*

Proof Proof by contradiction. Let $B_1, \dots, B_k$ be a partition of the set $E \setminus y$ into k pairwise disjoint bases. Without loss of generality $x \in B_1$. If the assertion is not true, then for each $i = 2, \dots, k$ the basis B_i can not be completed to a partition of $E \setminus x$ into k bases. Thus from the matroid union Theorem 2.1 it follows that for each $i = 2, \dots, k$ there is a set $A_i \subset E \setminus (x \cup B_i)$ such that $(k-1)r(A_i) < |A_i|$. On the other hand, since $E \setminus (y \cup B_i)$ has a partition into $k-1$ pairwise disjoint bases (namely $B_1, \dots, \hat{B}_i, \dots, B_k$), for every set $A \subset E \setminus (y \cup B_i)$ the inequality $(k-1)r(A) \geq |A|$ holds. Thus for each i we have $y \in A_i$, $r(A_i) = r(A_i \setminus y)$, and $(k-1)r(A_i \setminus y) = |A_i \setminus y|$. The last equality implies that for every basis B_j (for $j \neq i$) $|B_j \cap A_i| = r(A_i)$. Moreover, since there is equality in the inequality $(k-1)r(A_i \setminus y) \geq |A_i \setminus y|$, each A_i is closed in $E \setminus B_i$, and it is equal to the closure of $B_j \cap A_i$ (for $j \neq i$) in $E \setminus B_i$. Consider sets $B_1 \cap A_i \subset B_1 \setminus x$. None of them is empty, since otherwise $r(A_i) = 0$ and y would be a loop. Thus, since there are $k-1 \geq 2^{r-1}$ of them, for some $i \neq j$ the equality $B_1 \cap A_i = B_1 \cap A_j$ holds. But since A_i is the closure of $B_1 \cap A_i$ in $E \setminus B_i$ and A_j is the closure of $B_1 \cap A_j$ in $E \setminus B_j$, we get that the

set $A := A_i \cup A_j$ is the closure of $B_1 \cap A_i = B_1 \cap A_j$ in E , so it is closed, $y \in A$, and $|B_l \cap A| = r(A)$ for every $l = 1, \dots, k$. Therefore, $|A| = k \cdot r(A) + 1$ and $x \notin A$, which by the matroid union Theorem 2.1 contradicts the assumption that $E \setminus x$ can be partitioned into k pairwise disjoint bases. $\square$

3 Degree bounds and Gröbner bases

By Hilbert's basis theorem the ideal I_M is finitely generated. However, it is not easy to give any explicit bound on degree in which it is generated. A bound follows from a more general theorem about toric ideals. Theorem 13.14 from [29] asserts that if a graded set $\mathcal{A} \subset \mathbb{Z}^d$ generates a normal semigroup, then the corresponding toric ideal $I_{\mathcal{A}}$ is generated in degree at most d . For a matroid M we consider the set of indicator vectors of all bases, that is $\{e_B = \sum_{i \in B} e_i : B \in \mathfrak{B}\} \subset \mathbb{Z}^{|E|}$, and denote it also by $\mathfrak{B}$. By [30, Theorem 1] the semigroup $\mathbb{N}\mathfrak{B}$ generated by $\mathfrak{B}$ is normal (it is also an easy consequence of the matroid union theorem—Theorem 2.1). The toric ideal corresponding to $\mathfrak{B}$ is the ideal I_M . Hence, the toric ideal of a matroid is generated in degree at most the size of its ground set. If we fix the size of the ground set, then there are only finitely many matroids on it, so a common bound is not surprising.

When we fix only the rank, then the number of matroids of that rank is infinite. Theorem 3.2 asserts that in this case there is also a common bound on the degree in which the corresponding toric ideals are generated. Furthermore, there is a common bound on the degree of their Gröbner bases. In order to prove it we need the following structural result:

Theorem 3.1 *Let M be a k -matroid of rank r . Among every $k - s$ pairwise disjoint bases in M at most $r(r + 2)! + s(r + 1)!$ bases are not complementary.*

Proof We prove by induction on r , that among every $k - s$ pairwise disjoint bases there are at most $r(r + 2)! + s((r + 1)! - 2)$ not complementary bases. When $r = 1$, then the statement becomes trivial. Suppose $r \geq 2$, and fix k, s . Let $B_1, \dots, B_k$ be disjoint bases of the k -matroid M . Their union is the whole ground set E . Let $D_1, \dots, D_{k-s}$ be arbitrary pairwise disjoint bases in M . If every D_j is complementary, then the assertion clearly holds. So, we can assume that some basis D_j is not complementary.

Due to the matroid union theorem—Theorem 2.1, there exists a set $A \subset E \setminus D_j$, such that

$$(k - 1) \cdot r(A) < |A|.$$

Of course $0 < r(A) < r$. Indeed, otherwise either A would have to be empty (in a k -matroid M there are no loops) and we would have $0 < 0$, or we would have $(k - 1) \cdot r < |A| \leq |E \setminus D_j| = (k - 1) \cdot r$.

Let $A_i = A \cap B_i$ for every $i = 1, \dots, k$. Since every B_i is a basis, inequalities $|A_i| \leq r(A)$ hold. And, all together

$$(k - 1) \cdot r(A) < |A| = |A_1| + \dots + |A_k| \leq k \cdot r(A).$$

Therefore for every i , except at most $r(A) - 1 \leq r$, we have $|A_i| = r(A)$. Without loss of generality the equality holds for $i = 1, \dots, k - r$.

Let $E' = B_1 \cup \dots \cup B_{k-r}$ and let $A' = A_1 \cup \dots \cup A_{k-r}$. We are going to reduce the problem to the $(k - r)$ -matroid $M' = M|_{E'}$ (restriction of M to the set E'), and then use the set A' to split it into smaller instances—for $M'|_{A'}$ (restriction of M' to A') and for M'/A' (contraction of A' in M').

Notice that there are at most r^2 bases among bases D_i which have non-empty intersection with $B_{k-r+1} \cup \dots \cup B_k$. Thus, without loss of generality, bases $D_1, \dots, D_{k-s-r^2}$ are contained in E' .

Let $C_i = A' \cap D_i$ for $i = 1, \dots, k - s - r^2$. Since every D_i is a basis, inequalities $|C_i| \leq r(A') = r(A)$ hold. In order to split the problem for $M'|_{A'}$ and M'/A' we need bases D_i satisfying $|C_i| = |A' \cap D_i| = r(A)$.

Since $D_1, \dots, D_{k-s-r^2}$ cover all except $(s + r^2)r$ elements of E we get

$$\begin{aligned} (k - s - r^2)r(A) - (s + r^2)r &\leq (k - r)r(A) - (s + r^2)r = |A'| - (s + r^2)r \\ &\leq |A' \cap (D_1 \cup \dots \cup D_{k-s-r^2})| = |C_1| + \dots + |C_{k-s-r^2}| \leq (k - s - r^2)r(A). \end{aligned}$$

Therefore for every $i = 1, \dots, k - s - r^2$, except at most $(s + r^2)r$, the equality $|C_i| = r(A)$ holds. Without loss of generality it holds for $i = 1, \dots, k - (s + r^2)(r + 1)$. Denote $s' = (s + r^2)(r + 1) - r$. Now we can pass to the matroids $M'|_{A'}$ and M'/A' .

We have

- (1) $(k - r)$ -matroid $M'|_{A'}$ of rank $r(A) < r$ with $(k - r) - s'$ pairwise disjoint bases $D_1 \cap A', \dots, D_{k-r-s'} \cap A'$, and
- (2) $(k - r)$ -matroid M'/A' of rank $r - r(A) < r$ with $(k - r) - s'$ pairwise disjoint bases $D_1 \setminus A', \dots, D_{k-r-s'} \setminus A'$.

For both cases we use the inductive assumption. In the case (1) there are at most $r(A)(r(A) + 2)! + s'(r(A) + 1)! - 2s'$ bases not complementary in $M'|_{A'}$. In the case (2) there are at most $(r - r(A))(r - r(A) + 2)! + s'(r - r(A) + 1)! - 2s'$ bases not complementary in M'/A' .

Notice that if $D_i \cap A'$ is a basis complementary in $M'|_{A'}$ and $D_i \setminus A'$ is a basis complementary in M'/A' , then D_i is a basis complementary in M . Therefore all together there are at most

$$\begin{aligned} t &= (s' + r) + r(A)(r(A) + 2)! + s'(r(A) + 1)! - 2s' \\ &\quad + (r - r(A))(r - r(A) + 2)! + s'(r - r(A) + 1)! - 2s' \end{aligned}$$

bases not complementary in M . Denote $r' = r(A)$ and recall that $0 < r' < r$. Now,

$$\begin{aligned} t &= r + r'(r' + 2)! + (r - r')(r - r' + 2)! \\ &\quad + (r^2(r + 1) - r)(1 + (r' + 1)! - 2 + (r - r' + 1)! - 2) + \\ &\quad + s(r + 1)(1 + (r' + 1)! - 2 + (r - r' + 1)! - 2) \leq r(r + 2)! + s((r + 1)! - 2). \end{aligned}$$

It is easy to verify the last inequality, because for $s \geq 0$ and $r \geq 2$ we have both

$$s(r+1)(1+r!-2+2!-2) \leq s((r+1)!-2),$$

$$r+(r-1)(r+1)!+3!+(r^2(r+1)-r)(1+r!-2+2!-2) \leq r(r+2)!. \quad \square$$

We do not have faith in a positive answer to Question 1.4 (2) of Herzog and Hibi—that is, existence of a quadratic Gröbner basis for the toric ideal of a matroid. That would mean (see [29]) that the base polytope of a matroid possesses a quadratic regular unimodular triangulation. This property is nearly at the top of the whole hierarchy of possible combinatorial properties of a convex lattice polytope (consult [14]). It is known that the base polytope of a matroid has the integer Carathéodory property [13], and therefore it is normal (which was earlier shown in [30, Theorem 1]). However, using Theorem 3.1 we are able to prove a bound on the degree of a Gröbner basis that depends only on the rank of a matroid:

Theorem 3.2 *Let M be a matroid of rank r . The toric ideal I_M has a Gröbner basis of degree at most $(r+3)!$. In particular, it is generated in degree at most $(r+3)!$.*

Proof Let $<$ be any fixed order on variables y_B , and let $<_{\text{revlex}}$ be the corresponding reverse lexicographic order on monomials in the ring $\mathbb{K}[y_B : B \in \mathfrak{B}]$. We prove by induction on n , that if $b \in I_M$ is a binomial of degree n , then its leading monomial is divisible by the leading monomial of some binomial in I_M of degree at most $(r+3)!$. This implies that the set of monic binomials of degree at most $(r+3)!$ forms a Gröbner basis of I_M with respect to $<_{\text{revlex}}$.

If $n \leq (r+3)!$, then it is clear. Suppose $n > (r+3)!$ and that the assertion holds for all degrees less than n . Let $b = y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$ be a binomial, where the first monomial is leading. Suppose, without loss of generality, that y_{D_1} is a minimal variable among variables y_{D_i} . Notice that if for some i, j equality $B_i = D_j$ holds, then $b = y_{B_i} b'$ for some binomial $b' \in I_M$ and the assertion follows from the inductive assumption for b' which is of degree $n-1$. Therefore, since $y_{D_1} \cdots y_{D_n} <_{\text{revlex}} y_{B_1} \cdots y_{B_n}$, we have $y_{D_1} < y_{B_i}$ for every i .

Let M' be a n -matroid obtained from the matroid $M|_{B_1 \cup \dots \cup B_n}$ by replacing every element $e \in E$ by $w_e = \{|i : e \in B_i|\}$ parallel elements $(e, 1), \dots, (e, w_e)$. The ground set E' of M' has two natural partitions into n bases, namely $B'_1 \cup \dots \cup B'_n$ and $D'_1 \cup \dots \cup D'_n$, where $B'_i = \{(e, v_e) : e \in B_i, v_e = \{|j \leq i : e \in B_j|\}\}$ and D'_i are defined analogously. Denote by T the set of indices i such that B'_i intersects basis D'_1 , clearly $t = |T| \leq r$. Consider $(n-1)$ -matroid $M'|_{D'_2 \cup \dots \cup D'_n}$. By Theorem 3.1 among $n-t \geq n-r$ disjoint bases B'_i for $i \in \{1, \dots, n\} \setminus T$ there are at most $r(r+2)!(r-1)(r+1)! < (r+3)! - r < n-r$ not complementary bases. Thus, there exists a basis B'_k that can be completed by some bases $H'_1, \dots, H'_{n-2}$ to $E' \setminus D'_1$. Let $H_i = \{e \in E : \exists j(e, j) \in H'_i\}$ be the basis in M corresponding to H'_i . Now $b' = y_{B_1} \cdots y_{B_n} - y_{D_1} y_{B_k} y_{H_1} \cdots y_{H_{n-2}} \in I_M$ is a binomial in which $y_{B_1} \cdots y_{B_n}$ is the leading monomial. Moreover, $b = y_{B_k} b'$ for some binomial $b' \in I_M$ and the assertion follows by induction. $\square$

Corollary 3.3 *Checking if Conjecture 1.1, 1.2 or 1.3 is true for matroids of a fixed rank is decidable (it reduces to connectivity of a finite number of graphs).*

Proof Using Proposition 2.2, in order to check if Conjecture 1.1 is true for matroids of rank r it is enough to check if for every k -matroid M of rank r (for every $k > 2$), the k -base graph $\mathfrak{B}_k(M)$ is connected. By Theorem 3.2 it is enough to consider k from the range $(r+3)! \geq k > 2$, since for $k > (r+3)!$ the statement is true. That is, the problem reduces to checking connectivity of a finite number of graphs.

To check if Conjectures 1.1 and 1.3 are equivalent for matroids of rank r , by Proposition 2.3 it suffices to check connectivity of a finite number of graphs.

Analogously, to check if Conjectures 1.1 and 1.2 are equivalent for matroids of rank r it suffices to check connectivity of graphs from Proposition 2.3 modified by adding an edge between every complementary basis B and its complement B^c (these are bases of a 2-matroid). This completes the proof of Corollary 3.3. $\square$

We get a new class of discrete polymatroids for which White's Conjecture 1.1 is true (for an extension of White's conjectures to discrete polymatroids see [15]):

Corollary 3.4 *Let P be a discrete polymatroid which is a join of $c \geq \frac{1}{2}(r+3)!$ copies of a matroid M of rank r (a basis of P is a union, as a multiset, of c bases of M). Then the toric ideal I_P is generated in degree 2.*

Proof We will prove the following claim. Let P be a discrete polymatroid which is a join of c copies of a matroid M . Suppose that the toric ideal I_M is generated in degree at most $2c$. Then the toric ideal I_P is generated in degree 2.

Let $D_1, \dots, D_k, \tilde{D}_1, \dots, \tilde{D}_k$ be bases of P with $y_{D_1} \cdots y_{D_k} - y_{\tilde{D}_1} \cdots y_{\tilde{D}_k} \in I_P$. For $i = 1, \dots, k$ and $j = 1, \dots, c$ let B_i^j and $\tilde{B}_i^j$ be bases of M such that $D_i = \bigcup_j B_i^j$ and $\tilde{D}_i = \bigcup_j \tilde{B}_i^j$. Then $\prod_{i,j} y_{B_i^j} - \prod_{i,j} y_{\tilde{B}_i^j} \in I_M$.

When one exchanges bases B_i^j and $B_{i'}^{j'}$ between bases D_i and $D_{i'}$ of P , then the corresponding elements of I_P differ by an element generated in degree 2. Thus we can rearrange bases B_i^j (and $\tilde{B}_i^j$) into an arbitrary k multisets of c bases. Since I_M is generated in degree $2c$, one can pass between the multisets of bases $\{B_i^j : i, j\}$ and $\{\tilde{B}_i^j : i, j\}$ by a sequence of steps, in each step exchanging $2c$ bases for another $2c$ bases of the same union (as a multiset). We partition these $2c$ bases into an arbitrary 2 parts of c bases. Each part corresponds to a basis of P . This way we are able to pass between the multisets of bases $\{D_i\}_i$ and $\{\tilde{D}_i\}_i$ of P by a sequence of steps, in each step exchanging only 2 bases and preserving multiset union. $\square$

White's conjectures focus on generators of the toric ideal of a matroid. We believe the study of properties of its minimal free resolution may be very significant. The toric ideal of a matroid is graded, so it has a graded free resolution, invariants of which are summarized in the table of graded Betti numbers. The Betti table plays a central role in modern commutative algebra, as it provides a lot of algebraic information about the ideal, indicating e.g. regularity, projective dimension, the Cohen-Macaulay property, the Gorenstein property (for the toric ideal of a matroid see [16,21]), the Koszul property, and many others. Astonishingly little is known about the structure of Betti tables associated to arbitrary matroids. One of the possible explanations is the fact that providing the first step of the minimal free resolution is already as hard

as White's conjecture—as the zeroth graded Betti number of degree d is equal to the minimum number of generators of degree d .

We aim at a description of the shape of the nonzero part of the Betti tables of the toric ideals of matroids of a fixed rank:

Corollary 3.5 *Let M be a matroid of rank r . The i -th graded Betti number of the toric ideal I_M vanishes for degrees greater than $(r + 3)!(i + 1)$.*

Proof By Theorem 3.2 the toric ideal I_M has an initial ideal $in_{<}(I_M)$ generated in degree at most $(r + 3)!$. By the Taylor's resolution of monomial ideals, the i -th Betti number of degree j of the monomial ideal $in_{<}(I_M)$ vanishes for $j > (r + 3)!(i + 1)$. Now, by the standard deformation argument, the i -th Betti number of degree j of the original ideal I_M also vanishes for $j > (r + 3)!(i + 1)$. $\square$

However, we cannot bound the whole nonzero part of the Betti tables of the toric ideals of matroids of a fixed rank:

Remark 3.6 The projective dimension of the toric ideal I_M , that is the index of the largest nonzero Betti number, is not bounded with respect to the rank of M .

4 Ramsey-type results for blow-ups of bases

By k -th blow-up of a matroid M we mean a matroid obtained from M by replacing every element of its ground set E by k parallel elements. By k -th blow-up of a set $A \subset E$ in M we mean a matroid obtained from M by replacing every element of A by k parallel elements.

Let N, M be two matroids of the same rank r on the same ground set E . We say that N is a *submatroid* of M , if the complex of independent sets of N is a subcomplex of the complex of independent sets of M , or equivalently, if the set of bases of N is contained in the set of bases of M .

We define a convenient notion of morphisms between matroids. Let M and M' be two matroids of the same rank r on the corresponding ground sets E, E' . A *morphism* ψ from M to M' is a function $\psi : E \rightarrow E'$, such that if B' is a basis in M' , then any choice of representatives of sets $\{\psi^{-1}(b')\}_{b' \in B'}$ forms a basis in M . That is, a function ψ is a morphism if M contains a compatible submatroid which is obtained from M' by replacing every element $e \in E'$ by $|\psi^{-1}(e)|$ parallel copies of e . In particular, there is a natural morphism from the k -th blow up of a matroid to the original matroid. Let us formulate the key observation:

Observation 4.1 *Suppose $\psi : M \rightarrow M'$ is a morphism between matroids that sends variables of a binomial $b \in I_M$ to variables (i.e. images of the corresponding bases are bases). Let $\psi(b)$ be the binomial b with variables replaced by their images. Then $\psi(b) \in J_{M'}$ implies $b \in J_M$.*

Proof Suppose $b = y_{B_1} \cdots y_{B_k} - y_{D_1} \cdots y_{D_k}$. The condition $\psi(b) \in J_{M'}$ means that one can modify the monomial $y_{\psi(B_1)} \cdots y_{\psi(B_k)}$ using quadratic binomials corresponding to symmetric exchanges in M' to get $y_{\psi(D_1)} \cdots y_{\psi(D_k)}$. But, every symmetric

exchange in M' between bases $\psi(B_1), \psi(B_2)$ lifts to a symmetric exchange in M between B_1, B_2 . Therefore, we can modify $y_{B_1} \cdots y_{B_k}$ using symmetric exchanges in M to get $y_{B'_1} \cdots y_{B'_k}$ such that $\psi(B'_i) = \psi(D_i)$. Since any choice of representatives of bases from M' forms bases in M , and $b \in I_M$, we can modify $y_{B'_1} \cdots y_{B'_k}$, step by step using symmetric exchanges, to get $y_{D_1} \cdots y_{D_k}$. $\square$

We say that k disjoint bases $B_1, \dots, B_k$ of a matroid M contain the k -th blow-up of the basis B_1 if there exists a morphism $\psi : M|_{B_1 \cup \dots \cup B_k} \rightarrow M|_{B_1}$, such that $\psi(B_1) = \dots = \psi(B_k) = B_1$ and ψ is an identity on B_1 . That is, if one can label the elements of $B_1 \cup \dots \cup B_k$ with labels $l_1, \dots, l_r$, elements of every basis B_i with distinct labels, such that every set of r elements of distinct labels is a basis in M .

Our Ramsey-type result asserts that if a matroid contains sufficiently many disjoint bases, then it contains an arbitrarily large k -th blow-up of a basis. Moreover, if we modify these bases by only symmetric exchanges, we can guarantee that this k -th blow-up agrees with some k bases:

Lemma 4.2 *For every positive integers r, k there exists an integer $n = n(r, k)$, such that if M is an n -matroid of rank r with disjoint bases $B_1, \dots, B_n$, then there exists a modification of these bases by symmetric exchanges to $B'_1, \dots, B'_n$ from which one can pick k bases $B_{i_1}, \dots, B_{i_k}$ that contain the k -th blow-up of basis B_{i_1} .*

Proof The proof goes by induction on the rank r . If $r = 1$, then M itself is the n -th blow-up of a basis. In particular, we do not need to make a modification and any k bases contain the k -th blow-up of a basis. Thus $n(1, k) = k$.

Suppose $r \geq 2$, and fix also a positive integer k . We will show that for

$$n = n(r, k) = rn(r-1, k) + r^{rn(r-1, k)} 2^{2^{rn(r-1, k)}} n(r-1, k)$$

the desired property holds. Denote $s = rn(r-1, k)$, and $t = n - s$.

Let M be an n -matroid of rank r with disjoint bases $B_1, \dots, B_n$. Choose an element b_i in each basis B_i among $B_1, \dots, B_s$. Consider symmetric exchanges between bases $B_1, \dots, B_s$ and bases $B_{s+1}, \dots, B_{s+t}$. For $j = s+1, \dots, s+t$ and $i = 1, \dots, s$ let $b_{j,i}$ be an element of B_j that exchanges symmetrically with $b_i \in B_i$.

Label elements of each B_j among $B_{s+1}, \dots, B_{s+t}$ with distinct labels $l_1, \dots, l_r$. Then each B_j gets a label from $\{l_1, \dots, l_r\}^s$ which is a sequence of labels of $b_{j,1}, \dots, b_{j,s}$. From the pigeon hole principle at least $2^{2^{rn(r-1, k)}} n(r-1, k)$ bases B_j have the same label. Without loss of generality (statement of the lemma is independent on the order of bases) they are $B_{s+1}, \dots, B_{s+t'}$, for $t' = 2^{2^{rn(r-1, k)}} n(r-1, k)$. Now for $j = s+1, \dots, s+t'$ and $i = 1, \dots, s$ the label of $b_{j,i}$ is the same for all j 's. So, we label B_i with it. Again, from the pigeon hole principle at least $n(r-1, k)$ bases B_i have the same label, without of loss of generality they are $B_1, \dots, B_{n(r-1, k)}$. The label of $b_{j,i}$ is the same for all these i 's. Thus, we can define $b_j := b_{j,i}$. Now, for every $i = 1, \dots, n(r-1, k)$ and every $j = s+1, \dots, s+t'$, $b_i \in B_i$ exchanges symmetrically with $b_j \in B_j$.

Consider matroids $M_j := (M/b_j)|_{B_1 \cup \dots \cup B_{n(r-1, k)}}$ for $j = s+1, \dots, s+t'$. Since there are at most $2^{2^{rn(r-1, k)}}$ matroids on the ground set $B_1 \cup \dots \cup B_{n(r-1, k)}$ (of size

$rn(r-1, k)$), there are at least $n(r-1, k)$ indices j for which M_j is the same, without loss of generality for $j = s+1, \dots, s+n(r-1, k)$.

Now, we make a first modification of bases $B_1, \dots, B_n$. We exchange $b_i \in B_i$ symmetrically with $b_{s+i} \in B_{s+i}$ for every $i = 1, \dots, n(r-1, k)$, obtaining bases $D_1, \dots, D_n$. Each D_i for $i = 1, \dots, n(r-1, k)$ has a distinguished element d_i (former b_{s+i}), such that $d_i \in D_i$ exchanges symmetrically with $d_{i'} \in D_{i'}$ and matroids $N_i := (M/d_i)|_{(D_1 \setminus d_1) \cup \dots \cup (D_{n(r-1, k)} \setminus d_{n(r-1, k)})}$ are the same (as restrictions of matroids M_j).

Consider $n(r-1, k)$ -matroid N_i of rank $r-1$ with $n(r-1, k)$ disjoint bases $F_1 = D_1 \setminus d_1, \dots, F_{n(r-1, k)} = D_{n(r-1, k)} \setminus d_{n(r-1, k)}$ (here we use the fact that $d_i \in D_i$ exchanges symmetrically with $d_{i'} \in D_{i'}$). From the inductive assumption it follows that there are disjoint bases $F'_1, \dots, F'_{n(r-1, k)}$ obtained from $F_1, \dots, F_{n(r-1, k)}$ by symmetric exchanges, among which there are k bases (without loss of generality) $F'_1, \dots, F'_k$ that contain the k -th blow-up of a basis of rank $r-1$. That is, one can label the elements of $F'_1 \cup \dots \cup F'_k$ with labels $l_1, \dots, l_{r-1}$, each F'_j with distinct labels, such that every set of $r-1$ elements of distinct labels is a basis in N_i .

Notice that bases $B'_1 = F'_1 \cup d_1, \dots, B'_{n(r-1, k)} = F'_{n(r-1, k)} \cup d_{n(r-1, k)}$ are obtained from bases $D_1, \dots, D_{n(r-1, k)}$, hence also from bases $B_1, \dots, B_{n(r-1, k)}$, by symmetric exchanges in M . Moreover, bases $B'_1, \dots, B'_k$ contain the k -th blow-up of a basis of rank r . Namely, one can label the elements of $B'_1 \cup \dots \cup B'_k$ with labels $l_1, \dots, l_r$ (we use the former labeling of $F'_1 \cup \dots \cup F'_k$, additionally elements d_i get label l_r), each basis with distinct labels, such that every set of r elements of distinct labels forms a basis in M . This proves the inductive assertion. $\square$

We are going to prove a generalization of Lemma 4.2, which asserts that additionally the desired k -th blow-up can be compatible with a fixed subset of a matroid. For this purpose we will use Ramsey theory for hypergraphs. A result of Erdős [10] implies the following lemma (see [8] for possible generalizations):

Lemma 4.3 *For every integers r, k and c there exists an integer $R = R(r, k, c)$, such that if H is a c -colored (edges receive one of c colors) complete r -uniform r -partite hypergraph of size R (size of each part is R), then one can find in it a monochromatic complete subhypergraph H' of size k .*

Let $B_1, \dots, B_k$ be disjoint bases of a matroid M on the ground set E . Denote $F = E \setminus (B_1 \cup \dots \cup B_k)$. Furthermore, we say that bases $B_1, \dots, B_k$ contain the k -th blow-up of a basis B_1 in $B_1 \cup F$ if there exists a morphism $\psi : M|_{B_1 \cup \dots \cup B_k \cup F} \rightarrow M|_{B_1 \cup F}$, such that $\psi(B_1) = \dots = \psi(B_k) = B_1$ and ψ is an identity on $B_1 \cup F$.

Lemma 4.4 *For every positive integers r, k and nonnegative integer l there exists an integer $m = m(r, k, l)$, such that if M is a matroid of rank r on the ground set E , containing m disjoint bases $B_1, \dots, B_m$ whose complement $F = E \setminus (B_1 \cup \dots \cup B_m)$ is of size l , then there exists a modification of bases $B_1, \dots, B_m$ by symmetric exchanges to bases $B'_1, \dots, B'_m$ from which one can pick k bases $B'_{i_1}, \dots, B'_{i_k}$ that contain the k -th blow-up of basis B_1 in $B_1 \cup F$.*

Proof We will show that for $m = m(r, k, l) = n(r, R(r, k, 2^{r+l}))$ the desired property holds, where n, R are the functions from Lemmas 4.2 and 4.3.

Let M be a matroid of rank r on the ground set E , containing m disjoint bases $B_1, \dots, B_m$ whose complement $F = E \setminus (B_1 \cup \dots \cup B_m)$ has l elements. Due to Lemma 4.2 there exists a modification of bases $B_1, \dots, B_m$ by symmetric exchanges to bases $B'_1, \dots, B'_m$ from which one can pick $R := R(r, k, 2^{r+l})$ bases (without loss of generality) $B'_1, \dots, B'_R$ that contain the R -th blow-up of basis B'_1 . So, there exists a morphism $\psi : M|_{B'_1 \cup \dots \cup B'_R} \rightarrow M|_{B'_1}$, such that $\psi(B'_1) = \dots = \psi(B'_R) = B'_1$ and ψ is an identity on B'_1 .

There is only one possible extension of the morphism ψ to $\psi' : M|_{B'_1 \cup \dots \cup B'_R \cup F} \rightarrow M|_{B'_1 \cup F}$, such that ψ' is an identity on $B'_1 \cup F$. It is a morphism if for every $i = 1, \dots, r-1$, for every i -element proper subset $S \subset B'_1$, and for every choice of representatives of sets $\{\psi^{-1}(b)\}_{b \in S}$, their union with a $(r-i)$ -element subset $T \subset F$ is a basis of M if and only if $S \cup T$ is a basis in M . We will show that, but for a smaller blow-up.

Consider a complete r -uniform r -partite hypergraph H with parts $\psi^{-1}(e)$ for $e \in B'_1$, each part of size R . Edges of H are bases in M , since ψ is a morphism from M . Define a 2^{r+l} -coloring c of edges of H . For each i -element proper subset $S \subset B'_1$, for each $(r-i)$ -element subset $T \subset F$ and for an edge D of H , let the bit $c_{S,T}(D)$ of $c(D)$ be 1 if $(\psi^{-1}(S) \cap D) \cup T$ is a basis in M , and 0 otherwise.

Using Lemma 4.3 for H with coloring c we get a monochromatic subhypergraph H' of size k . Let $D_1, \dots, D_k$ be k disjoint edges of H' (they are also bases of M), and let $D_{k+1}, \dots, D_R$ be disjoint edges of H completing a partition of the ground set of H . Then, since H is a complete hypergraph, bases $D_1, \dots, D_R$ are obtained from $B'_1, \dots, B'_R$ by symmetric exchanges. Hence also bases $D_1, \dots, D_R, B'_{R+1}, \dots, B'_m$ are obtained from bases $B_1, \dots, B_m$ by symmetric exchanges. Directly from the construction, there exists a desired morphism $\psi : M|_{D_1 \cup \dots \cup D_k \cup F} \rightarrow M|_{D_1 \cup F}$. $\square$

5 White's conjecture for high degrees

Theorem 5.1 *For every r there exists a constant $c(r)$ such that for every matroid M of rank r the homogeneous parts of degree at least $c(r)$ of the toric ideal I_M are generated by quadratic binomials corresponding to symmetric exchanges.*

Proof Let $c(r) = (r+3)! + m(r, r(r+3)!, r(r+3)!)$, where m is the function from Lemma 4.4.

Let M be a matroid of rank r . We have to show $(J_M)_d = (I_M)_d$ for $d \geq c(r)$. The inclusion $J_M \subset I_M$ implies that $(J_M)_d \subset (I_M)_d$ for every d . To prove the opposite inclusion, let $b \in I_M$ be a binomial of degree $d \geq c(r)$. By Theorem 3.2 we have that $b = \sum_{i=1}^n a_i b_i$, where $b_i \in I_M$ is a binomial of degree $(r+3)!$ and a_i is a monomial of degree greater or equal to $m := m(r, r(r+3)!, r(r+3)!)$. We show that $ab \in J_M$, for every binomial $b \in I_M$ of degree $(r+3)!$ and every monomial a of degree m .

Suppose that $b = y_{D_1} \dots y_{D_{(r+3)!}} - y_{G_1} \dots y_{G_{(r+3)!}}$ and $a = y_{B_1} \dots y_{B_m}$. Denote the union $D_1 \cup \dots \cup D_{(r+3)!} \cup B_1 \cup \dots \cup B_m$ by S as a set, and by (S, μ) as a multiset in which μ is the multiplicity function. Let M' be a matroid obtained from $M|_S$ by replacing every element e by $\mu(e)$ parallel elements. That is, there is a morphism $\psi' : M' \rightarrow M|_S$. Let $D'_1, \dots, D'_{(r+3)!}, B'_1, \dots, B'_m$ be disjoint bases of M' such that

$\psi'(D'_i) = D_i$ and $\psi'(B'_i) = B_i$. Let $G'_1, \dots, G'_{(r+3)!}$ be disjoint bases of M' such that $D'_1 \cup \dots \cup D'_{(r+3)!} = G'_1 \cup \dots \cup G'_{(r+3)!}$ and $\psi'(G'_i) = G_i$. Let b', a' be the corresponding polynomials. Clearly, $b' \in I_{M'}$. Observe that $ab \in J_M$ if and only if $a'b' \in J_{M'}$. Therefore, we need to show that $a'b' \in J_{M'}$.

Due to Lemma 4.4 applied for m disjoint bases $B'_1, \dots, B'_m$ in M' whose complement $F = D'_1 \cup \dots \cup D'_{(r+3)!}$ is of size $l = r(r+3)!$, there exists a modification of bases $B'_1, \dots, B'_m$ by symmetric exchanges to bases $B''_1, \dots, B''_m$ among which $k = r(r+3)!$ bases (without loss of generality) $B''_1, \dots, B''_k$ contain the k -th blow-up of basis B'_1 in $B'_1 \cup F$. Let $\psi : M'|_{B'_1 \cup \dots \cup B''_k \cup F} \rightarrow M'|_{B''_1 \cup F}$ be a morphism, such that $\psi(B''_1) = \dots = \psi(B''_k) = B''_1$ and ψ is an identity on $B''_1 \cup F$. Denote $a'' = y_{B''_1} \cdots y_{B''_m}$. Then, $a' - a'' \in J_{M'}$. Hence, it is enough to show that $a''b' \in J_{M'}$, or even that $y_{B''_1} \cdots y_{B''_k} b' \in J_{M'}$.

We use Observation 4.1 for morphism ψ and $f = y_{B''_1} \cdots y_{B''_k} b' \in I_M$. We have $\psi(y_{B''_1} \cdots y_{B''_k} b') = y_{B''_1}^{r(r+3)!} b'$, so it is enough to show that $y_{B''_1}^{r(r+3)!} b' \in J_{M'}$.

Finally, we use [19, Claim 4]. It asserts that for every basis B of M' , if $b' \in I_{M'}$ is a binomial, then $y_B^{\deg_B(b) - \deg(b)} b' \in J_{M'}$ (where $\deg_B(y_{B'}) = |B' \setminus B|$). In our case $y_{B''_1}^{r(r+3)!} b' \in J_{M'}$, since the degree of b' is $(r+3)!$. This finishes the proof. $\square$

Acknowledgements We thank Professor Jürgen Herzog for a question during the conference in Osnabrück in 2015 that initiated this study. We thank Jan Draisma and Mateusz Michałek for stimulating discussions and helpful comments. We are also grateful to the anonymous referee for showing us the proof of Corollary 3.5.

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