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Donaldson–Thomas invariants from tropical disks

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Abstract

We prove that the quantum DT-invariants associated to quivers with genteel potential can be expressed in terms of certain refined counts of tropical disks. This is based on a quantum version of Bridgeland's description of cluster scattering diagrams in terms of stability conditions, plus a new version of the description of scattering diagrams in terms of tropical disk counts. The weights with which the tropical disks are counted are expressed in terms of motivic integrals of certain quiver flag varieties. We also show via explicit counterexample that Hall algebra broken lines do not result in consistent Hall algebra theta functions, i.e., they violate the extension of a lemma of Carl–Pumperla–Siebert from the classical setting.

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Contents

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1 Introduction

In [22], Gross–Hacking–Keel–Kontsevich used scattering diagrams to construct canonical bases for cluster algebras. Several articles [24,30,32,44,48] have developed connections between DT-invariants and various scattering diagrams or cluster transformations. Building off these ideas, Bridgeland [6] constructed Hall algebra scattering diagrams whose classical integrals often recover the cluster scattering diagrams (cf. our Proposition 3.16 for the quantum analog). On the other hand, [9,19,25,38] show how to express various scattering diagrams in terms of certain (refined) counts of tropical curves or disks. By extending and combining these ideas, we obtain new expressions for quantum DT-invariants in terms of refined counts of tropical disks.

1.1 Quantum DT-invariants from tropical ribbons

Let (Q, W) be a finite quiver Q without loops or oriented 2-cycles, plus a choice of finite potential W, i.e., a finite linear combination of oriented cycles in Q. Denote the vertex set of Q by Q_0. There are standard notions, reviewed in Sect. 2, of the associated category of representations $\mathrm{rep}(Q, W)$, the corresponding Grothendieck lattice $N = \mathbb{Z}^{Q_0}$, and the moduli stack $\mathcal{M}$ of objects in $\mathrm{rep}(Q, W)$. Points θ in $M_{\mathbb{R}} := \mathrm{Hom}(N, \mathbb{R})$ can be viewed as stability conditions on $\mathrm{rep}(Q, W)$, determining a substack $1_{\mathrm{ss}}(\theta)$ of θ-stemistable objects[1] in $\mathrm{rep}(Q, W)$, cf. Definition 3.7. We let $\mathcal{I}_t$ denote the quantum integration map taking varieties over $\mathcal{M}$ to elements of a quantum torus algebra $\mathbb{C}_t[N^{\oplus}]$, cf. Sect. 2.6. E.g., if $W = 0$, then $\mathcal{I}_t$ is the generalized Poincaré polynomial. We are interested are the invariants $\mathcal{I}_t(\log(1_{\mathrm{ss}}(\theta)))$.

Some additional notation and terminology regarding this setup will be needed. The lattice N is equipped with the natural basis $\{e_i\}_{i\in Q_0}$. Let B denote the skew-symmetric Euler form on N, cf. (2), and define[2] $p^* : N \to M$, $p^*(n) = B(\cdot, n)$. We say $\theta \in M_{\mathbb{R}}$ is general if it is not in the intersection of two distinct hyperplanes of the form $n^{\perp}$ for $n \in N\setminus\{0\}$, cf. Remark 3.9.

We will also need the following setup coming from the theory of tropical curves and scattering diagrams, cf. Sect. 4.1 for details. By a weight-vector, we will mean a tuple $\mathbf{w} = (\mathbf{w}_i)_{i\in Q_0}$ where each $\mathbf{w}_i = (w_{ij})_{j=1,\ldots,l_i}$ consists of positive integers $w_{i1} \leq w_{i2} \leq \cdots \leq w_{il_i}$. Denote the length $l(\mathbf{w}) := \sum_i l_i$, and let $\mathrm{Aut}(\mathbf{w})$ be the group of automorphisms of the second indices of the $\mathbf{w}_i$'s which act trivially on $\mathbf{w}$. Define the "multiple cover contributions" $R_{\mathbf{w}} := \prod_{ij} \frac{(-1)^{w_{ij}-1}}{w_{ij}(q^{w_{ij}}-1)} \in \mathbb{C}[t^{\pm 1}]$ where $q := t^2$.

One says that a tropical disk $h : \Gamma \to M_{\mathbb{R}}$ (cf. Sect. 4.1) has degree $\Delta_{\mathbf{w}}$ if the unbounded edges E_{ij} are labelled by the indices of $\mathbf{w}$, and if the weighted outgoing direction of $h(E_{ij})$ equals $w_{ij}p^*(e_i)$. Let $\mathbf{A}_{\mathbf{w}}$ be a collection of affine hyperplanes $\{A_{ij} \subset M_{\mathbb{R}}\}$ with A_{ij} a generic translate of $e_i^{\perp}$. For $\delta > 0$, we say that a tropical disk matches the constraints $\delta\mathbf{A}_{\mathbf{w}}$ if $h(E_{ij}) \subset \delta A_{ij}$ for each i, j. The type τ of a tropical

[1] More precisely, $1_{\mathrm{ss}}(\theta)$ is the Hall algebra element associated to the substack of θ-semistable objects.

[2] In [22], p^* is defined by inserting n into the first entry of the skew-symmemtrizable form. This discrepancy is because the form B here and in [6] is negative the form used in [22].

disk is the data of the underlying weighted graph Γ plus the data of directions of $h(E)$ for each edge E of Γ.

Each tropical disk includes the data of a special endpoint vertex $V_\infty \in \Gamma^{[0]}$, and we will impose an additional constraint on the image of V_∞. Specifically, given $\theta \in M_{\mathbb{R}}$ and a tropical disk type τ, we say that $\tau \in \mathfrak{T}_{\mathbf{w}}(\theta)$ if the following holds: given $\epsilon > 0$ and any sufficiently small $\delta > 0$ (small relative to ϵ), there exist tropical disks of degree $\Delta_{\mathbf{w}}$ and type τ which match the constraint $\delta\mathbf{A}_{\mathbf{w}}$ and have $h(V_\infty) \in B_\epsilon(\theta)$ (the radius ϵ open ball about θ). Let $\widehat{\mathfrak{T}}_{\mathbf{w}}(\theta)$ denote the corresponding space of tropical ribbon types, i.e., tropical disk types plus the additional data of a cyclic ordering of the edges at each vertex. We wish to count elements of $\widehat{\mathfrak{T}}_{\mathbf{w}}(\theta)$ with a multiplicity which we define next.

The form B descends to a form $\overline{B}$ on $p^*(N)$ given by $\overline{B}(p^*(n_1), p^*(n_2)) = B(n_1, n_2)$. For $\widehat{\tau} \in \widehat{\mathfrak{T}}_{\mathbf{w}}(\theta)$, let $\nu(\widehat{\tau})$ denote (-1) to the power of the number of vertices of $\widehat{\tau}$ where the ribbon structure does not agree with the orientation induced by $\overline{B}$, cf. Sect. 4.2.2. The ribbon structure induces an ordering $E_{i_1,j_1}, \ldots E_{i_{l(\mathbf{w})}j_{l(\mathbf{w})}}$ on the E_{ij}'s. Given such a tropical ribbon type $\widehat{\tau}$, let $\mathfrak{Flag}(\widehat{\tau})$ denote the variety over $\mathcal{M}$ whose fiber over a (stacky) point corresponding to a representation M is the space of composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{\sum_{ij} w_{ij}} = M$$

such that the first $w_{i_1 j_1}$ quotients M_i/M_{i-1} are isomorphic to the simple representation S_{i_1}, then the next $w_{i_2 j_2}$ quotients M_i/M_{i-1} are isomorphic to the simple representation S_{i_2}, and so on. The following is the quantum integral case of Theorem 4.10.

Theorem 1.1 *Suppose (Q, W) is genteel over $\mathfrak{g}^q$ (cf. Sect. 3.2.3). Then for general $\theta \in M_{\mathbb{R}}$,*

$$\mathcal{I}_t(\log(1_{\mathrm{ss}}(\theta))) = \sum_{\mathbf{w}} \left(\frac{1}{|\operatorname{Aut}(\mathbf{w})|} \sum_{\widehat{\tau} \in \widehat{\mathfrak{T}}_{\mathbf{w}}(\theta)} \nu(\widehat{\tau}) \mathcal{I}_t(R_{\mathbf{w}} \mathfrak{Flag}(\widehat{\tau})) \right). \qquad (1)$$

See Sect. 3.2.3 for details on our version of the genteel property and modifications thereof. We note here that genteelness of (Q, W) is known to at least hold for acyclic quivers with $W = 0$. More generally, the possibly weaker condition of genteelness over $\mathfrak{g}^q$ holds whenever Q admits a green-to-red sequence and W is non-degenerate [40, Cor. 1.2(i)].

The same statement applies with $\mathcal{I}_t$ replaced by the classical integration map $\mathcal{I}$ (i.e., the $t \mapsto 1$ limit, i.e., taking generalized Euler characteristics), and similarly for $\mathcal{I}_t$ replaced by other projections $\mathcal{I}^{\mathrm{i}}$ of the Hall algebra defined in Sect. 3.2.1. For the classical version though, one should view $\mathcal{I}(R_{\mathbf{w}}\mathfrak{Flag}(\widehat{\tau}))$ as living in $\mathfrak{A}^{\mathrm{cl}}$, a logarithmic version of the Weyl algebra, cf. Example 4.1. These factors $\mathcal{I}_t(R_{\mathbf{w}}\mathfrak{Flag}(\widehat{\tau}))$ and $\mathcal{I}(R_{\mathbf{w}}\mathfrak{Flag}(\widehat{\tau}))$ can be more easily computed as products in the quantum torus algebra or Weyl algebra, respectively, cf. Remark 4.11.

Alternatively, one can replace the sum over tropical ribbons with a sum over tropical disks, and then the tropical ribbon multiplicities $\nu(\widehat{\tau})\mathcal{I}_t(R_{\mathbf{w}}\mathfrak{Flag}(\widehat{\tau}))$ are replaced with

Block–Göttsche [2] style refined tropical disk multiplicities $R'_{\mathbf{w}} \prod_V [\mathrm{Mult}(V)]_t$, and similarly for the classical cases, cf. Remark 4.11 again. Our intention in this paper though is to give a representation-theoretic description of the tropical multiplicities, which is why we state Theorem 1.1 in terms of moduli of flags.

See Example 4.12 for a sample computation of a term on the right-hand side of (1).

1.2 Hall algebra broken lines violate the Carl–Pumperla–Siebert Lemma

One might hope (as we had hoped) that Theorem 1.1 holds without applying the integration maps, i.e., as an identity in the Hall algebra. Unfortunately, this fails as a result of the fact that elements of the Hall algebra with parallel dimension vectors need not commute (although we see that the result does hold after modding out by the ideal generated by these commutators). In Sect. 5, we show that similar issues cause problems for theta functions.

As in [9,21,23], the construction of theta functions in [22] is based on enumerating broken lines (an abridged version of tropical disks). This enumeration depends on the designated endpoint of the broken lines, but according to [9, § 4], different choices of endpoint are related by path-ordered product, essentially meaning that these choices glue to give well-defined global functions on the mirror. [38, Thm. 2.14] gives a refined version of this Carl–Pumperla–Siebert Lemma, implying that the analogous gluing property holds for quantum theta functions, cf. Lemma 5.2. Refining further, [7,8] defines Hall algebra broken lines, and from these one might hope to define Hall algebra theta functions. Unfortunately, this is not a well-behaved notion:

Proposition 1.2 (Proposition 5.4 in the main text) *the Carl–Pumperla–Siebert Lemma does not hold for Hall algebra broken lines.*

Our proof is via the explicit construction of a counterexample for an A_3-quiver, cf. Sect. 5.3.

1.3 Motivation

When B has rank 2, the tropical disk counts of Theorem 1.1 can be replaced with tropical curve counts, cf. [25, Thm. 2.8] and [19, Cor. 4.9]. In higher-dimensions this is only the case for certain limits of choices of θ, cf. [38, Thm 3.7]. The tropical curve versions are nice because in the classical limit they can be related via [45] to log Gromov–Witten invariants, cf. [25, Prop. 5.3]. As the authors have learned from Mark Gross, the classical versions of our tropical disk counts should also have an algebraic Gromov–Witten theoretic meaning: according to the announced result [27, Thm. 2.14], they should be related to certain *punctured* Gromov–Witten invariants (one also expects the existence of correspoding holomorphic disk counts defined from the perspective of open Gromov–Witten theory, e.g., as in [36] for the case of K3 surfaces). One expects DT/GW correspondence results to follow from [27, Thm. 2.14] combined with [6, Lem. 11.4].

On the other hand, the quantum tropical curve counts in rank 2 are Block–Göttsche invariants [2], which have been related to higher-genus Gromov–Witten invariants in [3,4] and to real curve counts in [39]. Upcoming work of the second author will extend the correspondence of [39] to higher-dimensions, although an extension to tropical disks is still more distant. Still, we hope that Theorem 1.1 will lead to new refined DT/GW correspondence results, and we further hope that this correspondence will be enriched by our interpretation of tropical ribbon multiplicities in terms of moduli of composition series.

A version of Theorem 1.1 for bipartite quivers was previously observed in [19, Thm. 5.3]. Their argument was based on the observation that in these cases, (1) is equivalent to a representation-theoretic formula of Manschot–Pioline–Sen [41]. We therefore hope that our result may be related to some generalization of this MPS formula. With this in mind, we strongly suspect that our tropical counts are closely related to the attractor flow trees studied by physicists, cf. [1] in particular, as well as [35].

We note that [37], which appeared immediately after this paper was first posted, deals with similar problems on scattering diagrams and tropical disks using a differential-geometric perspective.

1.4 Outline of the paper

In Sects. 2.1–2.2, we review Joyce's construction [28] of the Hall algebra associated to a quiver with potential, following [6, § 4–5]. Then in Sects. 2.3–2.4, we use [5, Lem. 4.4] to describe certain products in the Hall algebra in terms of moduli of composition series. We review the quantum and classical integration maps in Sects. 2.5–2.6.

We review the definition of scattering diagrams in Sect. 3.1, and in Theorem 3.5 we generalize previously known results about initial scattering diagrams uniquely determining consistent scattering diagrams. We then we review Bridgeland's Hall algebra scattering diagrams (and some variants) in Sect. 3.2. If the potential W is genteel, then the Hall algebra scattering diagram is determined by an easily understood initial scattering diagram which we describe explicitly in Sect. 3.3.

We review the notion of tropical disks in Sect. 4.1, and in Sect. 4.2 we focus on the tropical ribbons and multiplicities associated to an initial scattering diagram. The description of scattering diagrams in terms of tropical disks (Theorem 4.4) is given in Sect. 4.3 and proven in Sect. 4.4. This is applied to the Hall algebra scattering diagram in Sect. 4.5 to prove our main results, Theorems 4.9 and 4.10.

We turn our attention to theta functions in Sect. 5. We review the definitions of broken lines and theta functions in Sect. 5.1, explaining how these apply to various flavors of cluster varieties in Sect. 5.2. Finally, in Sect. 5.3, we work out an explicit counterexample to show that a foundational result of [9] (cf. Lemma 5.2) does not extend to the Hall algebra setting (Proposition 5.4).

2 The motivic Hall algebra of a quiver with potential

2.1 Preliminaries on quivers with potential and their representations

Let Q be a finite quiver. Denote the sets of vertices and arrows of Q as (Q_0, Q_1). Let $\mathbb{C}Q$ denote the path algebra of Q. Suppose that Q is equipped with a finite **potential**, i.e., a finite linear combination of cycles, denoted $W \in \mathbb{C}Q$. Define a two-sided ideal $I_W \subseteq \mathbb{C}Q$ on Q by

$$I_W = (\partial_a W : a \in Q_1).$$

Here, if $b_1 \dots b_k$ is a cycle of arrows in Q, then

$$\partial_a(b_1 \dots b_k) = \sum_{i=1}^{k} \delta_{ab_i} b_{i+1} \dots b_k b_1 \dots b_{i-1},$$

where δ_{ab_i} is 1 if $a = b_i$ and 0 otherwise. Then the Jacobi algebra for (Q, W) is the quotient algebra $\mathbb{C}Q/I_W$. Let $\mathrm{rep}(Q, W) := \mathrm{mod}\mathbb{C}Q/I_W$ be the abelian category of finite-dimensional representations of the quiver with potential (Q, W), i.e., finite-dimensional left $\mathbb{C}Q/I_W$-modules.

Set $N = \mathbb{Z}^{Q_0}$, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\{e_i\}_{i \in Q_0}$ be the natural basis indexed by the vertices of Q. Denote $N^{\oplus} := \{\sum_i a_i e_i \in N | a_i \in \mathbb{Z}_{\geq 0}\ \forall i\}$, and $N^+ := N^{\oplus} \setminus \{0\}$. There is a group homomorphism

$$\dim : K_0(\mathrm{rep}(Q, W)) \to N$$

sending a representation to its dimension vector. For vertices $i, j \in Q_0$, let a_{ij} denote the number of arrows from i to j. Let B denote the integral skew-symmetric bilinear form on N determined by setting

$$B(e_i, e_j) := a_{ji} - a_{ij}. \tag{2}$$

We note that our B is negative the pairing $\langle \cdot, \cdot \rangle$ used in [6]. We will also use a second $\mathbb{Z}$-valued bilinear form χ on N given by

$$\chi(e_i, e_j) := \delta_{ij} - a_{ij}. \tag{3}$$

Note that $B(a, b) := \chi(a, b) - \chi(b, a)$.

It is well-known (cf. [6, Lem 4.1]) that there is an algebraic moduli stack $\mathcal{M}$ parameterizing all objects of the category $\mathrm{rep}(Q, W)$. Briefly, objects of $\mathcal{M}$ over a scheme S are isomorphism classes of locally free finite-rank $\mathcal{O}_S$-modules E, together with morphisms $\rho : \mathbb{C}Q/I_W \to \mathrm{End}_S(E)$, cf. [6, § 4.2] for details. Furthermore, $\mathcal{M}$ decomposes as

$$\mathcal{M} = \bigsqcup_{d \in N^{\oplus}} \mathcal{M}_d \tag{4}$$

where $\mathcal{M}_d$ is the open and closed substack parametrizing objects of dimension vector d. There is a 2-category of algebraic stacks over $\mathcal{M}$, and we let $\mathrm{St}/\mathcal{M}$ denote the full subcategory consisting of objects $f : X \to \mathcal{M}$ for which X is of finite type over $\operatorname{Spec}\mathbb{C}$ and has affine stabilizers. We similarly write $\mathrm{St}/\mathbb{C}$ for the analogous category of stacks over $\operatorname{Spec}\mathbb{C}$.

2.2 Construction of the Hall algebra

We now review the motivic Hall algebra developed by Joyce [28], following the presentation of [6, § 5].

Let $K(\mathrm{St}/\mathcal{M})$ be the free abelian group with basis given by isomorphism classes of objects of $\mathrm{St}/\mathcal{M}$ modulo the relations given in [6, Def. 5.1]. In particular, one imposes the scissor relations

$$[f : X \to \mathcal{M}] = [f|_Y : Y \to \mathcal{M}] + [f|_U : U \to \mathcal{M}],$$

where $[f : X \to \mathcal{M}]$ is an object of $\mathrm{St}/\mathcal{M}$, $Y \subset X$ is a closed substack, and $U := X\backslash Y$.

One endows the group $K(\mathrm{St}/\mathcal{M})$ with a $K(\mathrm{St}/\mathbb{C})$-module structure by setting $[X] \cdot [Y \to \mathcal{M}] = [X \times Y \to \mathcal{M}]$ and extending linearly. There is a unique ring homomorphism

$$\Upsilon : K(\mathrm{St}/\mathbb{C}) \to \mathbb{C}(t) \tag{5}$$

taking the class of a smooth projective variety X over $\mathbb{C}$ to its Poincaré polynomial

$$\sum_{k=1}^{2d} \dim_{\mathbb{C}} H^k(X_{\mathrm{an}}, \mathbb{C})(-t)^k \in \mathbb{C}[q],$$

where $q := t^2$ and $H^k(X_{\mathrm{an}}, \mathbb{C})$ denotes singular cohomology. For $X \in K(\mathrm{St}/\mathbb{C})$, we will often denote

$$|X| := \Upsilon(X).$$

Let

$$K_\Upsilon(\mathrm{St}/\mathcal{M}) := K(\mathrm{St}/\mathcal{M}) \otimes_{K(\mathrm{St}/\mathbb{C})} \mathbb{C}(t).$$

As a $\mathbb{C}(t)$-module, the (motivic) **Hall algebra** $H(Q, W)$ is $K_\Upsilon(\mathrm{St}/\mathcal{M})$. To define the multiplication, the convolution product, on $H(Q, W)$ and make it into a $\mathbb{C}(t)$-algebra, we consider the stack $\mathcal{M}^{(2)}$ of short exact sequences in $\mathrm{rep}(Q, W)$. There is a diagram

$$\begin{tikzcd} \mathcal{M}^{(2)} \arrow[r, "b"] \arrow[d, "(a_1,a_2)"] & \mathcal{M} \\ \mathcal{M} \times \mathcal{M}, & \end{tikzcd} \tag{6}$$

where a_1, a_2, b sends a short exact sequence

$$0 \to A_1 \to B \to A_2 \to 0$$

to A_1, A_2, and B respectively. The convolution product is defined to be

$$m = b_* \circ (a_1, a_2)^* : H(Q, W) \times H(Q, W) \to H(Q, W).$$

This product can be expressed as

$$[X_1 \xrightarrow{f_1} \mathcal{M}] * [X_2 \xrightarrow{f_2} \mathcal{M}] = [Z \xrightarrow{b \circ h} \mathcal{M}],$$

where Z and h are defined by the Cartesian square

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d] & \mathcal{M}^{(2)} \arrow[r, "b"] \arrow[d, "(a_1, a_2)"] & \mathcal{M} \\ X_1 \times X_2 \arrow[r, "f_1 \times f_2"] & \mathcal{M} \times \mathcal{M} & \end{tikzcd}$$

The following is due to Joyce [28, Thm. 5.2], see also [5, Thm. 4.3].

Theorem 2.1 *The product m gives $H(Q, W)$ the structure of an associative unital algebra over $\mathbb{C}(t)$. The unit element is $1 = [\mathcal{M}_0 \subset \mathcal{M}]$.*

We note that the decomposition (4) of $\mathcal{M}$ induces an $N^{\oplus}$-grading

$$H(Q, W) = \bigoplus_{d \in N^{\oplus}} H(Q, W)_d, \tag{7}$$

where $H(Q, W)_d$ is the submodule of $K_{\Upsilon}(\mathrm{St}/\mathcal{M})$ generated by objects of the form $[X \to \mathcal{M}_d \subset \mathcal{M}]$.

2.3 *k*-fold products

We will also need a description of the k-fold product $m_k : H(Q, W)^{\otimes k} \to H(Q, W)$. For this we follow [5, § 4.1–4.2]. Let $\mathcal{M}^{(k)}$ denote the algebraic moduli stack of k-flags. That is, the objects of $\mathcal{M}^{(k)}$ over a scheme S are isomorphism classes of k-tuples of objects $(E_1, \rho_1), \ldots, (E_k, \rho_k)$ of $\mathcal{M}(S)$, together with monomorphisms

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_k \tag{8}$$

respecting the maps ρ_i and such that each factor $F_i := E_i/E_{i-1}$ is flat over S. Given another scheme T, an object $(E'_1, \rho'_1), \ldots, (E'_k, \rho'_k)$ over T, and a morphism $f : T \to S$, a morphism in $\mathcal{M}^{(k)}$ lying over f is a collection of isomorphisms of sheaves $\Phi_i : f^*(E_i) \to E'_i$ respecting the maps ρ_i and the maps in the sequences of monomorphisms as in (8).

For each $i = 1, \ldots, k$, we have a morphism of stacks $a_i : \mathcal{M}^{(k)} \to \mathcal{M}$ taking an object as in (8) to its i-th factor $F_i = E_i/E_{i-1}$. We also have another morphism $b : \mathcal{M}^{(k)} \to \mathcal{M}$ taking the object as in (8) to the final term (E_k, ρ_k) of the sequence. One easily sees that the stack $\mathcal{M}^{(2)}$, together with these morphisms a_1, a_2, b, is equivlaent to the data we had when defining $\mathcal{M}^{(2)}$ as the stack of short exact sequences above. We now obtain a diagram generalizing (6):

$$\begin{array}{ccc} \mathcal{M}^{(k)} & \xrightarrow{b} & \mathcal{M} \\ \big\downarrow{\scriptstyle (a_1,\ldots,a_k)} & & \\ \mathcal{M}^k & & \end{array}$$

Lemma 2.2 ([5], Lemma 4.4) *The k-**fold product** $m_k : H(Q, W)^{\otimes k} \to H(Q, W)$ is given by*

$$m_k := b_* \circ (a_1, \ldots, a_k)^*.$$

2.4 H_{reg} and the composition algebra

Next, recalling the notation $q = t^2$, let

$$\mathbb{C}_{\mathrm{reg}}(t) := \mathbb{C}[t, t^{-1}][(1 + q + q^2 + \cdots + q^k)^{-1} : k \geq 1] \subset \mathbb{C}(t).$$

Let $H_{\mathrm{reg}}(Q, W)$ be the $\mathbb{C}_{\mathrm{reg}}(t)$-submodule of $H(Q, W)$ generated by elements of the form

$$[f : X \to \mathcal{M}]$$

such that X is a variety over $\mathbb{C}$ (so in particular, $X \in \mathrm{St}/\mathbb{C}$, and so we can apply Υ to X).

Lemma 2.3 ([6], Thm. 5.2) *$H_{\mathrm{reg}}(Q, W)$ is closed under the Hall algebra product and thus forms an $N^{\oplus}$-graded $\mathbb{C}_{\mathrm{reg}}(t)$-subalgebra. Furthermore, $H_{\mathrm{reg}}(Q, W)$ forms a Poisson algebra under the bracket*

$$\{a, b\} := (t - t^{-1})^{-1}[a, b]. \tag{9}$$

Now, for any representation $A \in \mathrm{ob}(\mathrm{rep}(Q, W))$, let p_A denote the corresponding (stacky) point in $\mathcal{M}$, and let δ_A be the element of $H(Q, W)$ corresponding to the inclusion $[p_A \hookrightarrow \mathcal{M}]$. Let

$$\kappa_A := |\operatorname{Aut}(A)|\delta_A \in H(Q, W) \tag{10}$$

be the element $[\operatorname{Spec} \mathbb{C} \to p_A \in \mathcal{M}]$. Clearly, κ_A is in $H_{\mathrm{reg}}(Q, W)$.

Given a collection of objects $A_1, \ldots, A_k, M \in \mathrm{ob}(\mathrm{rep}(Q, W))$, let $\widetilde{\mathfrak{Flag}}(A_1, \ldots, A_k; M)$ denote the space of filtrations

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

of M such that $M_i/M_{i-1} \cong A_i$ for each i. We also consider the quotient stack $\mathfrak{Flag}(A_1, \ldots, A_k; M)$ in which the identification of M_k with M is no longer part of the data of an object. This has the effect of enlarging the automorphism groups since now automorphisms of M induce automorphisms of flags, so

$$\mathfrak{Flag}(A_1, \ldots, A_k; M) = [\widetilde{\mathfrak{Flag}}(A_1, \ldots, A_k; M)/\operatorname{Aut}(M)]. \tag{11}$$

By Lemma 2.2, we have the following:

Lemma 2.4 *Given a collection of objects $A_1, \ldots, A_k \in \mathrm{ob}(\mathrm{rep}(Q, W))$, let $d = \sum_{j=1}^k dim(A_j) \in N^{\oplus}$. The product $\kappa_{A_1} \cdots \kappa_{A_k}$ is represented by a complex variety $\mathfrak{Flag}(A_1, \ldots, A_k) \to \mathcal{M}_d$ whose fiber over a point p_M is $\widetilde{\mathfrak{Flag}}(A_1, \ldots, A_k; M)$. Equivalently, the fiber of $\mathfrak{Flag}(A_1, \ldots, A_k)$ over the geometric point $[\operatorname{Spec}\mathbb{C} \to p_M \in \mathcal{M}_d]$ is $\mathfrak{Flag}(A_1, \ldots, A_k; M)$.*

For each vertex $i \in Q_0$, we have an associated simple representation $S_i \in \mathrm{rep}(Q, W)$ of dimension vector e_i. We denote $\delta_i := \delta_{S_i}$ and $\kappa_i := \kappa_{S_i}$. More generally, for each $k \in \mathbb{Z}_{\geq 0}$, we will write the semisimple representation $S_i^{\oplus k}$ as S_{ki}, and we will write $\delta_{ki} := \delta_{S_{ki}}$ and $\kappa_{ki} := \kappa_{S_{ki}}$. As in [28, Ex. 5.20], we define the **composition algebra** $\mathcal{C}(Q, W)$ to be the subalgebra of $H_{\mathrm{reg}}(Q, W)$ generated by the elements κ_i for $i \in Q_0$. By Lemma 2.4, products of the elements κ_i are given in terms of spaces of composition series.

Example 2.5 For $i \in Q_0$, let us apply Lemma 2.4 to κ_i^k. The only point in $\mathcal{M}_{ke_i}$ is the one corresponding to the semisimple representation S_{ki}. Furthermore, $\mathfrak{Flag}(S_i, \ldots, S_i; S_{ki})$ (S_i occurring k times before the semicolon) contains only one (stacky) point—all maximal flags of S_{ki} are related by automorphisms of S_{ki}. The stabilizer group for this point (i.e., the space of automorphisms of $\mathbb{C}^k$ which fix a maximal flag) is the unipotent group $U_k(\mathbb{C})$. Thus,

$$\kappa_{ki} = |U_k(\mathbb{C})|\kappa_i^k = q^{k(k-1)/2}\kappa_i^k. \tag{12}$$

Using (10) and the fact that

$$|\operatorname{Aut}(S_{ki})| = |\operatorname{GL}_k(\mathbb{C})| = q^{k(k-1)/2}\prod_{j=1}^k (q^j - 1),$$

we can re-express (12) as

$$\delta_{ki} = \frac{1}{\prod_{j=1}^k (q^j - 1)}\kappa_i^k. \tag{13}$$

Alternatively, this could be realized directly as

$$\delta_{ki} = \kappa_i^k / |\widetilde{\mathfrak{Flag}}(S_i, \ldots, S_i; S_{ki})|$$

(S_i again appearing k times before the semicolon). (13) will be useful in Sect. 3.3.

2.5 The quantum torus algebra

Let $\mathbb{C}_t[N^{\oplus}]$ denote the **quantum torus algebra**, by which we mean the $N^{\oplus}$-graded algebra defined by:

$$\mathbb{C}_t[N^{\oplus}] := \mathbb{C}_{\mathrm{reg}}(t)[z^n : n \in N^{\oplus}] / \langle z^{n_1} z^{n_2} = t^{B(n_1,n_2)} z^{n_1+n_2} : n_1, n_2 \in N^{\oplus} \rangle$$

(the monomials z^n adjoined here are non-commuting). This forms a **Poisson algebra** under the bracket

$$\{a, b\} := \frac{[a, b]}{t - t^{-1}}. \tag{14}$$

Note that

$$\{z^{n_1}, z^{n_2}\} = [B(n_1, n_2)]_t z^{n_1+n_2},$$

where for any $a \in \mathbb{Z}$,

$$[a]_t := \frac{t^a - t^{-a}}{t - t^{-1}} = \mathrm{sgn}(a)(t^{-|a|+1} + t^{-|a|+3} + \cdots + t^{|a|-3} + t^{|a|-1}). \tag{15}$$

The usual commutative algebra $\mathbb{C}[N^{\oplus}]$ also forms a Poisson algebra, with bracket defined by

$$\{z^{n_1}, z^{n_2}\} := B(n_1, n_2) z^{n_1+n_2}. \tag{16}$$

Note that there is a surjective homomorphism of Poisson algebras defined by

$$\pi_{t \mapsto 1} : \mathbb{C}_t[N^{\oplus}] \to \mathbb{C}[N^{\oplus}], \quad t \mapsto 1, \; z^n \mapsto z^n.$$

Remark 2.6 Note that $\mathbb{C}_t[N^{\oplus}]$ viewed as a Lie algebra with its Poisson bracket is isomorphic as a Lie algebra to $(t - t^{-1})^{-1} \cdot \mathbb{C}_t[N^{\oplus}]$ with its commutator bracket via the map

$$x \mapsto \frac{x}{t - t^{-1}}.$$

We may thus view $\pi_{t \mapsto 1}$ as a Lie algebra homomorphism $(t - t^{-1})^{-1} \cdot \mathbb{C}_t[N^{\oplus}] \to \mathbb{C}[N^{\oplus}]$. Similarly, as noted in [6, § 5.9], $H_{\mathrm{reg}}(Q, W)$ with the bracket from (9) is isomorphic as a Lie algebra to $(t-t^{-1})^{-1}\cdot H_{\mathrm{reg}}(Q,W)$ with its commutator bracket. In Sect. 2.6, we will discuss the "integration map" $\mathcal{I}=\pi_{t\mapsto 1}\circ\mathcal{I}_t$ as a homomorphism of Poisson algebras $H_{\mathrm{reg}}(Q,W)\to\mathbb{C}[N^{\oplus}]$, but this can also be viewed as a homomorphism of Lie algebras $(t-t^{-1})^{-1}\cdot H_{\mathrm{reg}}(Q,W)\to\mathbb{C}[N^{\oplus}]$. Similarly, we may view the quantum integration map $\mathcal{I}_t : H_{\mathrm{reg}}(Q,W)\to\mathbb{C}_t[N^{\oplus}]$ as a Lie algebra homomorphism $(t-t^{-1})^{-1}\cdot H_{\mathrm{reg}}(Q,W)\to(t-t^{-1})^{-1}\mathbb{C}_t[N^{\oplus}]$.

In place of the quantum torus algebra $\mathbb{C}_t[N^{\oplus}]$ considered above, one may use the quantum tropical vertex group of [34, § 6.1] or the quantum torus Lie algebra of [13, § 2.2.3]. These alternatives are nice because they still admit well-defined Poisson algebra maps $\pi_{t\mapsto 1}$ to $\mathbb{C}[N^{\oplus}]$, but now the Poisson bracket for the domain is simply the commutator bracket. While this is often convenient, we shall not use this viewpoint here.

2.6 The integration map

There are several constructions of **(quantum) integration maps** in the literature, i.e., homomorphisms (of algebras, Lie algebras, or Poisson algebras) from $H(Q,W)$ or $H_{\mathrm{reg}}(Q,W)$ to the (quantum) torus algebra. Reineke [47, Lem. 6.1] first constructed the analog of such a quantum integration map for finitary Hall algebras associated to quivers without potential. Joyce [28, § 6] then constructed classical and quantum integration maps with domain $H_{\mathrm{reg}}(Q,0)$. The classical version of Joyce's map (of Lie algebras) was generalized to quivers with potential in [29, § 7] (cf. [6, Thm. 11.1] for an interpretation as a map of Poisson algebras). On the other hand, a very general construction of algebra homomorphisms from a full Hall algebra to the "motivic quantum torus algebra" (which can then be further integrated to the usual quantum torus algebra) has been outlined by Kontsevich and Soibelman [32, § 6]. Making this more precise and more algebraic, in [34, § 7], Kontsevich and Soibelman defined a (monodromic) mixed Hodge structure (building off Saito's theory of mixed Hodge modules [49]) on the equivariant cohomology of the vanishing cycle complex, and then [14] and [11] built on these ideas to rigorously define a quantum integration map $\mathcal{I}_t$.

We give a brief sketch of this **integration map**

$$\mathcal{I}_t : H_{\mathrm{reg}}(Q,W)\to\mathbb{C}_t[N^{\oplus}]$$

essentially as in [11, § 3.3]. We then use this to compute the integration in the simplest cases. We note that by the definitions of the Poisson structures in (9) and (14), it is clear that $\mathcal{I}_t$ being a map of algebras implies it is also a map of Poisson algebras, thus also giving maps of Lie algebras as in Remark 2.6.

Recall that $\mathcal{M}$ is the moduli stack of objects in $\mathrm{rep}(Q,W):=\mathrm{mod}\mathbb{C}Q/I_W$. Let $\mathcal{M}^{\circ}$ be the moduli stack of objects in $\mathrm{rep}(Q,0)$. Given an arrow $a\in Q_1$, let $t(a),h(a)\in Q_0$ denote the tail and head of a respectively. For any $i\in Q_0$ and $d\in N^{\oplus}$, let d_i denote the corresponding component of d. Denote

$$\widetilde{\mathcal{M}}_d^\circ := \prod_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{t(a)}}, \mathbb{C}^{d_{h(a)}}),$$

and

$$\mathrm{GL}_d := \prod_{i\in Q_0} \mathrm{GL}_{d_i}(\mathbb{C}).$$

Then $\mathcal{M}^\circ = \bigsqcup_{d\in N^\oplus} \mathcal{M}_d^\circ$ where $\mathcal{M}_d^\circ$ is the stack-theoretic quotient

$$\mathcal{M}_d^\circ = \widetilde{\mathcal{M}}_d^\circ / \mathrm{GL}_d, \tag{17}$$

where the action by GL_d is the one induced by the conjugation action of $\mathrm{GL}_{d_i}(\mathbb{C})$ on $\mathbb{C}^{d_i}$ for each $i \in Q_0$.

Viewing elements of $\widetilde{\mathcal{M}}_d^\circ$ as modules over the path-algebra $\mathbb{C}Q$, we see that multiplication by W gives an endomorphism of $\widetilde{\mathcal{M}}_d^\circ$. Since the trace is invariant under the action of GL_d, we obtain a function

$$\mathrm{Tr}(W) : \mathcal{M}^\circ \to \mathbb{C},$$

the critical locus of which recovers $\mathcal{M}$:

$$\mathcal{M} = \mathrm{crit}(\mathrm{Tr}(W)) \subset \mathcal{M}^\circ.$$

Let Y be a smooth complex variety and let $f : Y \to \mathbb{C}$ be a regular function on Y. The corresponding vanishing cycle functor φ_f is defined as follows (following [11, § 3.1], also cf. [34, § 7.2]). Let $Y_0 := f^{-1}(0)$, and let $Y_{\leq 0} := f^{-1}(\mathbb{R}_{\leq 0})$. For a sheaf $\mathcal{F}$ on Y and U an analytic open subset of Y, define

$$\Gamma_{X_{\leq 0}}\mathcal{F}(U) := \ker\left(\mathcal{F}(U) \to \mathcal{F}(U\backslash(U \cap X_{\leq 0}))\right).$$

Then $\varphi_f := (R\Gamma_{X\leq 0}\mathcal{F})[1]|_{X_0}$.

The stacks $\mathcal{M}_d^\circ$ for $d \in N^\oplus$ are not quite smooth complex varieties, but each is a quotient of a smooth complex variety by the action of an algebraic group, cf. (17). One can thus extend the definition of φ_f to regular functions f on $\mathcal{M}_d^\circ$ using an equivariant version of the vanishing cycle construction as in [15, § 2.2].

Let $\underline{\mathbb{Q}}_d$ denote the constant sheaf on $\mathcal{M}_d^\circ$. For each $u \in \mathbb{C}^*$, we can define $\varphi_{\mathrm{Tr}(W)/u}\underline{\mathbb{Q}}_d$. Now consider $[X \to \mathcal{M}_d] \in H_{\mathrm{reg}}(Q, W)$. Composing with the inclusion $\mathcal{M}_d \subset \mathcal{M}_d^\circ$, we can consider the pullback

$$\varphi^X_{\mathrm{Tr}(W)/u}\underline{\mathbb{Q}}_d := (X \to \mathcal{M}_d^\circ)^*\varphi_{\mathrm{Tr}(W)/u}\underline{\mathbb{Q}}_d.$$

This sheaf $\varphi^X_{\mathrm{Tr}(W)/u}\underline{\mathbb{Q}}_d$ on X in fact has the structure of a mixed Hodge module on X, and so the compactly supported cohomology $H_c^*(X, \varphi^X_{\mathrm{Tr}(W)/u}\underline{\mathbb{Q}}_d)$ has a cohomologically graded rational mixed Hodge structure. Recall here that a rational mixed Hodge structure is a finite-dimensional vector space V over $\mathbb{Q}$, plus the data of an ascending filtration W_* of V (the weight filtration) and a descending filtration F^* of $V \otimes_{\mathbb{Q}} \mathbb{C}$ (the Hodge filtration) such that the filtration induced by F^* on

$$\mathrm{Gr}_n^W(V) := W_n \otimes_{\mathbb{Q}} \mathbb{C}/W_{n-1} \otimes_{\mathbb{Q}} \mathbb{C}$$

determines a pure Hodge structure of weight n. By a cohomologically graded rational mixed Hodge structure on $H_c^*(X, \varphi^X_{\mathrm{Tr}(W)/u}\underline{\mathbb{Q}}_d)$, we mean a rational mixed Hodge structure on $H_c^i(X, \varphi^X_{\mathrm{Tr}(W)/u}\underline{\mathbb{Q}}_d)$ for each $i \in \mathbb{Z}$.

Let us abbreviate $H_c^*(X, \varphi^X_{\mathrm{Tr}(W)/u}\underline{\mathbb{Q}}_d)$ as simply H_d^*. Up to isomorphism, the sheaf $\varphi^X_{\mathrm{Tr}(W)/u}\underline{\mathbb{Q}}_d$ is independent of u. However, there may be non-trivial monodromy μ on H_d^* as u travels around the origin in $\mathbb{C}$. This μ is quasi-unipotent, i.e., the eigenvalues are roots of unity. Let $\mathrm{Gr}_n^W(H_d^i)_1$ denote the generalized eigenspace for the possible eigenvalue 1 of μ, and let $\mathrm{Gr}_n^W(H_d^i)_{\neq 1}$ denote the direct sum of the generalized eigenspaces for all eigenvalues of μ other than 1. Finally, the quantum integration map $\mathcal{I}_t$ is defined by taking the Serre polynomial (cf. [12, § 3.1.3], also [32, p. 69]) defined as follows:

$$\mathcal{I}_t([X \to \mathcal{M}_d]) := t^{\chi(d,d)} z^d \sum_{i,n\in\mathbb{Z}} (-1)^i \\ \left(\dim(\mathrm{Gr}_n^W(H_d^i)_1)(-t)^n + \dim(\mathrm{Gr}_n^W(H_d^i)_{\neq 1})(-t)^{n+1}\right), \tag{18}$$

where χ is defined as in (3).

The map of (18) above is essentially the same as that of [11, (18)], although the two look somewhat different. The $t^{\chi(d,d)}$-factor in our (18) is simply to account for the twisting of the monoidal structure in [11, (16)]. The extra factor of $(-t)$ on the $\dim(\mathrm{Gr}_n^W(H_d^i)_{\neq 1})(-t)^{n+1}$-term in our (18) is needed because [11] actually works with the category of monodromic mixed Hodge modules, a difference which results in a shift for part of the weight filtration. See [10, Prop. 2.5] for details on the relationship between these two perspectives. [11, Prop. 3.13] thus yields the following:

Proposition 2.7 ([11], Prop. 3.13) $\mathcal{I}_t : H_{\mathrm{reg}}(Q, W) \to \mathbb{C}_t[N^{\oplus}]$ *is a homomorphism of* $\mathbb{C}_{\mathrm{reg}}(t)$*-algebras.*

This construction simplifies quite a bit for $[f : X \to \mathcal{M}_d]$ with $\mathrm{Tr}(W)|_{\mathcal{M}_d^\circ} = 0$ and X a smooth projective variety. In this case, $\varphi_{\mathrm{Tr}(W)} = \mathrm{Id}$, and so $\mathrm{Gr}_n^W(H_d^i)$ equals $H_c^n(X, \underline{\mathbb{Q}})$ if $n = i$ and vanishes otherwise. Recalling the definition of Υ from (5), we thus recover the following:

Proposition 2.8 *If* $[f : X \to \mathcal{M}_d \subset \mathcal{M}] \in H_{\mathrm{reg}}(Q, W)$ *and* $\mathrm{Tr}(W)|_{\mathcal{M}_d^\circ} = 0$*, then*

$$\mathcal{I}_t([f : X \to \mathcal{M}]) = \Upsilon(X) t^{\chi(d,d)} z^d. \tag{19}$$

In particular, if $W = 0$ (e.g., for Q acyclic), $\mathcal{I}_t$ equals the quantum integration map of [28, § 6].

Example 2.9 Recall $\kappa_{ki} := [\mathrm{Spec}\,\mathbb{C} \to p_{S_i^{\oplus k}} = \mathcal{M}_{ke_i}]$. We have $W|_{\mathcal{M}^\circ_{ke_i}} = 0$, $\Upsilon(\kappa_{ki}) = 1$ (the Poincaré polynomial of a point), and $\chi(ke_i, ke_i) = k^2$. Hence, $\mathcal{I}_t(\kappa_{ki}) = t^{k^2} z^{ke_i}$. In particular,

$$\mathcal{I}_t(\kappa_i) = t z^{e_i}. \tag{20}$$

As a check, one can use (12) to confirm that $\mathcal{I}_t(\kappa_i^k) = \mathcal{I}_t(\kappa_i)^k$.

Composing $\mathcal{I}_t$ with $\pi_{t\mapsto 1}$ induces the classical integration map:

$$\mathcal{I} := \pi_{t\mapsto 1} \circ \mathcal{I}_t : H_{\mathrm{reg}}(Q, W) \to \mathbb{C}[N^{\oplus}].$$

The classical integration maps of [29, § 7] and [6, Thm. 11.1] are always (even for nonzero W) given by the $t \mapsto 1$ limit of (19), i.e., by taking Euler characteristics. Note that (20) is sufficient to completely determine the restrictions of $\mathcal{I}_t$ and $\mathcal{I}$ to the composition algebra $\mathcal{C}(Q, W)$ in which all our computations will lie. Since $\mathcal{I}$ agrees with the classical integration maps of [29, § 7] and [6, Thm. 11.1] on the generators κ_i, the maps necessarily agree on all of $\mathcal{C}(Q, W)$.

3 Scattering diagrams from Hall algebras

3.1 Background on scattering diagrams

Here we review the basic definitions and properties of scattering diagrams from the perspective useful for understanding the Hall algebra scattering diagrams of [6].

Let Λ denote a finite-rank lattice equipped with a $\mathbb{Z}$-valued skew-symmetric form $\{\cdot,\cdot\}$. Let $\Lambda^\vee := \mathrm{Hom}(\Lambda, \mathbb{Z})$ be the dual lattice, and let $\langle\cdot,\cdot\rangle : \Lambda \oplus \Lambda^\vee \to \mathbb{Z}$ denote the dual pairing. We have a map

$$\begin{aligned} p^* : \Lambda &\to \Lambda^\vee \\ n &\mapsto \{\cdot, n\}. \end{aligned} \tag{21}$$

Fix a strictly convex rational polyhedral cone $\sigma_{\Lambda^\oplus} \subset \Lambda_{\mathbb{R}}$. Let $\Lambda^\oplus := \sigma_{\Lambda^\oplus} \cap \Lambda$, and let $\Lambda^+ := \Lambda^\oplus \setminus \{0\}$.

Let $\mathfrak{g} := \bigoplus_{n\in\Lambda^+} \mathfrak{g}_n$ be a Lie algebra graded by Λ^+, meaning that $[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] \subseteq \mathfrak{g}_{n_1+n_2}$. We say that $\mathfrak{g}$ is **skew-symmetric** with respect to $\{\cdot,\cdot\}$ if

$$[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] = 0 \text{ whenever } \{n_1, n_2\} = 0. \tag{22}$$

For each $k \in \mathbb{Z}_{\geq 1}$, let

$$k\Lambda^+ := \{n_1 + \cdots + n_k \in \Lambda^+ | n_i \in \Lambda^+ \text{ for each } i = 1, \ldots, k\}.$$

Let $\mathfrak{g}^{\geq k} := \bigoplus_{n\in k\Lambda^+}\mathfrak{g}_n$. Note that $\mathfrak{g}^{\geq k}$ is a Lie subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}_k$ denote the nilpotent Lie algebra $\mathfrak{g}/\mathfrak{g}^{\geq k}$, and let $\widehat{\mathfrak{g}} := \varprojlim \mathfrak{g}_k$. We have corresponding Lie groups $G := \exp\mathfrak{g}$, $G_k := \exp\mathfrak{g}_k$, and $\widehat{G} := \exp\widehat{\mathfrak{g}} = \varprojlim G_k$.

For each $n \in \Lambda^+$, we have a Lie subalgebra $\mathfrak{g}_n^{\parallel} := \prod_{k\in\mathbb{Z}_{\geq 1}} \mathfrak{g}_{kn} \subset \widehat{\mathfrak{g}}$. We say that $\mathfrak{g}$ has **Abelian walls** if each $\mathfrak{g}_n^{\parallel}$ is Abelian. In particular, $\mathfrak{g}$ has Abelian walls whenever $\mathfrak{g}$ is skew-symmetric. Let $G_n^{\parallel} := \exp(\mathfrak{g}_n^{\parallel}) \subset \widehat{G}$.

The Abelian walls condition is usually assumed to hold when working with scattering diagrams, but when defining Hall algebra scattering diagrams, one needs a slight generalization as in [6, § 2].

Definition 3.1 A **wall** in $\Lambda_{\mathbb{R}}^{\vee}$ over $\widehat{\mathfrak{g}}$ is data of the form $(\mathfrak{d}, g_{\mathfrak{d}})$, where:

- $g_{\mathfrak{d}} \in \mathfrak{g}_{n_{\mathfrak{d}}}^{\parallel}$ for some primitive $n_{\mathfrak{d}} \in \Lambda^+$. The element $-p^*(n_{\mathfrak{d}})$ is called the **direction** of the wall. We call $g_{\mathfrak{d}}$ the **scattering function** associated to the wall.
- $\mathfrak{d}$ is a closed, convex (but not necessarily strictly convex), rational-polyhedral, codimension-one affine cone in $\Lambda_{\mathbb{R}}^{\vee}$, parallel to $n_{\mathfrak{d}}^{\perp}$. We call $\mathfrak{d}$ the **support** of the wall.

A **scattering diagram** $\mathfrak{D}$ over $\widehat{\mathfrak{g}}$ is a set of walls in $\Lambda_{\mathbb{R}}^{\vee}$ over $\widehat{\mathfrak{g}}$ such that for each $k > 0$, there are only finitely many $(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}$ with $g_{\mathfrak{d}}$ not projecting to 0 in $\mathfrak{g}_k$. If $(\mathfrak{d}_1, g_{\mathfrak{d}_1})$ and $(\mathfrak{d}_2, \mathfrak{g}_{\mathfrak{d}_2})$ are two walls of $\mathfrak{D}$, and if $\operatorname{codim}_{\Lambda_{\mathbb{R}}^{\vee}}(\mathfrak{d}_1 \cap \mathfrak{d}_2) = 1$, then we require that $[g_{\mathfrak{d}_1}, g_{\mathfrak{d}_2}] = 0$ (note that this is automatic for Abelian walls).

A wall with direction $-v$ is called **incoming** if it contains v. Otherwise, the wall is called **outgoing**.

We will sometimes denote a wall $(\mathfrak{d}, g_{\mathfrak{d}})$ by just $\mathfrak{d}$. Denote $\mathrm{Supp}(\mathfrak{D}) := \bigcup_{\mathfrak{d}\in\mathfrak{D}} \mathfrak{d}$, and

$$\mathrm{Joints}(\mathfrak{D}) := \bigcup_{\mathfrak{d}\in\mathfrak{D}} \partial\mathfrak{d} \cup \bigcup_{\substack{\mathfrak{d}_1,\mathfrak{d}_2\in\mathfrak{D} \\ \dim(\mathfrak{d}_1\cap\mathfrak{d}_2)=\mathrm{rank}(\Lambda)-2}} \mathfrak{d}_1 \cap \mathfrak{d}_2.$$

Note that for each $k > 0$, a scattering diagram $\mathfrak{D}$ over $\widehat{\mathfrak{g}}$ induces a finite scattering diagram $\mathfrak{D}^k$ over $\mathfrak{g}_k$ with walls corresponding to the $\mathfrak{d} \in \mathfrak{D}$ for which the projection of $g_{\mathfrak{d}}$ to $\mathfrak{g}_k$ is nonzero.

Consider a smooth immersion $\gamma : [0, 1] \to \Lambda_{\mathbb{R}}^{\vee} \setminus \mathrm{Joints}(\mathfrak{D})$ with endpoints not in $\mathrm{Supp}(\mathfrak{D})$ which is transverse to each wall of $\mathfrak{D}$ it crosses. Let $(\mathfrak{d}_i, g_{\mathfrak{d}_i})$, $i = 1, \ldots, s$, denote the walls of $\mathfrak{D}^k$ crossed by γ, and say they are crossed at times $0 < t_1 \leq \cdots \leq t_s < 1$, respectively.[3] Define

$$\Phi_{\mathfrak{d}_i} := \exp(g_{\mathfrak{d}_i})^{\mathrm{sgn}\langle n_{\mathfrak{d}_i}, -\gamma'(t_i)\rangle} \in G_k. \tag{23}$$

[3] If $t_i = t_{i+1}$, then the corresponding elements $g_{\mathfrak{d}_i}, g_{\mathfrak{d}_{i+1}} \in \mathfrak{g}_n^{\parallel}$ must commute, and so the ordering of the corresponding walls does not affect $\Phi_{\gamma,\mathfrak{D}}^k$.

Let $\Phi^k_{\gamma,\mathfrak{D}} := \Phi_{\mathfrak{d}_s} \cdots \Phi_{\mathfrak{d}_1} \in G_k$, and define the **path-ordered product**:

$$\Phi_{\gamma,\mathfrak{D}} := \varprojlim_k \Phi^k_{\gamma,\mathfrak{D}} \in \widehat{G}.$$

Definition 3.2 Two scattering diagrams $\mathfrak{D}$ and $\mathfrak{D}'$ are **equivalent** if $\Phi_{\gamma,\mathfrak{D}} = \Phi_{\gamma,\mathfrak{D}'}$ for each smooth immersion γ as above. $\mathfrak{D}$ is **consistent** if each $\Phi_{\gamma,\mathfrak{D}}$ depends only on the endpoints of γ.

We say that $x \in \Lambda^\vee_{\mathbb{R}}$ is **general** if it is contained in at most one hyperplane of the form $n^\perp$ for $n \in \Lambda$. For $\mathfrak{D}$ a scattering diagram over $\widehat{\mathfrak{g}}$ and $x \in \Lambda^\vee_{\mathbb{R}}$ general, denote

$$g_{x,\mathfrak{D}} := \sum_{\mathfrak{d} \ni x} g_{\mathfrak{d}} \in \widehat{\mathfrak{g}},$$

where the sum is over all walls $(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}$ with $\mathfrak{d} \ni x$. One easily sees the following standard fact (cf. [22, Lem. 1.9]):

Lemma 3.3 *Two scattering diagrams $\mathfrak{D}$ and $\mathfrak{D}'$ over $\widehat{\mathfrak{g}}$ are equivalent if and only if $g_{x,\mathfrak{D}} = g_{x,\mathfrak{D}'}$ for all general $x \in \Lambda^\vee_{\mathbb{R}}$.*

Example 3.4 (1) For $\mathfrak{D}$ a scattering diagram, consider a set of walls $\{(\mathfrak{d}, g_i) \in \mathfrak{g}^{\parallel}_{n_{\mathfrak{d}}}) \in \mathfrak{D} | i \in S\}$, where S is some countable index set and $n_{\mathfrak{d}}$ and $\mathfrak{d}$ are independent of i. Then replacing this set of walls with a single wall $(\mathfrak{d}, \sum_{i \in S} g_i)$ produces an equivalent scattering diagram.
(2) Replacing a wall $(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}$ with a pair of walls $(\mathfrak{d}_i, g_{\mathfrak{d}})$, $i = 1, 2$, such that $\mathfrak{d}_1 \cup \mathfrak{d}_2 = \mathfrak{d}$ and $\operatorname{codim}_{\Lambda^\vee_{\mathbb{R}}}(\mathfrak{d}_1 \cap \mathfrak{d}_2) = 2$ produces an equivalent scattering diagram.

The following theorem is fundamental to the study of scattering diagrams. The 2-dimensional version was first proved in [33], and this was generalized to higher dimensions in [26, § 3] for scattering diagrams over the module of log derivations. The higher-dimensional version for scattering diagrams over skew-symmetric Lie algebras follows from [35, Prop. 3.2.6, 3.3.2] (cf. [22, Thm. 1.21] for a review of this argument from our viewpoint). As pointed out to us by Lang Mou, this result had not previously been proven in the presence of non-Abelian walls.

Theorem 3.5 *Let $\mathfrak{g}$ be a Λ^+-graded Lie algebra, and let $\mathfrak{D}_{\text{in}}$ be a finite scattering diagram over $\widehat{\mathfrak{g}}$ whose walls are of the form $(n_i^\perp, g_i)$ for various primitive $n_i \in N^+$. If $\mathfrak{g}$ has Abelian walls, then there is a unique-up-to-equivalence scattering diagram $\mathfrak{D}$ such that $\mathfrak{D}$ is consistent, $\mathfrak{D} \supset \mathfrak{D}_{\text{in}}$, and $\mathfrak{D} \backslash \mathfrak{D}_{\text{in}}$ consists only of outgoing walls. Even if $\mathfrak{g}$ does not have Abelian walls, if there exists a consistent scattering diagram $\mathfrak{D} \supset \mathfrak{D}_{\text{in}}$ such that $\mathfrak{D} \backslash \mathfrak{D}_{\text{in}}$ consists only of outgoing walls as above, then this $\mathfrak{D}$ is the unique such scattering diagram, up to equivalence.*

We note that an earlier version of this paper claimed existence more generally, but we have since realized that proving the consistency of the scattering diagram $\mathfrak{D}^\infty_k$ in Sect. 4.4 requires the Abelian walls condition, and so our argument was flawed. Fortunately, the existence of the Hall algebra scattering diagram is already given by [6, Theorem 6.5], restated below as Theorem 3.8.

Proof As noted above, the only new statement is the uniqueness statement in the case of non-Abelian walls. We prove this using an argument inspired by that [22, Lem. C.7]. Let $\mathfrak{D}, \mathfrak{D}'$ be two consistent scattering diagrams over $\widehat{\mathfrak{g}}$ with incoming walls $\mathfrak{D}_{\mathrm{in}}$ as in the statement of the theorem. We shall prove by induction on k that $\mathfrak{D}^k$ and $(\mathfrak{D}')^k$ are equivalent over $\mathfrak{g}_k$ for each k, and then the equivalence of $\mathfrak{D}$ and $\mathfrak{D}'$ follows. Note that $\mathfrak{D}^1$ and $(\mathfrak{D}')^1$ are both equivalent to the trivial scattering diagram, hence to each other.

Now suppose that $\mathfrak{D}^k$ and $(\mathfrak{D}')^k$ are equivalent over $\mathfrak{g}_k$. Let $\mathfrak{D}''$ be a scattering diagram over $\mathfrak{g}_{k+1}$ such that

$$g_{x,\mathfrak{D}''} = g_{x,\mathfrak{D}^{k+1}} - g_{x,(\mathfrak{D}')^{k+1}}$$

for each general $x \in \Lambda_{\mathbb{R}}^{\vee}$. Since $\mathfrak{D}^k$ and $(\mathfrak{D}')^k$ are equivalent over $\mathfrak{g}_k$, we must have $g_{x,\mathfrak{D}''} \in \mathfrak{g}^{\geq k} \backslash \mathfrak{g}^{\geq k-1}$, hence $g_{x,\mathfrak{D}''}$ is central in $\mathfrak{g}_{k+1}$. Hence, $(\mathfrak{D}')^k \cup \mathfrak{D}''$ is a well-defined scattering diagram over $\mathfrak{g}_{k+1}$, and by Lemma 3.3 it is equivalent to $\mathfrak{D}^{k+1}$. Our goal now is to show that $\mathfrak{D}''$ is equivalent to the trivial scattering diagram.

Since both $\mathfrak{D}$ and $\mathfrak{D}'$ were assumed to be consistent, and the scattering functions of $\mathfrak{D}''$ are all central in $\mathfrak{g}_{k+1}$, $\mathfrak{D}''$ must also be consistent (over $\mathfrak{g}_{k+1}$). Furthermore, this consistency plus centrality of the scattering functions implies that, up to equivalence, the support of every wall of $\mathfrak{D}''$ is an entire affine hyperplane in $\Lambda_{\mathbb{R}}^{\vee}$. But then all walls of $\mathfrak{D}''$ (up to equivalence) are incoming, and since the incoming walls of $\mathfrak{D}$ and $\mathfrak{D}'$ are the same, this implies that $\mathfrak{D}''$ is equivalent to the trivial scattering diagram over $\mathfrak{g}_{k+1}$, as desired. □

A scattering diagram playing the role of $\mathfrak{D}_{\mathrm{in}}$ in Theorem 3.5 will be referred to as an **initial scattering diagram**. The consistent scattering diagram $\mathfrak{D}$ (up to equivalence) with incoming walls $\mathfrak{D}_{\mathrm{in}}$ as in the theorem will be denoted $\mathrm{Scat}(\mathfrak{D}_{\mathrm{in}})$.

Example 3.6 Consider $\Lambda = \mathbb{Z}^2$. Equip Λ with the skew-symmetric form $\{\cdot,\cdot\}$ represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and consider the quantum torus algebra $\mathbb{C}_t[\Lambda]$ as in Sect. 2.5. Take $\mathfrak{g}$ to be the Lie subalgebra (with respect to Poisson bracket) with basis $\{z^n : n \in \Lambda^+\}$. Let

$$\mathfrak{D}_{\mathrm{in}} := \{(e_1^{\perp}, -\mathrm{Li}(-z^{e_1};t)), (e_2^{\perp}, -\mathrm{Li}(-z^{e_2};t))\},$$

where $\mathrm{Li}(x,t)$ denotes the quantum dilogarithm as in (25) below. Then $\mathfrak{D} := \mathrm{Scat}(\mathfrak{D}_{\mathrm{in}})$ is obtained by adding a single outgoing wall $(\mathbb{R}_{\geq 0}(1,-1), -\mathrm{Li}(-z^{(1,1)};t))$, cf. Fig. 1.

The consistency of this scattering diagram is equivalent to a version of the quantum pentagon identity of [18]. The classical limit is essentially the $\ell_1 = \ell_2 = 1$ case of [25, Ex. 1.6] (with some small changes in sign conventions). We will see in Example 3.10 that this is the scattering diagram obtained when applying the quantum integration map to the Hall algebra scattering diagram associated to the A_2-quiver.

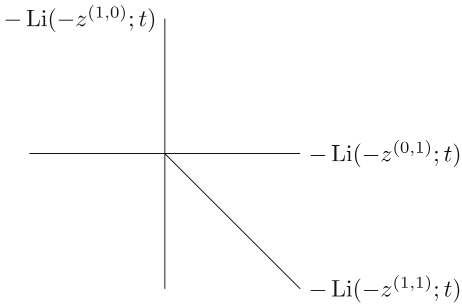

Fig. 1 The quantum A_2 scattering diagram

3.2 Hall algebra scattering diagrams

3.2.1 Setup for Hall algebra scattering diagrams and their variants

We now take $\Lambda = N$, $\Lambda^{\oplus} = N^{\oplus}$, and $\{\cdot,\cdot\} = B$. Recall that $H(Q,W)$ admits a grading by $N^{\oplus}$ as in (7). In particular, we can write $H_{\mathrm{reg}}(Q,W) = H_{\mathrm{reg}}(Q,W)_0 \oplus H_{\mathrm{reg}}(Q,W)_{>0}$ for $H_{\mathrm{reg}}(Q,W)_{>0} := \bigoplus_{d\in N^+} H_{\mathrm{reg}}(Q,W)_d$. Let $\mathfrak{g}^{\mathrm{Hall}} := (t-t^{-1})^{-1}\cdot H_{\mathrm{reg}}(Q,W)_{>0}$, viewed as a Lie algebra using the commutator bracket as in Remark 2.6.

The Lie algebra $\mathfrak{g}^{\mathrm{Hall}}$ typically is not skew-symmetric and does not have Abelian walls. To get around this issue, let $\mathfrak{i}^{\mathrm{skew}}$ denote the Lie ideal of $\mathfrak{g}^{\mathrm{Hall}}$ generated by the commutators we wish to vanish, i.e.,

$$\mathfrak{i}^{\mathrm{skew}} := \left\langle [\mathfrak{g}^{\mathrm{Hall}}_{d_1}, \mathfrak{g}^{\mathrm{Hall}}_{d_2}] : d_1, d_2 \in N^+, \{d_1, d_2\} = 0 \right\rangle.$$

Here, for S a subset of $\mathfrak{g}_{\mathrm{reg}}$, $\langle S\rangle$ denotes the Lie ideal generated by S, i.e., the intersection of all Lie ideals of $\mathfrak{g}^{\mathrm{Hall}}$ which contain S. Then for any Lie ideal $\mathfrak{i}$ which contains $\mathfrak{i}^{\mathrm{skew}}$, we define

$$\mathfrak{g}^{\mathfrak{i}} := \mathfrak{g}^{\mathrm{Hall}}/\mathfrak{i}.$$

Note that for any Lie algebra ideal $\mathfrak{i}$ of $\mathfrak{g}^{\mathrm{Hall}}$, $\mathfrak{g}^{\mathrm{Hall}}/\mathfrak{i}$ is skew-symmetric if and only if $\mathfrak{i} \supset \mathfrak{i}^{\mathrm{skew}}$. Since the commutator bracket on the quantum torus algebra makes it into a skew-symmetric Lie algebra, we in particular have

$$\ker(\mathcal{I}_t) \supset \mathfrak{i}^{\mathrm{skew}}.$$

The resulting Lie algebra $\mathfrak{g}^q := \mathfrak{g}^{\ker(\mathcal{I}_t)}$ is just the quantum torus algebra $(t-t^{-1})^{-1}\cdot \mathbb{C}_t[N^{\oplus}]$ with its commutator bracket as in (14). Similarly, $\ker(\mathcal{I}) \supset \mathfrak{i}^{\mathrm{skew}}$, and $\mathfrak{g}^{\mathrm{cl}} := \mathfrak{g}^{\ker(\mathcal{I})}$ is just $\mathbb{C}[N^{\oplus}]$ together with its Poisson bracket as in (16). In general, let $\mathcal{I}^{\mathfrak{i}} : \mathfrak{g}^{\mathrm{Hall}} \to \mathfrak{g}^{\mathfrak{i}}$ denote the projection.

For $\mathfrak{g}$ equal to $\mathfrak{g}^{\mathrm{Hall}}$, $\mathfrak{g}^{\mathfrak{i}}$, $\mathfrak{g}^q$, or $\mathfrak{g}^{\mathrm{cl}}$, we denote the corresponding Lie group G by G^{Hall}, $G^{\mathfrak{i}}$, G^q, or G^{cl}, respectively. The notation for the associated completions and scattering diagrams will be similarly obvious except for sometimes using “Hall” instead of “reg.”[4]

3.2.2 The Hall algebra scattering diagram

Definition 3.7 Given $\theta \in M_{\mathbb{R}}$, an object $E \in \mathrm{rep}(Q, W)$ is said to be θ**-semistable** if

- $\theta(E) = 0$,
- Every subobject $B \subset E$ satisfies $\theta(B) \leq 0$. If, furthermore, this inequality is strict, then we say that E is θ-stable.

The notion of semistability given above is due to [31]. Let $\mathcal{M}_{\mathrm{ss}}(\theta) \subset \mathcal{M}$ denote the substack of $\mathcal{M}$ representing the θ-semistable objects, and let $1_{\mathrm{ss}}(\theta) := [\mathcal{M}_{\mathrm{ss}}(\theta) \subset \mathcal{M}] \in \widehat{G}_{\mathrm{Hall}}$.

The scattering diagram defined in the following theorem of Bridgeland is what one calls the **Hall algebra scattering diagram**.

Theorem 3.8 [6, Theorem 6.5] *There exists a consistent scattering diagram $\mathfrak{D}^{\mathrm{Hall}}$ in $M_{\mathbb{R}}$ over $\mathfrak{g}^{\mathrm{Hall}}$ such that:*

(1) *The support* $\mathrm{Supp}(\mathfrak{D}^{\mathrm{Hall}})$ *consists of those $\theta \in M_{\mathbb{R}}$ for which there exist θ-semistable objects in* $\mathrm{rep}(Q, I)$*;*
(2) *For $\theta \subset \mathrm{Supp}(\mathfrak{D}^{\mathrm{Hall}}) \setminus \mathrm{Joints}(\mathfrak{D}^{\mathrm{Hall}})$, there is a unique wall $(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}^{\mathrm{Hall}}$ for which $\mathfrak{d} \ni \theta$. For this wall, we have* $\exp(g_{\mathfrak{d}}) = 1_{\mathrm{ss}}(\theta) \in \hat{G}_{\mathrm{Hall}}$.

Remark 3.9 We say $\theta \in M_{\mathbb{R}}$ is **general** if it is not in the intersection of two distinct hyperplanes of the form $n^{\perp}$ for $n \in N \setminus \{0\}$. Since the joints of $\mathfrak{D}^{\mathrm{Hall}}$ are codimension 2 subsets of $M_{\mathbb{R}}$ and have rational slope, Theorem 3.8 gives the scattering functions of $\mathfrak{D}^{\mathrm{Hall}}$ at all general points $\theta \in M_{\mathbb{R}}$. Alternatively, we could use a more refined notion of general. Call $\theta \in M_{\mathbb{R}}$ **special** if at least one of the following holds:

- There exists a pair of θ-semistable objects with non-parallel dimension vectors;
- Some $E \in \mathrm{rep}(Q, W)$ is θ-semistable, but for $0 < \epsilon \ll 1$, E is either not $(\theta + \epsilon p^*(\dim(E)))$-semistable or not $(\theta - \epsilon p^*(\dim(E)))$-semistable.

The former condition accounts for joints where two walls of different slopes intersect, while the latter accounts for intersections of walls with the same slope. That is, $\theta \in \mathrm{Joints}(\mathfrak{D}^{\mathrm{Hall}})$ if and only if θ is special. Theorem 1.1 will still hold and will be slightly stronger if we define general to mean not special.

Note that we obtain new scattering diagrams $\mathfrak{D}^{\mathrm{i}}$, $\mathfrak{D}^{q}$, and $\mathfrak{D}^{\mathrm{cl}}$ over $\mathfrak{g}^{\mathrm{i}}$, $\mathfrak{g}^{q}$, and $\mathfrak{g}^{\mathrm{cl}}$, respectively, by applying $\mathcal{I}^{\mathrm{i}}$, $\mathcal{I}^{q}$, or $\mathcal{I}^{\mathrm{cl}}$ to $\mathfrak{D}^{\mathrm{Hall}}$. The scattering diagram $\mathfrak{D}^{\mathrm{cl}}$ is what Bridgeland calls the **stability scattering diagram**. We call $\mathfrak{D}^{q}$ the **quantum stability scattering diagram**.

[4] Note that we could define the Hall algebra scattering diagram using the full Hall algebra $H(Q, W)$ in place of $H_{\mathrm{reg}}(Q, W)$ (as is done in [6]), or alternatively using just the composition algebra $\mathcal{C}(Q, W)$. The advantage of using $H_{\mathrm{reg}}(Q, W)$ or $\mathcal{C}(Q, W)$ instead of $H(Q, W)$ is just for convenience when we talk about applying integration maps.

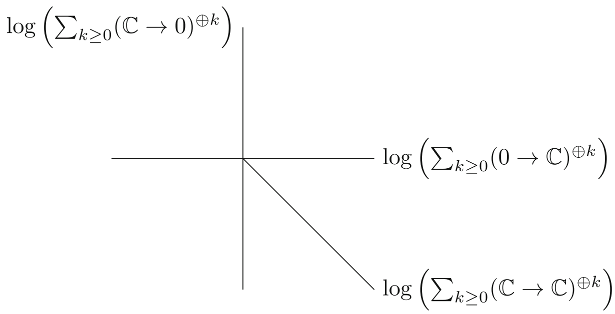

Fig. 2 The A_2 Hall algebra scattering diagram

Example 3.10 Let us consider the A_2 quiver $1 \to 2$ with $W = 0$. The corresponding matrix B is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as in Example 3.6. Let us explicitly describe the Hall algebra scattering diagram $\mathfrak{D}^{\mathrm{Hall}}$ from Theorem 3.8 in this case. Note that there are 3 indecomposable representations of A_2 up to isomorphism: $\mathbb{C} \to 0$, $0 \to \mathbb{C}$, and $\mathbb{C} \to \mathbb{C}$ (the last map being nonzero). Consider $\mathfrak{d} = (1,0)^{\perp}$. For any point $\theta \in \mathfrak{d}$, one can see that the representations $(\mathbb{C} \to 0)^{\oplus k}$ are θ-semistable for any positive integer k, and we find $1_{\mathrm{ss}}(\theta) = \sum_{k\geq 0}(\mathbb{C} \to 0)^{\oplus k}$. We similarly compute that for $\theta \in (0,1)^{\perp}$, $1_{\mathrm{ss}}(\theta) = \sum_{k\geq 0}(0 \to \mathbb{C})^{\oplus k}$, and for $\theta \in \mathbb{R}_{\geq 0}(1,-1)$, we have $1_{\mathrm{ss}}(\theta) = \sum_{k\geq 0}(\mathbb{C} \to \mathbb{C})^{\oplus k}$. Note that $(\mathbb{C} \to \mathbb{C})$ contains $(0 \to \mathbb{C})$ as a subrepresentation, and so $(\mathbb{C} \to \mathbb{C})^{\oplus k}$ is not $(-\alpha,\alpha)$-semistable for $\alpha \in \mathbb{R}_{>0}$. There are no other θ-semistable representations for any θ in this example, so the Hall algebra scattering diagram is as in Fig. 2. Note that $\mathfrak{D}^q$, obtained from applying the quantum integration map $\mathcal{I}$ to the scattering functions of $\mathfrak{D}^{\mathrm{Hall}}$ (cf. Sect. 3.3 for such computations) yields the consistent scattering diagram of Example 3.6.

3.2.3 Genteel potentials

We say that a quiver with potential (Q, W) is **genteel** (or that W is genteel) if the only incoming walls of $\mathfrak{D}^{\mathrm{Hall}}$ are

$$\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{in}} := \{e_i^{\perp}, \log 1_{\mathrm{ss}}(p^*(e_i))\}. \tag{24}$$

Theorems 3.5 and 3.8 together imply the following:

Lemma 3.11 *If (Q, W) is genteel, then $\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{Scat}} := \mathrm{Scat}(\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{in}})$ exists and equals $\mathfrak{D}^{\mathrm{Hall}}$ (up to equivalence).*

It is expected (cf. [35, Conj. 3.3.4]) that for every 2-acyclic quiver Q, a generic potential W will be genteel (at least over $\mathrm{ad}(\mathfrak{g}^q)$ in the sense explained below). The following is proved in [13, § 7.1]:

Lemma 3.12 *If Q is acyclic—or more generally, if the only cycles in Q are composed of loops (i.e., 1-cycles)—then $(Q, 0)$ is genteel.*

Remark 3.13 On its face, Lemma 3.12 is, in the cases without loops, the same as [6, Lem. 11.5] (and the proof in [13] is inspired by that in [6]). However, [6, § 11.5] uses a slightly different and possibly flawed definition of genteel. In [6, Def. 11.3], an object $E \in \mathrm{rep}(Q, W)$ is called self-stable if it is stable with respect to the stability condition $-p^*(\dim(E))$. Then (Q, W) is called genteel if the only self-stable objects are the simple objects S_i for $i \in Q_0$. Unfortunately, as pointed out to us by Lang Mou and acknowledged in [6, arXiv v4], it is not clear that this version of genteel really does imply the claim about incoming walls being as in (24). For this one would need to replace "self-stable" with "self-semistable," but doing so results in other problems, e.g., acyclic examples which would fail to be genteel. We have therefore taken the motivating property regarding incoming walls as our definition.

A potentially weaker (but for most purposes equally useful) version of genteel is as follows: we say that (Q, W) is genteel over $\mathfrak{g}^{\mathfrak{i}}$ if, up to equivalence, the only incoming walls of $\mathfrak{D}^{\mathfrak{i}}$ are

$$\mathfrak{D}^{\mathfrak{i}}_{\mathrm{in}} := \{e_i^{\perp}, \mathcal{I}^{\mathfrak{i}}(\log 1_{\mathrm{ss}}(p^*(e_i)))\}.$$

In general (even without genteelness), Theorem 3.5 guarantees the existence of

$$\mathfrak{D}^{\mathfrak{i}}_{\mathrm{Scat}} := \mathrm{Scat}(\mathfrak{D}^{\mathfrak{i}}_{\mathrm{in}}).$$

As with Lemma 3.11, W being genteel over $\mathfrak{g}^{\mathfrak{i}}$ means that $\mathfrak{D}^{\mathfrak{i}}_{\mathrm{Scat}} = \mathfrak{D}^{\mathfrak{i}}$. We note that genteel implies genteel over every $\mathfrak{g}^{\mathfrak{i}}$, and genteel over $\mathfrak{g}^{\mathfrak{i}}$ implies genteel over $\mathfrak{g}^{\mathfrak{i}'}$ for every $\mathfrak{i}' \supset \mathfrak{i}$.

Proposition 3.14 ([40], Cor. 1.2(i)) *Let (Q, W) be a quiver with potential (and no loops) such that W is non-degenerate and Q admits a green-to-red sequence.*[5] *Then (Q, W) is genteel over $\mathfrak{g}^q$ and $\mathfrak{g}^{\mathrm{cl}}$.*

We note that a version of Proposition 3.14 over $\mathrm{ad}(\mathfrak{g}^{\mathrm{cl}})$ (i.e., the quotient of $\mathfrak{g}^{\mathrm{cl}}$ by its center) was also proved in [46, Thm. 1.2.2]. Also, [40, Cor. 1.2(ii)] proves that non-degenerate potentials for the Markov quiver (which does not admit a green-to-red sequence) are genteel over $\mathrm{ad}(\mathfrak{g}^q)$ and $\mathrm{ad}(\mathfrak{g}^{\mathrm{cl}})$.

Example 3.15 For the A_2-quiver of Example 3.10, the simple representations are $\mathbb{C} \to 0$ and $0 \to \mathbb{C}$. Thus $\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{in}}$ would be as in Fig. 3. By Lemmas 3.11 and 3.12, we have that $\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{Scat}}$ exists and agrees with $\mathfrak{D}^{\mathrm{Hall}}$ from Fig. 2.

[5] See [16, Def. 7.2] for the definition of a non-degenerate potential, and see [43, Def. 3.1.1] for the definition of a quiver admitting a green-to-red sequence (or see [22, Def. 8.27] for the equivalent notion of a quiver with a "large cluster complex"). It is known (at least when allowing infinite potentials) that all quivers Q without 2-cycles admit non-degenerate potentials, cf. [16, Cor. 7.4]. In particular, $(Q, 0)$ with Q acyclic satisfy the hypotheses for Proposition 3.14.

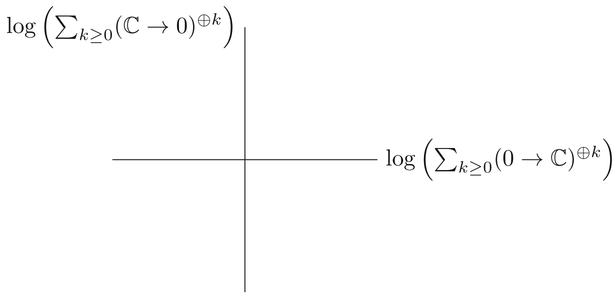

Fig. 3 The initial Hall algebra scattering diagram $\mathfrak{D}_{\mathrm{in}}^{\mathrm{Hall}}$ for the A_2-quiver

3.3 The initial Hall algebra scattering diagrams

We next wish to better understand the scattering functions of (24). We assume from now on that Q contains no loops or oriented 2-cycles, although a generalization to cases with loops is possible—cf. [13, Prop. 7.7] for a description of the incoming walls associated to vertices with a loop.

For each $i \in S$, we will find a nice expression for $\log 1_{\mathrm{ss}}(p^*(e_i))$ in terms of powers of κ_i. We will need the **quantum dilogarithm**

$$\begin{aligned}\Psi_t(x) &:= \sum_{k=0}^{\infty} \frac{t^{-k(k-1)/2}x^k}{(t-t^{-1})(t^2-t^{-2})\cdots(t^k-t^{-k})}\\ &= \sum_{k=0}^{\infty} \frac{(tx)^k}{\prod_{j=1}^k (t^{2j}-1)},\end{aligned}$$

and the standard fact that $\log \Psi_t(x) = -\mathrm{Li}(-x;t)$, where

$$\mathrm{Li}(x;t) := \sum_{k=1}^{\infty} \frac{x^k}{k(t^k-t^{-k})}. \tag{25}$$

Denote

$$f_i := 1_{\mathrm{ss}}(p^*(e_i)) = \sum_{k=0}^{\infty} \delta_{ki}.$$

By (13), we can rewrite f_i as[6]

$$f_i = \sum_{k=0}^{\infty} \frac{\kappa_i^k}{\prod_{j=1}^k (q^j-1)} = \Psi_t\left(\frac{\kappa_i}{t}\right).$$

[6] This expression seems to be well-known to experts, cf. [32, § 6.4].

Hence, using that $\log \Psi_t(x) = -\mathrm{Li}(-x; t)$, we find

$$\log f_i = -\mathrm{Li}(-\kappa_i/t; t) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(t^k - t^{-k})} \left(\frac{\kappa_i}{t}\right)^k$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(q^k - 1)} \kappa_i^k. \tag{26}$$

We denote

$$R_k := \frac{(-1)^{k-1}}{k(q^k - 1)} \tag{27}$$

so $\log f_i$ can be written as

$$\log f_i = \sum_{k=1}^{\infty} R_k \kappa_i^k. \tag{28}$$

It follows immediately from (26), (20), and Theorem 3.5 that applying $\mathcal{I}_t$ to $\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{Scat}}$ produces the quantum cluster scattering diagrams of [38, § 4.2]:

Proposition 3.16 *Applying $\mathcal{I}_t$ to $\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{Scat}}$ produces the scattering diagram $\mathfrak{D}^{q}_{\mathrm{Scat}} :=$ $\mathrm{Scat}(\mathfrak{D}^{q}_{\mathrm{in}})$ over the quantum torus algebra, where*

$$\mathfrak{D}^{q}_{\mathrm{in}} := \{e_i^{\perp}, -\mathrm{Li}(-z^{e_i}, t)\}. \tag{29}$$

Applying $\pi_{t\mapsto 1}$, it follows that $\mathcal{I}$ applied to $\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{Scat}}$ yields $\mathrm{Scat}(\mathfrak{D}^{\mathrm{cl}}_{\mathrm{in}})$, where

$$\mathfrak{D}^{\mathrm{cl}}_{\mathrm{in}} := \{e_i^{\perp}, -\mathrm{Li}(-z^{e_i})\}.$$

Here, $\mathrm{Li}(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is the classical dilogarithm. This is precisely [6, Lem. 11.4].

Remark 3.17 Instead of viewing $\mathfrak{g}^{\mathrm{Hall}}$ as $(t - t^{-1})^{-1} H_{\mathrm{reg}}(Q, W)$ with its commutator bracket, one might try to view it as simply $H_{\mathrm{reg}}(Q, W)$ with the Poisson bracket of (9), cf. Remark 2.6. In this version, instead of having $g = \log(1_{\mathrm{ss}}(\theta))$ in Theorem 3.8, one has $g = (t - t^{-1}) \log(1_{\mathrm{ss}}(\theta))$. In (28), this corresponds to redefining R_k to be $\frac{(-1)^{k-1}}{kt(1+q+q^2+\cdots+q^{k-1})}$.

4 Scattering diagrams in terms of tropical disks

4.1 Tropical disks

We now introduce the tropical disks whose enumerations will be related to the scattering diagrams of Sect. 3. For now, our tropical disks will live in $L_{\mathbb{R}} := L \otimes \mathbb{R}$ for an arbitrary finite-rank lattice L (later we will take $L = \Lambda^{\vee} = M$).

Let $\overline{\Gamma}$ be the topological realization of a finite connected tree without bivalent vertices, and let Γ denote the complement of all but one of its 1-valent vertices. Denote this remaining 1-valent vertex by V_∞, and denote the edge containing this vertex by E_∞. Let $\Gamma^{[0]}$, $\Gamma^{[1]}$, and $\Gamma^{[1]}_\infty$ denote the sets of vertices, edges, and non-compact edges of Γ, respectively. Let $e_\infty := \#\Gamma^{[1]}_\infty$. Equip Γ with a weighting $w : \Gamma^{[1]} \to \mathbb{Z}_{>0}$, plus a marking $\epsilon : S \xrightarrow{\sim} \Gamma^{[1]}_\infty$ for some index set S with $\#S = e_\infty$. For $s \in S$, we denote $E_s := \epsilon(s)$.

A **parametrized tropical disk** (Γ, w, ϵ, h) in $L_\mathbb{R}$ is data Γ, w, and ϵ as above, plus a proper continuous map $h : \Gamma \to L_\mathbb{R}$ such that:

- For each $E \in \Gamma^{[1]}$, $h|_E$ is an embedding into an affine line with rational slope;
- For any vertex V and edge $E \ni V$, denote by $u_{(V,E)}$ the primitive integral vector emanating from $h(V)$ into $h(E)$. For each $V \in \Gamma^{[0]} \setminus \{V_\infty\}$, the following **balancing condition** is satisfied:

$$\sum_{E \ni V} w(E) u_{(V,E)} = 0.$$

For unbounded edges $E_s \ni V$, we may denote $u_{(V,E_s)}$ simply as u_{E_s} or u_s. An **isomorphism** of parameterized tropical disks (Γ, h) and (Γ', h') is a homeomorphism $\Phi : \Gamma \to \Gamma'$ respecting the weights and markings such that $h = h' \circ \Phi$. A **tropical disk** is then defined to be an isomorphism class of parameterized tropical disks. We will let (Γ, h) denote the isomorphism class it represents, and we will often further abbreviate this as simply Γ or h.

A **tropical ribbon** $\widehat{\Gamma}$ is a tropical disk (Γ, w, ϵ, h) as above, together with the additional data of a cyclic ordering of the edges at each vertex. A tropical disk or ribbon is called **trivalent** if every vertex other than V_∞ is trivalent.

The **degree** Δ of a tropical disk (Γ, w, ϵ, h) is the map $\Delta : S \to L$ given by

$$\Delta(s) = w(E_s) u_{E_s}.$$

Let $\mathrm{Flags}(\Gamma)$ denote the set of flags (V, E), $V \in E$, of Γ. The **type** of a tropical disk is the data of Γ, w, and ϵ, along with the data of the map $u : \mathrm{Flags}(\Gamma) \to L$, $(V, E) \mapsto u_{(V,E)}$. Note that the type of a tropical disk determines its degree.

Similarly, the **type** of a tropical ribbon is the data of the type of the associated tropical disk, plus the data of the ribbon structure, i.e., the data of the cyclic orderings at each vertex.

Let $\mathbf{A} := (A_s)_{s \in S}$ be a tuple of affine-linear subspaces $A_s \subset L_\mathbb{R}$, each with rational slope. We say a tropical disk (Γ, w, ϵ, h) **matches the constraint A** if $h(E_s) \subset A_s$ for each $s \in S$.

4.2 Tropical degrees, constraints, and multiplicities associated to a scattering diagram

We now combine the setup of Sect. 4.1 with that of Sect. 3.1. Let $L = \Lambda^\vee$. Let $\{e_i\}_{i \in I}$ be a finite collection of vectors in Λ^+, indexed by a set I. Suppose we have an initial scattering diagram $\mathfrak{D}_{\text{in}}$ over $\mathfrak{g}$, with $\mathfrak{D}_{\text{in}}$ having the form

$$\mathfrak{D}_{\text{in}} = \{(\mathfrak{d}_i, g_i) | i \in I\},$$

where for each i, we have $\mathfrak{d}_i = e_i^{\perp}$ and

$$g_i = \sum_{w \geq 1} g_{i,w} \in \mathfrak{g}_{e_i}^{\parallel}, \tag{30}$$

where $g_{i,w} \in \mathfrak{g}_{we_i}$. Assume as in Theorem 3.5 that for each i, the terms $g_{i,w}$ pairwise commute. We denote

$$v_i = p^*(e_i)$$

for each $i \in S$, so $-v_i$ is the direction of the wall $\mathfrak{d}_i$.

4.2.1 Degrees and constraints

Let $\mathbf{w} := (\mathbf{w}_i)_{i \in S}$ be a tuple of weight vectors $\mathbf{w}_i := (w_{i1}, \ldots, w_{il_i})$ with $0 < w_{i1} \leq \cdots \leq w_{il_i}$, $w_{ij} \in \mathbb{Z}$. For Σ_{l_i} denoting the group of permutations of $\{1, \ldots, l_i\}$, let

$$\operatorname{Aut}(\mathbf{w}) \subset \prod_{i \in S} \Sigma_{l_i}$$

be the group of automorphisms of the second indices of the weights $\mathbf{w}_i$ which act trivially on $\mathbf{w}$. We also define

$$l(\mathbf{w}) = \sum_{i \in S} l_i.$$

Associated to $\mathbf{w}$, we consider the degree $\Delta_{\mathbf{w}} : S_{\mathbf{w}} \to L$ given by

$$S_{\mathbf{w}} := \{(i, j) | i \in I, j \in \{1, \ldots, l_i\}\}, \tag{31}$$

and

$$\Delta_{\mathbf{w}}((i, j)) = w_{ij} v_i. \tag{32}$$

For the associated constraints

$$\mathbf{A}_{\mathbf{w}} = (A_{ij})_{(i,j) \in S_{\mathbf{w}}}$$

we take the affine-linear space A_{ij} to be a generic translate of $e_i^{\perp}$. Here, the translates for different pairs (i, j) are generic relative to each other. We fix such a choice of $\mathbf{A}_{\mathbf{w}}$ for each $\mathbf{w}$. Given $\delta > 0$, let $\delta \mathbf{A}_{\mathbf{w}}$ denote the constraints obtained from $\mathbf{A}_{\mathbf{w}}$ by multiplying each A_{ij} by δ (i.e., the distance from the origin is multiplied by δ).

4.2.2 Multiplicities

For each $(i, j) \in S_{\mathbf{w}}$, we denote

$$g_{ij} := g_{i,w_{ij}} \in \mathfrak{g}_{w_{ij}e_i}. \tag{33}$$

Denote

$$n_{\mathbf{w}} := \sum_{(i,j)\in S_{\mathbf{w}}} w_{ij}e_i \in \Lambda.$$

Now consider a trivalent tropical disk Γ of degree $\Delta_{\mathbf{w}}$. We will denote $E_{(i,j)}$ simply as E_{ij}. We view Γ as flowing towards the univalent vertex V_∞, and we use this flow to inductively associate an element $g_E \in \mathfrak{g}_{n_E} \subset \mathfrak{g}$ to each edge E of Γ, where n_E is an element of Λ^+ such that $p^*(n_E) \in L$ is the weighted tangent vector to $h(E)$ pointing in the direction opposite the flow.

To each of the source edges E_{ij}, we associate the element g_{ij} from (33) above. Now consider a vertex $V \neq V_\infty$ with E_1, E_2 flowing into V and E_3 flowing out of V, and suppose that for $i = 1, 2$, we already have associated elements $g_{E_i} \in \mathfrak{g}_{n_{E_i}}$. By the balancing condition, we have $n_{E_3} = n_{E_1} + n_{E_2}$. Let us assume that the labelling of the edges E_1, E_2 is such that

$$\{n_{E_1}, n_{E_2}\} \geq 0 \tag{34}$$

(otherwise we re-label). We then define

$$g_{E_3} := [g_{E_1}, g_{E_2}] \in \mathfrak{g}_{n_{E_3}}.$$

We now define the **multiplicity** of Γ as

$$\mathrm{Mult}(\Gamma) := g_{E_\infty} \in \mathfrak{g}_{n_{\mathbf{w}}}.$$

Now suppose that $\mathfrak{g}$ is a Lie subalgebra of the commutator algebra of a Λ^+-graded associative algebra $\mathfrak{A}$, i.e., we have an associative product such that

$$[g_1, g_2] = g_1g_2 - g_2g_1.$$

Example 4.1 For $\mathfrak{g} = \mathfrak{g}^{\mathrm{Hall}}$, Remark 2.6 says that we can take

$$\mathfrak{g}^{\mathrm{Hall}} = (t - t^{-1})^{-1} \cdot H_{\mathrm{reg}}(Q, W) \subset \mathfrak{A}^{\mathrm{Hall}} := H_{\mathrm{reg}}(Q, W)[(t - t^{-1})^{-1}].$$

Similarly, we can take

$$\mathfrak{g}^q = (t - t^{-1})^{-1}\mathbb{C}_t[N^\oplus] \subset \mathfrak{A}^q := \mathbb{C}_t[N^\oplus][(t - t^{-1})^{-1}].$$

Moreover, for any $\mathfrak{i}$ such that $\mathfrak{i}^{\mathrm{skew}} \subseteq \mathfrak{i} \subseteq \ker(\mathcal{I}_t)$, since $(t - t^{-1}) \notin \mathfrak{i}$, we can take

$$\mathfrak{g}^{\mathfrak{i}} = (t - t^{-1})^{-1} H_{\mathrm{reg}}(Q, W)/\mathfrak{i} \subset \mathfrak{A}^{\mathfrak{i}} := (H_{\mathrm{reg}}(Q, W)/\mathfrak{i})[(t - t^{-1})^{-1}].$$

However, $t - t^{-1} = 0$ in $\mathfrak{g}^{\mathrm{cl}}$, so we cannot apply this localization in the classical setting. Instead, we take $\mathfrak{A}^{\mathrm{cl}}$ to be the universal enveloping algebra of $\mathfrak{g}^{\mathrm{cl}}$.

Alternatively, the Poisson algebra $\mathbb{C}[N^{\oplus}]$ can be identified with a subalgebra of the module of log derivations $\Theta(N^{\oplus}) := \mathbb{C}[N^{\oplus}] \otimes_{\mathbb{Z}} M$ as in [25, § 1.1]. Here, $z^n \otimes m$, typically denoted $z^n \partial_m$, is viewed as acting on $\mathbb{C}[N]$ via $z^{n'} \mapsto \langle n, m\rangle Z^{n+n'} \partial_m$. The commutator of these derivations makes $\Theta(N^{\oplus})$ into a Lie algebra with bracket given by

$$[z^{n_1}\partial_{m_1}, z^{n_2}\partial_{m_2}] = z^{n_1+n_2}\partial_{\langle n_2, m_1\rangle m_2 - \langle n_1, m_2\rangle m_1}.$$

Let $\mathfrak{h}$ be the Lie subalgebra spanned by elements of the form $z^n\partial_m$ for $\langle n, m\rangle = 0$. Then $\mathbb{C}[N^{\oplus}]$ embeds into $\mathfrak{h}$ via $z^n \mapsto z^n\partial_{B(n,\cdot)}$. Hence, instead of taking $\mathfrak{A}^{\mathrm{cl}}$ to be the universal enveloping algebra of $\mathbb{C}[N^{\oplus}]$, it is reasonable to take it to be the universal enveloping algebra of $\mathfrak{h}$ or $\Theta(N^{\oplus})$. The latter is simply a log version of the Weyl algebra in $\mathrm{rank}(N)$ variables. That is, we may view $\mathfrak{A}^{\mathrm{cl}}$ as an algebra of logarithmic differential forms.

We note that the usual classical multiplicities of tropical curves (as in correspondence theorems like those of [45]) can similarly be computed via iterated Lie brackets of polyvector fields, cf. [42]. Also, the quantum ribbon multiplicities computed using $\mathfrak{A}^q$ are related to certain counts of real curves, cf. [39].

For example, we can always take $\mathfrak{A}$ to be the universal enveloping algebra of $\mathfrak{g}$. Alternatively, for $\mathfrak{g}^{\mathrm{Hall}}$ or $\mathfrak{g}^q$, we can produce such an $\mathfrak{A}$ using Remark 2.6.

Suppose that Γ is equipped with a ribbon structure $\widehat{\Gamma}$. At each vertex $V \neq V_\infty$, let E_1, E_2 be the vertices flowing into V and E_3 the vertex flowing out of V, and assume the cyclic ordering of the labelling E_1, E_2, E_3 agrees with the ribbon structure of $\widehat{\Gamma}$ at V (otherwise we re-label). We say that the vertex $V \in \widehat{\Gamma}^{[0]}$ is **positive** if the edges E_1, E_2, labelled in this way with respect to the ribbon structure, satisfy the condition (34). Otherwise, we say V is **negative**.

We now describe a method of inductively associating an element of $\mathfrak{g}_{n_E} \subset \mathfrak{A}$ to each edge E of $\widehat{\Gamma}$, this time denoting the elements by g_E^{Υ}. The vectors n_E will be the same as before, but the elements g_E^{Υ} will be different and will depend on the ribbon structure. As before, we take $g_{E_{ij}}^{\Upsilon} := g_{ij}$ for the source edges. But now, for E_1, E_2 the edges flowing into a vertex V, E_3 the edge flowing out of V, and the labelling agreeing with the ribbon structure at V, we define

$$g_{E_3}^{\Upsilon} := \nu(V) g_{E_1}^{\Upsilon} g_{E_2}^{\Upsilon},$$

where

$$\nu(V) := \begin{cases} 1 & \text{if } V \text{ is positive} \\ -1 & \text{if } V \text{ is negative,} \end{cases}$$

and $g^{\Upsilon}_{E_1} g^{\Upsilon}_{E_2}$ is the associative product in $\mathfrak{A}$. Finally, we define

$$\mathrm{Mult}^{\Upsilon}(\widehat{\Gamma}) := g^{\Upsilon}_{E_\infty} \in \mathfrak{A}_{n_{\mathbf{w}}}. \tag{35}$$

Alternatively, define

$$\nu(\widehat{\Gamma}) := \prod_{V \in \widehat{\Gamma}} \nu(V).$$

The ribbon structure induces an ordering of the unbounded edges of $\widehat{\Gamma}$, starting with E_∞ and then continuing with $E_{i_1 j_1}, \ldots, E_{i_{l(\mathbf{w})} j_{l(\mathbf{w})}}$. Using the associativity of $\mathfrak{A}$, we can rewrite (35) as

$$\mathrm{Mult}^{\Upsilon}(\widehat{\Gamma}) = \nu(\widehat{\Gamma}) g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_{l(\mathbf{w})} j_{l(\mathbf{w})}} \in \mathfrak{A}. \tag{36}$$

One easily sees the following:

Lemma 4.2 *For each* Γ *as above,*

$$\mathrm{Mult}(\Gamma) = \sum \mathrm{Mult}^{\Upsilon}(\widehat{\Gamma}),$$

where the sum is over all possible tropical ribbons $\widehat{\Gamma}$ *with underlying tropical curve* Γ.

Note that $\mathrm{Mult}(\Gamma)$ and $\mathrm{Mult}^{\Upsilon}(\widehat{\Gamma})$ are completely determined by the type τ of Γ or $\widehat{\Gamma}$, respectively. We thus define the **multiplicity of a tropical disk or ribbon type** τ as the multiplicity of any of the tropical disks/ribbons of type τ.

4.3 Tropical ribbon counts and the consistent scattering diagram

We continue with the setup of Sect. 4.2. For each weight vector $\mathbf{w}$, each $\delta > 0$, and each $\theta \in L_{\mathbb{R}}$, let $\mathfrak{T}_{\mathbf{w},\delta}(\theta)$ denote the set of types of tropical disks of degree $\Delta_{\mathbf{w}}$ which match the constraint $\delta \mathbf{A}_{\mathbf{w}}$ and for which $h(V_\infty) = \theta$. For each $\epsilon > 0$ and $\theta \in L_{\mathbb{R}}$, let $B_\epsilon(\theta)$ denote the open radius ϵ ball centered at θ (with respect to the Euclidean metric associated to any fixed choice of basis for L). Let $\mathfrak{T}_{\mathbf{w}}(\theta)$ denote the set of tropical disk types[7] τ such that, for any $\epsilon > 0$ and all sufficiently small $\delta > 0$, there exist $\theta' \in B_\epsilon(\theta)$ with $\tau \in \mathfrak{T}_{\mathbf{w},\delta}(\theta')$. See Fig. 5 for an example.

Lemma 4.3 *Recall our assumption that* $\mathbf{A}_{\mathbf{w}}$ *is generic. For* θ *outside some locus of codimension* 2 *(in particular, for* θ *general in the sense of Sect.* 3.1*), every tropical disk type in* $\mathfrak{T}_{\mathbf{w}}(\theta)$ *is trivalent.*

[7] We believe these can be interpreted as types of "virtual tropical disks" as defined in [9, § 5], so that the classical case of Theorem 4.9 can be viewed as a special case of [9, Prop. 5.14].

Proof This follows from the correspondence between tropical disks and scattering walls in Lemma 4.4.2 below. More explicitly, consider a tropical disk Γ of degree $\Delta_{\mathbf{w}}$ matching the constraints $\delta\mathbf{A}_{\mathbf{w}}$ for some $\delta > 0$. Consider the flow of Γ towards V_∞ as in Sect. 4.2.2. Suppose $E_1, \ldots, E_s$ flow into a vertex V with E_V flowing out, and suppose that E_i lies in a generically translated affine hyperspace A_{E_i} for $i = 1, \ldots, s$. Then E_V lies in

$$A_{E_V} := \left(\bigcap_i A_{E_i}\right) + \mathbb{R}v_{E_i},$$

where $v_{E_V} = p^*(n_{E_V})$ is the direction of E_V. In particular,

$$\operatorname{codim}(A_{E_V}) = \left(\sum_{i=1}^{s} \operatorname{codim}(A_{E_i})\right) - 1.$$

For each unbounded edge E_{ij}, we can take $A_{E_{ij}} = A_{ij}$, which has codimension 1. It follows that, if there is a vertex of valence higher than three, then $h(V_\infty)$ will necessarily lie in a generically determined translate of some rational-slope subspace of codimension at least 2. The assumption on θ then implies that $h(V_\infty)$ cannot be in $B_\epsilon(\theta)$, and the result follows. $\square$

Lemma 4.3 ensures that we can define the multiplicities of elements of $\mathfrak{T}_{\mathbf{w}}(\theta)$ as in Sect. 4.2.2 whenever θ is outside some bad codimension 2 locus (which will be the joints of a scattering diagram). Define

$$N(\theta) := \sum_{\mathbf{w}} \frac{1}{|\operatorname{Aut}(\mathbf{w})|} \sum_{\tau \in \mathfrak{T}_{\mathbf{w}}(\theta)} \operatorname{Mult}(\tau) \in \widehat{\mathfrak{g}}.$$

Let $\mathfrak{T}^{\Upsilon}_{\mathbf{w}}(\theta)$ denote the set of tropical ribbons types $\widehat{\tau}$ such that the associated tropical disk type τ is in $\mathfrak{T}_{\mathbf{w}}(\theta)$. By Lemma 4.2, we can express $N(\theta)$ as

$$N(\theta) = \sum_{\mathbf{w}} \frac{1}{|\operatorname{Aut}(\mathbf{w})|} \sum_{\widehat{\tau} \in \mathfrak{T}^{\Upsilon}_{\mathbf{w}}(\theta)} \operatorname{Mult}^{\Upsilon}(\widehat{\tau}).$$

Note that for each $n \in \Lambda^+$, the strict convexity of Λ^+ ensures that there are only finitely many $\mathbf{w}$ such that $n = n_{\mathbf{w}}$. Furthermore, for each $\mathbf{w}$, there are clearly only finitely many types of tropical disks of degree $\Delta_{\mathbf{w}}$. The well-definedness of $N(\theta)$ follows, assuming that we have already fixed $\mathbf{A}_{\mathbf{w}}$. The fact that the generic choice of $\mathbf{A}_{\mathbf{w}}$ does not matter is part of the following theorem.

Theorem 4.4 *Assume* $\mathfrak{g}$ *has Abelian walls. Let* $\mathfrak{D} = \operatorname{Scat}(\mathfrak{D}_{\mathrm{in}})$, *and consider* $\theta \in L_{\mathbb{R}} \setminus \operatorname{Joints}(\mathfrak{D})$. *Up to equivalence, we may assume that* $\mathfrak{D}$ *has at most one wall* $(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}$ *with* $\theta \in \mathfrak{d}$. *If there is no such wall, then* $N(\theta) = 0$, *and otherwise,*

$$g_{\mathfrak{d}} = N(\theta).$$

4.4 Proof of Theorem 4.4

Theorem 4.4 is a modified version of [38, Thm 3.7], or a refinement of some cases of [9, Prop. 5.14]. The two-dimensional quantum version is [19, Cor. 4.9], and the two-dimensional classical version is [25, Thm. 2.8]. The proof is similar in each case. We repeat the setup here, following [38, § 3.2].

4.4.1 Perturbing the scattering diagram

Definition 4.5 For any scattering diagram $\mathfrak{D}$ over a Lie algebra $\mathfrak{g}$ with Abelian walls, the **asymptotic scattering diagram** $\mathfrak{D}_{\mathrm{as}}$ of $\mathfrak{D}$ is defined by replacing every wall $(n + \mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}$ with the wall $(\mathfrak{d}, g_{\mathfrak{d}})$. Here, $\mathfrak{d}$ denotes a rational polyhedral cone (with apex at the origin) and $n \in N_{\mathbb{R}}$ translates this cone.

Now let T denote the commutative polynomial ring $\mathbb{Z}[t_i|i \in I]$, and let $T_k := T/\langle t_i^{k+1}|i \in I\rangle$. Let $\mathfrak{D}_{\mathrm{in},T_k}$ and $\mathfrak{D}_{\mathrm{in},T}$ be the initial scattering diagrams over $\mathfrak{g} \otimes T_k$ and $\mathfrak{g} \otimes T$, respectively, given by replacing each $g_{\mathfrak{d}_i} = \sum_{j\geq 1} g_{ij}$ from $\mathfrak{D}_{\mathrm{in}}$ with $g'_{\mathfrak{d}_i} := \sum_{j\geq 1} t_i^j g_{ij}$. We will show that Theorem 4.4 holds for $\mathfrak{D}_{T_k} := \mathrm{Scat}(\mathfrak{D}_{\mathrm{in},T_k})$ for all k, hence for $\mathfrak{D}_T := \mathrm{Scat}(\mathfrak{D}_{\mathrm{in},T})$. Taking $t_i = 1$ for each i then recovers the theorem for $\mathfrak{D} = \mathrm{Scat}(\mathfrak{D}_{\mathrm{in}})$.

We have an inclusion of commutative rings

$$\begin{aligned} T_k \hookrightarrow T'_k &:= \mathbb{Z}[\overline{u}_{ij}|i \in I, 1 \leq j \leq k]/\langle \overline{u}_{ij}^2|i \in I, 1 \leq j \leq k\rangle \\ t_i &\mapsto \sum_{j=1}^{k} \overline{u}_{ij}. \end{aligned}$$

Using this inclusion to work in $\mathfrak{g} \otimes T'_k$, we have

$$g'_{\mathfrak{d}_i} = \sum_{w=1}^{k} t_i^w g_{iw} = \sum_{w=1}^{k} \sum_{\#J=w} w! g_{iw} \overline{u}_{iJ}, \tag{37}$$

where the second sum is over all subsets $J \subset \{1, \ldots, k\}$ of size w, and

$$\overline{u}_{iJ} := \prod_{j\in J} \overline{u}_{ij}.$$

Consider our scattering diagram $\mathfrak{D}_{\mathrm{in}} = \{(\mathfrak{d}_i, g_i)|i \in I\}$ with $\mathfrak{d}_i = e_i^{\perp}$ and $g_i = \sum_{w\geq 1} g_{i,w}$ as in (30). Applying the equivalence from Example 3.4(1) in reverse and then perturbing the walls (i.e., translating the walls by some generic amount), we obtain a scattering diagram

$$\overline{\mathfrak{D}}_k^0 := \{(\mathfrak{d}_{iJ}, w! g_{iw} \overline{u}_{iJ})|1 \leq w \leq k, J \subset \{1, \ldots, k\}, \#J = w\}, \tag{38}$$

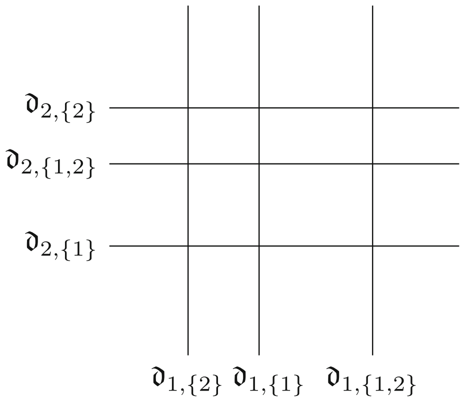

Fig. 4 $\mathfrak{D}_2^0$ perturbing the A_2 initial scattering diagram of Fig. 3

where $\mathfrak{d}_{iJ}$ is some generic translation of $\mathfrak{d}_i = e_i^{\perp}$. Note that $\mathrm{Scat}(\overline{\mathfrak{D}}_k^0)_{\mathrm{as}} = \mathfrak{D}_{\mathrm{in},T_k}$.

It will be useful for us to refine this setup a bit, working over a different commutative ring $\widetilde{T}_k$ defined by

$$\widetilde{T}_k := \mathbb{Z}[u_{iJ}|i \in I, J \subset \{1, \ldots, k\}]/\langle u_{iJ_1} u_{iJ_2} | J_1 \cap J_2 \neq \emptyset \rangle.$$

Note that we have a surjective homomorphisms

$$\widetilde{\pi} : \widetilde{T}_k \to T_k', \qquad u_{iJ} \mapsto \overline{u}_{iJ}. \tag{39}$$

Let $\mathfrak{D}_k^0$ denote the initial scattering diagram over $\mathfrak{g} \otimes \widetilde{T}_k$ defined as in (38), but with the factors $\overline{u}_{iJ}$ replaced by u_{iJ}, i.e.,

$$\mathfrak{D}_k^0 := \{(\mathfrak{d}_{iJ}, w! g_{iw} u_{iJ}) | 1 \leq w \leq k, J \subset \{1, \ldots, k\}, \#J = w\}. \tag{40}$$

Example 4.6 For $\mathfrak{D}_{\mathrm{in}}$ as in Example 3.15, the corresponding $\mathfrak{D}_k^0$ for $k = 2$ may look like Fig. 4.

Note that $\widetilde{\pi}$ takes the scattering functions of $\mathfrak{D}_k^0$ to those of $\overline{\mathfrak{D}}_k^0$, and so the same will be true for the corresponding consistent scattering diagrams and their asymptotic versions. We will write our walls in the form $(\mathfrak{d}, g_{\mathfrak{d}} u_{\mathbf{J}_{\mathfrak{d}}})$, where $g_{\mathfrak{d}} \in \mathfrak{g}_{n_{\mathfrak{d}}}$ for some $n_{\mathfrak{d}} \in N^+$, $\mathbf{J}_{\mathfrak{d}}$ is a collection of pairwise-disjoint subsets of $I \times \{1, \ldots, k\}$ of the form (i, J) for various $i \in I$ and $J \subset \{1, \ldots, k\}$, and

$$u_{\mathbf{J}_{\mathfrak{d}}} := \prod_{(i,J) \in \mathbf{J}_{\mathfrak{d}}} u_{iJ}. \tag{41}$$

We now inductively produce a scattering diagram $\mathfrak{D}_k^{\infty} = \mathrm{Scat}(\mathfrak{D}_k^0)$ from $\mathfrak{D}_k^0$ as follows: whenever two walls $(\mathfrak{d}_1, g_{\mathfrak{d}_1} u_{\mathbf{J}_{\mathfrak{d}_1}})$ and $(\mathfrak{d}_2, g_{\mathfrak{d}_2} u_{\mathbf{J}_{\mathfrak{d}_2}})$ intersect and satisfy $u_{\mathbf{J}_{\mathfrak{d}_1}} u_{\mathbf{J}_{\mathfrak{d}_2}} \neq 0$, we add a new wall $\mathfrak{d}(\mathfrak{d}_1, \mathfrak{d}_2)$ defined as follows: assume $\{n_{\mathfrak{d}_1}, n_{\mathfrak{d}_2}\} \geq 0$

(otherwise reorder), and then set

$$\mathfrak{d}(\mathfrak{d}_1, \mathfrak{d}_2) := ((\mathfrak{d}_1 \cap \mathfrak{d}_2) + \mathbb{R}_{\leq 0} p^*(n_{\mathfrak{d}_1} + n_{\mathfrak{d}_2}), [g_{\mathfrak{d}_1}, g_{\mathfrak{d}_2}] u_{\mathbf{J}_{\mathfrak{d}_1}} u_{\mathbf{J}_{\mathfrak{d}_2}}). \tag{42}$$

This indeed terminates in finitely many steps and produces a consistent scattering diagram $\mathfrak{D}_k^\infty$, cf. [38, § 3.2.2–3.2.3] for details. We note that the consistency argument in [38, § 3.2.3] requires the Abelian walls assumption.

Definition 4.7 If $\mathfrak{d} = \mathfrak{d}(\mathfrak{d}_1, \mathfrak{d}_2)$, define $\mathrm{Parents}(\mathfrak{d}) := \{\mathfrak{d}_1, \mathfrak{d}_2\}$, and if $\mathfrak{d} \in \mathfrak{D}_k^0$, define $\mathrm{Parents}(\mathfrak{d}) := \emptyset$. Recursively define $\mathrm{Ancestors}(\mathfrak{d})$ by $\mathrm{Ancestors}(\mathfrak{d}) := \{\mathfrak{d}\} \cup \bigcup_{\mathfrak{d}' \in \mathrm{Parents}(\mathfrak{d})} \mathrm{Ancestors}(\mathfrak{d}')$. Define

$$\mathrm{Leaves}(\mathfrak{d}) := \{\mathfrak{d}' \in \mathrm{Ancestors}(\mathfrak{d}) | \quad \mathfrak{d}' \text{ is the support of a wall in } \mathfrak{D}_k^0\}.$$

4.4.2 The tropical description of $\mathfrak{D}_k^\infty$

We continue to write $\mathbf{J}$ to denote a collection of pairwise-disjoint subsets of $I \times \{1, \ldots, k\}$ of the form (i, J) for various $i \in I$ and $J \subset \{1, \ldots, k\}$. Now, as in Sect. 4.2.1, fix a weight vector $\mathbf{w} := (\mathbf{w}_i)_{i \in I}$, $\mathbf{w}_i := (w_{i1}, \ldots, w_{il_i})$ with $0 < w_{i1} \leq \cdots \leq w_{il_i}$. Let $\mathbf{J}_\mathbf{w}$ denote the set of all possible $\mathbf{J}$ which can be written in the form

$$\mathbf{J} = \{(i, J_{ij}) : i \in I, j = 1, \ldots, l_i\}$$

with $\# J_{ij} = w_{ij}$. Note that each $\mathbf{J} \in \mathbf{J}_\mathbf{w}$ corresponds to a set of walls

$$\mathfrak{D}_{k,\mathbf{J}}^0 = \{\mathfrak{d}_{iJ_{ij}}\}_{(i, J_{ij}) \in \mathbf{J}} \subset \mathfrak{D}_k^0,$$

and two choices of $\mathbf{J}$ correspond to the same $\mathfrak{D}_{k,\mathbf{J}}^0$ exactly if they are related by an element of $\mathrm{Aut}(\mathbf{w})$. Given $\mathbf{J}$, let $\mathbf{w}_\mathbf{J}$ denote the corresponding weight vector $\mathbf{w}$ for which $\mathbf{J} \in \mathbf{J}_\mathbf{w}$.

Let $\mathfrak{D}_{k,\mathbf{J}}^\infty$ denote the set of walls in $\mathfrak{D}_k^\infty$ whose leaves are precisely the walls of $\mathfrak{D}_{k,\mathbf{J}}^0$. Note that, for $\mathbf{J} \in \mathbf{J}_\mathbf{w}$ and $(\mathfrak{d}, g_\mathfrak{d} u_\mathbf{J}) \in \mathfrak{D}_{k,\mathbf{J}}^\infty$, we must have $g_\mathfrak{d} \in \mathfrak{g}_{n_\mathbf{w}}$. We will write $\mathfrak{T}_{\mathbf{w},\delta}(\theta, \mathbf{A}_\mathbf{J})$ to indicate $\mathfrak{T}_{\mathbf{w},\delta}(\theta)$ as in Sect. 4.3 with the the representatives of the incidence conditions $\mathbf{A}_\mathbf{w}$ chosen so that $A_{ij} = \mathfrak{d}_{iJ_{ij}}$.

Lemma 4.8 *For every wall $(\mathfrak{d}, g_\mathfrak{d} u_\mathbf{J}) \in \mathfrak{D}_{k,\mathbf{J}}^\infty$ and every θ in the interior of $\mathfrak{d}$, there exists a unique tropical disk $h : \Gamma \to L_\mathbb{R}$ in $\mathfrak{T}_{\mathbf{w}_\mathbf{J},1}(\theta, \mathbf{A}_\mathbf{J})$ with $h(V_\infty) = \theta$. Furthermore, we have*

$$g_\mathfrak{d} = \mathrm{Mult}(\Gamma) \prod_{ij} (w_{ij}!). \tag{43}$$

Proof We construct the tropical disk by starting at $h(V_\infty) = \theta \in \mathfrak{d}$ and following $\mathfrak{d}$ in the direction $p^*(n_\mathfrak{d})$ until we reach a point $p \in \mathfrak{d}_1 \cap \mathfrak{d}_2$, where $\{\mathfrak{d}_1, \mathfrak{d}_2\} = \mathrm{Parents}(\mathfrak{d})$. The resulting segment is given weight $|n_\mathfrak{d}|$ (the index of $n_\mathfrak{d}$, i.e., $n_\mathfrak{d}$ equals $|n_\mathfrak{d}| > 0$ times a primitive vector). From p, extend the tropical curve in the directions $n_{\mathfrak{d}_1}$ and

$n_{\mathfrak{d}_2}$ with weights $|n_{\mathfrak{d}_1}|$ and $|n_{\mathfrak{d}_2}|$, respectively, until reaching the boundaries of the walls $\mathfrak{d}_1$ and $\mathfrak{d}_2$. The balancing condition at p follows easily from (42) and the fact that commutators in $\mathfrak{g}$ respect the N^+-grading. The process is repeated for each of these branches, and continues until every branch extends to infinity in some leaf. This gives the desired tropical disk. The formula for $g_{\mathfrak{d}}$ follows easily from (42) and the definition of g_Γ, noting that the $\prod w_{ij}!$ factor appears because of the fact that g_{iw} is multiplied by $w!$ in the definition of $\mathfrak{D}_k^0$ in (40), and similarly for the $u_{\mathbf{J}}$ factor. □

4.4.3 Proof of Theorem 4.4

Given a weight vector $\mathbf{w}$, let $|\mathbf{w}_i| := \sum_{j=1}^{l_i} w_{ij}$, and let $t^{\mathbf{w}} = \prod_{i,j} t_i^{w_{ij}} = \prod_i t_i^{|\mathbf{w}_i|}$. Also, for $\mathbf{J} = \{(i, J_{ij}) \subset I \times \{1, \ldots, k\}\}_{(i,j)\in S_{\mathbf{w}}}$, let

$$\overline{u}_{\mathbf{J}} := \prod_{i,j} \overline{u}_{iJ_{ij}}.$$

We will use the following formula, cf. [38, (45)]:

$$t^{\mathbf{w}} = \sum_{\mathbf{J}\in\mathbf{J}_{\mathbf{w}}} \left(\overline{u}_{I_{\mathbf{J}}} \prod_{i,j} w_{ij}! \right). \tag{44}$$

For a scattering diagram $\mathfrak{D}$ and $\delta \in \mathbb{R}_{>0}$, let $\delta\mathfrak{D}$ denote the scattering diagram obtained by multiplying the supports of the walls of $\mathfrak{D}$ by δ (i.e., multiplying their distances from the origin by 0).

Now fix a point $\theta \in L_{\mathbb{R}} \setminus \mathrm{Joints}(\mathfrak{D}_{T_k})$. Recall that $\mathfrak{D}_{T_k} = (\widetilde{\pi}(\mathfrak{D}_k^\infty))_{\mathrm{as}}$. Hence, if $\theta \notin \mathrm{supp}(\mathfrak{D}_{T_k})$, then for sufficiently small $\delta > 0$, no walls of $\delta\mathfrak{D}_k^\infty = \mathrm{Scat}(\delta\mathfrak{D}_k^0)$ will intersect a small ϵ-neighborhood of θ. So then by Lemma 4.8, no tropical disks representing a type in any $\mathfrak{T}_{\mathbf{w_J},1}(\theta, \mathbf{A_J})$ will intersect such an ϵ-neighborhood either, and so we obtain $N(\theta) = 0$.

Now suppose $\theta \in \mathrm{supp}(\mathfrak{D}_k)$, and for convenience, use Example 3.4(1) to combine all walls containing θ into a single wall $\mathfrak{d}$. Then since $\mathfrak{D}_{T_k} = (\widetilde{\pi}(\mathfrak{D}_k^\infty))_{\mathrm{as}}$, we know that $g_{\mathfrak{d}} = \sum \widetilde{\pi}(g_{\mathbf{J}} u_{\mathbf{J}})$, where the sum is over all walls $(\mathfrak{d}_{\mathbf{J}}, g_{\mathbf{J}} u_{\mathbf{J}}) \in \mathfrak{D}_k^\infty$ such that for any $\epsilon > 0$, there exists a $\delta > 0$ for which $\delta\mathfrak{d}_{\mathbf{J}}$ intersects $B_\epsilon(\theta)$. By Lemma 4.8, this is the same as

$$\widetilde{\pi}\left(\sum_{\mathbf{w}} \frac{1}{|\mathrm{Aut}(\mathbf{w})|} \sum_{\mathbf{J}\in\mathbf{J}_{\mathbf{w}}} \sum_{\tau\in\mathfrak{T}_{\mathbf{w}}(\theta,\mathbf{A_J})} \mathrm{Mult}(\tau) u_{\mathbf{J}} w_{ij}! \right),$$

where here we write $\mathfrak{T}_{\mathbf{w}}(\theta, \mathbf{A_J})$ to indicate $\mathfrak{T}_{\mathbf{w}}(\theta)$ for our particular choice of $\mathbf{A_w}$ as $\mathbf{A_J}$ (since a priori $\mathfrak{T}_{\mathbf{w}}(\theta)$ might depend on this choice). Here we use our observation that two choices of $\mathbf{J}$ correspond to the same $\mathfrak{D}_{k,\mathbf{J}}^0$, hence the same $\mathfrak{D}_{k,\mathbf{J}}^\infty$, if and only if they are related by an element of $\mathrm{Aut}(\mathbf{w})$.

Now, note that for each $\mathbf{w}$, $(\mathfrak{D}_k^0)_{\mathrm{as}}$ is symmetric with respect to permuting the elements of $\mathbf{J_w}$, i.e., for $\mathbf{J}_1, \mathbf{J}_2 \in \mathbf{J_w}$, swapping the supports of $\mathfrak{d}_{i\mathbf{J}_1}$ and $\mathfrak{d}_{i\mathbf{J}_2}$ in (40) does not affect $(\mathfrak{D}_k^0)_{\mathrm{as}}$. Hence, $\sum_{\tau\in\mathfrak{T}_{\mathbf{w}}(\theta,\mathbf{A_J})}\mathrm{Mult}(\tau)$ is independent of $\mathbf{J}\in\mathbf{J_w}$, and so we obtain

$$g_{\mathfrak{d}} = \sum_{\mathbf{w}} \frac{1}{|\operatorname{Aut}(\mathbf{w})|} \sum_{\tau\in\mathfrak{T}_{\mathbf{w}}(\theta)} \mathrm{Mult}(\tau) \left(\sum_{\mathbf{J}\in\mathbf{J_w}} \overline{u}_{\mathbf{J}} w_{ij}!\right).$$

Finally, applying (44) yields the desired result. □

4.5 The main theorem

We cannot apply Theorem 4.4 directly to $\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{Scat}}$ because $\mathfrak{g}^{\mathrm{Hall}}$ is not skew-symmetric. However, the theorem does apply to any of our $\mathfrak{D}^{\mathfrak{i}}_{\mathrm{Scat}} := \mathcal{I}^{\mathfrak{i}}(\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{Scat}})$ for $\mathfrak{i}\supseteq\mathfrak{i}^{\mathrm{skew}}$ as in Sect. 3.2. Here, we take the associative algebra $\mathfrak{A}^{\mathfrak{i}}$ to be as in Remark 4.1, with $\mathfrak{A}^{\mathfrak{i}}$ meaning $\mathfrak{A}^{\mathrm{cl}}$ as Remark 4.1 in the case where $\mathfrak{i}=\ker(\mathcal{I})$.

Given a weight vector $\mathbf{w}$, define

$$R_{\mathbf{w}} := \prod_{i,j} R_{w_{ij}},$$

where we recall from (27) that $R_k := \frac{(-1)^{k-1}}{k(q^k-1)}$.

Now, let us fix a quiver with potential (Q,W) and consider the corresponding $\mathfrak{D}^{\mathrm{Hall}}_{\mathrm{Scat}}$. Consider a weight vector $\mathbf{w}$ and a choice of $\widehat{\tau}\in\mathfrak{T}^{\Upsilon}_{\mathbf{w}}(\theta)$. Recall that the ribbon structure induces an ordering of the unbounded edges of $\widehat{\tau}$, starting with E_∞ and then continuing with $E_{i_1j_1},\ldots,E_{i_{l(\mathbf{w})}j_{l(\mathbf{w})}}$. Using (36) and (28), we have

$$\mathrm{Mult}^{\Upsilon}(\widehat{\tau}) = \nu(\widehat{\tau})R_{\mathbf{w}}\kappa_{i_1}^{w_{i_1j_1}}\cdots\kappa_{i_{l(\mathbf{w})}}^{w_{i_{l(\mathbf{w})}j_{l(\mathbf{w})}}}.$$

By Lemma 2.4, we have

$$\kappa_{i_1}^{w_{i_1j_1}}\cdots\kappa_{i_{l(\mathbf{w})}}^{w_{i_{l(\mathbf{w})}j_{l(\mathbf{w})}}} = \mathfrak{Flag}(w_{i_1j_1}S_1,\ldots,w_{i_{l(\mathbf{w})}j_{l(\mathbf{w})}}S_{i_{l(\mathbf{w})}}) =: \mathfrak{Flag}(\widehat{\tau}), \tag{45}$$

where we write $w_{i_kj_k}S_{i_k}$ to indicate that the entry S_{i_k} appears $w_{i_kj_k}$ times, and we neglect writing the data of the map to $\mathcal{M}_{n_{\mathbf{w}}}\subset\mathcal{M}$. Finally, applying Theorem 4.4 to the image under $\mathcal{I}^{\mathfrak{i}}$, we obtain:

Theorem 4.9 *Let $\mathfrak{D}=\mathfrak{D}^{\mathfrak{i}}_{\mathrm{Scat}}$ for $\mathfrak{i}\supset\mathfrak{i}^{\mathrm{skew}}$, and consider $\theta\in L_{\mathbb{R}}\setminus\mathrm{Joints}(\mathfrak{D})$. Up to equivalence, we may assume that $\mathfrak{D}$ has at most one wall $(\mathfrak{d},g_{\mathfrak{d}})\in\mathfrak{D}$ with $\theta\in\mathfrak{d}$. If there is no such wall, then $N(\theta)=0$, and otherwise, $g_{\mathfrak{d}}=N(\theta)$, where $N(\theta)$ is defined as*

$$N(\theta) := \sum_{\mathbf{w}}\left(\frac{1}{|\operatorname{Aut}(\mathbf{w})|}\sum_{\widehat{\tau}\in\widehat{\mathfrak{T}}_{\mathbf{w}}(\theta)}\nu(\widehat{\tau})\mathcal{I}^{\mathfrak{i}}(R_{\mathbf{w}}\mathfrak{Flag}(\widehat{\tau}))\right).$$

Combining Theorem 4.9 with Theorem 3.8 and Lemma 3.11, we immediately obtain the following:

Theorem 4.10 [Main result] *Let (Q, W) be a quiver with genteel potential over $\mathfrak{g}^{\mathfrak{i}}$ for some $\mathfrak{i} \supset \mathfrak{i}^{\text{skew}}$. Let $\theta \in M_{\mathbb{R}}$ be general. Then*

$$\mathcal{I}^{\mathfrak{i}}(\log(1_{\text{ss}}(\theta))) = \sum_{\mathbf{w}} \left(\frac{1}{|\operatorname{Aut}(\mathbf{w})|} \sum_{\widehat{\tau} \in \widehat{\mathfrak{T}}_{\mathbf{w}}(\theta)} \nu(\widehat{\tau}) \mathcal{I}^{\mathfrak{i}}(R_{\mathbf{w}} \mathfrak{Flag}(\widehat{\tau})) \right). \tag{46}$$

Theorem 1.1 is the special case where $\mathfrak{i} = \ker(\mathcal{I}_t)$. The classical limit is the case where $\mathfrak{i} = \ker(\mathcal{I})$.

Remark 4.11 While we find the expression of $\operatorname{Mult}^{\Upsilon}(\widehat{\tau})$ in terms of moduli of flags to be interesting, it is of course not generally simple to compute $\mathfrak{Flag}(\widehat{\tau})$. However, it is not difficult to describe the quantum and classical integrals of the terms $R_{\mathbf{w}} \mathfrak{Flag}(\widehat{\tau})$.

First, for the quantum cases, recall from (45) that $\mathfrak{Flag}(\widehat{\tau})$ arose as a product $\kappa_{i_1}^{w_{i_1 j_1}} \cdots \kappa_{i_{l(\mathbf{w})}}^{w_{i_{l(\mathbf{w})} j_{l(\mathbf{w})}}}$. From (2.9), $\mathcal{I}_t(\kappa_{i_j}) = t z^{e_{i_j}} \in \mathbb{C}_t[N^{\oplus}]$. Hence, defining $R'_k := \frac{(-1)^{k-1}}{k(t^k - t^{-k})}$ and $R'_{\mathbf{w}} := \prod_{i,j} R'_{w_{ij}}$, we have

$$\mathcal{I}_t(R_{\mathbf{w}} \mathfrak{Flag}(\widehat{\tau})) = R'_{\mathbf{w}} \cdot z^{w_{i_1 j_1} e_{i_1}} \cdots z^{w_{i_{l(\mathbf{w})} j_{l(\mathbf{w})}} e_{i_{l(\mathbf{w})}}} \in \mathfrak{A}^q. \tag{47}$$

Now for the classical version, recall that the map $\pi_{t \mapsto 1} : \mathfrak{g}^q \to \mathfrak{g}^{\text{cl}}$ takes $\frac{z^n}{t - t^{-1}}$ to z^n. Let us embed $\mathfrak{g}^{\text{cl}}$ into the Weyl algebra $\mathfrak{A}^{\text{cl}}$ as in Example 4.1, so z^n becomes $z^n \partial_{B(n,\cdot)}$. Let $R^{\text{cl}}_k := \frac{(-1)^k}{k^2}$ and $R^{\text{cl}}_{\mathbf{w}} := \prod_{i,j} R^{\text{cl}}_{w_{ij}}$. We then obtain

$$\mathcal{I}(R_{\mathbf{w}} \mathfrak{Flag}(\widehat{\tau})) = R^{\text{cl}}_{\mathbf{w}} \cdot z^{w_{i_1 j_1} e_{i_1}} \partial_{B(w_{i_1 j_1} e_{i_1}, \cdot)} \cdots z^{w_{i_{l(\mathbf{w})} j_{l(\mathbf{w})}} e_{i_{l(\mathbf{w})}}} \partial_{B(w_{i_{l(\mathbf{w})} j_{l(\mathbf{w})}} e_{i_{l(\mathbf{w})}}, \cdot)} \in \mathfrak{A}^{\text{cl}}.$$

We wrote Theorems 4.9 and 4.10 in terms of tropical disk counts because we do not know a nice moduli-theoretic description of the tropical curve multiplicities. However, there are again nice interpretations for the quantum and classical integrals. Consider a tropical curve type $\tau \in \mathfrak{T}_{\mathbf{w}}(\theta)$. For each vertex V of $\tau \setminus \{V_\infty\}$, let u_1 and u_2 be any two of the weighted tangent vectors of edges emanating from V. Define $\operatorname{Mult}(V) := |B(u_1, u_2)|$. Then the classical multiplicity $\operatorname{Mult}(\tau)$ is given by

$$\operatorname{Mult}(\tau) = R^{\text{cl}}_{\mathbf{w}} \prod_V \operatorname{Mult}(V),$$

and the quantum multiplicity $\operatorname{Mult}^q(\tau)$, by which we mean $\operatorname{Mult}(\tau)$ in the quantum cases, is given by

$$\operatorname{Mult}^q(\tau) = R'_{\mathbf{w}} \prod_V [\operatorname{Mult}(V)]_t,$$

where $[\operatorname{Mult}(V)]_t$ is defined as in (15).

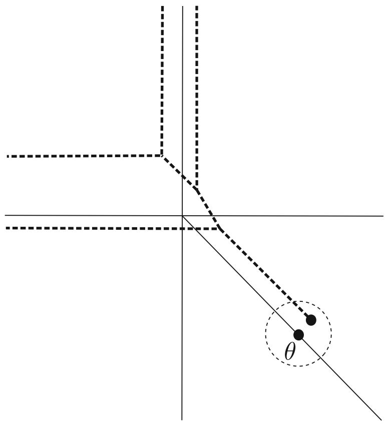

Fig. 5 A tropical disk (given by the dashed segments) representing a type $\tau \in \mathfrak{T}_{\mathbf{w}}(\theta)$. The unbounded segments of the tropical curve are in generically specified lines. The dashed circle around θ is an ϵ-ball $B_\epsilon(\theta)$. The solid rays are the support of the scattering diagram

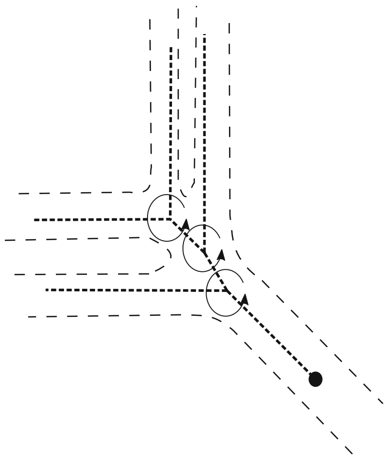

Fig. 6 A tropical ribbon type $\widehat{\tau}$ associated to the disk from Fig. 5. The dashed curves around the tropical ribbon outline a topological realization of the ribbon, the induced orientation being clockwise

Example 4.12 Let us continue our ongoing example of the A_2 quiver $1 \to 2$ with $W = 0$ as in Example 3.10. Recall that in this case, $L \cong \mathbb{Z}^2$, $I = \{1, 2\}$, and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Consider the weight-vector $\mathbf{w} = ((1, 1), (1, 1))$, so $\mathrm{Aut}(\mathbf{w}) = 2^2 = 4$. For $\theta \in \mathbb{R}_{>0}(1, -1)$, a possible tropical disk type $\tau \in \mathfrak{T}_{\mathbf{w}}(\theta)$ is illustrated in Fig. 5:

Since there are 3 vertices, there are 2^3 possible ribbon structures on this tropical disk. One such ribbon structure $\widehat{\tau}$ is illustrated in Fig. 6, namely, the ribbon structure for which $\nu(V) = 1$ for each V.

The multiplicity $\mathrm{Mult}^{\Upsilon}(\widehat{\tau})$ is given by $\mathcal{I}^{\mathrm{i}}(\nu(\widehat{\tau}) R_{\mathbf{w}} \kappa_1 \kappa_1 \kappa_2 \kappa_2)$. We easily see $\nu(\widehat{\tau}) = (-1)^3 = -1$, and $R_{\mathbf{w}} = \left(\frac{1}{q-1}\right)^3$. The space $\mathfrak{Flag}(\widehat{\tau}) = \kappa_1 \kappa_1 \kappa_2 \kappa_2$ is the space of composition series of the following form:

$$(0 \to 0) \subset (\mathbb{C} \to 0) \subset (\mathbb{C}^2 \to 0) \subset (\mathbb{C}^2 \to \mathbb{C}) \subset (\mathbb{C}^2 \to \mathbb{C}^2).$$

All maps $V \to W$ here, where V, W are $\mathbb{C}$-vector spaces, are necessarily the 0 map, and so by Lemma 2.4 we have $\mathfrak{Flag}(\widehat{\tau}) = \mathfrak{Flag}(S_2, S_2, S_1, S_1; \mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2)\kappa_{\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2}$. As in Example 2.5, since we work modulo automorphisms of $\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2$, there is only one flag, and its stabilizer group is $U_2(\mathbb{C})^2$ (one copy of the unitary group $U_2(\mathbb{C})$ for each $\mathbb{C}^2$). Hence, we find

$$\mathfrak{Flag}(\widehat{\tau}) = \frac{1}{|U_2(\mathbb{C})|^2}\kappa_{\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2} = \frac{1}{q^2}\kappa_{\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2}$$

So the contribution of this tropical ribbon $\widehat{\tau}$ to (46) is

$$\frac{1}{|\operatorname{Aut}(\mathbf{w})|}\nu(\widehat{\tau})\mathcal{I}^{\mathfrak{i}}(R_{\mathbf{w}}\mathfrak{Flag}(\widehat{\tau})) = \left(\frac{-1}{4}\right)\mathcal{I}^{\mathfrak{i}}\left(\frac{1}{(q-1)^3}\cdot\frac{1}{q^2}\kappa_{\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2}\right). \tag{48}$$

In the quantum case $\mathfrak{i} = \ker(\mathcal{I}_t)$, we find using Proposition 2.8 that

$$\mathcal{I}_t(\kappa_{\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2}) = \Upsilon(\mathrm{pt})t^{\chi((2,2),(2,2))}z^{(2,2)} = t^4 z^{(2,2)},$$

and so (48) becomes

$$\frac{-z^{(2,2)}}{4(q-1)^3}.$$

Alternatively, this may be computed using (47).

5 Broken lines and theta functions

5.1 Definitions of broken lines and theta functions

Recall the notation and setup of Sect. 3.1. Fix a scattering diagram $\mathfrak{D}$. Suppose we have a commutative ring R and a Λ-graded R-algebra $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ with $A_0 = R$ on which $\mathfrak{g}$ acts via Λ-graded R-algebra derivations. We say that this action is **skew-symmetric** if $\mathfrak{g}_n \cdot A_\lambda = 0$ whenever $\{n, \lambda\} = 0$. Let $\widehat{A}$ denote the (Λ^+)-adic completion of A. Note that $\widehat{G}$ acts on $\widehat{A}$ via Λ-graded R-algebra automorphisms.

Definition 5.1 Let $\lambda \in \Lambda\setminus\{0\}$, $\mathcal{Q} \in \Lambda^\vee_{\mathbb{R}}\setminus \operatorname{Supp}(\mathfrak{D})$. A **broken line** γ with ends $(\lambda, \mathcal{Q})$ is the data of a continuous map $\gamma : (-\infty, 0] \to \Lambda^\vee_{\mathbb{R}}\setminus \operatorname{Joints}(\mathfrak{D})$, values $-\infty =: t_{-1} < t_0 \le t_1 \le \cdots \le t_\ell = 0$, and for each $i = 0, \ldots, \ell$, an associated homogeneous element $a_i \in A_{\lambda_i}$ for some $\lambda_i \in \Lambda\setminus\{0\}$, such that:

(i) $\lambda_0 = \lambda$ and $\gamma(0) = \mathcal{Q}$.
(ii) For $i = 0\ldots, \ell$, $\gamma'(t) = -p^*(\lambda_i)$ for all $t \in (t_{i-1}, t_i)$.
(iii) $a_0 = z^\lambda$.

(iv) For $i = 0, \ldots, \ell - 1$, $\gamma(t_i) \in \mathrm{Supp}(\mathfrak{D})$. Let

$$g_i := \prod_{\substack{(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D} \\ \mathfrak{d} \ni \gamma(t_i)}} \exp(g_{\mathfrak{d}})^{\mathrm{sgn}\langle n_{\mathfrak{d}}, p^*(\lambda)\rangle} \in \widehat{G}. \tag{49}$$

I.e., g_i is the $\epsilon \to 0$ limit of the wall-crossing automorphism $\Phi_{\gamma|_{(t_i-\epsilon, t_i+\epsilon)}}$ defined in (23) (using a smoothing of γ). Then a_{i+1} is a homogeneous term of $g_i \cdot a_i$, other than a_i.

The **theta function** $\vartheta_{\lambda, \mathcal{Q}} \in \widehat{A}$ is defined by

$$\vartheta_{\lambda, \mathcal{Q}} := \sum_{\gamma} a_\gamma,$$

where the sum is over all broken lines γ with ends $(\lambda, \mathcal{Q})$, and a_γ denotes the element of A associated to the last straight segment of γ.

If $\mathfrak{g}$ is skew-symmetric and the action on A is skew-symmetric, then [38, Thm. 2.14] (a refinement of [9, Lemmas 4.7, 4.9]) states the following:

Lemma 5.2 (The Carl–Pumperla–Siebert Lemma) *Suppose $\mathfrak{g}$ is skew-symmetric with skew-symmetric action on A, and suppose $\mathfrak{D} = \mathrm{Scat}(\mathfrak{D}_{\mathrm{in}})$ as in Theorem 3.5. Let γ be a smooth path in $\Lambda_{\mathbb{R}}^{\vee} \setminus \mathrm{Joints}(\mathfrak{D})$ from $\mathcal{Q}_1$ to $\mathcal{Q}_2$, with $\mathcal{Q}_1, \mathcal{Q}_2 \notin \mathrm{Supp}(\mathfrak{D})$. Then for any $\lambda \in \Lambda$,*

$$\vartheta_{\lambda, \mathcal{Q}_2} = \Phi_{\gamma, \mathfrak{D}}(\vartheta_{\lambda, \mathcal{Q}_1}).$$

In any case, we have a copy $\widehat{A}_{\mathcal{Q}}$ of $\widehat{A}$ and a collection of elements $\{\vartheta_{\lambda, \mathcal{Q}} | \lambda \in \Lambda\} \subset \widehat{A}_{\mathcal{Q}}$ associated to every $\mathcal{Q} \in \Lambda_{\mathbb{R}}^{\vee} \setminus \mathrm{Supp}(\mathfrak{D})$. If $\mathfrak{D}$ is consistent, then the identifications of the $\widehat{A}_{\mathcal{Q}}$'s with $\widehat{A}$ are all compatible with the path-ordered products. Furthermore, if Lemma 5.2 holds, it says that the elements $\vartheta_{\lambda, \mathcal{Q}} \in \widehat{A}_{\mathcal{Q}}$ are also compatible with the path-ordered products, thus giving a canonical collection of elements $\vartheta_\lambda \in \widehat{A}$. We may therefore simply denote $\vartheta_{\lambda, \mathcal{Q}}$ as ϑ_λ.

5.2 Hall algebra, quantum, and classical broken lines

Take $\Lambda = N^{\mathrm{prin}} := N \oplus M$, and take $\Lambda^{\oplus} := (N^{\oplus}, 0)$. Denote $M^{\mathrm{prin}} := (N^{\mathrm{prin}})^{\vee} = M \oplus N$. Take $\{\cdot, \cdot\}$ to be the $\mathbb{Z}$-valued skew-symmetric form B^{prin} on N^{prin} defined via

$$B^{\mathrm{prin}}((n_1, m_1), (n_2, m_2)) = B(n_1, n_2) - \langle n_1, m_2 \rangle + \langle n_2, m_1 \rangle. \tag{50}$$

We will write

$$\begin{aligned} \pi^* : N &\to M \\ n &\mapsto B(\cdot, n), \end{aligned}$$

while the map p^* of (21) will be denoted by

$$\begin{aligned} p^{*,\mathrm{prin}} : N^{\mathrm{prin}} &\to M^{\mathrm{prin}} \\ (n, m) &\mapsto B^{\mathrm{prin}}(\cdot, (n, m)) = (\pi^*(n) - m, n). \end{aligned} \tag{51}$$

One can show that B^{prin} is unimodular, so the map $p^{*,\mathrm{prin}}$ is an isomorphism.

Now for our Λ-graded algebras, we take the following:

$$\begin{aligned} A^{\mathrm{Hall,prin}} &:= H_{\mathrm{reg}}(Q, W) \otimes_{\mathbb{C}_{\mathrm{reg}}(t)} \mathbb{C}_{\mathrm{reg}}(t)[M], \\ A^{q,\mathrm{prin}} &:= \mathbb{C}_t[N^{\oplus}] \otimes_{\mathbb{C}_{\mathrm{reg}}(t)} \mathbb{C}_{\mathrm{reg}}(t)[M], \end{aligned}$$

and

$$A^{\mathrm{cl,prin}} := \mathbb{C}[N^{\oplus}] \otimes_{\mathbb{C}} \mathbb{C}[M].$$

The algebra structure on $A^{\mathrm{Hall,prin}}$ is determined by specifying that for $a_d \in H_{\mathrm{reg}}(Q, W)_d$ and $m \in M$, we have

$$a_d * z^m = q^{-\langle d,m \rangle} z^m * a_d. \tag{52}$$

Similarly, for $A^{q,\mathrm{prin}}$ we specify that

$$z^d * z^m = q^{-\langle d,m \rangle} z^m * z^d.$$

Equivalently, $A^{q,\mathrm{prin}}$ is the quantum torus algebra $\mathbb{C}_t[\Lambda]$ with respect to the form B^{prin}. Finally, $A^{\mathrm{cl,prin}}$ is just given the usual algebra structure, making it into $\mathbb{C}[\Lambda]$.

For the action of $\mathfrak{g}^{\mathrm{Hall}}$ on $A^{\mathrm{Hall,prin}}$ we take the adjoint action, i.e.,

$$g \cdot a := [g, a] = ga - ag,$$

using the natural inclusion of $\mathfrak{g}^{\mathrm{Hall}}$ into $(t - t^{-1})^{-1} \cdot A^{\mathrm{Hall,prin}}$ to make sense of the multiplication. Similarly for the action of $\mathfrak{g}^q$ on $A^{q,\mathrm{prin}}$. The action of $\mathfrak{g}^{\mathrm{cl}}$ is the action with respect to the Poisson bracket, i.e.,

$$z^d \cdot z^{(n,m)} := \{z^{(d,0)}, z^{(n,m)}\} = B^{\mathrm{prin}}((d, 0), (n, m)) z^{(n+d,m)}.$$

One sees that the maps $\mathcal{I}_t$, $\mathcal{I}$, and $\pi_{t\mapsto 1}$ extend to homomorphisms between these algebras which commute with the corresponding Lie algebra actions.

We note that we could also define $A^{\mathfrak{i},\mathrm{prin}}$ for any other $\mathfrak{i} \supseteq \mathfrak{i}^{\mathrm{skew}}$ by applying $\mathcal{I}^{\mathfrak{i}}$ to $A^{\mathrm{Hall,prin}}$. The induced $\mathfrak{g}^{\mathfrak{i}}$-actions on $A^{\mathfrak{i},\mathrm{prin}}$ are skew-symmetric, thus yielding new examples of algebras for which Lemma 5.2 holds.

Let $\square$ represent Hall, $\mathfrak{i}$, q, or cl. We can consider scattering diagrams in $M_{\mathbb{R}}^{\mathrm{prin}}$ over $\mathfrak{g}^{\square}$. We take $\mathfrak{D}_{\mathrm{Scat}}^{\square,\mathrm{prin}} := \mathrm{Scat}(\mathfrak{D}_{\mathrm{in}}^{\square,\mathrm{prin}})$, where $\mathfrak{D}_{\mathrm{in}}^{\square,\mathrm{prin}}$ is defined as in (24), but with $(e_i, 0)^{\perp}$ in place of $e_i^{\perp}$, and with $\log 1_{\mathrm{ss}}(p^*(e_i))$ replaced with its image under $\mathcal{I}^{\mathfrak{i}}$,

$\mathcal{I}_t$ or $\mathcal{I}$ if $\square$ represents $\mathfrak{i}$, q or cl, respectively. Note that the intersection of $\mathfrak{D}^{\square}$ with $(M_{\mathbb{R}}, 0) \subset M_{\mathbb{R}}^{\mathrm{prin}}$ agrees with what we previously called $\mathrm{Scat}(\mathfrak{D}_{\mathrm{in}}^{\square})$.

Remark 5.3 Note that all scattering walls have supports of the form $(n, 0)^{\perp}$ for $n \in N$, so they are invariant under translation by $(0, N_{\mathbb{R}})$. It follows that $\vartheta_{\lambda,\mathcal{Q}}$ is invariant under translation of $\mathcal{Q}$ by elements of $(0, N_{\mathbb{R}})$, and when enumerating broken lines, it suffices to consider their projections modulo $(0, N_{\mathbb{R}})$.

Note that $\mathfrak{g}^{\square}$ and the action on $A^{\square,\mathrm{prin}}$ are skew-symmetric if $\square = \mathfrak{i}$, q or cl, but typically not for $\square =$ Hall. With this setup and for $\square =$ Hall, the broken lines with ends $(\lambda, \mathcal{Q})$ with $\mathcal{Q} \in (M \oplus N)_{\mathbb{R}}$ and $\lambda \in \Lambda$ are precisely the **Hall algebra broken lines** discussed in [7]. These will be examined in Sect. 5.3.

We now briefly explain how the above theta functions for $\square =$ cl relate to those of [22]. In the usual cluster algebras language, z^m for $m \in M$ gives the A cluster variables, while z^n for $n \in N$ gives the X cluster variables.[8] In the principle coefficients setting, we have $z^{(m,n)} = \prod_i A_i^{m_i} X_i^{n_i}$, where $m = \sum_i m_i e_i^*$ and $n = \sum_i n_i e_i$. The theta functions on $\mathcal{A}^{\mathrm{prin}}$, $\mathcal{A}$, and $\mathcal{X}$ are obtained as follows:

- Allowing any $\lambda \in \Lambda$, the resulting theta functions $\vartheta_{\lambda}^{\mathrm{prin}} := \vartheta_{\lambda}$ are the theta functions which [22] constructs on the cluster variety with principal coefficients $\mathcal{A}^{\mathrm{prin}}$ (or rather, on some formal version of this in general). The theta functions $\vartheta_{\lambda}^{\mathrm{prin}}$ for $\lambda \in (N^{\oplus})^{\vee}$ (i.e., the positive span of the vectors e_i^*) are the ones examined by Bridgeland [6, Thm. 1.4].
- One obtains the theta function $\vartheta_{\lambda}^{\mathcal{A}}$ on the cluster $\mathcal{A}$-variety via the projection $(n, m) \mapsto m$ of $\vartheta_{\lambda}^{\mathrm{prin}}$ (i.e. setting all the X-variables in ϑ_{λ} equal to 1), assuming that this projection is well-defined, i.e., that it converges. The middle cluster algebra of [22] is defined to be the span of all the $\vartheta_{\lambda}^{\mathcal{A}}$ for which the convergence holds. The corresponding elements λ form a cone $\Xi \subset M$ which contains the Fock–Goncharov [17] cluster complex C. The $\vartheta_{\lambda}^{\mathcal{A}}$ for $\lambda \in C$ give the cluster monomials.
- By applying a change of variables $z^{(m,n)} \mapsto z^n$ to $\{\vartheta_{(n,m)} | m = \pi^*(n)\}$, one obtains [22]'s theta functions $\vartheta_n^{\mathcal{X}}$ for the $\mathcal{X}$-space (or a formal version thereof). Note that $m = \pi^*(n)$ implies $p^{*,\mathrm{prin}}((n, m)) = (0, n)$, so this change of variables essentially amounts to applying $p^{*,\mathrm{prin}}$.

The theta functions for $\square = q$ are among those considered in [38]. It was recently shown in [13] that these form bases for the quantum cluster varieties (or formal versions thereof), thus giving quantum analogs for the results of [22].

5.3 Hall algebra theta functions and the CPS lemma

As noted in Sect. 5.2, $\mathfrak{g}^{\mathrm{Hall}}$ and its action on $A^{\mathrm{Hall,prin}}$ typically fail to be skew-symmetric, and so [38]'s proof of Lemma 5.2 does not apply to Hall algebra broken lines. In fact, we provide here a counterexample, thus showing that:

[8] The A and X notation is due to Fock and Goncharov, cf. [17], and was also used by [22]. Some other authors use the Fomin-Zelevinsky [20] convention of denoting the A-variables by X and the X-variables by Y.

Proposition 5.4 *The analog of Lemma 5.2 does not generally hold for theta functions constructed from Hall algebra broken lines.*

We note though that Hall algebra broken lines are still useful for understanding theta functions. For example, by studying Hall algebra broken lines and then integrating, we can understand the quantum or classical broken lines in terms of quiver Grassmannians, cf. [7].

Recall from Remark 5.3 that we may compute theta functions using the images of broken lines under the projection $M_{\mathbb{R}}^{\mathrm{prin}} = M_{\mathbb{R}} \oplus N_{\mathbb{R}} \to M_{\mathbb{R}}$. **We will work in this projection throughout this subsection.** Furthermore, the theta function we will consider will be of the form $\vartheta_{(0,m),\mathcal{Q}}$ with $m \in \pi^*(N)$. Suppose $a_i \in A_{\lambda_i}$ is the homogeneous element element attached to some straight segment of a broken line contributing to $\vartheta_{(0,m),\mathcal{Q}}$. Then λ_i has the form (n_i, m) for $n_i \in N$. Using (51), we see that the projection of $p^{*,\mathrm{prin}}(\lambda_i)$ modulo $(0, N)$ is $\pi^*(n_i) - m \in \pi^*(N)$. Hence, by Definition 5.1(ii) (modulo $(0, N)$), we have

$$\gamma'(t) = m - \pi^*(n_i) \in \pi^*(N) \tag{53}$$

for all t in the corresponding straight segment of γ. We thus obtain the following:

Lemma 5.5 *For $m \in \pi^*(N)$, all broken lines contributing to $\vartheta_{(0,m),\mathcal{Q}}$ are contained in $\mathcal{Q} + \pi^*(N_{\mathbb{R}})$.*

For our counterexample, we use the A_3-quiver $1 \to 2 \leftarrow 3$. In the corresponding (standard) basis e_1, e_2, e_3 for N, the matrix for B is

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

In general, the map $\pi^* : N \to M$ takes e_i to the i-th row of B. In particular, we see that $\ker(\pi^*)$ is in this case generated by $e_1 - e_3$, and $\mathrm{Image}(\pi^*) = (e_1 - e_3)^{\perp}$, or the span of the first two rows of B (viewed as vectors in the dual basis).

The walls of the initial scattering diagram $\mathfrak{D}_{\mathrm{in}}^{\mathrm{Hall}}$ (i.e., $\mathfrak{D}_{\mathrm{in}}^{\mathrm{Hall,prin}}$ projected to $M_{\mathbb{R}}$) are $\mathfrak{d}_i := (e_i^{\perp}, 1_{\mathrm{ss}}(\pi^*(e_i)))$. Figure 7 depicts a slice of the resulting consistent scattering diagram $\mathfrak{D}_{\mathrm{Scat}}^{\mathrm{Hall}}$ (which exists and agrees with $\mathfrak{D}^{\mathrm{Hall}}$ by Lemmas 3.11 and 3.12), parallel to $\pi^*(N_{\mathbb{R}})$ (taking advantage of Lemma 5.5), with the upward-pointing direction in the figure being parallel to $\pi^*(e_1) = \pi^*(e_3) = (0, 1, 0)$, and the leftward-pointing direction being parallel to $\pi^*(e_2) = (-1, 0, -1)$. In the figure, whenever two walls with attached scattering functions g_{i_1}, g_{i_2} collide with the corresponding n_{i_1}, n_{i_2} satisfying $B(n_{i_1}, n_{i_2}) > 0$ (otherwise reorder), the picture is locally the same, up to a change of variables, as the picture in Example 3.2. Hence, consistency results in one new wall with attached scattering function given up to first order by $[g_{i_1}, g_{i_2}]$. Recall from (26) that $\log(1_{\mathrm{ss}}(\pi^*(e_i))) = (q-1)^{-1}\kappa_i +$(higher order terms). So in particular, the element of $\widehat{\mathfrak{g}}^{\mathrm{Hall}}$ attached to $\mathfrak{d}_{12}$ is $(q-1)^{-2}[\kappa_2, \kappa_1] +$ (higher order terms).

The bold lines in Fig. 7 represent broken lines with ends $(m, \mathcal{Q}_1)$ and $(m, \mathcal{Q}_2)$, where $m = \pi^*(e_1) + \pi^*(e_2)$, i.e., $m = (-1, 1, -1)$ in the basis e_1^*, e_2^*, e_3^* for M. Here we keep in mind that by (53), $\gamma'(t) = m$ on the first straight segment of γ.

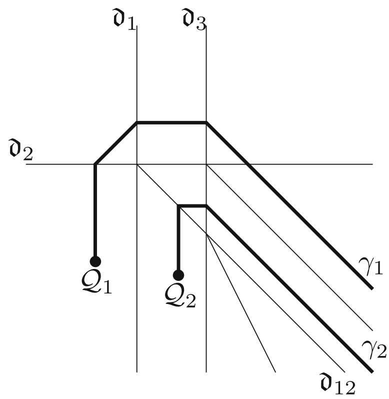

Fig. 7 Counterexample to the Carl–Pumperla–Siebert Lemma for Hall algebra broken lines

By inspection, these are the only broken lines contributing to $\vartheta_{m,\mathcal{Q}_1}$ or $\vartheta_{m,\mathcal{Q}_2}$ whose attached element of $\widehat{A}$ is in $A_{m+e_1+e_2+e_3}$ (the subscript denoting the degree in the N^{prin}-grading). We claim that for our counterexample, it suffices to check that these two attached elements are different from one another. Indeed, while $\mathcal{Q}_1$ and $\mathcal{Q}_2$ do lie on opposite sides of $\mathfrak{d}_1$, the wall-crossing from $\mathcal{Q}_2$ to $\mathcal{Q}_1$ can only affect the grading in the e_1-direction. So since there are no other broken lines with ends $(m, \mathcal{Q}_2)$ whose final monomial has degree $(m + ke_1 + e_2 + e_3)$ for $k \in \mathbb{Z}$, the degree $(m + e_1 + e_2 + e_3)$ part of $\vartheta_{m,\mathcal{Q}_2}$ after crossing $\mathfrak{d}_1$ must still be the final attached monomial of the bottom broken line from the figure.

Recall from Sect. 5.2 that when a broken line with attached element a crosses a wall with attached element $g \in \widehat{\mathfrak{g}}^{\mathrm{Hall}}$, the result of the action of g on a is $\exp[g, a]$. In particular, if $g = \sum_{k\geq 1} g_k$ with $g_k \in \mathfrak{g}^{\mathrm{Hall}}_{kn}$ for some $n \in N^+$, then the action yields $[g, a] +$ (higher order terms). Note that for each of the two broken lines in Fig. 7, it is only the first-order terms of the scattering functions that contribute. Also note that all the signs in the exponents as in (49) are positive for the two broken lines under consideration. We thus compute that the final attached element for the broken line γ_1 ending at $\mathcal{Q}_1$ is

$$a_{\gamma_1} = (q-1)^{-3}[\kappa_2, [\kappa_1, [\kappa_3, z^m]]],$$

Similarly, the final attached element for the broken line γ_2 ending at $\mathcal{Q}_2$ is

$$\begin{aligned} a_{\gamma_2} &= (q-1)^{-3}[[\kappa_2, \kappa_1], [\kappa_3, z^m]] \\ &= (q-1)^{-3}[\kappa_2, [\kappa_1, [\kappa_3, z^m]]] - (q-1)^{-3}[\kappa_1, [\kappa_2, [\kappa_3, z^m]]], \end{aligned}$$

where in the last step we applied the Jacobi identity. So the difference between a_{γ_2} and a_{γ_1} is

$$a_{\gamma_1} - a_{\gamma_2} = (q-1)^{-3}[\kappa_1, [\kappa_2, [\kappa_3, z^m]]]$$

After applying $\mathcal{I}^{\mathfrak{i}}$ for $\mathfrak{i} \supset \mathfrak{i}^{\mathrm{skew}}$, the skew-symmetry of the brackets implies that $[\kappa_2, [\kappa_3, z^m]]$ will vanish, and so this difference is indeed 0 as implied by Lemma 5.2.

But in $A^{\mathrm{Hall,prin}}$ this is not the case, as we will now check. Using (52), we compute that

$$[\kappa_3, z^m] = (q^{-1} - 1)z^m\kappa_3,$$

and then

$$[\kappa_2, (q^{-1} - 1)z^m\kappa_3] = (q^{-1} - 1)z^m(q^{-1}\kappa_2\kappa_3 - \kappa_3\kappa_2). \tag{54}$$

Let $\kappa_{23,f}$ denote the κ-element of the Hall algebra corresponding to the representation $(0 \to \mathbb{C} \xleftarrow{f} \mathbb{C})$. Up to isomorphism, f here is either 0 or Id. Then using Lemma 2.4, we compute

$$\kappa_2\kappa_3 = \kappa_{23,0} + (q-1)\kappa_{23,\mathrm{Id}}$$

and

$$\kappa_3\kappa_2 = \kappa_{23,0}.$$

Now the right-hand side of (54) becomes:

$$(q^{-1} - 1)^2 z^m(\kappa_{23,0} - \kappa_{23,\mathrm{Id}})$$

As a check, note that $\mathcal{I}_t(\kappa_{23,0} - \kappa_{23,\mathrm{Id}}) = 0$, so we do not violate Lemma 5.2 after integrating.

Finally, we must check that that

$$[\kappa_1, z^m(\kappa_{23,0} - \kappa_{23,\mathrm{Id}})]$$

is nonzero. Moving κ_1 past z^m yields

$$[\kappa_1, z^m(\kappa_{23,0} - \kappa_{23,\mathrm{Id}})] = z^m\left(q\kappa_1(\kappa_{23,0} - \kappa_{23,\mathrm{Id}}) - (\kappa_{23,0} - \kappa_{23,\mathrm{Id}})\kappa_1\right). \tag{55}$$

Let $\kappa_{123,a,b}$ denote the κ-element corresponding to the representation $\mathbb{C} \xrightarrow{a} \mathbb{C} \xleftarrow{b} \mathbb{C}$. Rather than completely computing $q\kappa_1(\kappa_{23,0} - \kappa_{23,\mathrm{Id}}) - (\kappa_{23,0} - \kappa_{23,\mathrm{Id}})\kappa_1$, let us just look at the coefficient of $\kappa_{123,0,0}$ in the product. The product $q\kappa_1\kappa_{23,0}$ yields a contribution of $q\kappa_{123,0,0}$, while the product $-\kappa_{23,0}\kappa_1$ yields a contribution of $-\kappa_{123,0,0}$. Thus, (55) includes a term of the form

$$z^m(q-1)\kappa_{123,0,0}$$

which will not cancel with any other terms. In particular, (55) is nonzero, as desired. This proves Proposition 5.4. $\square$

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