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Reflexive polytopes arising from bipartite graphs with γ-positivity associated to interior polynomials

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Abstract

In this paper, we introduce polytopes $\mathscr{B}_G$ arising from root systems B_n and finite graphs G, and study their combinatorial and algebraic properties. In particular, it is shown that $\mathscr{B}_G$ is reflexive if and only if G is bipartite. Moreover, in the case, $\mathscr{B}_G$ has a regular unimodular triangulation. This implies that the h^*-polynomial of $\mathscr{B}_G$ is palindromic and unimodal when G is bipartite. Furthermore, we discuss stronger properties, namely the γ-positivity and the real-rootedness of the h^*-polynomials. In fact, if G is bipartite, then the h^*-polynomial of $\mathscr{B}_G$ is γ-positive and its γ-polynomial is given by an interior polynomial (a version of the Tutte polynomial for a hypergraph). The h^*-polynomial is real-rooted if and only if the corresponding interior polynomial is real-rooted. From a counterexample to Neggers–Stanley conjecture, we construct a bipartite graph G whose h^*-polynomial is not real-rooted but γ-positive, and coincides with the h-polynomial of a flag triangulation of a sphere.

Keywords Reflexive polytope · Unimodular covering · Regular unimodular triangulation · γ-Positive · Real-rooted · Interior polynomial · Matching generating polynomial · P-Eulerian polynomial

Mathematics Subject Classification 05A15 · 05C31 · 13P10 · 52B12 · 52B20

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Introduction

A *lattice polytope* $\mathscr{P} \subset \mathbb{R}^d$ is a convex polytope all of whose vertices have integer coordinates. Ardila et al. [1] constructed a unimodular triangulation of the convex hull of the roots of the classical root lattices of type A_n, B_n, C_n and D_n, and gave an alternative proof for the known growth series of these root lattices by using the triangulation.

In [23], lattice polytopes arising from the root system of type A_n and finite graphs were introduced. Let G be a finite simple undirected graph on the vertex set $[d] = \{1, \ldots, d\}$ with the edge set $E(G)$. We denote $\mathscr{A}_G \subset \mathbb{R}^d$ the convex hull of the set

$$A(G) = \{\mathbf{0}\} \cup \{\pm(\mathbf{e}_i - \mathbf{e}_j) : \{i, j\} \in E(G)\},$$

where $\mathbf{e}_i$ is the i-th unit coordinate vector in $\mathbb{R}^d$ and $\mathbf{0}$ is the origin of $\mathbb{R}^d$. Then $\mathscr{A}_G$ is centrally symmetric, i.e., for any facet $\mathscr{F}$ of $\mathscr{A}_G$, $-\mathscr{F}$ is also a facet of $\mathscr{A}_G$, and $\mathbf{0}$ is the unique (relative) interior lattice point of $\mathscr{A}_G$. Note that, if G is a complete graph, then $\mathscr{A}_G$ coincides with the convex hull of the roots of the root lattices of type A_n studied in [1]. This polytope $\mathscr{A}_G$ is called the *symmetric edge polytope* of G and several combinatorial properties of $\mathscr{A}_G$ are well-studied [22,23,30].

In this paper, we introduce lattice polytopes arising from the root system of type B_n and finite graphs, and study their algebraic and combinatorial properties. Let $\mathscr{B}_G \subset \mathbb{R}^d$ denote the convex hull of the set

$$B(G) = \{\mathbf{0}, \pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d\} \cup \{\pm\mathbf{e}_i \pm \mathbf{e}_j : \{i, j\} \in E(G)\}.$$

Then $\dim \mathscr{B}_G = d$, $\mathscr{B}_G$ is centrally symmetric and $\mathbf{0}$ is the unique interior lattice point of $\mathscr{B}_G$. Note that, if G is a complete graph, then $\mathscr{B}_G$ coincides with the convex hull of the roots of the root lattices of type B_n studied in [1]. Several classes of lattice polytopes arising from graphs have been studied from viewpoints of combinatorics, graph theory, geometric and commutative algebra. In particular, *edge polytopes* give interesting examples in commutative algebra ([18,33–35,43]). Note that edge polytopes of bipartite graphs are called *root polytopes* and play important roles in the study of generalized permutohedra [41] and interior polynomials [25].

There is a strong relation between $\mathscr{B}_G$ and edge polytopes. In fact, one of the key properties of $\mathscr{B}_G$ is that $\mathscr{B}_G$ is divided into 2^d edge polytopes of certain non-simple graphs $\widetilde{G}$ (Proposition 1.1). This fact helps us to find and show interesting properties of $\mathscr{B}_G$. In Sect. 1, by using this fact, we will classify graphs G such that $\mathscr{B}_G$ has a unimodular covering (Theorem 1.3). We remark that $\mathscr{A}_G$ always has a unimodular covering.

On the other hand, the fact that $\mathscr{B}_G$ has a unique interior lattice point $\mathbf{0}$ leads us to consider when $\mathscr{B}_G$ is reflexive. A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d is called *reflexive* if the origin of $\mathbb{R}^d$ is a unique lattice point belonging to the interior of $\mathscr{P}$ and its dual polytope

$$\mathscr{P}^\vee := \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{x} \in \mathscr{P}\}$$

is also a lattice polytope, where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the usual inner product of $\mathbb{R}^d$. It is known that reflexive polytopes correspond to Gorenstein toric Fano varieties, and they are related to mirror symmetry (see, e.g., [2,9]). In each dimension there exist only finitely many reflexive polytopes up to unimodular equivalence [28] and all of them are known up to dimension 4 [27]. Here two lattice polytopes $\mathscr{P} \subset \mathbb{R}^d$ and $\mathscr{P}' \subset \mathbb{R}^{d'}$ are said to be *unimodularly equivalent* if there exists an affine map from the affine span $\text{aff}(\mathscr{P})$ of $\mathscr{P}$ to the affine span $\text{aff}(\mathscr{P}')$ of $\mathscr{P}'$ that maps $\mathbb{Z}^d \cap \text{aff}(\mathscr{P})$ bijectively onto $\mathbb{Z}^{d'} \cap \text{aff}(\mathscr{P}')$ and that maps $\mathscr{P}$ to $\mathscr{P}'$. Every lattice polytope is unimodularly equivalent to a full-dimensional one. In Sect. 2, we will classify graphs G such that $\mathscr{B}_G$ is a reflexive polytope. In fact, we will show the following.

Theorem 0.1 *Let G be a finite graph. Then the following conditions are equivalent*:

(i) *$\mathscr{B}_G$ is reflexive and has a regular unimodular triangulation*;
(ii) *$\mathscr{B}_G$ is reflexive*;
(iii) *G is a bipartite graph.*

We remark that not all unimodular equivalence classes of reflexive polytopes are represented by a polytope of the form $\mathscr{B}_G$. Indeed, $\mathscr{B}_G$ is always centrally symmetric but not all reflexive polytopes are centrally symmetric. On the other hand, $\mathscr{A}_G$ is always (unimodularly equivalent to) a centrally symmetric reflexive polytope. However, the class of reflexive polytopes $\mathscr{A}_G$ is different from that of $\mathscr{B}_G$. For example, if G is a complete bipartite graph, then $\mathscr{B}_G$ is not unimodularly equivalent to $\mathscr{A}_{G'}$ for any graph G'.

Next, by characterizing when the toric ideal of $\mathscr{B}_G$ has a Gröbner basis consisting of squarefree quadratic binomials for a bipartite graph G, we can classify graphs G such that $\mathscr{B}_G$ is a reflexive polytope with a flag regular unimodular triangulation. In fact,

Theorem 0.2 *Let G be a bipartite graph. Then the following conditions are equivalent*:

(i) *The reflexive polytope $\mathscr{B}_G$ has a flag regular unimodular triangulation*;
(ii) *Any cycle of G of length ≥ 6 has a chord* (*"chordal bipartite graph"*).

Now, we turn to the discussion of the h^*-polynomial $h^*(\mathscr{B}_G, x)$ of $\mathscr{B}_G$. Thanks to the key property (Proposition 1.1), we can compute the h^*-polynomial of $\mathscr{B}_G$ in terms of that of edge polytopes of some graphs. On the other hand, since it is known that the h^*-polynomial of a reflexive polytope with a regular unimodular triangulation is palindromic and unimodal [7], Theorem 0.1 implies that the h^*-polynomial of $\mathscr{B}_G$ is palindromic and unimodal if G is bipartite. In Sect. 3, we will show a stronger result, namely for any bipartite graph G, the h^*-polynomial $h^*(\mathscr{B}_G, x)$ is γ-positive. The theory of interior polynomials (a version of the Tutte polynomials for hypergraphs) introduced by Kálmán [24] and the theory of generalized permutohedra [31,41] play important roles.

Theorem 0.3 *Let G be a bipartite graph on $[d]$. Then h^*-polynomial of the reflexive polytope $\mathscr{B}_G$ is*

$$h^*(\mathscr{B}_G, x) = (x+1)^d I_{\widehat{G}} \left(\frac{4x}{(x+1)^2} \right),$$

where $\widehat{G}$ is a connected bipartite graph defined in (1) later and $I_{\widehat{G}}(x)$ is the interior polynomial of $\widehat{G}$. In particular, $h^*(\mathscr{B}_G, x)$ is γ-positive. Moreover, $h^*(\mathscr{B}_G, x)$ is real-rooted if and only if $I_{\widehat{G}}(x)$ is real-rooted.

In addition, we discuss the following relations between interior polynomials and other important polynomials in combinatorics:

- If G is bipartite, then the interior polynomial of $\widehat{G}$ is described in terms of k-matchings of G (Proposition 3.4);
- If G is a forest, then the interior polynomial of $\widehat{G}$ coincides with the matching generating polynomial of G (Proposition 3.5);
- If G is a bipartite permutation graph associated with a poset P, then the interior polynomial of $\widehat{G}$ coincides with the P-Eulerian polynomial of P (Proposition 3.6).

By using these results and a poset appearing in [47] as a counterexample to Neggers–Stanley conjecture, we will construct an example of a centrally symmetric reflexive polytope such that the h^*-polynomial is γ-positive and not real-rooted (Example 3.7). This h^*-polynomial coincides with the h-polynomial of a flag triangulation of a sphere (Proposition 3.8). Hence this example is a counterexample to "Real Root Conjecture" that has been already disproved by Gal [11]. Finally, inspired by a simple description for the h^*-polynomials of symmetric edge polytopes of complete bipartite graphs [23], we will compute the h^*-polynomial of $\mathscr{B}_G$ when G is a complete bipartite graph (Example 3.9).

1 A key property of $\mathscr{B}_G$ and unimodular coverings

In this section, we see a relation between $\mathscr{B}_G$ and edge polytopes. First, we recall what edge polytopes are. Let G be a graph on $[d]$ (only here we do not assume that G has no loops) with the edge (including loop) set $E(G)$. Then the *edge polytope* P_G of G is the convex hull of $\{\mathbf{e}_i + \mathbf{e}_j : \{i, j\} \in E(G)\}$. Note that P_G is a (0, 1)-polytope if and only if G has no loops. Given a graph G on $[d]$, let $\widetilde{G}$ be a graph on $[d + 1]$ whose edge set is

$$E(G) \cup \{\{1, d + 1\}, \{2, d + 1\}, \ldots, \{d + 1, d + 1\}\}.$$

Here, $\{d + 1, d + 1\}$ is a loop (a cycle of length 1) at $d + 1$. See Fig. 1. If G is a

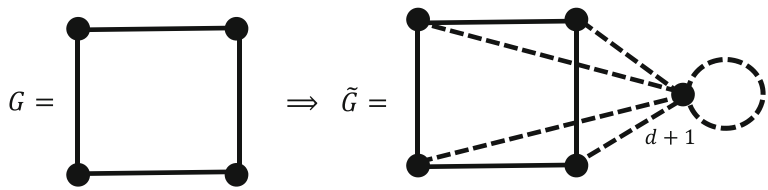

Fig. 1 Example of $\widetilde{G}$

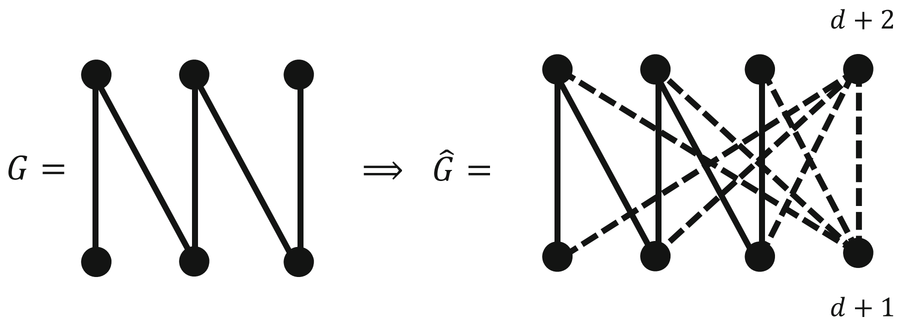

Fig. 2 Example of $\widehat{G}$

bipartite graph with a bipartition $V_1 \cup V_2 = [d]$, let $\widehat{G}$ be a connected bipartite graph on $[d+2]$ whose edge set is

$$E(\widehat{G}) = E(G) \cup \{\{i, d+1\} : i \in V_1\} \cup \{\{j, d+2\} : j \in V_2 \cup \{d+1\}\}. \quad (1)$$

See Fig. 2.

Now, we show the key proposition of this paper. Given $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d$, let $\mathscr{O}_\varepsilon$ denote the closed orthant $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \varepsilon_i \geq 0 \text{ for all } i \in [d]\}$. For a lattice polytope $\mathscr{P}$, let $\mathrm{Vol}(\mathscr{P})$ denote the normalized volume of $\mathscr{P}$.

Proposition 1.1 *Work with the same notation as above. Then we have the following*:

(a) *Each $\mathscr{B}_G \cap \mathscr{O}_\varepsilon$ is the convex hull of the set $B(G) \cap \mathscr{O}_\varepsilon$ and unimodularly equivalent to the edge polytope $P_{\widetilde{G}}$ of $\widetilde{G}$. Moreover, if G is bipartite, then $\mathscr{B}_G \cap \mathscr{O}_\varepsilon$ is unimodularly equivalent to the edge polytope $P_{\widehat{G}}$ of $\widehat{G}$. In particular, one has $Vol(\mathscr{B}_G) = 2^d Vol(P_{\widetilde{G}})$.*
(b) *The edge polytope of G is a face of $\mathscr{B}_G$.*

Proof (a) Let $\mathscr{P}$ be the convex hull of the set $B(G) \cap \mathscr{O}_\varepsilon$. The inclusion $\mathscr{B}_G \cap \mathscr{O}_\varepsilon \supset \mathscr{P}$ is trivial. Let $\mathbf{x} = (x_1, \ldots, x_d) \in \mathscr{B}_G \cap \mathscr{O}_\varepsilon$. Then $\mathbf{x} = \sum_{i=1}^{s} \lambda_i \mathbf{a}_i$, where $\lambda_i > 0$, $\sum_{i=1}^{s} \lambda_i = 1$, and each $\mathbf{a}_i$ belongs to $B(G)$. Suppose that the k-th component of $\mathbf{a}_i$ is positive and the k-th component of $\mathbf{a}_j$ is negative. Then $\mathbf{a}_i$ and $\mathbf{a}_j$ satisfy

$$\mathbf{a}_i + \mathbf{a}_j = (\mathbf{a}_i - \mathbf{e}_k) + (\mathbf{a}_j + \mathbf{e}_k),$$

where $\mathbf{a}_i - \mathbf{e}_k, \mathbf{a}_j + \mathbf{e}_k \in B(G)$. By using the above relations for $\mathbf{a}_i + \mathbf{a}_j$ finitely many times, we may assume that the k-th component of each vector $\mathbf{a}_i$ is nonnegative (resp. nonpositive) if $x_k \geq 0$ (resp. $x_k \leq 0$). Then each $\mathbf{a}_i$ belongs to $B(G) \cap \mathscr{O}_\varepsilon$ and hence $\mathbf{x} \in \mathscr{P}$.

Next, we show that each $\mathscr{B}_G \cap \mathscr{O}_\varepsilon$ is unimodularly equivalent to the edge polytope $P_{\widetilde{G}}$. Set $\mathscr{Q} = \mathscr{B}_G \cap \mathscr{O}_{(1,\ldots,1)}$. It is easy to see that each $\mathscr{B}_G \cap \mathscr{O}_\varepsilon$ is unimodularly equivalent to $\mathscr{Q}$ for all ε. Moreover, one has

$$B(G) \cap \mathscr{O}_{(1,\ldots,1)} = \{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d\} \cup \{\mathbf{e}_i + \mathbf{e}_j : \{i, j\} \in E(G)\}.$$

Hence $P_{\widetilde{G}}$ is unimodularly equivalent to $\mathscr{Q} \times \{2\}$. In particular, $\mathscr{B}_G$ consists of 2^d copies of the edge polytope $P_{\widetilde{G}}$. This implies $\mathrm{Vol}(\mathscr{B}_G) = 2^d \mathrm{Vol}(P_{\widetilde{G}})$. Similarly, if G is bipartite, it follows that $P_{\widehat{G}}$ is unimodularly equivalent to $\mathscr{Q} \times \{1\}^2$.

(b) The edge polytope P_G of G is the face of $\mathscr{B}_G$ with the supporting hyperplane $\mathscr{H} = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 + \cdots + x_d = 2\}$. □

Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope. A (lattice) *covering* of $\mathscr{P}$ is a finite collection Δ of lattice simplices such that the union of the simplices in Δ is $\mathscr{P}$, i.e., $\mathscr{P} = \bigcup_{\delta \in \Delta} \delta$. A (lattice) *triangulation* of $\mathscr{P}$ is a covering Δ of $\mathscr{P}$ such that

(i) every face of a member of Δ is in Δ, and
(ii) any two elements of Δ intersect in a common (possibly empty) face.

Note that their interiors may overlap in coverings but not in triangulations. A *unimodular simplex* is a lattice simplex whose normalized volume equals 1. We now focus on the following properties.

(VA) We say that $\mathscr{P}$ is *very ample* if for all sufficiently large $k \in \mathbb{Z}_{\geq 1}$ and for all $\mathbf{x} \in k\mathscr{P} \cap \mathbb{Z}^d$, there exist $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathscr{P} \cap \mathbb{Z}^d$ with $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$.
(IDP) We say that $\mathscr{P}$ possesses the *integer decomposition property* (or is *IDP* for short) if for all $k \in \mathbb{Z}_{\geq 1}$ and for all $\mathbf{x} \in k\mathscr{P} \cap \mathbb{Z}^d$, there exist $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathscr{P} \cap \mathbb{Z}^d$ with $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$.
(UC) We say that $\mathscr{P}$ has a unimodular covering if $\mathscr{P}$ admits a lattice covering consisting of unimodular simplices.
(UT) We say that $\mathscr{P}$ has a unimodular triangulation if $\mathscr{P}$ admits a lattice triangulation consisting of unimodular simplices.

These properties satisfy the implications

$$(\mathrm{UT}) \Rightarrow (\mathrm{UC}) \Rightarrow (\mathrm{IDP}) \Rightarrow (\mathrm{VA}),$$

see, e.g., [13, Section 4.2]. On the other hand, it is known that the opposite implications are false. However, for edge polytopes, the first three properties are equivalent ([8,33, 43]). We say that a graph G satisfies the *odd cycle condition* if, for any two odd cycles C_1 and C_2 that belong to the same connected component of G and have no common vertices, there exists an edge $\{i, j\}$ of G such that i is a vertex of C_1 and j is a vertex of C_2.

Proposition 1.2 [8,33,43] *Let G be a finite (not necessarily simple) graph. Suppose that there exists an edge $\{i, j\}$ of G whenever G has loops at i and j with $i \neq j$. Then the following conditions are equivalent*:

(i) *P_G has a unimodular covering*;
(ii) *P_G is IDP*;
(iii) *P_G is very ample*;
(iv) *G satisfies the odd cycle condition.*

We now show that the same assertion holds for $\mathscr{B}_G$. Namely, we prove the following.

Theorem 1.3 *Let G be a finite simple graph. Then the following conditions are equivalent*:

(i) *$\mathscr{B}_G$ has a unimodular covering*;
(ii) *$\mathscr{B}_G$ is IDP*;
(iii) *$\mathscr{B}_G$ is very ample*;
(iv) *G satisfies the odd cycle condition.*

Before proving Theorem 1.3, we recall a well-known result.

Lemma 1.4 *Every face of a very ample polytope is very ample.*

Proof Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d. Assume that there exists a facet $\mathscr{F}$ of $\mathscr{P}$ such that $\mathscr{F}$ is not very ample. Then we should show that $\mathscr{P}$ is not very ample. Let $\mathscr{H} \subset \mathbb{R}^d$ be the hyperplane in $\mathbb{R}^d$ defined as $\mathscr{H} = \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{a} \rangle = b\}$, with some lattice point $\mathbf{a} \in \mathbb{Z}^d$ and some integer b such that $\mathscr{F} = \mathscr{H} \cap \mathscr{P}$ and $\mathscr{P} \subset \mathscr{H}^{(+)}$, where $\mathscr{H}^{(+)} = \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{a} \rangle \leq b\}$. Since $\mathscr{F}$ is not very ample, there exist a positive integer k and a lattice point $\mathbf{x} \in k\mathscr{F} \cap \mathbb{Z}^d$ such that there are no lattice points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\mathscr{F}$ with $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$. Suppose that there are lattice points $\mathbf{y}_1, \ldots, \mathbf{y}_k$ in $\mathscr{P}$ with $\mathbf{x} = \mathbf{y}_1 + \cdots + \mathbf{y}_k$. Then for each $\mathbf{y}_i$, one has $\langle \mathbf{y}_i, \mathbf{a} \rangle \leq b$. Since $\langle \mathbf{x}, \mathbf{a} \rangle = kb$, it follows that for each i, $\langle \mathbf{y}_i, \mathbf{a} \rangle = b$, hence, $\mathbf{y}_i \in \mathscr{F}$. This is a contradiction. Therefore, $\mathscr{P}$ is not very ample. □

Remark 1.5 Similarly, we can prove that every face of an IDP polytope is IDP. Moreover, it is clear that if a lattice polytope has a unimodular triangulation (resp. a unimodular covering), then so does any face.

Now, we prove Theorem 1.3.

Proof of Theorem 1.3 First, implications (i) ⇒ (ii) ⇒ (iii) hold in general.

(iii) ⇒ (iv): Suppose that G does not satisfy the odd cycle condition. By Proposition 1.2, the edge polytope P_G of G is not very ample. Since P_G is a face of $\mathscr{B}_G$ by Proposition 1.1 (b) and Lemma 1.4, $\mathscr{B}_G$ is not very ample.

(iv) ⇒ (i): Suppose that G satisfies the odd cycle condition. Then so does $\widetilde{G}$. Hence Proposition 1.2 guarantees that $P_{\widetilde{G}}$ has a unimodular covering. By Proposition 1.1 (a), $\mathscr{B}_G$ has a unimodular covering. □

Example 1.6 Let G be a graph in Fig. 3. Since G satisfies the odd cycle condition, $\mathscr{B}_G$ has a unimodular covering. However, since the edge polytope P_G has no regular unimodular triangulations [34], so does $\mathscr{B}_G$ by Proposition 1.1 (b). We do not know whether $\mathscr{B}_G$ has a (nonregular) unimodular triangulation or not.

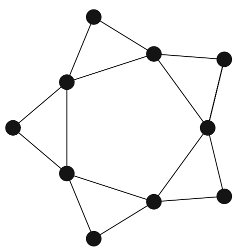

Fig. 3 A graph in [34]

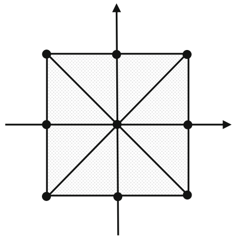

Fig. 4 Regular unimodular triangulation of $\mathscr{B}_G$ in Examples 2.1 (b)

2 Reflexive polytopes and flag triangulations of $\mathscr{B}_G$

In the present section, we classify graphs G such that

- $\mathscr{B}_G$ is a reflexive polytope.
- $\mathscr{B}_G$ is a reflexive polytope with a flag regular unimodular triangulation.

In other words, we prove Theorems 0.1 and 0.2. First, we see some examples that $\mathscr{B}_G$ is reflexive.

Examples 2.1 (a) If G is an empty graph, then $\mathscr{B}_G$ is a *cross polytope*.

(b) Let G be a complete graph with 2 vertices. Then $\mathscr{B}_G \cap \mathbb{Z}^2$ is the column vectors of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & -1 & -1 & 1 \end{pmatrix},$$

and $\mathscr{B}_G$ is a reflexive polytope having a regular unimodular triangulation. See Fig. 4. Since the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

is not unimodular, we cannot apply [36, Lemma 2.11] to show this fact.

In order to show that a lattice polytope is reflexive, we can use an algebraic technique on Gröbner bases. We recall basic materials and notation on toric ideals. Let $K[\mathbf{t}^{\pm 1}, s] = K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, s]$ be the Laurent polynomial ring in $d+1$ variables over a field K. If $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$, then $\mathbf{t}^{\mathbf{a}} s$ is the Laurent monomial $t_1^{a_1} \cdots t_d^{a_d} s \in K[\mathbf{t}^{\pm 1}, s]$. Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope and $\mathscr{P} \cap \mathbb{Z}^d = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Then, the *toric ring* of $\mathscr{P}$ is the subring $K[\mathscr{P}]$ of $K[\mathbf{t}^{\pm 1}, s]$ generated by $\{\mathbf{t}^{\mathbf{a}_1} s, \ldots, \mathbf{t}^{\mathbf{a}_n} s\}$ over K. We regard $K[\mathscr{P}]$ as a graded ring by setting each $\deg t_i = 0$ and $\deg s = 1$. Note that, if two lattice polytopes are unimodularly equivalent, then their toric rings are isomorphic as graded rings. Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over K. The *toric ideal* $I_{\mathscr{P}}$ of $\mathscr{P}$ is the kernel of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[\mathscr{P}]$ defined by $\pi(x_i) = \mathbf{t}^{\mathbf{a}_i} s$ for $1 \le i \le n$. It is known that $I_{\mathscr{P}}$ is generated by homogeneous binomials. See, e.g., [13, Chapter 3] and [48, Chapter 4]. Let $<$ be a monomial order on $K[\mathbf{x}]$ and $\mathrm{in}_<(I_{\mathscr{P}})$ the initial ideal of $I_{\mathscr{P}}$ with respect to $<$. The initial ideal $\mathrm{in}_<(I_{\mathscr{P}})$ is called *squarefree* (resp. *quadratic*) if

$\mathrm{in}_<(I_{\mathscr{P}})$ is generated by squarefree (resp. quadratic) monomials. We recall a relation between initial ideal $\mathrm{in}_<(I_{\mathscr{P}})$ and a triangulation of $\mathscr{P}$. Set

$$\Delta(\mathscr{P}, <) = \left\{ \mathrm{conv}(S) : S \subset \mathscr{P} \cap \mathbb{Z}^d, \prod_{\mathbf{a}_i \in S} x_i \notin \sqrt{\mathrm{in}_<(I_{\mathscr{P}})} \right\}.$$

Then we have

Proposition 2.2 [48, Theorem 8.3] *Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope and $<$ a monomial order on $K[\mathbf{x}]$. Then $\Delta(\mathscr{P}, <)$ is a regular triangulation of $\mathscr{P}$.*

Here we say that a lattice triangulation Δ of a lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ is *regular* if it arises as the projection of the lower hull of a lifting of the lattice points of $\mathscr{P}$ into $\mathbb{R}^{d+1}$. A simplicial complex is called *flag* if all minimal non-faces contain only two elements. We say that a lattice triangulation is *flag* if it is flag as a simplicial complex. Then we have

Proposition 2.3 [48, Corollary 8.9] *Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope and $<$ a monomial order on $K[\mathbf{x}]$. Suppose that any lattice point in $\mathbb{Z}^{d+1}$ is a linear integer combination of the lattice points in $\mathscr{P} \times \{1\}$. Then $\Delta(\mathscr{P}, <)$ is unimodular (resp. flag unimodular) if and only if $\mathrm{in}_<(I_{\mathscr{P}})$ is squarefree (resp. squarefree quadratic).*

By definition, a lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ is reflexive if and only if the lattice distance between the origin $\mathbf{0}$ and all affine hyperplanes generated by facets of $\mathscr{P}$ equals 1. Hence if $\mathbf{0}$ is contained in the interior of $\mathscr{P}$, and $\mathscr{P}$ has a unimodular triangulation such that $\mathbf{0}$ is a vertex of any maximal simplex of the triangulation, then $\mathscr{P}$ is reflexive. Now, we introduce an algebraic technique to show that a lattice polytope is reflexive.

Lemma 2.4 [16, Lemma 1.1] *Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d such that the origin of $\mathbb{R}^d$ is contained in its interior and $\mathscr{P} \cap \mathbb{Z}^d = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Suppose that $\mathbb{Z}^d = \{\sum_{i=1}^n z_i \mathbf{a}_i : z_i \in \mathbb{Z}\}$ and there exists an ordering of the variables $x_{i_1} < \cdots < x_{i_n}$ for which $\mathbf{a}_{i_1} = \mathbf{0}$ such that the initial ideal $\mathrm{in}_<(I_{\mathscr{P}})$ of $I_{\mathscr{P}}$ with respect to the reverse lexicographic order $<$ on $K[\mathbf{x}]$ induced by the ordering is squarefree. Then $\mathscr{P}$ is reflexive and has a regular unimodular triangulation.*

Proof The assertion follows since $\mathbf{0}$ is contained in the interior of $\mathscr{P}$ and since the triangulation $\Delta(\mathscr{P}, <)$ is unimodular and $\mathbf{0}$ is a vertex of any maximal simplex of the triangulation. □

By using this technique, several families of reflexive polytopes with regular unimodular triangulations are constructed in [15–17,19–21,37]. In order to apply Lemma 2.4 to show Theorem 0.1, we see a relation between the toric ideal of $\mathscr{B}_G$ and that of $P_{\widetilde{G}}$. Let G be a simple graph on $[d]$ with edge set $E(G)$ and let R_G denote the polynomial ring in $2d + 1 + 4|E(G)|$ variables

$$z, \quad x_{i+}, x_{i-} \;\; (1 \le i \le d), \quad y_{ij++}, y_{ij--}, y_{ij+-}, y_{ij-+} \;\; (\{i, j\} \in E(G))$$

over a field K. Then the toric ideal $I_{\mathscr{B}_G}$ of $\mathscr{B}_G$ is the kernel of a ring homomorphism $\pi : R_G \to K[t_1^{\pm}, \ldots, t_d^{\pm}, s]$ defined by $\pi(z) = s$, $\pi(x_{i+}) = t_i s$, $\pi(x_{i-}) = t_i^{-1}s$, $\pi(y_{ij++}) = t_i t_j s$, $\pi(y_{ij--}) = t_i^{-1}t_j^{-1}s$, $\pi(y_{ij+-}) = t_i t_j^{-1}s$, and $\pi(y_{ij-+}) = t_i^{-1}t_j s$. Let S_G denote the subring of R_G generated by the $d + 1 + |E(G)|$ variables

$$z, \quad x_{i+} \ (1 \le i \le d), \quad y_{ij++} \ (\{i, j\} \in E(G)).$$

Then the toric ideal $I_{P_{\widetilde{G}}}$ of $P_{\widetilde{G}}$ is the kernel of $\pi|_{S_G}$. For each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d$, we define a ring homomorphism $\varphi_\varepsilon : S_G \to R_G$ by $\varphi_\varepsilon(x_{i+}) = x_{i\alpha}$ and $\varphi_\varepsilon(y_{ij++}) = y_{ij\alpha\beta}$ where α is the sign of ε_i and β is the sign of ε_j. In particular, $\varphi_{(1,\ldots,1)} : S_G \to R_G$ is an inclusion map.

Lemma 2.5 *Let $\mathscr{G}$ be a Gröbner basis of $I_{P_{\widetilde{G}}}$ with respect to a reverse lexicographic order $<_S$ on S_G such that $z < \{x_{i+}\} < \{y_{ij++}\}$. Let $<_R$ be a reverse lexicographic order such that $z < \{x_{i+}, x_{i-}\} < \{y_{ij++}, y_{ij--}, y_{ij+-}, y_{ij-+}\}$ and that* (i) *$\varphi_\varepsilon(x_{i+}) <_R \varphi_\varepsilon(x_{j+})$ if $x_{i+} <_S x_{j+}$ and* (ii) *$\varphi_\varepsilon(y_{ij++}) <_R \varphi_\varepsilon(y_{k\ell++})$ if $y_{ij++} <_S y_{k\ell++}$ for all $\varepsilon \in \{-1, 1\}^d$. Then*

$$\begin{aligned}\mathscr{G}' = &\left(\bigcup_{\varepsilon \in \{-1,1\}^d} \varphi_\varepsilon(\mathscr{G})\right) \cup \{\underline{x_{i\alpha} y_{ij\beta\gamma}} - x_{j\gamma} z : \{i, j\} \in E(G), \alpha \neq \beta\} \\ &\cup \{\underline{y_{ij\alpha\gamma} y_{ik\beta\delta}} - x_{j\gamma} x_{k\delta} : \{i, j\}, \{i, k\} \in E(G), \alpha \neq \beta\} \\ &\cup \{\underline{x_{i+} x_{i-}} - z^2 : 1 \le i \le d\}\end{aligned}$$

is a Gröbner basis of $I_{\mathscr{B}_G}$ with respect to $<_R$, where the underlined monomial is the initial monomial of each binomial. (*Here we identify $y_{ij\alpha\beta}$ with $y_{ji\beta\alpha}$.*) *In particular, if* $\mathrm{in}_{<_S}(I_{P_{\widetilde{G}}})$ *is squarefree* (*resp. quadratic*)*, then so is* $\mathrm{in}_{<_R}(I_{\mathscr{B}_G})$.

Proof It is easy to see that $\mathscr{G}'$ is a subset of $I_{\mathscr{B}_G}$. Assume that $\mathscr{G}'$ is not a Gröbner basis of $I_{\mathscr{B}_G}$ with respect to $<_R$. Let $\mathrm{in}(\mathscr{G}') = \langle \mathrm{in}_{<_R}(g) : g \in \mathscr{G}' \rangle$. By [13, Theorem 3.11], there exists a non-zero irreducible homogeneous binomial $f = u - v \in I_{\mathscr{B}_G}$ such that neither u nor v belongs to $\mathrm{in}(\mathscr{G}')$. Since both u and v are divided by none of $x_{i+}x_{i-}$, $x_{i\alpha} y_{ij\beta\gamma}$, $y_{ij\alpha\gamma} y_{ik\beta\delta}$ ($\alpha \neq \beta$), they are of the form

$$u = \prod_{i \in I} (x_{i\alpha_i})^{u_i} \prod_{\{j,k\} \in E_1} (y_{jk\alpha_j\alpha_k})^{u_{jk}}, \quad v = \prod_{i \in I'} (x_{i\alpha'_i})^{v_i} \prod_{\{j,k\} \in E_2} (y_{jk\alpha'_j\alpha'_k})^{v_{jk}},$$

where $I, I' \subset [d]$, $E_1, E_2 \subset E(G)$ and $0 < u_i, u_{jk}, v_i, v_{jk} \in \mathbb{Z}$. Since f belongs to $I_{\mathscr{B}_G}$, the exponent of t_ℓ in $\pi(u)$ and $\pi(v)$ are the same. Hence, one of $x_{\ell\alpha_\ell}$ and $y_{\ell m\alpha_\ell\alpha_m}$ appears in u if and only if one of $x_{\ell\alpha'_\ell}$ and $y_{\ell n\alpha'_\ell\alpha'_n}$ appears in v with $\alpha_\ell = \alpha'_\ell$. Let ε be a vector in $\{-1, 1\}^d$ such that the sign of the i-th component of ε is α_i if one of $x_{i\alpha_i}$ and $y_{ij\alpha_i\alpha_j}$ appears in u. Then f belongs to the ideal $\varphi_\varepsilon(I_{P_{\widetilde{G}}})$. Let $f' \in I_{P_{\widetilde{G}}}$ be a binomial such that $\varphi_\varepsilon(f') = f$. Since $\mathscr{G}$ is a Gröbner basis of $I_{P_{\widetilde{G}}}$, there exists a binomial $g \in \mathscr{G}$ whose initial monomial $\mathrm{in}_{<_R}(g)$ divides the one of the monomials in f'. By the definition of $<_R$, we have $\mathrm{in}_{<_R}(\varphi_\varepsilon(g)) = \varphi_\varepsilon(\mathrm{in}_{<_R}(g))$. Hence $\mathrm{in}_{<_R}(\varphi_\varepsilon(g))$ divides one of the monomials in $f = \varphi_\varepsilon(f')$. This is a contradiction. □

Using this Gröbner basis with respect to a reverse lexicographic order, we verify which $\mathscr{B}_G$ is a reflexive polytope. Namely, we prove Theorem 0.1.

Proof of Theorem 0.1 The implication (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): Suppose that G is not bipartite. Let $G_1, \ldots, G_s$ be the connected components of G and let d_i be the number of vertices of G_i. In particular, we have $d = \sum_{i=1}^{s} d_i$. Since G is not bipartite, we may assume that G_1 is not bipartite. Let $\mathbf{w} = \sum_{i=1}^{s} \mathbf{w}_i$, where $\mathbf{w}_i = \sum_{k=1}^{\ell} \mathbf{e}_{p_k} \in \mathbb{R}^d$ if G_i is a non-bipartite graph on the vertex set $\{p_1, \ldots, p_\ell\}$, and $\mathbf{w}_i = \sum_{k=1}^{\ell} 2\mathbf{e}_{p_k} \in \mathbb{R}^d$ if G_i is a bipartite graph whose vertices are divided into two independent sets $\{p_1, \ldots, p_\ell\}$ and $\{q_1, \ldots, q_m\}$. It then follows that $\mathscr{H} = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{w}, \mathbf{x} \rangle = 2\}$ is a supporting hyperplane of $\mathscr{B}_G$ and the corresponding face $\mathscr{F}_{\mathbf{w}} = \mathscr{B}_G \cap \mathscr{H}$ is the convex hull of $H = \bigcup_{i=1}^{s} H_i$, where $H_i = \{\mathbf{e}_u + \mathbf{e}_v : \{u, v\} \in E(G_i)\}$ if G_i is not bipartite, and $H_i = \{\mathbf{e}_u + \mathbf{e}_v : \{u, v\} \in E(G_i)\} \cup \{\mathbf{e}_{p_1}, \ldots, \mathbf{e}_{p_\ell}\} \cup \{\mathbf{e}_{p_k} - \mathbf{e}_v : \{p_k, v\} \in E(G_i)\}$ if G_i is a bipartite graph with $\mathbf{w}_i = \sum_{k=1}^{\ell} 2\mathbf{e}_{p_k} \in \mathbb{R}^d$. We will show that $\mathscr{F}_{\mathbf{w}}$ is a facet of $\mathscr{B}_G$. The convex hull of $\{\mathbf{e}_u + \mathbf{e}_v : \{u, v\} \in E(G_i)\}$ is the edge polytope P_{G_i} of G_i and it is known [33, Proposition 1.3] that

$$\dim P_{G_i} = \begin{cases} d_i - 1 & \text{if } G_i \text{ is not bipartite,} \\ d_i - 2 & \text{otherwise.} \end{cases}$$

If G_i is not bipartite, then the dimension of $\mathrm{conv}(H_i) = P_{G_i}$ is $d_i - 1$. If G_i is bipartite, then $d_i - 2 = \dim P_{G_i} < \dim \mathrm{conv}(H_i)$ since $P_{G_i} \subset \mathrm{conv}(H_i)$ and a hyperplane $\mathscr{H} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 + \cdots + x_d = 2\}$ satisfies $\mathbf{e}_{p_1} \notin \mathscr{H} \supset P_{G_i}$. Hence the dimension of the face $\mathscr{F}_{\mathbf{w}}$ is at least $s - 1 + \sum_{i=1}^{s}(d_i - 1) = d - 1$, i.e., $\mathscr{F}_{\mathbf{w}}$ is a facet of $\mathscr{B}_G$. Since G_1 is not bipartite, we have $\frac{1}{2} \cdot \mathbf{w} \notin \mathbb{Z}^d$. Thus $\mathscr{B}_G$ is not reflexive.

(iii) $\Rightarrow$ (i): Suppose that G is bipartite. Let $<_S$ and $<_R$ be any reverse lexicographic orders satisfying the condition in Lemma 2.5. It is known [13, Theorem 5.24] that any triangulation of the edge polytope of a bipartite graph is unimodular. By [48, Corollary 8.9], the initial ideal of the toric ideal of $P_{\widehat{G}}$ with respect to $<_S$ is squarefree. Thanks to Lemmas 2.4 and 2.5, we have the desired conclusion. □

We now give a theorem on quadratic Gröbner bases of $I_{\mathscr{B}_G}$ when G is bipartite. This theorem implies that Theorem 0.2. The same result is known for edge polytopes ([35]).

Theorem 2.6 *Let G be a bipartite graph. Then the following conditions are equivalent*:

(i) *The toric ideal $I_{\mathscr{B}_G}$ of $\mathscr{B}_G$ has a squarefree quadratic initial ideal (i.e., $\mathscr{B}_G$ has a flag regular unimodular triangulation)*;
(ii) *The toric ring $K[\mathscr{B}_G]$ of $\mathscr{B}_G$ is a Koszul algebra*;
(iii) *The toric ideal $I_{\mathscr{B}_G}$ of $\mathscr{B}_G$ is generated by quadratic binomials*;
(iv) *Any cycle of G of length ≥ 6 has a chord ("chordal bipartite graph")*.

Proof Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold in general.

(iii) $\Rightarrow$ (iv): Suppose that G has a cycle of length ≥ 6 without chords. By the theorem in [35], the toric ideal of P_G is not generated by quadratic binomials. Since the edge polytope P_G is a face of $\mathscr{B}_G$, the toric ring $K[P_G]$ is a combinatorial pure subring [32] of $K[\mathscr{B}_G]$. Hence $I_{\mathscr{B}_G}$ is not generated by quadratic binomials.

(iv) ⇒ (i): Suppose that any cycle of G of length ≥ 6 has a chord. By Lemma 2.5, it is enough to show that the initial ideal of $I_{P_{\widehat{G}}}$ is squarefree and quadratic with respect to a reverse lexicographic order $<_S$ such that $z < \{x_{i+}\} < \{y_{k\ell++}\}$. Let $A = (a_{ij})$ be the incidence matrix of G whose rows are indexed by V_1 and whose columns are indexed by V_2. Then the incidence matrix of $\widehat{G}$ is

$$A' = \begin{pmatrix} & & 1 \\ & A & \vdots \\ & & 1 \\ 1 & \cdots\ 1 & 1 \end{pmatrix}$$

By the same argument as in the proof of the theorem in [35], we may assume that A' contains no submatrices $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ if we permute the rows and columns of A in A'. Each quadratic binomial in $I_{P_{\widehat{G}}}$ corresponds to a submatrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ of A'. The proof of the theorem in [35] guarantees that the initial ideal is squarefree and quadratic if the initial monomial of each quadratic binomial corresponds to $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. It is easy to see that there exists a such reverse lexicographic order which satisfies $z < \{x_{i+}\} < \{y_{k\ell++}\}$. □

3 γ-positivity and real-rootedness of the h^*-polynomial of $\mathscr{B}_G$

In this section, we study the h^*-polynomial of $\mathscr{B}_G$ for a graph G. First, we recall what h^*-polynomials are. Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d. Given a positive integer n, we define

$$L_{\mathscr{P}}(n) = |n\mathscr{P} \cap \mathbb{Z}^d|.$$

The study on $L_{\mathscr{P}}(n)$ originated in Ehrhart [10] who proved that $L_{\mathscr{P}}(n)$ is a polynomial in n of degree d with the constant term 1. We say that $L_{\mathscr{P}}(n)$ is the *Ehrhart polynomial* of $\mathscr{P}$. The generating function of the lattice point enumerator, i.e., the formal power series

$$\mathrm{Ehr}_{\mathscr{P}}(x) = 1 + \sum_{k=1}^{\infty} L_{\mathscr{P}}(k)x^k$$

is called the *Ehrhart series* of $\mathscr{P}$. It is well known that it can be expressed as a rational function of the form

$$\mathrm{Ehr}_{\mathscr{P}}(x) = \frac{h^*(\mathscr{P}, x)}{(1-x)^{d+1}}.$$

The polynomial $h^*(\mathscr{P}, x)$ is a polynomial in x of degree at most d with nonnegative integer coefficients ([44]) and it is called the h^*-*polynomial* (or the δ-*polynomial*) of $\mathscr{P}$. Moreover, one has $\mathrm{Vol}(\mathscr{P}) = h^*(\mathscr{P}, 1)$. Refer the reader to [3] for the detailed information about Ehrhart polynomials and h^*-polynomials.

Thanks to Proposition 1.1 (a), we give a formula for h^*-polynomial of $\mathscr{B}_G$ in terms of that of edge polytopes of some graphs. By the following formula, we can calculate the h^*-polynomial of $\mathscr{B}_G$ if we can calculate each $h^*(P_{\widetilde{H}}, x)$.

Proposition 3.1 *Let G be a graph on $[d]$. Then the h^*-polynomial of $\mathscr{B}_G$ satisfies*

$$h^*(\mathscr{B}_G, x) = \sum_{j=0}^{d} 2^j (x-1)^{d-j} \sum_{H \in S_j(G)} h^*(P_{\widetilde{H}}, x), \tag{2}$$

where $S_j(G)$ denote the set of all induced subgraph of G with j vertices.

Proof By Proposition 1.1 (a), $\mathscr{B}_G$ is divided into 2^d lattice polytopes of the form $\mathscr{B}_G \cap \mathscr{O}_\varepsilon$. Each $\mathscr{B}_G \cap \mathscr{O}_\varepsilon$ is unimodularly equivalent to $P_{\widetilde{G}}$. In addition, the intersection of $\mathscr{B}_G \cap \mathscr{O}_\varepsilon$ and $\mathscr{B}_G \cap \mathscr{O}_{\varepsilon'}$ is of dimension $d-1$ if and only if $\varepsilon - \varepsilon' \in \{\pm 2\mathbf{e}_1, \dots, \pm 2\mathbf{e}_d\}$. If $\varepsilon - \varepsilon' = 2\mathbf{e}_k$, then $(\mathscr{B}_G \cap \mathscr{O}_\varepsilon) \cap (\mathscr{B}_G \cap \mathscr{O}_{\varepsilon'}) = \mathscr{B}_G \cap \mathscr{O}_\varepsilon \cap \mathscr{O}_{\varepsilon'} \simeq \mathscr{B}_{G'} \cap \mathscr{O}_{\varepsilon''}$, where G' is the induced subgraph of G obtained by deleting the vertex k, and ε'' is obtained by deleting the k-th component of ε. Hence the Ehrhart polynomial $L_{\mathscr{B}_G}(n)$ satisfies the following:

$$L_{\mathscr{B}_G}(n) = \sum_{j=0}^{d} 2^j (-1)^{d-j} \sum_{H \in S_j(G)} L_{P_{\widetilde{H}}}(n).$$

Thus the Ehrhart series satisfies

$$\frac{h^*(\mathscr{B}_G, x)}{(1-x)^{d+1}} = \sum_{j=0}^{d} 2^j (-1)^{d-j} \sum_{H \in S_j(G)} \frac{h^*(P_{\widetilde{H}}, x)}{(1-x)^{j+1}},$$

as desired. □

Let $f = \sum_{i=0}^{d} a_i x^i$ be a polynomial with real coefficients and $a_d \neq 0$. We now focus on the following properties.

(RR) We say that f is *real-rooted* if all its roots are real.
(LC) We say that f is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all i.
(UN) We say that f is *unimodal* if $a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_d$ for some k.

If all its coefficients are positive, then these properties satisfy the implications

$$(\mathrm{RR}) \Rightarrow (\mathrm{LC}) \Rightarrow (\mathrm{UN}).$$

On the other hand, the polynomial f is said to be *palindromic* if $f(x) = x^d f(x^{-1})$. It is γ-*positive* if there are $\gamma_0, \gamma_1, \dots, \gamma_{\lfloor d/2 \rfloor} \geq 0$ such that $f(x) = \sum_{i \geq 0} \gamma_i x^i (1+x)^{d-2i}$.

The polynomial $\sum_{i \geq 0} \gamma_i x^i$ is called *γ-polynomial of f*. We can see that a γ-positive polynomial is real-rooted if and only if its γ-polynomial had only real roots ([40, Observation 4.2]).

By the following proposition, we are interested in connected bipartite graphs.

Proposition 3.2 *Let G be a bipartite graph and $G_1, \ldots, G_s$ the connected components of G. Then the h^*-polynomial of $\mathscr{B}_G$ is palindromic, unimodal and*

$$h^*(\mathscr{B}_G, x) = h^*(\mathscr{B}_{G_1}, x) \cdots h^*(\mathscr{B}_{G_s}, x).$$

Proof It is known [14] that the h^*-polynomial of a lattice polytope P with the interior lattice point $\mathbf{0}$ is palindromic if and only if P is reflexive. Moreover, if a reflexive polytope P has a unimodular triangulation, then the h^*-polynomial of P is unimodal (see [7]). It is easy to see that, $\mathscr{B}_G$ is the free sum of $\mathscr{B}_{G_1}, \ldots, \mathscr{B}_{G_s}$. Thus we have a desired conclusion by Theorem 0.1 and [6, Theorem 1]. □

In the rest of the present paper, we discuss the γ-positivity and the real-rootedness of the h^*-polynomial of $\mathscr{B}_G$ when G is a bipartite graph. The edge polytope P_G of a bipartite graph G is called the *root polytope* of G, and it is shown [25] that the h^*-polynomial of P_G coincides with the interior polynomial $I_G(x)$ of a hypergraph induced by G. First, we discuss interior polynomials introduced by Kálmán [24] and developed in many papers.

A *hypergraph* is a pair $\mathscr{H} = (V, E)$, where $E = \{e_1, \ldots, e_n\}$ is a finite multiset of non-empty subsets of $V = \{v_1, \ldots, v_m\}$. Elements of V are called vertices and the elements of E are the hyperedges. Then we can associate $\mathscr{H}$ to a bipartite graph Bip$\mathscr{H}$ with a bipartition $V \cup E$ such that $\{v_i, e_j\}$ is an edge of Bip$\mathscr{H}$ if $v_i \in e_j$. Assume that Bip$\mathscr{H}$ is connected. A *hypertree* in $\mathscr{H}$ is a function $\mathbf{f} : E \to \mathbb{Z}_{\geq 0}$ such that there exists a spanning tree Γ of Bip$\mathscr{H}$ whose vertices have degree $\mathbf{f}(e) + 1$ at each $e \in E$. Then we say that Γ induce $\mathbf{f}$. Let $\mathrm{HT}(\mathscr{H})$ denote the set of all hypertrees in $\mathscr{H}$. A hyperedge $e_j \in E$ is said to be *internally active* with respect to the hypertree $\mathbf{f}$ if it is not possible to decrease $\mathbf{f}(e_j)$ by 1 and increase $\mathbf{f}(e_{j'})$ $(j' < j)$ by 1 so that another hypertree results. We call a hyperedge *internally inactive* with respect to a hypertree if it is not internally active and denote the number of such hyperedges of $\mathbf{f}$ by $\bar{\iota}(\mathbf{f})$. Then the *interior polynomial* of $\mathscr{H}$ is the generating function $I_{\mathscr{H}}(x) = \sum_{\mathbf{f} \in \mathrm{HT}(\mathscr{H})} x^{\bar{\iota}(\mathbf{f})}$. It is known [24, Proposition 6.1] that $\deg I_{\mathscr{H}}(x) \leq \min\{|V|, |E|\} - 1$. If $G = $ Bip$\mathscr{H}$, then we set $I_G(x) = I_{\mathscr{H}}(x)$. In [25] Kálmán and Postnikov proved the following.

Proposition 3.3 [25, Theorems 1.1 and 3.10] *Let G be a connected bipartite graph. Then we have*

$$I_G(x) = h^*(P_G, x).$$

Interior polynomials of disconnected bipartite graphs are defined by Kato [26] and the same assertion of Proposition 3.3 holds for all bipartite graphs. Note that if G is a bipartite graph, then the bipartite graph $\widehat{G}$ arising from G is connected. Hence we can use this formula to study the Eq. (2) in Proposition 3.1.

A *k-matching* of G is a set of k pairwise non-adjacent edges of G. Let

$$M(G,k) = \left\{ \{v_{i_1}, \ldots, v_{i_k}, e_{j_1}, \ldots, e_{j_k}\} : \begin{array}{c} \text{there exists a } k\text{-matching of } G \\ \text{whose vertex set is } \{v_{i_1}, \ldots, v_{i_k}, e_{j_1}, \ldots, e_{j_k}\} \end{array} \right\}.$$

For $k = 0$, we set $M(G, 0) = \{\emptyset\}$. Using the theory of generalized permutohedra [31,41], we have the following important fact on interior polynomials:

Proposition 3.4 *Let G be a bipartite graph. Then we have*

$$I_{\widehat{G}}(x) = \sum_{k \geq 0} |M(G,k)|\, x^k. \tag{3}$$

Proof Let $V \cup E$ denote a bipartition of G, where $V = \{v_2, \ldots, v_m\}$ and $E = \{e_2, \ldots, e_n\}$ with $d = m + n - 2$. Then $\widehat{G}$ is a connected bipartite graph with a bipartition $V' \cup E'$ with $V' = \{v_1\} \cup V$ and $E' = \{e_1\} \cup E$. Recall that $\{v_i, e_j\}$ is an edge of $\widehat{G}$ if and only if either $(i-1)(j-1) = 0$ or $\{v_i, e_j\}$ is an edge of G. Let $\mathrm{HT}(\widehat{G})$ be the set of all hypertrees in the hypergraph associated with $\widehat{G}$. Given a hypertree $\mathbf{f} \in \mathrm{HT}(\widehat{G})$, let Γ be a spanning tree that induces $\mathbf{f}$. From [24, Lemma 3.3], we may assume that $\{v_1, e_j\}$ is an edge of Γ for all $1 \leq j \leq n$. Note that the degree of each v_i $(2 \leq i \leq m)$ is 1.

By definition, e_1 is always internally active. We show that, e_j $(j \geq 2)$ is internally active if and only if $\mathbf{f}(e_j) = 0$. By definition, if $\mathbf{f}(e_j) = 0$, then e_j is internally active. Suppose $\mathbf{f}(e_j) > 0$. Then there exists $i \geq 2$ such that $\{v_i, e_j\}$ is an edge of Γ. Let $\mathbf{f}' \in \mathrm{HT}(\widehat{G})$ be a hypertree induced by a spanning tree obtained by replacing $\{v_i, e_j\}$ with $\{v_i, e_1\}$ in Γ. Then we have $\mathbf{f}'(e_j) = \mathbf{f}(e_j) - 1$, $\mathbf{f}'(e_1) = \mathbf{f}(e_1) + 1$ and $\mathbf{f}'(e_k) = \mathbf{f}(e_k)$ for all $1 < k \neq j$. Hence e_j is not internally active. Thus $\bar{\iota}(\mathbf{f})$ is the number of e_j $(j \geq 2)$ such that there exists an edge $\{v_i, e_j\}$ of Γ for some $i \geq 2$.

In order to prove the equation (3), it is enough to show that, for fixed hyperedges $e_{j_1}, \ldots, e_{j_k}$ with $2 \leq j_1 < \cdots < j_k \leq n$, the cardinality of

$$\mathscr{S}_{j_1,\ldots,j_k} = \{\mathbf{f} \in \mathrm{HT}(\widehat{G}) : e_{j_1}, \ldots, e_{j_k} \text{ are not internally active and } \bar{\iota}(\mathbf{f}) = k\}$$

is equal to the cardinality of

$$\mathscr{M}_{j_1,\ldots,j_k} = \left\{ \{v_{i_1}, \ldots, v_{i_k}\} : \begin{array}{c} \text{there exists a } k\text{-matching of } G \\ \text{whose vertex set is } \{v_{i_1}, \ldots, v_{i_k}, e_{j_1}, \ldots, e_{j_k}\} \end{array} \right\}.$$

Let $G_{j_1,\ldots,j_k}$ be the induced subgraph of G on the vertex set $V \cup \{e_{j_1}, \ldots, e_{j_k}\}$. If e_{j_ℓ} is an isolated vertex in $G_{j_1,\ldots,j_k}$, then both $\mathscr{S}_{j_1,\ldots,j_k}$ and $\mathscr{M}_{j_1,\ldots,j_k}$ are empty sets. If v_i is an isolated vertex in $G_{j_1,\ldots,j_k}$, then there is no relations between v_i and two sets, and hence we can ignore v_i. Thus we may assume that $G_{j_1,\ldots,j_k}$ has no isolated vertices.

It is known that $\mathscr{M}_{j_1,\ldots,j_k}$ is the set of bases of a transversal matroid associated with $G_{j_1,\ldots,j_k}$. See, e.g., [39, Section 1.6]. For $i = 2, \ldots, m$, let

$$I_i = \{0\} \cup \{\ell : \{v_i, e_{j_\ell}\} \text{ is an edge of } G_{j_1,\ldots,j_k}\} \subset \{0, 1, \ldots, k\}.$$

Oh [31] defines a lattice polytope $P_{\mathscr{M}_{j_1,\dots,j_k}}$ to be the generalized permutohedron [41] of the induced subgraph of $\widehat{G}$ on the vertex set $V \cup \{e_1, e_{j_1}, \dots, e_{j_k}\}$, i.e., $P_{\mathscr{M}_{j_1,\dots,j_k}}$ is the Minkowski sum $\Delta_{I_2} + \cdots + \Delta_{I_m}$, where $\Delta_I = \mathrm{conv}(\{\mathbf{e}_j : j \in I\}) \subset \mathbb{R}^{k+1}$ and $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_k$ are unit coordinate vectors in $\mathbb{R}^{k+1}$. By [31, Lemma 22 and Proposition 26], the cardinality of $\mathscr{M}_{j_1,\dots,j_k}$ is equal to the number of the lattice point $(x_0, x_1, \dots, x_k) \in P_{\mathscr{M}_{j_1,\dots,j_k}} \cap \mathbb{Z}^{k+1}$ with $x_1, x_2, \dots, x_k \geq 1$. In addition, by [41, Proposition 14.12], any lattice point $(x_0, x_1, \dots, x_k) \in P_{\mathscr{M}_{j_1,\dots,j_k}} \cap \mathbb{Z}^{k+1}$ is of the form $\mathbf{e}_{i_2} + \cdots + \mathbf{e}_{i_m}$, where $i_\ell \in I_\ell$ for $2 \leq \ell \leq m$. By a natural correspondence $(x_0, x_1, \dots, x_k) \in P_{\mathscr{M}_{j_1,\dots,j_k}} \cap \mathbb{Z}^{k+1}$ with $x_1, x_2, \dots, x_k \geq 1$ and $(\mathbf{f}(e_1), \mathbf{f}(e_{j_1}), \dots, \mathbf{f}(e_{j_k}))$ with $\mathbf{f} \in \mathscr{S}_{j_1,\dots,j_k}$, it follows that the number of the lattice point $(x_0, x_1, \dots, x_k) \in P_{\mathscr{M}_{j_1,\dots,j_k}} \cap \mathbb{Z}^{k+1}$ with $x_1, x_2, \dots, x_k \geq 1$ is equal to the cardinality of $\mathscr{S}_{j_1,\dots,j_k}$, as desired. □

Now, we show that the h^*-polynomial of $\mathscr{B}_G$ is γ-positive if G is a bipartite graph. In fact, we prove Theorem 0.3.

Proof of Theorem 0.3 By Propositions 3.1 and 3.4, the h^*-polynomial of $\mathscr{B}_G$ is

$$\begin{aligned} h^*(\mathscr{B}_G, x) &= \sum_{j=0}^{d} 2^j (x-1)^{d-j} \sum_{H \in S_j(G)} I_{\widehat{H}}(x) \\ &= \sum_{j=0}^{d} 2^j (x-1)^{d-j} \sum_{H \in S_j(G)} \sum_{k \geq 0} |M(H,k)|\, x^k. \end{aligned}$$

Note that, for each $\{v_{i_1}, \dots, v_{i_k}, e_{j_1}, \dots, e_{j_k}\} \in M(G,k)$ there exist $\binom{d-2k}{j-2k}$ induced subgraphs $H \in S_j(G)$ such that $\{v_{i_1}, \dots, v_{i_k}, e_{j_1}, \dots, e_{j_k}\} \in M(H,k)$ for $j = 2k, 2k+1, \dots, d$. Thus we have

$$\begin{aligned} h^*(\mathscr{B}_G, x) &= \sum_{k \geq 0} \sum_{j=2k}^{d} 2^j (x-1)^{d-j} |M(G,k)| \binom{d-2k}{j-2k} x^k \\ &= \sum_{k \geq 0} |M(G,k)|\, 2^{2k} x^k (2 + (x-1))^{d-2k} \\ &= (x+1)^d \sum_{k \geq 0} |M(G,k)| \left(\frac{4x}{(x+1)^2} \right)^k \\ &= (x+1)^d I_{\widehat{G}} \left(\frac{4x}{(x+1)^2} \right), \end{aligned}$$

as desired. □

By Proposition 3.4, it follows that, if G is a forest, then $I_{\widehat{G}}(x)$ coincides with the *matching generating polynomial* of G.

Proposition 3.5 *Let G be a forest. Then we have*

$$I_{\widehat{G}}(x) = \sum_{k \geq 0} m_k(G)\, x^k,$$

where $m_k(G)$ is the number of k-matchings in G. In particular, $I_{\widehat{G}}(x)$ is real-rooted.

Proof Let M_1 and M_2 be k-matchings of G. Suppose that M_1 and M_2 have the same vertex set $\{v_{i_1}, \dots, v_{i_k}, e_{j_1}, \dots, e_{j_k}\}$. If $M = (M_1 \cup M_2) \setminus (M_1 \cap M_2)$ is not empty, then M corresponds to a subgraph of G such that the degree of each vertex is 2. Hence M has at least one cycle. This contradicts that G is a forest. Hence we have $M_1 = M_2$. Thus $m_k(G)$ is the cardinality of $M(G, k)$. In general, it is known that $\sum_{k \geq 0} m_k(G)\, x^k$ is real-rooted for any graph G. See, e.g., [12,29]. □

Next we will show that, if a bipartite graph G is a "permutation graph" associated with a poset P, then the interior polynomial $I_{\widehat{G}}(x)$ coincides with the P-Eulerian polynomial $W(P)(x)$. A *permutation graph* is a graph on $[d]$ with edge set

$$\{\{i, j\} : L_i \text{ and } L_j \text{ intersect each other}\},$$

where there are d points $1, 2, \dots, d$ on two parallel lines $\mathscr{L}_1$ and $\mathscr{L}_2$ in the plane, and the straight lines L_i connect i on $\mathscr{L}_1$ and i on $\mathscr{L}_2$. If G is a bipartite graph with a bipartition $V_1 \cup V_2$, the following conditions are equivalent:

(i) G is a permutation graph;
(ii) The complement of G is a comparability graph of a poset;
(iii) There exist orderings $<_1$ on V_1 and $<_2$ on V_2 such that

$$i, i' \in V_1, i <_1 i', \quad j, j' \in V_2, j <_2 j', \quad \{i, j\}, \{i', j'\} \in E(G) \\ \Longrightarrow \{i, j'\}, \{i', j\} \in E(G);$$

(iv) For any three vertices, there exists a pair of them such that there exists no path containing the two vertices that avoids the neighborhood of the remaining vertex.

See [5, p. 93] for details. On the other hand, let P be a naturally labeled poset on $[d]$. Then the *order polynomial* $\Omega(P, m)$ of P is defined for $0 < m \in \mathbb{Z}$ to be the number of order-preserving maps $\sigma : P \to [m]$. It is known that

$$\sum_{m \geq 0} \Omega(P, m+1) x^m = \frac{\sum_{\pi \in \mathscr{L}(P)} x^{d(\pi)}}{(1-x)^{d+1}},$$

where $\mathscr{L}(P)$ is the set of linear extensions of P and $d(\pi)$ is the number of descents of π. The *P-Eulerian polynomial* $W(P)(x)$ is defined by

$$W(P)(x) = \sum_{\pi \in \mathscr{L}(P)} x^{d(\pi)}.$$

See, e.g., [46] for details. We now see a relation between the interior polynomial of a bipartite permutation graph and the P-Eulerian polynomial of a finite poset.

Proposition 3.6 *Let G be a bipartite permutation graph and let P be a poset whose comparability graph is the complement of G. Then we have*

$$I_{\widehat{G}}(x) = W(P)(x).$$

Proof In this case, $\mathscr{B}_G \cap \mathscr{O}_{(1,\ldots,1)}$ is the chain polytope $\mathscr{C}_P$ of P. It is known that the h^*-polynomial of $\mathscr{C}_P$ is the P-Eulerian polynomial $W(P)(x)$. See [45,46] for details. Thus we have $I_{\widehat{G}}(x) = h^*(P_{\widehat{G}}, x) = W(P)(x)$, as desired. □

It was conjectured by Neggers–Stanley that $W(P)(x)$ is real-rooted. However this is false in general. The first counterexample was constructed in [4] (not naturally labeled posets). Counterexamples of naturally labeled posets were given in [47]. Counterexamples in these two papers are *narrow* posets, i.e., elements of posets are partitioned into two chains. It is easy to see that P is narrow poset if and only if the comparability graph of P is the complement of a bipartite graph. Since Stembridge found many counterexamples which are naturally labeled narrow posets, there are many bipartite permutation graphs G such that $h^*(\mathscr{B}_G, x)$ are not real-rooted. We give one of them as follows.

Example 3.7 Let P be a naturally labeled poset in Fig. 5 given in [47]. Then

$$W(P)(x) = 3x^8 + 86x^7 + 658x^6 + 1946x^5 + 2534x^4 + 1420x^3 + 336x^2 + 32x + 1$$

has a conjugate pair of zeros near $-1.85884 \pm 0.149768i$ as explained in [47]. Let G be the complement of the comparability graph of P. Then G is a bipartite graph with 17 vertices and 32 edges. The h^*-polynomial of $\mathscr{B}_G$ is

$$\begin{aligned} h^*(\mathscr{B}_G, x) &= (x+1)^{17} W(P)\left(\frac{4x}{(x+1)^2}\right) \\ &= x^{17} + 145x^{16} + 7432x^{15} + 174888x^{14} + 2128332x^{13} \\ &\quad + 14547884x^{12} + 59233240x^{11} \\ &\quad + 148792184x^{10} + 234916470x^9 + 234916470x^8 + 148792184x^7 \\ &\quad + 59233240x^6 + 14547884x^5 + 2128332x^4 + 174888x^3 \\ &\quad + 7432x^2 + 145x + 1 \end{aligned}$$

and has conjugate pairs of zeros near $-3.88091 \pm 0.18448i$ and $-0.257091 \pm 0.0122209i$. (We used `Mathematica` to compute approximate values.) On the other hand, $h^*(\mathscr{B}_G, x)$ is log-concave.

By the following proposition, it turns out that this example is a counterexample to “Real Root Conjecture” that has been already disproved by Gal [11].

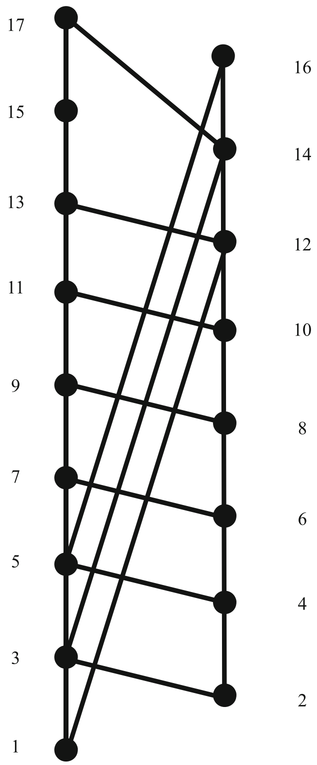

Fig. 5 A counterexample to Neggers–Stanley conjecture [47]

Proposition 3.8 *Let G be a bipartite permutation graph. Then $h^*(\mathscr{B}_G, x)$ coincides with the h-polynomial of a flag complex that is a triangulation of a sphere.*

Proof It is known [5, p.94] that any bipartite permutation graph satisfies the condition (iv) in Theorem 2.6. Hence there exists a squarefree quadratic initial ideal with respect to a reverse lexicographic order such that the smallest variable corresponds to the origin. This means that there exists a flag regular unimodular triangulation Δ such that the origin is a vertex of any maximal simplex in Δ. Then $h^*(\mathscr{B}_G, x)$ coincides with the h-polynomial of a flag triangulation of the boundary of a convex polytope $\mathscr{B}_G$ arising from Δ. □

In [23], for a (p, q)-complete bipartite graph $K_{p,q}$, a simple description for the h^*-polynomial of $\mathscr{A}_{K_{p,q}}$ was given. In fact, one has

$$h^*(\mathscr{A}_{K_{p+1,q+1}}, x) = \sum_{i=0}^{\min\{p,q\}} \binom{2i}{i}\binom{p}{i}\binom{q}{i} x^i (x+1)^{p+q+1-2i}. \tag{4}$$

Moreover, it was shown that $h^*(\mathscr{A}_{K_{p+1,q+1}}, x)$ is γ-positive and real-rooted. Similarly, we can obtain a simple description for the h^*-polynomial of $\mathscr{B}_{K_{p,q}}$ and show that $h^*(\mathscr{B}_{K_{p,q}}, x)$ is γ-positive and real-rooted.

Example 3.9 Let $K_{p,q}$ be a (p,q)-complete bipartite graph. Then the comparability graph of a poset P consisting of two disjoint chains $1 < 2 < \cdots < p$ and $p+1 < p+2 < \cdots < p+q$ is the complement of $K_{p,q}$. It is easy to see that

$$W(P)(x) = \sum_{i=0}^{\min\{p,q\}} \binom{p}{i}\binom{q}{i} x^i.$$

Hence we have

$$h^*(\mathscr{B}_{K_{p,q}}, x) = (x+1)^{p+q} W(P)\left(\frac{4x}{(x+1)^2}\right) = \sum_{i=0}^{\min\{p,q\}} 4^i \binom{p}{i}\binom{q}{i} x^i (x+1)^{p+q-2i}. \tag{5}$$

Simion [42] proved that $W(P)(x)$ is real-rooted if P is a naturally labeled and disjoint union of chains. Thus the h^*-polynomial $h^*(\mathscr{B}_{K_{p,q}}, x)$ is real-rooted.

Remark 3.10 In a very recent work [38], it is shown that the h^*-polynomials of symmetric edge polytopes of certain graphs can be computed by that of $\mathscr{B}_G$. For example, the Eq. (4) can be obtained from the Eq. (5) ([38, Example 5.10]).

In [23], for the proof of the real-rootedness of $h^*(\mathscr{A}_{K_{p,q}}, x)$, interlacing polynomials techniques were used. Let f and g be real-rooted polynomials with roots $a_1 \geq a_2 \geq \cdots$, respectively, $b_1 \geq b_2 \geq \cdots$. Then g is said to *interlace* f if

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots.$$

In this case, we write $f \preceq g$. In [23], it is shown that

$$h^*(\mathscr{A}_{K_{p,q}}, x) \preceq h^*(\mathscr{A}_{K_{p,q+1}}, x).$$

By a similar way of [23], we can prove the following.

Proposition 3.11 *For all* $p, q \geq 1$, *one has*

$$h^*(\mathscr{B}_{K_{p,q}}, x) \preceq h^*(\mathscr{B}_{K_{p,q+1}}, x).$$

Proof Set $\gamma(\mathscr{B}_{K_{p,q}}, x) = \sum_{i \geq 0} 4^i \binom{p}{i}\binom{q}{i} x^i$. Since $\{\binom{p}{i}\}_{i \geq 0}$ is a multiplier sequence (see [23]) and since $(4x+1)^q \preceq (4x+1)^{q+1}$, one has $\gamma(\mathscr{B}_{K_{p,q}}, x) \preceq \gamma(\mathscr{B}_{K_{p,q+1}}, x)$. By [23, Lemma 4.10], we obtain $h^*(\mathscr{B}_{K_{p,q}}, x) \preceq h^*(\mathscr{B}_{K_{p,q+1}}, x)$. □

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