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The pure cohomology of multiplicative quiver varieties

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Abstract
To a quiver Q and choices of nonzero scalars q_i, non-negative integers α_i, and integers θ_i labeling each vertex i, Crawley-Boevey–Shaw associate a *multiplicative quiver variety* $\mathcal{M}^q_\theta(\alpha)$, a trigonometric analogue of the Nakajima quiver variety associated to Q, α, and θ. We prove that the pure cohomology, in the Hodge-theoretic sense, of the stable locus $\mathcal{M}^q_\theta(\alpha)^s$ is generated as a $\mathbb{Q}$-algebra by the tautological characteristic classes. In particular, the pure cohomology of genus g twisted character varieties of GL_n is generated by tautological classes.

Mathematics Subject Classification 16G20 · 14L30

1 Introduction

A quiver $Q = (I, \Omega)$ is a directed graph with vertex set I and edge set Ω. Despite its simple definition, from the datum of a quiver one can build, using various geometric quotient constructions, rich families of symplectic algebraic varieties. The best-known examples of this are [25] Nakajima's quiver varieties. In this paper, however, we will study their cousins, the multiplicative quiver varieties, first introduced by Crawley-Boevey and Shaw [10]. Just as Nakajima's quiver varieties can be understood as (coarse) moduli spaces of semistable representations of a class of algebras known as *preprojective algebras* their multiplicative analogues can be viewed similarly as moduli spaces of representations of a noncommutative algebra Λ^q, the *multiplicative preprojective algebra*.

The significance of multiplicative quiver varieties is rapidly growing: Crawley-Boevey and Shaw were led to them through their work on the celebrated Deligne–

Tom Nevins passed away on February 1, 2020. He was always generous as both a friend and a collaborator. He is sorely missed.

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Simpson problem. Subsequently they have been shown to arise as moduli spaces of irregular connections in the work [3,4] of Boalch and Yamakawa (indeed Boalch's work [3] lead him to define an even more general notion of multiplicative quiver variety than that considered here). Bezrukavnikov and Kapranov [2] realise them as moduli of microlocal sheaves on nodal curves (see also the work of Crawley-Boevey [11]), while in symplectic topology the work of Etgü and Lekili [15] shows that the Fukaya categories of certain symplectic four-manifolds, which are built from quiver-type data, are controlled by a derived version of the associated multiplicative preprojective algebra. Moreover, results of Chalykh and Fairon [7] and Braverman–Etingof–Finkelberg [5] reveal exciting new connections between multiplicative quiver varieties and new families of integrable systems which have also been constructed using double affine Hecke algebras.

Recently a number of authors [27,30] have studied the geometry of multiplicative quiver varieties. The present paper is a contribution to the study of their topology, and, as we discuss later, we expect its results will help shed light on questions raised by Hausel and collaborators in [20].

1.1 Results

Just as for a Nakajima quiver variety, a multiplicative quiver variety $\mathcal{M}_\theta^q(\alpha)$, where $\alpha \in \mathbb{N}^I$ is a dimension vector, is defined as a GIT quotient (at a character $\chi_\theta : \mathbb{G} \to \mathbb{G}_m$) of the affine algebraic variety $\mathrm{Rep}(\Lambda^q, \alpha)$ of (framed) representations of $\Lambda^q(\alpha)$ by the group $\mathbb{G} = \left(\prod_i GL(\alpha_i)\right)/\Delta(\mathbb{G}_m)$, a product of general linear groups modulo the diagonal copy of $\mathbb{G}_m$; when it is a *free* quotient, this endows $\mathcal{M}_\theta^q(\alpha)$ with a map $c\colon \mathcal{M}_\theta^q(\alpha) \to B\mathbb{G}$.

The rational cohomology $H^*(B\mathbb{G}, \mathbb{Q})$ is pure in the sense of Hodge theory: $H^m(B\mathbb{G}, \mathbb{Q}) = W_m H^m(B\mathbb{G}, \mathbb{Q})$ and $W_{m-1}H^m B\mathbb{G}, \mathbb{Q})) = 0$, where $W_k H^m(B\mathbb{G}, \mathbb{Q})$ denotes the k-th piece of the weight filtration.

It follows that if we set

$$PH^*(\mathcal{M}_\theta^q(\alpha)) \stackrel{\mathrm{def}}{=} \bigoplus_m \mathrm{gr}W_m\left(H^m\left(\mathcal{M}_\theta^q(\alpha), \mathbb{Q}\right)\right),$$

to be the "pure part" of the cohomology, where $\mathrm{gr}W$ denotes the associated graded with respect to the weight filtration, then the image of the pullback map c^* on cohomology must inject into the pure part of the cohomology of $\mathcal{M}_\theta^q(\alpha)$.

Remark 1.1 Note that for a smooth variety X the weight filtration on $H^m(X, \mathbb{Q})$ vanishes below degree m, so that $\mathrm{gr}W_m(H^m(X, \mathbb{Q})) = W_m(H^m(X, \mathbb{Q}))$. Thus for such spaces the pure part of cohomology is a subspace of the ordinary cohomology.

The main result of the present paper is:

Theorem 1.2

(1) *Suppose that* $U \subseteq \mathcal{M}_\theta^q(\alpha)^{\mathrm{s}}$ *is any connected open subset of the stable locus of the multiplicative quiver variety* $\mathcal{M}_\theta^q(\alpha)$. *Then the induced map on cohomology*

$$H^*(B\mathbb{G}, \mathbb{Q}) \to H^*\big(U, \mathbb{Q}\big)$$

defines a surjection onto the pure cohomology $PH^*(U) = \bigoplus_m \mathrm{gr}W_m\big(H^m\big(U, \mathbb{Q}\big)\big)$.

(2) *In particular, if* $\mathcal{M}_\theta^q(\alpha) = \mathcal{M}_\theta^q(\alpha)^{\mathrm{s}}$ *and* $\mathcal{M}_\theta^q(\alpha)$ *is connected, then*

$$H^*(B\mathbb{G}, \mathbb{Q}) \to H^*\big(\mathcal{M}_\theta^q(\alpha), \mathbb{Q}\big)$$

surjects onto $PH^*\big(\mathcal{M}_\theta^q(\alpha)\big)$.

In light of Theorem 1.2 of [26], Theorem 1.2 is nicely consonant with Hausel's "purity conjecture" (cf. [17] as well as [19, Theorem 1.3.1 and Corollary 1.3.2], and the discussion around Conjecture 1.1.3 of [20]), which predicts that when $\mathcal{M}_\theta^q(\alpha) = \mathcal{M}_\theta^q(\alpha)^{\mathrm{s}}$, one should have an isomorphism $PH^*(\mathcal{M}_\theta^q(\alpha)^{\mathrm{s}}) \cong H^*\big(\mathfrak{M}_\theta(\alpha)^{\mathrm{s}}, \mathbb{Q}\big)$, where $\mathfrak{M}_\theta(\alpha)^{\mathrm{s}}$ denotes the corresponding Nakajima quiver variety.

In the special case in which Q is a quiver with a single node and $g \geq 1$ loops, the dimension vector is $\alpha = n$, and $q \in \mathbb{C}^\times$ is a primitive nth root of unity, the multiplicative quiver variety $\mathcal{M}_\theta^q(\alpha)$ is identified with the GL_n-character variety $\mathrm{Char}(\Sigma_g, GL_n, q\,\mathrm{Id})$ of a genus g surface with a single puncture with residue $q\,\mathrm{Id}$, sometimes called a genus g *twisted character variety* [18]. We obtain:

Corollary 1.3 *The pure cohomology* $PH^*\big(\mathrm{Char}(\Sigma_g, GL_n, q\,\mathrm{Id})\big)$ *is generated by tautological classes.*

Corollary 1.3 has already appeared in [29], where it was deduced, via the non-abelian Hodge theorem, from Markman's theorem [24] that the cohomology of the moduli space of GL_n-Higgs bundles of degree 1 on a smooth projective genus g curve is generated by tautological classes. A novelty of our result, compared to [29], is that we avoid invoking non-abelian Hodge theory: instead, we deduce Corollary 1.3 (as well as Theorem 1.2) via a more direct and concrete method that invokes only basic facts of ordinary mixed Hodge theory as in [12].

Unlike the situation of quiver varieties in [26], we know of no obvious generalizations of Theorems 1.2 to other even-oriented cohomology theories (such as topological K-theory or elliptic cohomology). However, we do obtain the following analogue of Theorem 1.6 of [26].

Theorem 1.4 *Suppose there is some vertex* $i \in I$ *for which the dimension vector* α *satisfies* $\alpha_i = 1$, *and let* $\mathcal{M} = \mathcal{M}_\theta^q(\alpha)^{\mathrm{s}}$. *Let* $D(\mathcal{M})$ *denote the unbounded quasicoherent derived category of* $\mathcal{M}$, *and* $D^b_{\mathrm{coh}}(\mathcal{M})$ *its bounded coherent subcategory.*

(1) *The category* $D(\mathcal{M})$ *is generated by tautological bundles.*

(2) *There is a finite list of tautological bundles from which every object of* $D^b_{\mathrm{coh}}(\mathcal{M})$ *is obtained by finitely many applications of (i) direct sum, (ii) cohomological shift, and (iii) cone.*

As for the analogous result in [26], we emphasize that Theorem 1.4(2) is *not* simply a formal consequence of Theorem 1.4(1), since we do *not* include taking direct summands (i.e., retracts) among the operations (i)–(iii). It would be interesting to know generators for $D^b_{\mathrm{coh}}(\mathcal{M})$ for more general dimension vectors α than in Theorem 1.4.

As mentioned above, one source of interest in the cohomology of twisted character varieties (see [18]) and more generally of multiplicative quiver varieties lies in the $P = W$ conjecture and Hausel's purity conjecture. One categorical level higher, Theorems 1.2 and 1.4 may also be expected to have relevance to versions of mirror symmetry for multiplicative quiver varieties (cf. Section 7B of [2] as well as [16]) and the Betti geometric Langlands program [1].

1.2 Method of proof

Theorem 1.2 has the following slightly different but equivalent formulation. Choose a subgroup $\mathbb{S} \subset \prod_i GL(\alpha_i)$ whose projection $\mathbb{S} \to \mathbb{G}$ is a finite covering. Then one can form the stack quotient $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$, which comes with a morphism $\pi : \mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \to \mathcal{M}^q_\theta(\alpha)^s$ that is a gerbe, in fact a torsor over the commutative group stack BH where $H = \ker(\mathbb{S} \to \mathbb{G})$. We have an isomorphism $H^*(B\mathbb{S}, \mathbb{Q}) \cong H^*(B\mathbb{G}, \mathbb{Q})$ and π induces an isomorphism $H^*\big(\mathcal{M}^q_\theta(\alpha)^s, \mathbb{Q}\big) \cong H^*\big(\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}, \mathbb{Q}\big)$. Thus Theorem 1.2 can be restated as:

Theorem 1.5 *Let* U *be a connected open substack* $U \subseteq \mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$. *The pure cohomology* $PH^*(U)$ *is generated as a* $\mathbb{Q}$*-algebra by the Chern classes of tautological bundles* $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}} \times_{\mathbb{S}} V$ *associated to finite-dimensional representations* V *of* $\mathbb{S}$.

It is Theorem 1.5 that we prove directly: the tautological bundles $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}} \times_{\mathbb{S}} V$ that appear naturally and geometrically in our proof do not themselves descend to the multiplicative quiver variety in general, so it is more convenient to work on the Deligne–Mumford stack $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$.

Our proof of Theorem 1.5 follows the broad outline of that used in [26] to establish that tautological classes generate the cohomology of Nakajima quiver varieties, however there are considerable additional technical difficulties not present in that setting: A first stage of the proof is devoted to producing a suitable modular compactification of the multiplicative quiver variety (or rather its Deligne–Mumford stack analogue). A major difference from the Nakajima quiver variety case arises already at this stage: one frequently relies on q being an appropriate tuple of primitive roots of unity to deduce that $\mathcal{M}^q_\theta(\alpha)$ parameterizes only stable representations, independently of the choice of θ; whereas in [26], we assumed, without significant loss of generality, that θ was a generic stability condition. We note that such a genericity assumption here would exclude the possibility of applications to the character variety $\mathrm{Char}(\Sigma_g, GL_n, q\,\mathrm{Id})$; hence we avoid it. Instead we identify a compactification by a "projective Artin stack" $\overline{\mathcal{M}}$, a quotient of a quasiprojective scheme by a reductive group whose coarse moduli space is a projective scheme. Known techniques [14,22] allow us to replace the

Artin stack compactification by a projective Deligne–Mumford stack at no cost to the validity of our approach.

The second stage is to identify a complex on $\mathcal{M}_\theta^q(\alpha) \times \overline{\mathcal{M}}$ that, roughly speaking, resolves the graph of the embedding $\mathcal{M}_\theta^q(\alpha) \hookrightarrow \overline{\mathcal{M}}$. Again, while this is morally similar to [26], the actual construction and proofs are more complicated and subtle. This is essentially because our compactification of the Nakajima quiver variety relied on a graded 3-Calabi–Yau algebra, whereas the compactification of $\mathcal{M}_\theta^q(\alpha)$ uses an algebra, denoted by A in the body of this paper, which may (conjecturally) be in most cases what one might call a "relative $2g$-Koszul algebra", but which (as far as we know) is not known to be so. Fortunately it turns out that we can proceed as if the algebra A were known to have various desired properties in order to build a suitable complex, and then carry out some (occasionally delicate) calculations to check by hand that it has the properties we need. We note that the good behaviour of this complex is closely related to the question of whether the multiplicative preprojective algebra is 2-Calabi–Yau—a conjecture known in many important cases thanks to [21]. It can be hoped that our graded algebra A may thus be of some independent interest in relation to this question. Since in the generality in which we work here (and again unlike [26]), we do not know if the complex actually provides a resolution of the structure sheaf of the graph of the embedding, we instead rely on work of Markman [24] to show that an appropriate Chern class of the complex we build is the Poincaré dual of the fundamental class of the graph.

The final step is to deduce the theorem via usual integral transform arguments. In [26], we used Nakajima's result that the (integral) cohomology of a quiver variety is generated by algebraic cycles, hence is naturally isomorphic to a quotient of the cohomology of any compactification. Such an assertion is not true of the multiplicative quiver varieties $\mathcal{M}_\theta^q(\alpha)$. Instead, we rely on the beautiful fact that the cohomology of any reasonable smooth compactification—which is always Hodge-theoretically pure—surjects onto the pure part of the cohomology of any open subset. This yields the assertion of the theorem, which in any case would be the best possible result, given that the cohomology $H^*(B\mathbb{G}, \mathbb{Q})$ is pure. The Hodge-theoretic nature of this result however necessitates working with rational cohomology. It is thus an interesting question to characterize the image of $H^*(B\mathbb{G}, \mathbb{Z})$ in $H^*\big(\mathcal{M}_\theta^q(\alpha), \mathbb{Z}\big)$.

1.3 Notation

Throughout, k denotes a field of characteristic 0. In Sects. 1 and 6, $k = \mathbb{C}$.

2 Quivers and multiplicative preprojective algebras

2.1 Truncations of graded algebras

We will frequently use certain "truncations" of a $\mathbb{Z}_{\geq 0}$-graded algebra A in what follows. For a $\mathbb{Z}$-graded vector space V and integer n, we write $V_{\geq n} = \oplus_{m \geq n} V_m$, a vector space graded by $\{n, n+1, \dots\}$. We note the vector space injection $V_{\geq n} \to V$ that is the identity on the mth graded piece for $m \geq n$.

Definition 2.1 For a $\mathbb{Z}_{\geq 0}$-graded algebra A and each $N \geq 0$, we define: $A_{[0,N]} := A/A_{\geq N+1}$.

2.2 Quivers, doubles, and triples

Let $Q = (I, \Omega)$ be a finite quiver, so that $s, t : \Omega \rightrightarrows I$ are the source and target maps: for $a \in \Omega$ we have $\overset{s(a)}{\bullet} \xrightarrow{a} \overset{t(a)}{\bullet}$.

The *double* of Q is a quiver $Q^{\mathrm{dbl}} = (I, H = \Omega \sqcup \overline{\Omega})$ with the same vertex set I as for Q and the set of arrows $H = \Omega \sqcup \overline{\Omega}$ where Ω is the arrow set of Q and $\overline{\Omega}$ is a set equipped with a bijection to Ω, written $\Omega \ni a \leftrightarrow a^* \in \overline{\Omega}$. We extend this bijection canonically to an involution on $H = \Omega \sqcup \overline{\Omega}$, still written $a \mapsto a^*$, and decree $s(a^*) = t(a), t(a^*) = s(a)$. For each arrow $a \in H$ we write

$$\epsilon(a) = \begin{cases} 1 & \text{if } a \in \Omega, \\ -1 & \text{if } a \in \overline{\Omega}. \end{cases}$$

Fix an integer $N \geq 1$. The *graded tripled quiver* Q^{gtr} associated to Q (cf. Section 4 of [26]) is a quiver defined as follows. We give Q^{gtr} the vertex set $I^{\mathrm{gtr}} = I \times [0, N]$ where I is the vertex set of Q. If Ω is the edge set of Q and $H = \Omega \sqcup \overline{\Omega}$ the associated set of pairs of an edge together with an orientation, we give Q^{gtr} the arrow set

$$\big(H \times [0, N-1]\big) \sqcup \big(I \times [0, N-1]\big). \qquad \text{Thus,}$$

(1) for each $h \in H, n \in [0, N-1]$ we have arrows (h, n) with $\overset{(s(h),n)}{\bullet} \xrightarrow{(h,n)} \overset{(t(h),n+1)}{\bullet}$, i.e.

$$s(h, n) = (s(h), n) \quad \text{and} \quad t(h, n) = (t(h), n+1);$$

(2) for each $i \in I, n \in [0, N-1]$ we have arrows $t_{(i,n)}$ with $\overset{(i,n)}{\bullet} \xrightarrow{t_{(i,n)}} \overset{(i,n+1)}{\bullet}$, i.e.

$$s(t_{i,n}) = (i, n) \quad \text{and} \quad t(t_{(i,n)}) = (i, n+1).$$

More discussion can be found in [26, §4].

2.3 Path algebras

Let $S = \bigoplus_i ke_i$ be a commutative semisimple algebra over a field k, with orthogonal system of idempotents $\{e_i\}$. Suppose A is an algebra with homomorphism $S \to A$. We say that $x \in A$ *has diagonal Peirce decomposition* if $x \in \bigoplus_{i \in I} e_i A e_i$, or equivalently if it lies in the centralizer $Z_A(S)$.

Let $Q = (I, H)$ be a quiver. The path algebra kQ of Q is defined as follows: Let S denote the finite-dimensional (semisimple commutative) k-algebra $S = \bigoplus_{i \in I} ke_i$ with idempotents e_i labelled by the vertices $i \in I$. We define an S-bimodule $B = B(Q)$, with k-basis labelled by the arrows, where the S-bimodule structure takes arrows to be directed "left-to-right," so $e_i a e_j = 0$ unless $i = s(a)$, $j = t(a)$, and so that $e_{s(a)} a e_{t(a)} = a$. Then the path algebra kQ is defined to be the tensor algebra $T_S(B(Q))$.

It is natural to grade the path algebra kQ of any quiver $Q = (I, H)$—for example, kQ^{dbl}— using the normal grading on a tensor algebra, thus the semisimple algebra S lies in degree 0 and the arrows $h \in H$ lie in degree 1.

If A is any S-algebra, we write $A[t]$ for the associative S algebra obtained by adjoining a central variable t (thus every element of A commutes with t). The algebra $A[t]$ is graded by t-degree, and hence if we take $A = kQ^{\mathrm{dbl}}$, the algebra $kQ^{\mathrm{dbl}}[t]$ is naturally bi-graded, but we will only use the total grading, with respect to which $\deg(t) = 1$. Using the above grading for the path algebra kQ^{gtr}, we obtain a graded algebra homomorphism

$$kQ^{\mathrm{dbl}}[t] \longrightarrow kQ^{\mathrm{gtr}}/J \qquad \text{by taking}$$
$$e_i \mapsto \sum_n e_{(i,n)} + J, \quad i \in I, \quad h \mapsto \sum_n (h,n) + J, \quad h \in H, \quad t \mapsto \sum_{(i,n)} t_{(i,n)} + J,$$

where J denotes the two-sided ideal

$$J = \Big(\big\{ t_{(s(h),n)} \cdot (h, n+1) - (h,n) \cdot t_{(t(h),n+1)} \;\big|\; h \in H, \quad n \in [0, N-2] \big\} \Big). \tag{2.1}$$

The graded algebra kQ^{gtr} has the property $kQ^{\mathrm{gtr}}_{\geq N+1} = 0$, so we obtain a homomorphism

$$kQ^{\mathrm{dbl}}[t]_{[0,N]} := kQ^{\mathrm{dbl}}[t]/kQ^{\mathrm{dbl}}[t]_{\geq N+1} \longrightarrow kQ^{\mathrm{gtr}}/J. \tag{2.2}$$

The above discussion thus shows:

Lemma 2.2 *Let kQ^{gtr}-mod$_J$ denote the category of finite-dimensional left modules for kQ^{gtr} annihilated by J, and furthermore let $kQ^{\mathrm{dbl}}[t]_{[0,N]}$-gr$_{[0,N]}$ denote the category of finite-dimensional graded left $kQ^{\mathrm{dbl}}[t]$-modules concentrated in degrees $[0, N]$. Then the homomorphism (2.2) determines an equivalence of categories:*

$$kQ^{\mathrm{gtr}}\text{-mod}_J \xrightarrow{\simeq} kQ^{\mathrm{dbl}}[t]_{[0,N]}\text{-gr}_{[0,N]}.$$

2.4 Universal localizations

We briefly review some aspects of universal localizations that may be unfamiliar to the reader, using Chapter 4 of [28] as our reference; see also [9].

Suppose that R is a ring with 1 and Σ is a set of elements of R. Then there is a ring R_Σ with a homomorphism $R \to R_\Sigma$ that is universal with respect to the property that for every $r \in \Sigma$, r becomes invertible in R_Σ, i.e. r has a two-sided inverse r^{-1} in R_Σ. The ring R_Σ is called the *universal localization* of R at Σ; an alternative notation that is sometimes preferable is $\Sigma^{-1}R$. The universal localization is constructed as follows: letting Σ^{-1} denote the set of symbols a^{-1} for $a \in \Sigma$, we define

$$\Sigma^{-1}R = R_\Sigma := R\langle\Sigma^{-1}\rangle/(\{a^{-1}a - 1, aa^{-1} - 1 \mid a \in \Sigma\}).$$

This evidently has the universal property claimed, though very little else can be deduced about R_Σ from this construction. Another more illuminating construction (which allows one to invert morphisms between arbitrary projective modules) is given in Theorem 4.1 of [28].

We however will only need the following properties, which follow immediately from the universal property.

Proposition 2.3 *Suppose R is a ring with 1.*

(1) *If $t \in Z(R)$ is central, then $R_{\{t\}}$ is isomorphic to the Øre localization of R at t.*
(2) *If $\Sigma, \Sigma' \subseteq R$ are subsets, let $\overline{\Sigma}'$ denote the image of Σ' in R_Σ. Then $(R_\Sigma)_{\overline{\Sigma}'} \cong R_{\Sigma\cup\Sigma'}$.*
(3) *Given a two-sided ideal $I \subseteq R$, let $\overline{\Sigma}$ denote the image of Σ in R/I and I_Σ denote the two-sided ideal in R_Σ generated by I. Then $(R/I)_{\overline{\Sigma}} \cong R_\Sigma/I_\Sigma$.*

2.5 Multiplicative preprojective algebras

We review the multiplicative preprojective algebra of a quiver Q as defined in [10].

Given a quiver Q with double $Q^{\mathrm{dbl}} = (I, H)$, for each arrow $a \in H$ of Q^{dbl}, we define $g_a = 1 + aa^* \in kQ^{\mathrm{dbl}}$. Write L_Q for the algebra obtained by universal localization of kQ^{dbl} inverting $\Sigma = \{g_a \mid a \in H\}$. Identify the tuple $q \in (k^\times)^I$ with the element $\sum_{i\in I} q_i e_i \in S$. Crawley-Boevey and Shaw choose an ordering of the arrows in H and define $\rho_{\mathrm{CBS}} = \prod_{a\in H}^{\rightarrow} g_a^{\epsilon(a)} - q$ (the arrow over the product indicates that it is taken in the chosen order). It is proven in [10, §2] that, up to isomorphism, the quotient algebra $L_Q/(\rho_{\mathrm{CBS}})$ does not depend on the choice of ordering. Thus, in this paper we specifically fix an ordering $\Omega = \{a_1, \ldots, a_g\}$ on the arrows in Q, and let

$$\rho_{\mathrm{CBS}} = g_{a_1} g_{a_2} \cdots g_{a_g} g_{a_1^*}^{-1} \cdots g_{a_g^*}^{-1} - q. \tag{2.3}$$

Definition 2.4 Following [10, Definition 1.2], the associated multiplicative preprojective algebra is defined to be

$$\Lambda^q = \Lambda^q(Q) = L_Q/(\rho_{\mathrm{CBS}}),$$

where ρ_{CBS} is defined as in (2.3).

2.6 Homogenized multiplicative preprojective algebras

A principal tool in this paper is a certain graded algebra A that "homogenizes" the multiplicative preprojective algebra Λ^q. Here we construct the algebra A and collect some basic facts about A and its relation to the multiplicative preprojective algebra Λ^q.

Thus, fix a quiver Q. Recall that we consider $kQ^{\mathrm{dbl}}[t]$ as a nonnegatively graded algebra, with the generators $a \in H, t$ all in degree 1, and $S = \oplus_{i\in I} ke_i$ in degree 0. We let

$$G_a := t^2 + aa^* \in kQ^{\mathrm{dbl}}[t] \quad \text{for all} \quad a \in H.$$

Remark 2.5 Each G_a has diagonal Peirce decomposition: more precisely,

$$e_{s(a)}G_a = e_{s(a)}t^2 + aa^* = G_a e_{s(a)}, \quad \text{and} \quad e_i G_a = e_i t^2 = t^2 e_i = G_a e_i \quad \text{for } i \neq s(a).$$

We note the obvious equalities

$$G_a a = aG_{a^*}, \quad a^* G_a = G_{a^*} a^*. \tag{2.4}$$

Given $q \in (k^\times)^I$, we identify q with $q := \sum_{i\in I} q_i e_i \in kQ^{\mathrm{dbl}}$, a sum of idempotents in the path algebra (which thus also has diagonal Peirce decomposition).

Analogously to [10], we write L_t for the universal localization of the algebra $kQ^{\mathrm{dbl}}[t]$ in which the elements $\{G_a, a \in H\}$, and t are inverted. The algebra L_t is graded and contains invertible elements $g_a = t^{-2}G_a = 1 + \frac{a}{t}\frac{a^*}{t}$ in graded degree 0. Moreover we have $(\mathsf{L}_t)_0 \cong L_Q$, where as reviewed above L_Q is the universal localization of $kQ^{\mathrm{dbl}} \cong kQ^{\mathrm{dbl}}[t^{\pm 1}]_0$ at the elements $g_a, a \in H$. As above, fix an ordering $\Omega = \{a_1, \ldots, a_g\}$ on the arrows in Q. Write

$$D = G_{a_1} \ldots G_{a_g}, \quad D^* = q(G_{a_g^*} \ldots G_{a_1^*}), \tag{2.5}$$

$$\rho = D - D^* = (G_{a_1} \ldots G_{a_g}) - q(G_{a_g^*} \ldots G_{a_1^*}) \in kQ^{\mathrm{dbl}}[t]. \tag{2.6}$$

Definition 2.6 We write $\mathsf{A} = kQ^{\mathrm{dbl}}[t]/(\rho)$, where (ρ) denotes the two-sided ideal generated by ρ.

The element ρ has diagonal Peirce decomposition, and so $\rho e_i = e_i \rho$, and $(\rho) = (\{\rho e_i | i \in I\})$.

Proposition 2.7 *Write* $\Sigma = \{G_a \mid a \in H\} \cup \{t\}$. *We have:*

(1) A *is a graded algebra where* a_i, a_i^* *and* t *have degree 1 (and* $S = \sum_{i \in I} ke_i$ *lies in degree 0).*
(2) *The universal localization*

$$\Lambda_t := \Sigma^{-1}\mathsf{A} \tag{2.7}$$

of A *obtained by inverting all* $G_a, a \in H$, *and* t, *is a graded algebra, and* $\Lambda_t \cong \Lambda^q(Q)[t^{\pm 1}]$ *where* $\Lambda^q(Q) =: \Lambda^q$ *denotes the multiplicative preprojective algebra of [10].*

Proof This almost all follows from the above discussion. The isomorphism (2.7) of part (2) of Proposition 2.7 follows from Proposition 2.3. □

3 Representations and their moduli

3.1 Representations of kQ^{dbl} and kQ^{gtr}

Fixing some $N \geq 2g$, where g is the number of arrows in Q, we form the graded-tripled quiver Q^{gtr} associated to Q as above.[1] Given a dimension vector $\alpha \in \mathbb{Z}_{\geq 0}^I$ for the quiver Q^{dbl}, we write $\alpha^{\mathrm{gtr}} \in \mathbb{Z}_{\geq 0}^{I \times [0,N]}$ for the dimension vector for kQ^{gtr} for which $\alpha^{\mathrm{gtr}}_{i,n} = \alpha_i$ for all $n \in [0, N]$. We write $\mathrm{Rep}(kQ^{\mathrm{dbl}}, \alpha)$ for the space of representations of kQ^{dbl} with on the I-graded vector space V with $V_i = k^{\alpha_i}$ for all $i \in I$, so that V has dimension vector α and $\mathbb{G} = \prod_i GL(\alpha_i)$ for the automorphism group; thus

$$\mathrm{Rep}(kQ^{\mathrm{dbl}}, \alpha) := \prod_{h \in H} \mathrm{Hom}(k^{\alpha_{s(h)}}, k^{\alpha_{t(h)}}).$$

Similarly we write $\mathrm{Rep}(kQ^{\mathrm{gtr}}, \alpha^{\mathrm{gtr}})$ for the space of representations of kQ^{gtr} with dimension vector α^{gtr}, and $\mathbb{G}^{\mathrm{gtr}}$ for the automorphism group.

As in the construction of Section 4.3 of [26], there is a natural "induction functor" from the category of representations of kQ^{dbl} with dimension vector α to the category of representations of kQ^{gtr} of dimension vector α^{gtr}. The construction proceeds as follows. To a representation V of kQ^{dbl} we may associate the $\mathbb{Z}_{\geq 0}$-graded vector space $V[t]$, and let arrows h of Q^{dbl} act as multiplication followed by shift-of-grading. This makes $V[t]$ into a graded left $kQ^{\mathrm{dbl}}[t]$-module. We then form $V[t]/V[t]_{\geq N+1}$, a graded left $kQ^{\mathrm{dbl}}[t]_{[0,N]}$-module, and finally apply Lemma 2.2 to get a representation of kQ^{gtr}: in fact, a representation of the quotient kQ^{gtr}/J where J is as in (2.1).

More concretely, the above construction is the following. Suppose we have a representation $V = (V_i)_{i \in I}$ of kQ^{dbl} of dimension vector α. We obtain a representation of $kQ^{\mathrm{dbl}}[t]$ on a vector space $V_{\bullet,\bullet}$ of dimension vector α^{gtr} defined by setting $V_{\bullet,\bullet} = V \times [0, N]$ where

[1] Thus, in particular, N is at least as large as the degree of the relation ρ.

(1) the grading is given by $V_{i,n} := V_i \times \{n\}$ for all $n \in [0, N]$;
(2) the action of the generators $t_{i,n}$ of kQ^{gtr} for $(i, n) \in I \times [0, N-1]$ is given by $\mathrm{id}_{V_i} \times \sigma$, where $\sigma : [0, N-1] \to [0, N]$ is given by $\sigma(n) = n+1$; and
(3) the action of each generator $(h, n) \in H \times [0, N-1]$ is given by

$$h \times \sigma : V_{s(h),n} \to V_{t(h),n+1}$$

The construction determines a morphism of algebraic varieties ("induction")

$$\mathsf{Ind}^\circ : \mathrm{Rep}(kQ^{\mathrm{dbl}}, \alpha) \longrightarrow \mathrm{Rep}(kQ^{\mathrm{gtr}}, \alpha^{\mathrm{gtr}}).$$

Write $\mathbb{G} = \prod_i GL(V_i)$ and $\mathbb{G}^{\mathrm{gtr}} = \prod_{(i,n)\in I\times[a,b]} GL(V_{i,n}) \cong \prod_{n\in[a,b]} \mathbb{G}$,

with the diagonal homomorphism $\mathrm{diag} : \mathbb{G} \to \mathbb{G}^{\mathrm{gtr}} \cong \prod_{n\in[a,b]} \mathbb{G}$. Then the morphism Ind° is $(\mathbb{G}, \mathbb{G}^{\mathrm{gtr}})$-equivariant. We thus get a natural $\mathbb{G}^{\mathrm{gtr}}$-equivariant morphism

$$\mathsf{Ind} : \mathbb{G}^{\mathrm{gtr}} \times_{\mathbb{G}} \mathrm{Rep}(kQ^{\mathrm{dbl}}, \alpha) \longrightarrow \mathrm{Rep}(kQ^{\mathrm{gtr}}, \alpha^{\mathrm{gtr}}). \tag{3.1}$$

Thus, given a representation $(a_h : V_{s(h)} \to V_{t(h)})_{h\in H}$ of kQ^{dbl} on V, and $(g_{i,n}) \in \mathbb{G}^{\mathrm{gtr}}$, we have

$$\mathsf{Ind}\big((g_{i,n}), a_h\big) = \big((h, n), t_{i,n}\big) \text{ where } (h, n) = g_{t(h),n+1} a_h g_{s(h),n}^{-1} \text{ and } t_{i,n} = g_{i,n+1} g_{i,n}^{-1}.$$

Proposition 3.1 *The map* Ind *of* (3.1) *defines a* $\mathbb{G}^{\mathrm{gtr}}$*-equivariant open immersion of* $\mathbb{G}^{\mathrm{gtr}} \times_{\mathbb{G}} \mathrm{Rep}(kQ^{\mathrm{dbl}}, \alpha)$ *in* $\mathrm{Rep}(kQ^{\mathrm{gtr}}/J, \alpha^{\mathrm{gtr}})$*, whose image consists of those* $\big((h, n), t_{i,n}\big)$ *for which:*

$$t_{i,n} \text{ is an isomorphism for all } n \in [0, N-1]. \quad (\dagger)$$

Proof This follows from the discussion above, indeed one can readily adapt the proof of the corresponding statment for Nakajima quiver varieties, see [26, Proposition 4.1]. □

3.2 Representations of A and $\mathsf{A}_{[0,N]}$

Let A-Gr denote the category of graded left A-modules. We also consider the category $\mathsf{A}_{[0,N]}$-$\mathrm{Gr}_{\geq 0}$ of those graded left $\mathsf{A}_{[0,N]}$-modules M for which $M_i = 0$ for $i \notin [0, N]$. We remark that $\mathsf{A}_{[0,N]}$-$\mathrm{Gr}_{\geq 0}$ can naturally be viewed as a full subcategory of the category $\mathsf{A}_{[0,N]}$-Gr of all graded left $\mathsf{A}_{[0,N]}$-modules, hence also of A-Gr. Define a functor of truncation,

$$\tau_{[0,N]} : \mathsf{A}\text{-Gr} \longrightarrow \mathsf{A}_{[0,N]}\text{-}\mathrm{Gr}_{\geq 0},$$

by $M \mapsto \tau_{[0,N]}M := M_{\geq 0}/M_{\geq N+1}$. As above, we have a graded vector space injection $\tau_{[0,N]}(M) \to M$ that is the identity on the mth graded piece for $m \in [0, N]$ and is zero elsewhere; this map is $\mathsf{A}_{\leq m}$-linear on M_{N-m}.

3.3 Representations of A and Λ^q

We note:

Remark 3.2 The functor Λ_t-Gr $\longrightarrow \Lambda^q$-Mod, $M \mapsto M_0$, is an equivalence of categories.

Recall from (2.7) that, letting $\Sigma = \{G_a \mid a \in H\} \cup \{t\}$, we have a graded algebra isomorphism $\Sigma^{-1}\mathsf{A} \cong \Lambda_t = \Lambda^q[t^{\pm 1}]$, and hence a graded algebra homomorphism $\mathsf{A} \to \Lambda^q[t^{\pm 1}]$. Given a left or right Λ^q-module $\overline{\mathcal{M}}$, we form a graded left or right Λ_t-module $\mathcal{M} = \overline{\mathcal{M}}[t^{\pm 1}]$, and thus a graded A-module $M = \mathcal{M}[t] = \mathcal{M}[t^{\pm 1}]_{\geq 0}$. This defines a functor

$$R : \Lambda^q\text{-Mod} \longrightarrow \mathsf{A}\text{-Gr}_{\geq 0}\,.$$

In the opposite direction, we have a functor $(\Lambda_t \otimes_{\mathsf{A}} -)_0 : \mathsf{A}\text{-Gr}_{\geq 0} \to \Lambda^q$-Mod. We have:

Lemma 3.3

(1) *The functors* $(\Lambda_t \otimes_{\mathsf{A}} -)_0 : \mathsf{A}\text{-Gr}_{\geq 0} \rightleftarrows \Lambda^q\text{-Mod} : R$ *form an adjoint pair.*

(2) *If* $\overline{\mathcal{M}}$ *is a finite-dimensional left* Λ^q*-module then the graded left* A*-module* $M = \overline{\mathcal{M}}[t]$ *is finitely generated and projective as a left* S_t*-module and as a left* $k[t]$*-module. Moreover, we have* $\mathrm{Hom}_{k[t]}(M, k[t]) \cong \mathrm{Hom}_k(\overline{\mathcal{M}}, k)[t]$ *as a graded right* A*-module.*

Proof The first statement follows from the isomorphism $\Lambda_t \cong \Lambda^q[t^{\pm 1}]$ along with the equivalence of categories noted in the above remark. Indeed

$$\begin{aligned}\mathrm{Hom}(M, N[t]) &\cong \mathrm{Hom}(M, N[t^{\pm 1}]) \cong \mathrm{Hom}(\Lambda_t \otimes_A M, N[t^{\pm 1}])\\ &\cong \mathrm{Hom}((\Lambda_t \otimes_A M)_0, N),\end{aligned}$$

where the second isomorphism follow from universality. The second statement follows immediately from the definitions. □

3.4 Representation spaces and group actions

Because the multiplicative preprojective algebra Λ^q is the quotient $L_Q/(\rho_{\mathrm{CBS}})$ of the localization L_Q of kQ^{dbl} by the ideal generated by ρ_{CBS}, the space $\mathrm{Rep}(\Lambda^q, \alpha)$ of left Λ^q-modules with dimension vector α is naturally a locally closed subscheme of $\mathrm{Rep}(kQ^{\mathrm{dbl}}, \alpha)$: it is the closed subset, defined by vanishing of ρ_{CBS}, of the open set defined by invertibility of the elements g_a.

Similarly, the algebra $\mathsf{A}_{[0,N]}$ is a quotient of $kQ^{\mathrm{dbl}}[t]_{[0,N]}$ and thus, via Lemma 2.2, the space $\mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})$ of graded left $\mathsf{A}_{[0,N]}$-modules concentrated in degrees $[0, N]$ is identified with a closed subscheme of $\mathrm{Rep}(kQ^{\mathrm{gtr}}, \alpha^{\mathrm{gtr}})$ defined by the vanishing of the images of ρ and J in kQ^{gtr}.

It is immediate from the construction of Sect. 3.1 that:

Proposition 3.4 (cf. Prop. 4.7 of [26]) *The morphism* Ind *of* (3.1) *restricts to an open immersion:*

$$\mathsf{Ind} : \mathbb{G}^{\mathrm{gtr}} \times_{\mathbb{G}} \mathrm{Rep}(\Lambda^q, \alpha) \to \mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}}).$$

Its image consists of those representations on which the elements t, G_a *act invertibly whenever their domain and target lie in the range* $[0, N]$.

Corollary 3.5 *The map* Ind *defines an open immersion of moduli stacks*

$$\mathrm{Rep}(\Lambda^q, \alpha)/\mathbb{G} \to \mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})/\mathbb{G}^{\mathrm{gtr}}.$$

3.5 Semistability and stability

We next discuss (semi)stability of representations and the corresponding GIT quotients.

For any quiver $Q = (I, \Omega)$ with dimension vector $\alpha \in \mathbb{Z}^I_{\geq 0}$, a GIT stability condition is given by $\theta \in \mathbb{Z}^I_{\geq 0}$ satisfying $\sum_i \theta_i \alpha_i = 0$. The vector θ determines a character $\chi_\theta : \prod_i GL(\alpha_i) \to \mathbb{G}_m$, $\chi(g_i)_{i\in I} = \prod_i \det(g_i)^{\theta_i}$, and the condition $\sum_i \theta_i \alpha_i = 0$ guarantees that the diagonal copy $\Delta(\mathbb{G}_m)$ of $\mathbb{G}_m$ in $\prod_i GL(\alpha_i)$ lies in the kernel of χ; we require this because $\Delta(\mathbb{G}_m)$ acts trivially on $\mathrm{Rep}(Q, \alpha)$. Given dimension vectors β, α, we write $\beta < \alpha$ if $\beta \neq \alpha$ and $\beta_i \leq \alpha_i$ for all $i \in I$.

We now turn to stability conditions for the doubled and tripled quivers Q^{dbl} and Q^{gtr} for a fixed quiver Q. Suppose θ is a stability condition for Q^{dbl} and dimension vector α. We construct a stability condition θ^{gtr} for Q^{gtr} with dimension vector α^{gtr} as follows. For a representation M of kQ^{gtr} of dimension vector α^{gtr}, we write $\delta_{i,n}(M) := \dim(M_{i,n})$; we will write θ^{gtr} as a linear combination of the $\delta_{i,n}$. Also, we note that it suffices to construct a *rational* linear functional θ^{gtr}, since any positive integer multiple of θ^{gtr} evidently defines the same stable and semistable loci. We fix an ordering on the vertices of Q, identifying $I = \{1, \ldots, r\}$. and a positive integer $T \gg 0$. We define:

$$\theta^{\mathrm{gtr}} := \sum_{i=1}^{r} T^i \left[\delta_{i,N} - \delta_{i,0}\right] + \sum_{i\in I} \theta_i \delta_{i,0}.$$

Proposition 3.6 *Suppose* $M = \mathsf{Ind}(N)$ *for some representation* N *of* kQ^{dbl} *with dimension vector* α. *Then* M *is semistable, respectively stable, with respect to* θ^{gtr} *if and only if* N *is semistable, respectively stable, with respect to* θ.

The proof is an easy adaptation of that of Proposition 4.12(4) of [26].

We remark that the above construction does *not* match [26]: there we chose to construct a stability θ^{gtr} for Q^{gtr} that would be nondegenerate if θ was, whereas here we ignore this possible requirement. While it would be possible to copy the construction of a stability θ^{gtr} from [26] and prove analogues of the statements of [26], there are cases important to multiplicative quiver varieties in which it is not possible to find a stability condition for kQ^{dbl} that is nondegenerate in the sense used in [26]: for example, the case when Q has a single vertex and loops based at that vertex, with dimension vector $\alpha = n > 1$. However, again for multiplicative quiver varieties, in some interesting cases the choice of the parameter q can guarantee that every semistable representation of Λ^q is automatically stable (though not for numerical reasons, as nondegeneracy guarantees). Indeed, we say $q = (q_i)_{i\in I} \in (k^\times)^I$ is a *primitive αth root of unity* if $q^\alpha := \prod q_i^{\alpha_i} = 1$ and $q^\beta \neq 1$ for all $0 < \beta < \alpha$. We have:

Lemma 3.7 ([10], Lemma 1.5)

(1) *Suppose that M is a representation of Λ^q with dimension vector α. Then $q^\alpha = 1$.*
(2) *In particular, if q is a primitive αth root of 1, then every representation of Λ^q of dimension vector α is θ-stable for every θ.*

For example, if $Q = (\{*\}, E)$ where E has g loops at $*$, $\alpha = n$, and q is a primitive nth root of 1, then every representation of Λ^q of dimension n is stable for every θ; the corresponding moduli space of representations of Λ^q is the character variety $\mathrm{Char}(\Sigma_g, GL_n, q\,\mathrm{Id})$ of the introduction.

Remark 3.8 It would be interesting to characterize those stability conditions θ^{gtr} for kQ^{gtr} with the property that there is a stability condition θ for kQ^{dbl} so that if $M = \mathsf{Ind}(N)$ then M is θ^{gtr}-(semi)stable if and only if N is θ-(semi)stable.

Notation 3.9 *We write*

$$\mathcal{M}_\theta^q(\alpha) := \mathrm{Rep}(\Lambda^q, \alpha)/\!\!/_\theta \mathbb{G} \quad \textit{and} \quad \mathcal{M}_\theta^q(\alpha)^{\mathrm{s}} := \mathrm{Rep}(\Lambda^q, \alpha)^{\theta-\mathrm{s}}/\!\!/_\theta \mathbb{G}$$

for the coarse moduli spaces determined by a stability condition θ.

3.6 Moduli stacks and resolutions

The moduli stacks

$$\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-ss}}/\mathbb{G} \text{ and } \mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}/\mathbb{G}^{\mathrm{gtr}}$$

are never Deligne–Mumford stacks: the diagonal copy of $\mathbb{G}_m$ in $\mathbb{G}$, respectively $\mathbb{G}^{\mathrm{gtr}}$, always acts trivially on $\mathrm{Rep}(\Lambda^q, \alpha)$, respectively $\mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}$. Thus, the moduli stack of stable representations $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{G}$ is always a $\mathbb{G}_m$-gerbe over the moduli space $\mathcal{M}_\theta^q(\alpha)^{\mathrm{s}}$ of stable representations.

However, one can make a choice of subgroup $\mathbb{S} \subset \mathbb{G}$ that ensures that the quotient stack $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta-\mathrm{s}}/\mathbb{S}$ is a Deligne–Mumford stack and that $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta-\mathrm{s}}/\mathbb{S} \to \mathcal{M}_\theta^q(\alpha)^{\mathrm{s}}$ is a finite gerbe (indeed a principal BH-bundle for a finite abelian group

H). Indeed, for example, we can choose any character $\rho : \mathbb{G}^{\mathrm{gtr}} \to \mathbb{G}_m$ for which the composite with the diagonal embedding $\rho \circ \Delta : \mathbb{G}_m \to \mathbb{G}_m$ is nontrivial, hence surjective. Then $\mathbb{S}^{\mathrm{gtr}} := \ker(\rho)$ has the property that $\mathbb{G}^{\mathrm{gtr}} = \mathbb{S}^{\mathrm{gtr}} \cdot \Delta(\mathbb{G}_m)$ and similarly letting $\mathbb{S} = \mathbb{G} \cap \mathbb{S}^{\mathrm{gtr}}$ we have $\mathbb{G} = \mathbb{S} \cdot \Delta(\mathbb{G}_m)$. Moreover, since $\Delta(\mathbb{G}_m)$ is the stabilizer of every point of $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}$ and $H := \Delta(\mathbb{G}_m) \cap \mathbb{S}$ is finite, we get:

Lemma 3.10 *The quotient* $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$ *is a Deligne–Mumford stack and the natural morphism*

$$\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \to \mathcal{M}_\theta^q(\alpha)^{\mathrm{s}}$$

is a torsor for the commutative group stack BH *(in particular, is a finite gerbe over* $\mathcal{M}_\theta^q(\alpha)^{\mathrm{s}}$*).*

By construction, we have an open immersion:

$$\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \hookrightarrow \mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}/\mathbb{S}^{\mathrm{gtr}},$$

and the coarse space of the target $\mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}/\mathbb{S}^{\mathrm{gtr}}$ is the projective moduli scheme $\mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})/\!\!/_{\theta^{\mathrm{gtr}}}\mathbb{G}^{\mathrm{gtr}}$: it is a closed subscheme of $\mathrm{Rep}(k\,Q^{\mathrm{gtr}}, \alpha^{\mathrm{gtr}})/\!\!/_{\theta^{\mathrm{gtr}}}\mathbb{G}^{\mathrm{gtr}}$, which (as in [26]) is projective because $k\,Q^{\mathrm{gtr}}$ has no oriented cycles, and hence is itself projective.

As in [26], our goal is to compactify $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$ appropriately. We therefore replace the quotient stack $\mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}/\mathbb{S}^{\mathrm{gtr}}$ by its closed substack defined as the closure of $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$:

Notation 3.11 *We denote the closure of* $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$ *in* $\mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}/\mathbb{S}^{\mathrm{gtr}}$ *by* $\overline{\mathcal{M}}_{\mathrm{st}}$.

Lemma 3.12 *The stack* $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$ *is smooth. The stack* $\overline{\mathcal{M}}_{\mathrm{st}}$ *is integral and its coarse moduli space is a projective scheme. The natural morphism* $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \hookrightarrow \overline{\mathcal{M}}_{\mathrm{st}}$ *is an open immersion.*

Proof The smoothness of $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$ is Theorem 1.10 of [10]. The remaining assertions are immediate. □

We may apply the results of [22] or [14] to $\mathrm{Rep}(k\,Q^{\mathrm{gtr}}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}/\mathbb{S}^{\mathrm{gtr}}$ and its closed substack $\overline{\mathcal{M}}_{\mathrm{st}}$ to obtain a projective Deligne–Mumford stack (i.e., a Deligne–Mumford stack whose coarse space is a projective scheme) $\overline{\mathcal{M}}'_{\mathrm{st}}$ equipped with a projective morphism $\overline{\mathcal{M}}'_{\mathrm{st}} \to \overline{\mathcal{M}}_{\mathrm{st}}$ that is an isomorphism over $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$. The stack $\overline{\mathcal{M}}'_{\mathrm{st}}$ is itself, by construction, a global quotient of a quasiprojective variety by $\mathbb{S}$, and thus we may apply equivariant resolution to resolve the singularities of $\overline{\mathcal{M}}'_{\mathrm{st}}$, to obtain:

Proposition 3.13 *The smooth Deligne–Mumford stack* $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$ *admits an open immersion*

$$\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \hookrightarrow \overline{\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}}$$

in a smooth projective Deligne–Mumford stack which is equipped with a projective morphism

$$\overline{\mathrm{Rep}(\Lambda^q,\alpha)^{\theta\text{-s}}/\mathbb{S}} \to \overline{\mathcal{M}}_{\mathrm{st}}$$

that extends the open immersion $\mathrm{Rep}(\Lambda^q,\alpha)^{\theta\text{-s}}/\mathbb{S} \hookrightarrow \overline{\mathcal{M}}_{\mathrm{st}}$.

4 The diagonal of the algebra A

4.1 Bimodule of derivations

Recall that we have fixed an ordering $\Omega = \{a_1, \dots, a_g\}$ on the arrows in Q. For $j = 1, \dots, g$ we write

$$\begin{aligned} L_{a_j} &= G_{a_1} \dots G_{a_{j-1}}, \quad R_{a_j} = G_{a_{j+1}} \dots G_{a_g}, \quad \text{so} \quad D = L_{a_j} G_{a_j} R_{a_j}, \\ L_{a_j^*} &= G_{a_g^*} \dots G_{a_{j+1}^*}, \quad R_{a_j^*} = G_{a_{j-1}^*} \dots G_{a_1^*}, \quad \text{so} \quad D^* = q L_{a_j^*} G_{a_j^*} R_{a_j^*}. \end{aligned}$$

Let B denote the sub-$(S[t], S[t])$-bimodule of $kQ^{\mathrm{dbl}}[t]$ spanned by the arrows, so that $kQ^{\mathrm{dbl}}[t]$ is identified with the tensor algebra $T_{S[t]}(B)$. As in [10, p. 190], the bimodule that is the target of the universal S-linear bimodule derivation of $kQ^{\mathrm{dbl}}[t]$ satisfies

$$\Omega_{S[t]}(kQ^{\mathrm{dbl}}[t]) \cong kQ^{\mathrm{dbl}}[t] \otimes_{S[t]} B \otimes_{S[t]} kQ^{\mathrm{dbl}}[t],$$

under which the universal derivation $\delta_{kQ^{\mathrm{dbl}}[t]/S[t]} : kQ^{\mathrm{dbl}}[t] \to \Omega_{S[t]}(kQ^{\mathrm{dbl}}[t])$ is identified with $a \mapsto 1 \otimes a \otimes 1$. As in [10, p. 190], for the universal localization L_t we also get $\Omega_{S[t]}(\mathsf{L}_t) \cong \mathsf{L}_t \otimes_{kQ^{\mathrm{dbl}}[t]} \Omega_{S[t]}(kQ^{\mathrm{dbl}}[t]) \otimes_{kQ^{\mathrm{dbl}}[t]} \mathsf{L}_t$ with the obvious identification of the universal derivation $\delta_{\mathsf{L}_t/S[t]}$. We write:

$$P_1 = \mathsf{A} \otimes_{S[t]} B \otimes_{S[t]} \mathsf{A} \cong \mathsf{A} \underset{kQ^{\mathrm{dbl}}[t]}{\otimes} \Omega_{S[t]}(kQ^{\mathrm{dbl}}[t]) \underset{kQ^{\mathrm{dbl}}[t]}{\otimes} \mathsf{A}. \tag{4.1}$$

The module P_1 is evidently projective as a bimodule. Via the above description, we obtain a collection of bimodule basis elements

$$\eta_a, a \in H, \quad \text{via} \quad \eta_a = 1 \otimes a \otimes 1 \in \mathsf{A} \otimes_{S[t]} B \otimes_{S[t]} \mathsf{A} = P_1.$$

4.2 An exact sequence

We write

$$P_0 = \mathsf{A} \otimes_{S[t]} \mathsf{A}.$$

Write $\eta_i = e_i \otimes 1 = 1 \otimes e_i, i \in I$, for the obvious bimodule generators of P_0. Define graded bimodule maps

$$P_0(-2g) \xrightarrow{\alpha} P_1(-1) \xrightarrow{\beta} P_0 \tag{4.2}$$

by $\beta(\eta_a) = a\eta_{s(a)} - \eta_{t(a)}a$ for arrows a of Q^{gtr}, and

$$\alpha(\eta_i) = \sum_{a\in\Omega, s(a)=i} L_a \Delta_a R_a - \sum_{a\in\Omega, t(a)=i} q L_{a^*} \Delta_{a^*} R_{a^*}, \tag{4.3}$$

where $\Delta_a = \delta(G_a)$ (where δ denotes the universal derivation). It is then immediate that $\alpha(\eta_i) = e_i \cdot \delta(\rho)$; in particular, letting $\theta : P_0 \to (\rho)/(\rho^2)$ denote the map defined by $\theta(p \otimes q) = p\rho q$ and writing ϕ for the isomorphism defined by (4.1), we have:

$$\phi \circ \alpha = \delta \circ \theta. \tag{4.4}$$

Imitating the proof of Lemma 3.1 of [10] gives:

Lemma 4.1 *The sequence*

$$P_0(-2g) \xrightarrow{\alpha} P_1(-1) \xrightarrow{\beta} P_0 \xrightarrow{\gamma} \mathsf{A} \to 0, \tag{4.5}$$

where $\gamma(p \otimes q) = pq$, is an exact sequence of $\mathbb{Z}$-graded bimodules.

Proof As in [28, Theorem 10.3], one gets an exact sequence

$$(\rho)/(\rho^2) \xrightarrow{\delta} \Omega_{S[t]}\big(kQ^{\mathrm{dbl}}[t]\big) \to \Omega_{S[t]}\mathsf{A} \to 0.$$

As in [10], splicing this sequence and the defining sequence for $\Omega_{S[t]}\big(\mathsf{A}\big)$ and applying (4.4) gives a commutative diagram

$$\begin{array}{ccccccccc}
P_0(-2g) & \xrightarrow{\alpha} & P_1(-1) & \xrightarrow{\beta} & P_0 & \longrightarrow & \mathsf{A} & \longrightarrow & 0 \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\cong}{\scriptstyle\phi} & & \downarrow{\scriptstyle\cong}{\scriptstyle\psi} & & \downarrow{\scriptstyle=} & & \\
(\rho)/(\rho^2) & \xrightarrow{\delta} & \mathsf{A} \underset{kQ^{\mathrm{dbl}}[t]}{\otimes} \Omega_{S[t]}(kQ^{\mathrm{dbl}}[t]) \underset{kQ^{\mathrm{dbl}}[t]}{\otimes} \mathsf{A} & \xrightarrow{\xi} & \mathsf{A} \otimes_{S[t]} \mathsf{A} & \longrightarrow & \mathsf{A} & \longrightarrow & 0.
\end{array}$$

The vertical arrows ϕ, ψ are isomorphisms and θ is surjective, yielding the assertion. □

4.3 Dual of the map $P_0(-2g) \xrightarrow{\alpha} P_1(-1)$

Recall that the *enveloping algebra* of A over $k[t]$ is

$$\mathsf{A}^e := \mathsf{A} \otimes_{k[t]} \mathsf{A}^{\mathrm{op}}.$$

We consider A^e as a left A^e-module where $a \otimes a' \in \mathsf{A}^e$ acts by

$$a \otimes a' \cdot (x \otimes x') = ax \otimes x'a'.$$

We remark that A^e naturally also has a *right* A^e-module structure commuting with the left A^e-action, where $a \otimes a' \in \mathsf{A}^e$ acts on the right by

$$(x \otimes x') \cdot a \otimes a' = xa \otimes a'x'.$$

Given a finitely generated left A^e-module, we form $P^\vee = \mathrm{Hom}_{\mathsf{A}^e}(P, \mathsf{A}^e)$, the dual over the enveloping algebra; by the above discussion, this module has a right A^e-module structure, which we can identify with a left A^e-module structure via the isomorphism

$$(\mathsf{A}^e)^{\mathrm{op}} \to \mathsf{A}^e, \quad a \otimes a' \mapsto a' \otimes a.$$

We now want to calculate the dual $\alpha^\vee$ of the map α of (4.2) using the formula (4.3). Note that

$$\Delta(G_a) = a\delta(a^*) + \delta(a)a^* = a\eta_{a^*} + \eta_a a^*.$$

We thus find from Formula (4.3) that the η_a-component of α is given by

$$\alpha(\eta_i)_{\eta_a} = \begin{cases} L_a \eta_a a^* R_a - q L_{a^*} a^* \eta_a R_{a^*} & \text{if } a \in \Omega, i = s(a), \\ L_{a^*} a^* \eta_a R_{a^*} - q L_a \eta_a a^* R_a & \text{if } a \in \overline{\Omega}, i = t(a) \end{cases}$$

and zero otherwise. Let $\{\eta_a^\vee\}$ denote the basis of $P_1^\vee$ dual to the basis $\{\eta_a\}$ of P_1; we note that

$$\eta_a^\vee \in e_{t(a)} P_1^\vee e_{s(a)}. \tag{4.6}$$

The above formulas then imply:

$$\alpha^\vee(\eta_a^\vee) = \begin{cases} a^* R_a \eta_{s(a)}^\vee L_a - q R_{a^*} \eta_{t(a)}^\vee L_{a^*} a^* & \text{if } a \in \Omega, \\ R_a^* \eta_{t(a)}^\vee L_{a^*} a^* - q a^* R_a \eta_{s(a)}^\vee L_a & \text{if } a \in \overline{\Omega}. \end{cases} \tag{4.7}$$

Lemma 4.2 *For all $a \in \Omega$, we have*

$$\alpha^\vee\big(\eta_a^\vee a - a^* \eta_{a^*}^\vee\big) = G_{a^*}\big(q R_{a^*} \eta_{t(a)}^\vee L_{a^*}\big) - \big(q R_{a^*} \eta_{t(a)}^\vee L_{a^*}\big) G_{a^*}, \tag{4.8}$$

$$\alpha^\vee\big(a\eta_a^\vee - \eta_{a^*}^\vee a^*\big) = G_a\big(R_a \eta_{s(a)}^\vee L_a\big) - \big(R_a \eta_{s(a)}^\vee L_a\big) G_a. \tag{4.9}$$

Proof These formulas follow by direct calculation using 4.7. □

Lemma 4.3 *If $a \in H$, $s(a) \neq i$, then $G_a D\eta_i^\vee = D\eta_i^\vee G_a$ in $P_0^\vee$.*

Proof The element D is a product of elements of diagonal Peirce type, hence itself is of diagonal Peirce type. Thus, using $e_{s(a)}\eta_i^\vee = 0 = \eta_i^\vee e_{s(a)}$, we get

$$\begin{aligned} G_a D\eta_i^\vee &= \big(G_a e_{s(a)} + (1-e_{s(a)})t^2\big)D\eta_i^\vee = (1-e_{s(a)}t^2)D\eta_i^\vee \\ &= D\eta_i^\vee(1-e_{s(a)}t^2) = D\eta_i^\vee\big(e_{s(a)}G_a + (1-e_{s(a)})t^2\big) = D\eta_i^\vee G_a. \end{aligned}$$

This completes the proof. □

Suppose now that $\mathcal{M}$ is a graded right Λ_t-module; then $M = \mathcal{M}_{\geq 0}$ is a graded right A-submodule of $\mathcal{M}$. For example, we could take $\mathcal{M} = \Lambda_t$ itself, as in (2.7). We consider the map

$$M \otimes_{\mathsf{A}} P_1^\vee(1) \xrightarrow{1_M \otimes \alpha^\vee} M \otimes_{\mathsf{A}} P_0^\vee(2g).$$

Remark 4.4 We note that, under the above hypothesis on M, for any product Q of elements G_a, $a \in H$, of degree $\deg(Q)$, the elements $Qt^{-\deg(Q)}$ and $t^{\deg(Q)}Q^{-1}$ of Λ_t give well defined operators of right multiplication on M that satisfy all relations in Λ_t.

Proposition 4.5 *Suppose that $M = \mathcal{M}_{\geq 0}$ for a graded right Λ_t-module $\mathcal{M}$. Then for all $m \in M$ and all $i \in I$ and $1 \leq j \leq g$,*

(1) *the elements $m\big(G_{a_j}D\eta_i^\vee - D\eta_i^\vee G_{a_j}\big)$, $m\big(G_{a_j^*}D\eta_i^\vee - D\eta_i^\vee G_{a_j^*}\big)$, and*
(2) *the elements $m\big(a_j^* Dt^{-2}\eta_{s(a_j)}^\vee - Dt^{-2}\eta_{t(a_j)}^\vee a_j^*\big)$, $m\big(a_j Dt^{-2}\eta_{t(a_j)}^\vee - Dt^{-2}\eta_{s(a_j)}^\vee a_j\big)$*

lie in $\mathrm{Im}(1_{\Lambda_t} \otimes \alpha^\vee) \subseteq M \otimes_{\mathsf{A}} P_0^\vee(2g)$.

Proof (1) We first prove that $m\big(G_{a_j}D\eta_i^\vee - D\eta_i^\vee G_{a_j}\big) \in \mathrm{Im}(1_M \otimes \alpha^\vee)$ by (strong) induction on j.
Base case $j = 1$. By Lemma 4.3, the assertion is true for $i \neq s(a_1)$. From Lemma 4.2, we have

$$\begin{aligned} mG_{a_1}\alpha^\vee\big(a_1\eta_{a_1}^\vee - \eta_{a_1^*}^\vee a_1^*\big) &= mG_{a_1}G_{a_1}\big(R_{a_1}\eta_{s(a_1)}^\vee L_{a_1}\big) - mG_{a_1}\big(R_{a_1}\eta_{s(a_1)}^\vee L_{a_1}\big)G_{a_1} \\ &= mG_{a_1}D\eta_{s(a_1)} - mD\eta_{s(a_1)}G_{a_1}. \end{aligned}$$

This completes the base case.
Induction step Assume $m\big(G_{a_k}D\eta_i^\vee - D\eta_i^\vee G_{a_k}\big) \in \mathrm{Im}(1_M \otimes \alpha^\vee)$ for all $i \in I$ and $k < j$. Again, by Lemma 4.3, we have $mG_{a_j}D\eta_i^\vee - mD\eta_i^\vee G_{a_j} \in \mathrm{Im}(1_M \otimes \alpha^\vee)$ for $i \neq s(a_j)$. Applying Lemma 4.2 gives

$$\begin{aligned} mG_{a_j}\alpha^\vee\big(a_j\eta_{s(a_j)}^\vee - \eta_{s(a_j)}^\vee a_j^*\big) &= mG_{a_j}G_{a_j}\big(R_{a_j}\eta_{s(a_j)}^\vee L_{a_j}\big) - mG_{a_j}\big(R_{a_j}\eta_{s(a_j)}^\vee L_{a_j}\big)G_{a_j} \\ &= mG_{a_j}\big(t^{\deg(L_{a_j})}L_{a_j}^{-1}Dt^{-\deg(L_{a_j})}\eta_{s(a_j)}^\vee L_{a_j}\big) \\ &\quad - \big(t^{\deg(L_{a_j})}L_{a_j}^{-1}Dt^{-\deg(L_{a_j})}\eta_{s(a_j)}^\vee L_{a_j}\big)G_{a_j} \\ &\in m\big(G_{a_j}D\eta_{s(a_j)}^\vee - D\eta_{s(a_j)}^\vee G_{a_j}\big) + \mathrm{Im}(1_M \otimes \alpha^\vee) \end{aligned}$$

where the last equality applies the inductive hypothesis for each $k < j$ (and various $m \in M$, see Remark 4.4). This completes the induction step, thus proving the assertion for the elements $G_{a_j} D\eta_i^\vee - D\eta_i^\vee G_{a_j}$.

The proof for $G_{a_j^*} D\eta_i^\vee - D\eta_i^\vee G_{a_j^*}$ follows the analogous descending induction on j.

(2) Taking note of Remark 4.4, from (4.7) we have

$$\alpha^\vee(mG_{a_j}t^{-2}\eta_{a_j}) = mG_{a_j}t^{-2}a_j^* R_{a_j}\eta^\vee_{s(a_j)} L_{a_j} - mG_{a_j}t^{-2}q\, R_{a_j^*}\eta^\vee_{t(a_j)} L_{a_j^*}a_j^*.$$

Applying part (1) of the proposition to the right-hand side of this formula gives

$$\begin{aligned}\alpha^\vee(mG_{a_j}t^{-2}\eta_{a_j}) &= mG_{a_j}t^{-2}a_j^* R_{a_j} L_{a_j}\eta^\vee_{s(a_j)} - mqG_{a_j}t^{-2} R_{a_j^*} L_{a_j^*}\eta^\vee_{t(a_j)} a_j^* \\ &\quad + \mathrm{Im}(1_{\Lambda_t}\otimes\alpha^\vee) \\ &= mG_{a_j}a_j^* G_{a_j}^{-1} Dt^{-2}\eta^\vee_{s(a_j)} - mG_{a_j}G_{a_j^*}^{-1} Dt^{-2}\eta^\vee_{t(a_j)} a_j^* \\ &\quad + \mathrm{Im}(1_{\Lambda_t}\otimes\alpha^\vee) \\ &= mG_{a_j}G_{a_j^*}^{-1}\big(a_j^* Dt^{-2}\eta^\vee_{s(a_j)} - Dt^{-2}\eta^\vee_{t(a_j)}a_j^*\big) + \mathrm{Im}(1_{\Lambda_t}\otimes\alpha^\vee)\end{aligned}$$

where the last equality uses (2.4); in particular this gives the first assertion of Part (2) of the proposition. The second assertion follows similarly. □

5 Analysis of the ext-complex

5.1 The complex (4.5) and the Hom-functor

Let M, N be graded left A-modules such that M is finitely generated and projective as a $k[t]$-module. To the exact sequence

$$P_0(-2g)\otimes_{\mathsf{A}} M \xrightarrow{\alpha\otimes 1} P_1(-1)\otimes_{\mathsf{A}} M \xrightarrow{\beta\otimes 1} P_0\otimes_{\mathsf{A}} M \xrightarrow{\gamma\otimes 1} M \to 0$$

we apply the functor $\mathrm{Hom}_{\mathsf{A}}(-, N)$ to obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathsf{A}}(M,N) \to \mathrm{Hom}_{\mathsf{A}}(P_0\otimes_{\mathsf{A}} M, N) \xrightarrow{(\beta\otimes 1)^*} \mathrm{Hom}_{\mathsf{A}}(P_1(-1)\otimes_{\mathsf{A}} M, N) \quad (5.1)$$

We continue the sequence (5.1) using

$$\mathrm{Hom}_{\mathsf{A}}(P_1(-1)\otimes_{\mathsf{A}} M, N) \xrightarrow{(\alpha\otimes 1)^*} \mathrm{Hom}_{\mathsf{A}}(P_0(-2g)\otimes_{\mathsf{A}} M, N). \quad (5.2)$$

Thus, we would like to compute the cokernel of the map (5.2).

Proposition 5.1 *Let M, N be graded left A-modules such that M is finitely generated and projective as a $k[t]$-module, and write $M^* = \mathrm{Hom}_{k[t]}(M, k[t])$. Consider the*

contravariant functors of finitely generated projective A^e*-modules* P*,*

$$P \mapsto \left(N \otimes_{k[t]} M^*\right) \otimes_{\mathsf{A}^e} P^\vee \quad \textit{and} \quad P \mapsto \mathrm{Hom}_{\mathsf{A}}(P \otimes_{\mathsf{A}} M, N).$$

The natural transformation $\left(N \otimes_{k[t]} M^*\right) \otimes_{\mathsf{A}^e} P^\vee \xrightarrow{\Psi} \mathrm{Hom}_{\mathsf{A}}(P \otimes_{\mathsf{A}} M, N)$ *of these functors of projective* A^e*-modules* P *is a natural isomorphism.*

Proof By projectivity, it suffices to check for $P = \mathsf{A}^e$, where it follows by adjunction. □

Corollary 5.2 *Under the hypotheses of Proposition 5.1, the cokernel of the map (5.2) is*

$$\mathrm{coker}\left(1_{M^*} \otimes \alpha^\vee \otimes 1_N : M^* \otimes_{\mathsf{A}} P_1^\vee(1) \otimes_{\mathsf{A}} N \to M^* \otimes_{\mathsf{A}} P_0^\vee(2g) \otimes_{\mathsf{A}} N\right).$$

We note the following identities, which are immediate from adjunction:

Lemma 5.3 *Suppose that* $M = \overline{M}[t]$ *is the graded left* A*-module associated to a finite-dimensional left* Λ^q*-module* $\overline{M}$*. Then:*

$$\begin{aligned}\mathrm{Hom}_{\mathsf{A}}(P_1 \otimes_{\mathsf{A}} M, N) &\cong \mathrm{Hom}_{\mathsf{A}}(\mathsf{A} \otimes_{S_t} B[t] \otimes_{S_t} \mathsf{A} \otimes_{\mathsf{A}} M, N) \cong \mathrm{Hom}_{S_t}(B \otimes_S M, N)\\ &\cong \mathrm{Hom}_S(B \otimes_S \overline{M}, N),\\ \mathrm{Hom}_{\mathsf{A}}(P_0 \otimes_{\mathsf{A}} M, N) &\cong \mathrm{Hom}_{\mathsf{A}}(\mathsf{A} \otimes_{S_t} \mathsf{A} \otimes_{\mathsf{A}} M, N) \cong \mathrm{Hom}_{S_t}(M, N) \cong \mathrm{Hom}_S(\overline{M}, N).\end{aligned}$$

5.2 The Ext-complex

Fix $N \geq 2g$. Let $\overline{V}$ be a finite-dimensional representation of Λ^q of dimension vector α, and let $V = \overline{V}[t]$ be the corresponding graded A-module as in Sect. 3.2, and specifically as in Lemma 3.3. Suppose W is a $\mathbb{Z}_{\geq 0}$-graded $\mathsf{A}_{[0,N]} = \mathsf{A}/\mathsf{A}_{\geq N+1}$-module, identified with a representation of Q^{gtr} that has dimension vector α^{gtr}. Thus $\tau_{[0,N]}V$ is also identified with a representation of Q^{gtr} that has dimension vector α^{gtr}.

Let $P_\bullet$ denote the complex of (4.2). We consider the complex $\mathrm{Hom}_{\mathsf{A}}(P_\bullet \otimes_{\mathsf{A}} V, W)$. Since the sources and target of the Homs in this complex are graded A-modules, each Hom-space can be regarded as a graded vector space; we write

$$\begin{aligned}\mathsf{Ext} = \Big[&\mathrm{Hom}_{\mathsf{A}\text{-}\mathrm{Gr}}(P_0 \otimes_{\mathsf{A}} V, W) \xrightarrow{\beta^\vee} \mathrm{Hom}_{\mathsf{A}\text{-}\mathrm{Gr}}(P_1 \otimes_{\mathsf{A}} V, W(1))\\ &\xrightarrow{\alpha^\vee} \mathrm{Hom}_{\mathsf{A}\text{-}\mathrm{Gr}}(P_0 \otimes_{\mathsf{A}} V, W(2g))\Big]\end{aligned}$$

for its degree 0 graded piece. As in [26], using Lemma 5.3 we may identify Ext with:

$$L(V_0, W_0) \xrightarrow{\partial_0} E(V_0, W_1) \xrightarrow{\partial_1} L(V_0, W_{2g}), \tag{5.3}$$

where $\partial_0 = \beta_0^\vee$ and $\partial_1 = \alpha_0^\vee$.

Proposition 5.4 *Suppose that $\tau_{[0,N]}V$ and W are graded $\mathsf{A}_{[0,N]}$-modules. Then:*

(1) *We have an isomorphism* $\operatorname{coker}(\partial_1) \cong \operatorname{Hom}_k\left(\operatorname{Hom}_{\mathsf{A}_{[0,N]}\text{-Gr}}(W, \tau_{[0,N]}V), k\right)$.

If, in addition, $\tau_{[0,N]}V$ is θ-stable and W is θ-semistable, both of dimension vector α^{gtr}, then:

(2) *We have* $\ker(\partial_0) = 0$ *unless* $\tau_{[0,N]}V \cong W$, *in which case* $\ker(\partial_0) \cong k$.
(3) *We have that* $\operatorname{coker}(\partial_1)$ *is zero unless* $\tau_{[0,N]}V \cong W$, *in which case* $\operatorname{coker}(\partial_1) \cong k$.

Proof Assertion (2) follows from the exactness of (5.1) and stability. Similarly, assertion (3) is immediate from assertion (1) by stability of $\tau_{[0,N]}V$ and semistability of W.

Thus it remains to prove assertion (1). Similarly to Lemma 5.3, we use Proposition 5.1 to identify

$$\operatorname{Hom}_{\mathsf{A}\text{-Gr}}(P_0 \otimes_{\mathsf{A}} V, W(2g)) \cong V^* \otimes_{S_t} W \cong V_0^* \otimes_S W_{2g} \cong \operatorname{Hom}_S(V_0, W_{2g}), \tag{5.4}$$

$$\operatorname{Hom}_{\mathsf{A}\text{-Gr}}(P_1 \otimes_{\mathsf{A}} V, W(1)) \cong (B \otimes_S V_0)^* \otimes_S W_1 \cong \operatorname{Hom}_S(B \otimes_S V_0, W_1). \tag{5.5}$$

Specifically, we use (5.4) to identify $\sum_r \lambda_r \otimes w_r \in V_0^* \otimes_S W_{2g}$ with an element $\phi \in L(V_0, W_{2g})$, i.e., an I-graded homomorphism $(\phi_i) : V_0 \to W_{2g}$; and we use (5.5) to identify $\sum_r \lambda_r \otimes w_r \in (B \otimes_S V_0)^* \otimes_S W_1$ with an element $\psi \in E(V_0, W_1)$. Under these identifications, the elements

$$\sum_r \lambda_r \left(a_j^* D t^{-2} \eta^\vee_{s(a_j)} - D t^{-2} \eta^\vee_{t(a_j)} a_j^*\right) w_r, \quad \sum_r \lambda_r \left(a_j D t^{-2} \eta^\vee_{t(a_j)} - D t^{-2} \eta^\vee_{s(a_j)} a_j\right) w_r$$

of Proposition 4.5 are identified with

$$\psi_{a_j} a_j^* t^{-2} D - a_j^* \psi_{a_j} t^{-2} D \quad \text{and} \quad \psi_{a_j^*} a_j t^{-2} D - a_j \psi_{a_j^*} t^{-2} D$$

for $\psi \in E(V_0, W_1) \cong \operatorname{Hom}_{\mathsf{A}\text{-Gr}}(P_1 \otimes_{\mathsf{A}} V, W(1))$.

Via the trace pairings, the k-linear dual of ∂_1 is a map $L(W_{2g}, V_0) \xrightarrow{\partial_1^*} E(W_1, V_0)$; an element $\phi^* \in L(W_{2g}, V_0)$ satisfies $\partial_1^*(\phi^*) = 0$ only if

$$\operatorname{tr}\left[\phi^* \psi_{a_j} a_j^* t^{-2} D - \phi^* a_j^* \psi_{a_j} t^{-2} D\right] = 0$$
$$\text{and} \quad \operatorname{tr}\left[\phi^* \psi_{a_j^*} a_j t^{-2} D - \phi^* a_j \psi_{a_j^*} t^{-2} D\right] = 0$$

for all $\psi \in E(V_0, W_1)$. Since each $G_{a_j} t^{-2}$ acts as an isomorphism on V^*, the elements $\lambda G_{a_j} t^{-2} \eta_{a_j} w$ and $\lambda G_{a_j^*} t^{-2} \eta_{a_j^*} w$, for $\lambda \in V_0^*$, $w \in W_1$, collectively generate $\operatorname{Hom}_{\mathsf{A}\text{-Gr}}(P_1 \otimes_{\mathsf{A}} V, W(1))$; it follows that an element $\phi^* \in L(W_{2g}, V_0)$ satisfies $\partial_1^*(\phi^*) = 0$ if and only if the above conditions are satisfied for all $\psi \in E(V_0, W_1)$.

Cyclically permuting, these conditions become

$$a_j^* t^{-2} D\phi^* - t^{-2} D\phi^* a_j^* = 0 \quad \text{and} \quad a_j t^{-2} D\phi^* - t^{-2} D\phi^* a_j = 0. \tag{5.6}$$

Given $\phi^* \in L(W_{2g}, V_0)$ satisfying these conditions, define $\Phi^* : W \to \tau_{[0,N]}V$ by taking $\Phi^*|_{W_{2g-m}} = t^{-m}D\phi^*t^m$. It is immediate from the conditions (5.6) that on W_{2g-m}, $m \geq 2$, we have that Φ^* commutes with all a_j and a_j^*, whereas for $m = 1$ we may write $\Phi^*|_{W_{2g-1}} = tt^{-2}D\phi^*t$ and again Φ^* commutes with a_j, a_j^*. Thus Φ^* defines an $\mathsf{A}_{[0,N]}$-linear homomorphism $W \to \tau_{[0,N]}V$, yielding a linear map $\ker(\partial_1^*) \hookrightarrow \mathrm{Hom}_{\mathsf{A}_{[0,N]}\text{-Gr}}(W, \tau_{[0,N]}V)$. Conversely, given a graded $\mathsf{A}_{[0,N]}$-module homomorphism $\Phi^* : W \to \tau_{[0,N]}V$, defining $\phi^* : W_{2g} \to V_0$ by $\phi^* = D^{-1}\Phi^*|_{W_{2g}}$, we see that $\phi^* \in \ker(\partial_1^*)$. This completes the proof. □

6 Cohomology of varieties and stacks

In the remainder of the paper, the base field k is assumed to be $\mathbb{C}$.

Here as throughout the paper, we use $H^*(X)$ to denote cohomology with $\mathbb{Q}$-coefficients, and $H^{\mathrm{BM}}_*(X)$ to denote Borel-Moore homology with $\mathbb{Q}$-coefficients; if X is a smooth Deligne–Mumford stack, there is a canonical isomorphism $H^*(X) \cong H^{\mathrm{BM}}_*(X)$.

6.1 Mixed hodge structure on the cohomology of an algebraic stack

Suppose that X is an algebraic stack of finite type over $\mathbb{C}$. It follows from Example 8.3.7 of [13] that the cohomology $H^*(X)$ comes equipped with a functorial mixed Hodge structure.

Proposition 6.1 *Suppose X is a complex Deligne–Mumford stack with the action of the commutative group stack BH for some finite group H, and that X has a coarse moduli space $X \to \mathrm{sp}(X)$ with an isomorphism $X \to \mathrm{sp}(X) = X/BH$. Then $H^*(X, \mathbb{Q}) = H^*\big(\mathrm{sp}(X), \mathbb{Q}\big)$ as mixed Hodge structures.*[2]

Proof Use the Leray spectral sequence and the fact that $H^*(BH, \mathbb{Q}) = \mathbb{Q}$ for a finite group H. □

6.2 Pushforwards and the projection formula

Suppose $f : X \to Y$ is a proper morphism of relative dimension d of smooth, connected Deligne–Mumford stacks. Then there is a pushforward, or Gysin, map $f_* : H^*(X) \to H^{*-d}(Y)$.

Proposition 6.2 ([6]) *If X and Y are of finite type (so their cohomologies support canonical mixed Hodge structures), the Gysin map f_* is a morphism of mixed Hodge structures.*

The Gysin map satisfies the projection formula: for classes $c \in H^*(X), c' \in H^*(Y)$, we have

$$f_*(c \cup f^*c') = f_*(c) \cup c'. \tag{6.1}$$

[2] We explicitly write the $\mathbb{Q}$-coefficients to emphasize that they are essential.

Suppose X and Y are smooth Deligne–Mumford stacks and $C \in H^*(X \times Y)$ is a cohomology class. By the Künneth theorem we have $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$, and thus we may write $C = \sum x_i \otimes y_i$ with $x_i \in H^*(X)$, $y_i \in H^*(Y)$. The classes x_i, y_i are the *Künneth components* of C (with respect to X or Y respectively).

Now suppose that $f : X \to Y$ is a representable morphism from a smooth Deligne–Mumford stack X to a smooth, proper Deligne–Mumford stack Y. The graph morphism $X \xrightarrow{(1,f)} X \times Y$ is not usually a closed immersion.

Proposition 6.3 (cf. Proposition 2.1 of [26]) *The image of $f^* : H^*(Y) \to H^*(X)$ is contained in the span of the Künneth components of $(1, f)_*[X]$ with respect to the left-hand factor X.*

Proof Write $X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$ for the projections. Write $p_* : Y \to \mathrm{Spec}(\mathbb{C})$ for the projection to a point; then $(p_X)_*$ exists since Y is proper. We have $f^* = (1, f)^* p_Y^*$ and $(p_X)_*(1, f)_* = \mathrm{id}$. Using the projection formula, then, we get

$$f^* = (p_X)_*(1, f)_* f^* = (p_X)_*(1, f)_*(1, f)^* p_Y^* = (p_X)_*\big((1, f)_*[X] \cap p_Y^*(-)\big).$$

This proves the claim. □

6.3 Cohomology of compactifications

We say that a finite-type Deligne–Mumford stack $\mathfrak{X}$ is *quasi-projective* if its coarse space $\mathrm{sp}(\mathfrak{X})$ is a quasi-projective scheme. For example, if a reductive group $\mathbb{S}$ acts on a polarized quasiprojective variety $\mathbb{M}$, then any open substack of $\mathbb{M}^s/\mathbb{S}$ is a quasi-projective Deligne–Mumford stack.[3]

The cohomology $H^k(\mathfrak{X})$ is *pure* if its mixed Hodge structure is pure of weight k: that is, $W_k\big(H^k(\mathfrak{X})\big) = H^k(\mathfrak{X})$ and $W_{k-1}\big(H^k(\mathfrak{X})\big) = 0$. We say $H^*(\mathfrak{X})$ *is pure* if each $H^k(\mathfrak{X})$ is pure.

Proposition 6.4 *Suppose $\mathfrak{Y} = Y/\mathbb{G}$ is a quotient stack (i.e., the quotient of an algebraic space by a linear algebraic group scheme) and that $\mathfrak{X}^\circ \subset \mathfrak{X} \subset \mathfrak{Y}$ are open, separated, quasi-projective, smooth Deligne–Mumford substacks of $\mathfrak{Y}$. Then the image of the restriction map $H^k(\mathfrak{X}) \to H^k(\mathfrak{X}^\circ)$ contains $W_k\big(H^k(\mathfrak{X}^\circ)\big)$; in particular, if $H^*(\mathfrak{X}^\circ)$ is pure, then the restriction map is surjective.*

Proof Consider first the case of smooth quasi-projective varieties $\mathfrak{X}^\circ \subset \mathfrak{X}$. Then, for any smooth projective compactification $\overline{\mathfrak{X}}$ of $\mathfrak{X}$, the image of $H^*(\overline{\mathfrak{X}}) \to H^*(\mathfrak{X}^\circ)$ is independent of the choice of $\overline{\mathfrak{X}}$: for example, by the Weak Factorization theorem, any two such $\overline{\mathfrak{X}}, \overline{\mathfrak{X}}'$ are related by a sequence of blow-ups and blow-downs along smooth centers in the complement of $\mathfrak{X}^\circ$, and the claimed independence follows from the usual formula for the cohomology of a blow-up. Since the image of $H^k(\overline{\mathfrak{X}})$ in $H^k(\mathfrak{X}^\circ)$ is $W_k\big(H^*K(\mathfrak{X}^\circ)\big)$ by Corollaire 3.2.17 of [12], the claim follows in this case.

[3] Here $\mathbb{M}^s$ means stable points in the GIT sense: in particular, stabilizers are finite.

We now consider the general case. By the assumptions, $\mathfrak{X}$ and $\mathfrak{X}^\circ$ are (separated) quasi-projective smooth Deligne–Mumford stacks that are global quotients. By Theorem 1 of [23], there exist a smooth quasi-projective scheme W and a finite flat LCI morphism $\mathsf{W} \to \mathfrak{X}$; the fiber product $\mathfrak{X}^\circ \times_{\mathfrak{X}} \mathsf{W} \to \mathfrak{X}^\circ$ is then also finite, flat, and LCI. Using the commutative square

$$\begin{array}{ccc} \mathfrak{X}^\circ \times_{\mathfrak{X}} \mathsf{W} & \xrightarrow{\widetilde{j}} & \mathsf{W} \\ \downarrow{\scriptstyle q^\circ} & & \downarrow{\scriptstyle q} \\ \mathfrak{X}^\circ & \xrightarrow{j} & \mathfrak{X} \end{array}$$

and base change, we find:

(1) $H^k(\mathsf{W}) \xrightarrow{q_*} H^k(\mathfrak{X})$ and $H^k(\mathfrak{X}^\circ \times_{\mathfrak{X}} \mathsf{W}) \xrightarrow{q_*^\circ} H^k(\mathfrak{X}^\circ)$ are surjective (indeed, $q_* q^*$ and $q_*^\circ (q^\circ)^*$ are multiplication by the degree of q).
(2) Since the Gysin maps q_*°, q_* are morphisms of mixed Hodge structures by Proposition 6.2,

$$W_k\big(H^k(\mathfrak{X}^\circ \times_{\mathfrak{X}} \mathsf{W})\big) \xrightarrow{q_*^\circ} W_k\big(H^k(\mathfrak{X}^\circ)\big) \quad \text{is surjective.}$$

(3) The image of $H^k(\mathsf{W})$ in $H^k(\mathfrak{X}^\circ \times_{\mathfrak{X}} \mathsf{W})$ contains $W_k\big(H^k(\mathfrak{X}^\circ \times_{\mathfrak{X}} \mathsf{W})\big)$, by the conclusion of the previous paragraph.

The assertion is now immediate. □

6.4 Markman's formula for Chern classes of complexes

Suppose that $\mathfrak{M}$ is a smooth Deligne–Mumford stack and

$$C : V_{-1} \xrightarrow{g} V_0 \xrightarrow{f} V_1 \tag{6.2}$$

is a complex of locally free sheaves on $\mathfrak{M}$ of ranks r_{-1}, r_0, r_1 respectively.

Proposition 6.5 (Lemma 4 of [24]) *Suppose that $\Gamma \subset \mathfrak{M}$ is a smooth closed substack of pure codimension m, and that the complex C of (6.2) satisfies:*

(1) $\mathcal{H}^{-1}(C) = 0$,
(2) $\mathcal{H}^1(C)$ *and* $\mathcal{H}^1(C^\vee)$ *are line bundles on* Γ,
(3) $m \geq 2$ *and* $\mathrm{rk}(C) = m - 2$.

Then if m is even, $c_m(C) = [\Gamma]$ and $c_m\big(\mathcal{H}^0(C)\big) = (1 - (m-1)!)\,[\Gamma]$.

Remark 6.6 Markman's Lemma 4 is ostensibly stated for smooth varieties M, but Section 3 of *op. cit.* generalizes the assertion to smooth Deligne–Mumford stacks.

6.5 Proofs of Theorems 1.5 and 1.2

Fix a quiver Q, stability condition θ for Q^{dbl} and the corresponding stability condition θ^{gtr} for Q^{gtr} as in Sect. 3.5. Choosing a subgroup $\mathbb{S} \subset \mathbb{G}$ as in Sect. 3.6, we obtain a "graph immersion" in a product of Deligne–Mumford stacks

$$\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \xrightarrow{\iota} \mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \times \mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}/\mathbb{S}^{\mathrm{gtr}}. \quad (6.3)$$

We write ι for the immersion and $\Gamma = \mathrm{Im}(\iota)$ for its image, a smooth closed substack. We remark that ι is *not* a closed immersion unless H is trivial; however, the morphism ι identifies

$$\Gamma \cong \mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \times BH.$$

It follows that $(1 \times \iota)_*[\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}]$ is a nonzero rational multiple of $[\Gamma]$, and thus we may apply Proposition 6.3 with $(1 \times \iota)_*[\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}]$ replaced by $[\Gamma]$, and we do this below.

The factors $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$ and $\mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}/\mathbb{S}^{\mathrm{gtr}}$ come equipped with universal representations V, W respectively. The complex Ext defined in Sect. 5.2 descends to the product $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \times \mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}/\mathbb{S}^{\mathrm{gtr}}$. We recall from Proposition 3.13 the compactification $\overline{\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}}$ of $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$. This carries a natural map to $\mathrm{Rep}_{\mathrm{gr}}(\mathsf{A}_{[0,N]}, \alpha^{\mathrm{gtr}})^{\theta^{\mathrm{gtr}}\text{-ss}}/\mathbb{S}^{\mathrm{gtr}}$ which induces an isomorphism on the open substack $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}$. Pulling the complex Ext back to the product $\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \times \overline{\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}}$, we get a complex that we will denote C.

Direct calculation shows that the rank of C is $m - 2 = \mathrm{codim}(\Gamma) - 2$ (we note that its rank depends only on Q and α: only the differentials distinguish between the ordinary and multiplicative preprojective algebras). It follows from Proposition 5.4 that C has the following properties:

(1) $\mathcal{H}^{-1}(C) = 0$,
(2) $\mathcal{H}^1(C)$ and $\mathcal{H}^1(C^\vee)$ are set-theoretically supported on Γ, and their scheme-theoretic restrictions to Γ are line bundles.

Thus, in order to show that Γ satisfies the hypotheses of Proposition 6.5, it suffices to show that Γ is the scheme-theoretic support of both $\mathcal{H}^1(C)$ and $\mathcal{H}^1(C^\vee)$. We do this by considering a morphism

$$\mathrm{Spec}(k[\epsilon]) \to \mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S} \times \overline{\mathrm{Rep}(\Lambda^q, \alpha)^{\theta\text{-s}}/\mathbb{S}}$$

(where here and throughout the remainder of the proof, $k[\epsilon]$ denotes the ring of dual numbers) with the property that the closed point maps to Γ. Then it will suffice to show that either $\mathrm{Spec}(k[\epsilon])$ maps scheme-theoretically to Γ, or that the pullbacks of $\mathcal{H}^1(C)$ and $\mathcal{H}^1(C^\vee)$ to $\mathrm{Spec}(k[\epsilon])$ are scheme-theoretically supported at $\mathrm{Spec}(k) \subset \mathrm{Spec}(k[\epsilon])$.

We thus consider a representations $\overline{V}_\epsilon$, $\overline{V}'_\epsilon$ of $\Lambda^q[\epsilon]$ that are flat over $k[\epsilon]$ and having dimension vector α after tensoring with $k \otimes_{k[\epsilon]} -$; and let $V_\epsilon = \overline{V}_\epsilon[t]$, $V'_\epsilon = \overline{V}'_\epsilon[t]$. Assume $\tau_{[0,N]}V_\epsilon$, $\tau_{[0,N]}V'_\epsilon$ are θ^{gtr}-stable. The complex C_ϵ defined as in (5.3) becomes a complex of free $k[\epsilon]$-modules, and $\mathcal{H}^{-1}(C_\epsilon) = \mathrm{Hom}_{\mathsf{A}_\epsilon\text{-Gr}}\left(\tau_{[0,N]}V_\epsilon, \tau_{[0,N]}V'_\epsilon\right)$. This cohomology is isomorphic to $k[\epsilon]$ if and only if $\tau_{[0,N]}V_\epsilon \cong \tau_{[0,N]}V'_\epsilon$. Thus, $\mathcal{H}^1(C_\epsilon^\vee)$ is isomorphic to $k[\epsilon]$ if and only if $\tau_{[0,N]}V_\epsilon \cong \tau_{[0,N]}V'_\epsilon$. It follows that the scheme-theoretic support of $\mathcal{H}^1(C^\vee)$ is the reduced diagonal Γ.

It remains to check that the same is true of $\mathcal{H}^1(C)$. To do that, we again start with $\tau_{[0,N]}V_\epsilon$, $\tau_{[0,N]}V'_\epsilon$ as above, but consider them as graded A-modules (i.e., forgetting the $k[\epsilon]$-module structure) and form the complex C. We have a short exact sequence of graded A-modules

$$0 \to \epsilon\tau_{[0,N]}V_\epsilon \to \tau_{[0,N]}V_\epsilon \to k \otimes_{k[\epsilon]} \tau_{[0,N]}V_\epsilon \to 0, \tag{6.4}$$

where by $k[\epsilon]$-flatness we have $\epsilon\tau_{[0,N]}V_\epsilon \cong k \otimes_{k[\epsilon]} \tau_{[0,N]}V_\epsilon$, both stable; and similarly for V'. Assume without loss of generality that $k \otimes_{k[\epsilon]} \tau_{[0,N]}V_\epsilon \cong k \otimes_{k[\epsilon]} \tau_{[0,N]}V'_\epsilon$ as graded A-modules. Suppose there is a nonzero map of graded A-modules, $\phi : \tau_{[0,N]}V_\epsilon \to \tau_{[0,N]}V'_\epsilon$. If the composite

$$\epsilon\tau_{[0,N]}V_\epsilon \hookrightarrow \tau_{[0,N]}V_\epsilon \xrightarrow{\phi} \tau_{[0,N]}V'_\epsilon \twoheadrightarrow k \otimes_{k[\epsilon]} \tau_{[0,N]}V'_\epsilon \tag{6.5}$$

is nonzero, it is an isomorphism, since both its domain and target are stable of dimension vector α^{gtr}; in which case both (6.4) and its analogue for $\tau_{[0,N]}V'_\epsilon$ are split extensions. This means that the tangent vector to $\mathrm{Rep}(\Lambda^q,\alpha)^{\theta\text{-s}}/\mathbb{S} \times \mathrm{Rep}(\Lambda^q,\alpha)^{\theta\text{-s}}/\mathbb{S}$ determined by $(\overline{V}_\epsilon, \overline{V}'_\epsilon)$ is zero, and thus irrelevant to our analysis of the scheme-theoretic support of $\mathcal{H}^1(C)$. Thus we may assume that the composite (6.5) is zero, and so the morphism ϕ is a homomorphism of 1-extensions. Now if $\phi(\epsilon\tau_{[0,N]}V_\epsilon) \neq 0$, then again by stability it maps isomorphically onto $\epsilon\tau_{[0,N]}V'_\epsilon$. Since (6.4) is non-split, it follows that ϕ is an isomorphism, implying that the tangent vector determined by $(\overline{V}_\epsilon, \overline{V}'_\epsilon)$ is tangent to Γ, and again irrelevant to our analysis of the scheme-theoretic support of $\mathcal{H}^1(C)$. Finally then, we may assume that $\phi(\epsilon\tau_{[0,N]}V_\epsilon) = 0$. It follows that ϕ factors through the quotient $k \otimes_{k[\epsilon]} \tau_{[0,N]}V_\epsilon$; similarly its image lies in $\epsilon\tau_{[0,N]}V'_\epsilon$. It follows that $\mathrm{Hom}_{\mathsf{A}\text{-Gr}}\left(\tau_{[0,N]}V_\epsilon, \tau_{[0,N]}V'_\epsilon\right)$ is scheme-theoretically supported over $\mathrm{Spec}(k) \subset \mathrm{Spec}\, k[\epsilon]$, and hence by Proposition 5.4(1) that the same is true of $\mathcal{H}^1(C)$. Since this is true for every $\mathrm{Spec}\, k[\epsilon] \to \mathrm{Rep}(\Lambda^q,\alpha)^{\theta\text{-s}}/\mathbb{S} \times \mathrm{Rep}(\Lambda^q,\alpha)^{\theta\text{-s}}/\mathbb{S}$ not tangent to Γ, we conclude that $\mathcal{H}^1(C)$ has scheme-theoretic support equal to Γ, as required.

By Proposition 6.5, then, we conclude that $[\Gamma] = c_m(C)$. By Proposition 6.3, the Künneth components of $c_m(C)$ thus span the image of the restriction map

$$H^*\left(\overline{\mathrm{Rep}(\Lambda^q,\alpha)^{\theta\text{-s}}/\mathbb{S}}\right) \longrightarrow H^*\left(\mathrm{Rep}(\Lambda^q,\alpha)^{\theta\text{-s}}/\mathbb{S}\right),$$

which by Proposition 6.4 is exactly $\oplus_m W_m\left(H^m\left(\mathrm{Rep}(\Lambda^q,\alpha)^{\theta\text{-s}}/\mathbb{S}\right)\right)$. Since the Chern classes of C are polynomials in the Chern classes of the tautological bundles (see the

proof of Proposition 2.4(ii) of [26]), this completes the proof of Theorem 1.5, hence also of Theorem 1.2. □

6.6 Proof of Theorem 1.4

The proof of Theorem 1.4 is essentially identical to that of Theorem 1.6 of [26] (and we note that Theorem 1.4 holds whenever k is any field of characteristic zero and $q \in k^\times$). Indeed, the assumption that there is a vertex $i_0 \in I$ for which $\alpha_{i_0} = 1$ guarantees the following. First, we may take $\mathbb{S} = \prod_{i \neq i_0} GL(\alpha_i)$, which acts freely on the stable locus: thus, $\mathcal{M}^q_\theta(\alpha)^s$ is a fine moduli space for stable representations of Λ^q. Second, exactly as in the proof of Theorem 1.6 of [26], in the complex (5.3), there are direct sum decompositions

$$L(V_0, W_0) = \mathrm{Hom}(V_{0,i_0}, W_{0,i_0}) \oplus \big(\oplus_{i \neq i_0} \mathrm{Hom}(V_{0,i}, W_{0,i}) \big) \text{ and}$$
$$L(V_0, W_{2g}) = \mathrm{Hom}(V_{0,i_0}, W_{2g,i_0}) \oplus \big(\oplus_{i \neq i_0} \mathrm{Hom}(V_{0,i}, W_{2g,i}) \big),$$

so that the complex obtained by modifying (5.3) given by

$$\oplus_{i \neq i_0} \mathrm{Hom}(V_{0,i}, W_{0,i}) \xrightarrow{\partial_0} E(V_0, W_1) \xrightarrow{\partial_1} L(V_0, W_{2g}) / \mathrm{Hom}(V_{0,i_0}, W_{2g,i_0})$$

has no cohomology at the ends, and in the middle has cohomology $\mathcal{H}$ that is a rank $m = \mathrm{codim}(\Gamma)$ vector bundle. Moreover, the remaining map $k = \mathrm{Hom}(V_{0,i_0}, W_{0,i_0}) \to E(V_0, W_1)$ defines a section s of $\mathcal{H}$ whose scheme-theoretic zero locus is $Z(s) = \Gamma$. The remainder of the proof now copies that of Theorem 1.6 of [26]. □

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