Factorisation of quasi K -matrices for quantum symmetric pairs

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Abstract

The theory of quantum symmetric pairs provides a universal K -matrix which is an analog of the universal R -matrix for quantum groups. The main ingredient in the construction of the universal K -matrix is a quasi K -matrix which has so far only been constructed recursively. In this paper we restrict to the cases where the underlying Lie algebra is $\mathfrak{sl}_n$ or the Satake diagram has no black dots. In these cases we give an explicit formula for the quasi K -matrix as a product of quasi K -matrices for Satake diagrams of rank one. This factorisation depends on the restricted Weyl group of the underlying symmetric Lie algebra in the same way as the factorisation of the quasi R -matrix depends on the Weyl group of the Lie algebra. We conjecture that our formula holds in general.

Keywords Quantum groups · Quantum symmetric pairs · Quasi K -matrix · Restricted Weyl group · Restricted root system

Mathematics Subject Classification 17B37 · 81R50

1 Introduction

1.1 Background

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $U_q(\mathfrak{g})$ the corresponding Drinfeld-Jimbo quantised enveloping algebra with positive and negative parts U^+ and U^- , respectively. The quasi R -matrix for $U_q(\mathfrak{g})$ is a canonical element in a completion

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of $U^- \otimes U^+$ which plays a pivotal role in many applications of quantum groups. In the theory of canonical or crystal bases for $U_q(\mathfrak{g})$ developed by G. Lusztig and M. Kashiwara, the quasi R -matrix appears as an intertwiner of two bar involutions on $\Delta(U_q(\mathfrak{g}))$, where Δ denotes the coproduct of $U_q(\mathfrak{g})$. The quasi R -matrix is used to define canonical bases of tensor products of $U_q(\mathfrak{g})$ -modules and of the modified quantised enveloping algebra $\hat{U}$, see [23, Part IV].

Moreover, the quasi R -matrix for $U_q(\mathfrak{g})$ is used in [23, Chapter 32] to construct a family of commutativity isomorphisms. These maps turn suitable categories of $U_q(\mathfrak{g})$ -modules into braided monoidal categories and hence have applications in low-dimensional topology, in particular the construction of invariants of knots and links, see [26]. Up to completion, the commutativity isomorphisms come from a universal R -matrix for $U_q(\mathfrak{g})$.

As in [3] we denote the quasi R -matrix of $U_q(\mathfrak{g})$ by R . One of the key properties of R is that it admits a factorisation as a product of quasi R -matrices for $U_q(\mathfrak{sl}_2)$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots for $\mathfrak{g}$. The quasi R -matrix corresponding to $i \in I$ is given by

$$R_i = \sum_{r \geq 0} (-1)^r q_i^{-r(r-1)/2} \frac{(q_i - q_i^{-1})^r}{[r]_{q_i}!} F_i^r \otimes E_i^r \quad (1.1)$$

where $E_i, F_i \in U_q(\mathfrak{g})$ are generators of the copy of $U_q(\mathfrak{sl}_2)$ labelled by i , and $q_i = q^{(\alpha_i, \alpha_i)/2}$. Let σ_i for $i \in I$ be the generators of the Weyl group W of $\mathfrak{g}$, and let $T_i : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ for $i \in I$ be the corresponding Lusztig automorphisms. For any reduced expression $w_0 = \sigma_{i_1} \cdots \sigma_{i_t}$ of the longest element $w_0 \in W$ define

$$R^{[j]} = (T_{i_1} \cdots T_{i_{j-1}} \otimes T_{i_1} \cdots T_{i_{j-1}})(R_{i_j}) \quad \text{for } j = 1, \dots, t.$$

With this notation the quasi R -matrix of $U_q(\mathfrak{g})$ can be written as

$$R = R^{[t]} \cdot R^{[t-1]} \cdots R^{[2]} \cdot R^{[1]}, \quad (1.2)$$

see [14, 15, 21], [11, 8.30].

In the present paper we aim to find a similar factorisation in the theory of quantum symmetric pairs. Let $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be an involutive Lie algebra automorphism and let $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$ be the corresponding fixed Lie subalgebra. We refer to $(\mathfrak{g}, \mathfrak{k})$ as a symmetric pair. A comprehensive theory of quantum symmetric pairs was developed by Letzter [16, 17]. This theory provides families of subalgebras $B_{\mathbf{c}, \mathbf{s}} \subset U_q(\mathfrak{g})$ with parameters $\mathbf{c}$ and $\mathbf{s}$, which are quantum group analogs of $U(\mathfrak{k})$. Crucially, $B_{\mathbf{c}, \mathbf{s}}$ is a right coideal subalgebra of $U_q(\mathfrak{g})$, that is

$$\Delta(B_{\mathbf{c}, \mathbf{s}}) \subseteq B_{\mathbf{c}, \mathbf{s}} \otimes U_q(\mathfrak{g}).$$

Initially, the theory of quantum symmetric pairs was used to perform harmonic analysis on quantum group analogs of symmetric spaces, see [19, 25]. Recent pioneering work by Bao and Wang [4] and by Ehrig and Stroppel [7] has placed quantum symmetric

pairs in a much broader representation theoretic context. Both papers consider a bar involution for specific quantum symmetric pair coideal subalgebras of type $AIII/AIV$. Moreover, Bao and Wang construct an intertwiner $\mathfrak{X}$ (denoted by Υ in [4]) between the bar involutions on $B_{c,s}$ and on $U_q(\mathfrak{g})$. The intertwiner $\mathfrak{X}$ is an analog for quantum symmetric pairs of the quasi R -matrix for $U_q(\mathfrak{g})$. Using the intertwiner, Bao and Wang show that large parts of Lusztig's theory of canonical bases [23, Part IV] extend to the setting of quantum symmetric pairs. While [4] restricts to the specific quantum symmetric pairs of type $AIII/AIV$, the more recent work [5] develops a theory of canonical bases for all quantum symmetric pairs of finite type.

Following the program outlined in [4], the existence of the bar involution and the intertwiner $\mathfrak{X}$ was proved for general quantum symmetric pairs in [2] and [3], respectively. The intertwiner $\mathfrak{X}$ was then used in [3] to construct a universal K -matrix for $B_{c,s}$ which is an analog of the universal R -matrix for $U_q(\mathfrak{g})$. For this reason we call the intertwiner $\mathfrak{X}$ the *quasi K -matrix* for $B_{c,s}$. The universal K -matrix gives suitable categories of $B_{c,s}$ -modules the structure of a braided module category and allows similar applications in low-dimensional topology as braided monoidal categories, see [13].

In [4] and [3] the quasi K -matrix is defined recursively by the intertwiner property for the two bar involutions. It is noted at the end of [5, 4.4] that this recursion can be solved in principle. However, to this date no closed formula for $\mathfrak{X}$ is known.

1.2 Results

In the present paper we provide a general closed formula for the quasi K -matrix $\mathfrak{X}$ in many cases, and we conjecture that our formula holds for all quantum symmetric pairs of finite type. In doing so, we take guidance from the known factorisation (1.2) of the quasi R -matrix of $U_q(\mathfrak{g})$.

Recall that the involutive automorphism θ is determined up to conjugation by a Satake diagram (I, X, τ) . Here, X is a subset of I and $\tau : I \rightarrow I$ is a diagram automorphism. The τ -orbits in $I \setminus X$ correspond to rank one subdiagrams of the Satake diagram (I, X, τ) . Associated to the symmetric Lie algebra $(\mathfrak{g}, \theta)$ is a restricted root system Σ with Weyl group $\tilde{W} = W(\Sigma)$ which can be considered as a subgroup of W . The Coxeter generators $\tilde{\sigma}_i$ of $\tilde{W}$ are parametrised by the τ -orbits in $I \setminus X$. We introduce the notion of a partial quasi K -matrix $\mathfrak{X}_{\tilde{w}}$ for any $\tilde{w} \in \tilde{W}$ with a reduced expression $\tilde{w} = \tilde{\sigma}_{i_1} \dots \tilde{\sigma}_{i_t}$. More precisely, for $j = 1, \dots, t$ let $\mathfrak{X}_i$ denote the quasi K -matrix corresponding to the rank one Satake subdiagram $(\{i, \tau(i)\} \cup X, X, \tau|_{\{i, \tau(i)\} \cup X})$ of (I, X, τ) . The Lusztig automorphisms $T_w : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ for $w \in W$ allow us to define automorphisms $\tilde{T}_i := T_{\tilde{\sigma}_i}$ for all $i \in I \setminus X$. For $j = 1, \dots, t$ we set

$$\mathfrak{X}_{\tilde{w}}^{[j]} = \Psi \circ \tilde{T}_{i_1} \dots \tilde{T}_{i_{j-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_j})$$

where Ψ denotes an algebra automorphism of an extension of $\tilde{U}^+ = \bigoplus_{\mu \in Q^+(2\Sigma)} U_{\mu}^+$ defined in Eq. (3.45). In analogy to (1.2) we now define the partial quasi K -matrix corresponding to $\tilde{w} \in \tilde{W}$ by

$$\mathfrak{X}_{\tilde{w}} = \mathfrak{X}_{\tilde{w}}^{[t]} \cdot \mathfrak{X}_{\tilde{w}}^{[t-1]} \dots \mathfrak{X}_{\tilde{w}}^{[2]} \cdot \mathfrak{X}_{\tilde{w}}^{[1]}. \quad (1.3)$$

The following theorem is the main result of the present paper. It gives an explicit formula for $\mathfrak{X}$ in the case that $\mathbf{s} = \mathbf{0} = (0, 0, \dots, 0)$ for certain Satake diagrams. Let $\tilde{w}_0 \in \tilde{W}$ denote the longest element.

Theorem A (Corollary 3.21) *Let $\mathfrak{g}$ be of type A_n or $X = \emptyset$. Then the quasi K -matrix $\mathfrak{X}$ for $B_{\mathbf{c}, \mathbf{0}}$ is given by $\mathfrak{X} = \mathfrak{X}_{\tilde{w}_0}$ for any reduced expression of the longest element $\tilde{w}_0 \in \tilde{W}$.*

We conjecture that Theorem A holds for all Satake diagrams of finite type, see Conjecture 3.22. The proof of Theorem A proceeds in three steps and resembles the construction of the quasi R -matrix in [15]. First we construct the quasi K -matrices corresponding to all rank one Satake diagrams of type A_n in the case where $\mathbf{s} = \mathbf{0}$. The difficulty here is that there are many rank one cases, see Table 1. Secondly, we prove Theorem A in rank two by direct calculation. In rank two the longest element $\tilde{w}_0 \in \tilde{W}$ has two reduced expressions. We show for each reduced expression that the element $\mathfrak{X}_{\tilde{w}_0}$ defined by (1.3) satisfies the defining recursive relations for the quasi K -matrix. This involves tedious calculations which we have banned to Appendix A. The calculations require explicit knowledge of the rank one quasi K -matrices. The restriction to type A_n or $X = \emptyset$ in Theorem A hence stems from the fact that the rank one quasi K -matrices are only known in type A_n .

In the cases considered in Appendix A, the calculations imply in particular that $\mathfrak{X}_{\tilde{w}_0}$ is independent of the chosen reduced expression for $\tilde{w}_0$. We conjecture that this is true in general.

Conjecture B (Conjecture 3.14) *Assume that (I, X, τ) is a Satake diagram of rank two. Then the element $\mathfrak{X}_{\tilde{w}}$ defined by (1.3) depends only on $\tilde{w} \in \tilde{W}$ and not on the chosen reduced expression.*

Conjecture B is all that is needed to prove Theorem A for all Satake diagrams of finite type. Indeed, assume that Conjecture B holds for all rank two Satake subdiagrams of the given Satake diagram. Then we can use braid relations for the operators $\tilde{T}_i$ to show that $\mathfrak{X}_{\tilde{w}}$ is independent of the chosen reduced expression for $\tilde{w} \in \tilde{W}$. In the case of the longest element $\tilde{w}_0 \in \tilde{W}$ we choose different reduced expressions for $\tilde{w}_0$ to show that $\mathfrak{X}_{\tilde{w}_0}$ satisfies the defining recursive relations for the quasi K -matrix for $B_{\mathbf{c}, \mathbf{0}}$. In summary, we obtain the following result in the case $\mathbf{s} = \mathbf{0}$.

Theorem C (Theorems 3.17, 3.20) *Let (I, X, τ) be a Satake diagram such that all rank two Satake subdiagrams satisfy Conjecture B. Then the following hold:*

- (1) *The partial quasi K -matrix $\mathfrak{X}_{\tilde{w}}$ depends only on $\tilde{w} \in \tilde{W}$ and not on the chosen reduced expression.*
- (2) *The quasi K -matrix $\mathfrak{X}$ for $B_{\mathbf{c}, \mathbf{0}}$ is given by $\mathfrak{X} = \mathfrak{X}_{\tilde{w}_0}$ where $\tilde{w}_0 \in \tilde{W}$ denotes the longest element.*

In the case $\mathbf{s} \neq \mathbf{0}$ it is harder to give an explicit formula for the quasi K -matrix $\mathfrak{X}_{\mathbf{c}, \mathbf{s}}$ of $B_{\mathbf{c}, \mathbf{s}}$. However, we can make use of the fact that $B_{\mathbf{c}, \mathbf{s}}$ is obtained from $B_{\mathbf{c}, \mathbf{0}}$ via a

twist by a character χ_s of $B_{\mathbf{c},\mathbf{0}}$. We consider the element $R_{\mathbf{c},s}^\theta = \Delta(\mathfrak{X}_{\mathbf{c},s})R(\mathfrak{X}_{\mathbf{c},s}^{-1} \otimes 1)$ which was introduced in [4] under the name quasi R -matrix for $B_{\mathbf{c},s}$, and which lives in a completion of $B_{\mathbf{c},s} \otimes U^+$, see also [13, 3.4]. We show that the quasi K -matrix $\mathfrak{X}_{\mathbf{c},s}$ for $B_{\mathbf{c},s}$ satisfies the relation

$$\mathfrak{X}_{\mathbf{c},s} = (\chi_s \otimes \text{id})(R_{\mathbf{c},\mathbf{0}}^\theta). \quad (1.4)$$

Hence the explicit formulas (1.2) and (1.3) for R and $\mathfrak{X}_{\mathbf{c},\mathbf{0}}$, respectively, provide a formula for the quasi K -matrix of $B_{\mathbf{c},s}$ also in the case $s \neq \mathbf{0}$. However, in this case we do not obtain a factorisation as in Eq. (1.3).

1.3 Organisation

In Sect. 2 we recall background material on the restricted root system Σ and its Weyl group $W(\Sigma)$. We show in particular how $W(\Sigma)$ can be identified with a subgroup $\tilde{W}$ of the Weyl group W of $\mathfrak{g}$, see Proposition 2.7. Section 3 forms the heart of the paper. In Sects. 3.1 and 3.2 we fix notation for quantised enveloping algebras and quantum symmetric pairs, respectively. In Sect. 3.3 we determine explicit closed formulas for the quasi K -matrices $\mathfrak{X}$ for $B_{\mathbf{c},\mathbf{0}}$ in all cases where the Satake diagram is of rank one and of type A_n . We plan to return to the remaining rank one cases in the future. The theory of partial quasi K -matrices $\mathfrak{X}_{\tilde{w}}$ for $\tilde{w} \in \tilde{W}$ is developed in Sect. 3.4. Here the rank two case is crucial. The explicit calculations which prove Theorem A in rank two are left for Appendix A. Based on the rank two results we prove in Theorem 3.17 that $\mathfrak{X}_{\tilde{w}}$ is independent of the chosen reduced expression for $\tilde{w}$. This gives the first part of Theorem C, and the second part as well as Theorem A follow, see Theorem 3.20. The case $s \neq \mathbf{0}$ is treated in Sect. 3.5 where Formula (1.4) is proved in Proposition 3.25.

2 The restricted Weyl group

In this section we recall the construction of involutive automorphisms $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ following [12] and [19]. This allows us to construct a subgroup W^θ of the Weyl group W consisting of elements fixed under the corresponding group automorphism of W . Of particular importance is a subgroup $\tilde{W}$ of W^θ which has an interpretation as the Weyl group of the corresponding restricted root system. The results in this section do not claim originality, but are all in some form contained in [22,24] and also [8].

2.1 Involutive automorphisms of semisimple Lie algebras

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and $\Phi \subset \mathfrak{h}^*$ the corresponding root system. Choose a set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$ where I is an index set labelling the nodes of the Dynkin diagram of $\mathfrak{g}$. Let Φ^+ be the corresponding set of positive roots and set $V = \mathbb{R}\Phi$. For $i \in I$, let $\sigma_i : V \rightarrow V$ denote the reflection at the hyperplane orthogonal to α_i . We write W to

denote the Weyl group generated by the simple reflections σ_i . We fix the W -invariant non-degenerate bilinear form $(-, -)$ on V such that $(\alpha, \alpha) = 2$ for all short roots $\alpha \in \Phi$ in each component. Let $\{e_i, f_i, h_i \mid i \in I\}$ be the Chevalley generators for $\mathfrak{g}$.

Involutive automorphisms of $\mathfrak{g}$ are classified up to conjugation by pairs (X, τ) where $X \subset I$ and $\tau : I \rightarrow I$ is a diagram automorphism. More precisely, for any subset $X \subset I$ let $\mathfrak{g}_X$ denote the Lie subalgebra of $\mathfrak{g}$ generated by $\{e_i, f_i, h_i \mid i \in X\}$. Let Q_X denote the sublattice of the root lattice $Q = \mathbb{Z}\Pi$ generated by $\{\alpha_i \mid i \in X\}$. This is the root lattice for $\mathfrak{g}_X$. Let $\rho_X \in V$ and $\rho_X^\vee \in V^*$ denote the half sum of positive roots and positive coroots for $\mathfrak{g}_X$. The Weyl group W_X of $\mathfrak{g}_X$ is the parabolic subgroup of W generated by $\{\sigma_i \mid i \in X\}$. Let $w_X \in W_X$ denote the longest element of W_X .

Definition 2.1 ([27, p. 109], see also [12, Definition 2.3]) Let $X \subset I$ and $\tau : I \rightarrow I$ a diagram automorphism such that $\tau(X) = X$. The pair (X, τ) is called a Satake diagram if it satisfies the following properties:

- (1) $\tau^2 = \text{id}_I$.
- (2) The action of τ on X coincides with the action of $-w_X$, that is

$$-w_X(\alpha_i) = \alpha_{\tau(i)} \quad \text{for all } i \in X.$$

- (3) If $j \in I \setminus X$ and $\tau(j) = j$, then $\alpha_j(\rho_X^\vee) \in \mathbb{Z}$.

Remark 2.2 When we need to identify the underlying Lie algebra, we write (I, X, τ) to indicate the Satake diagram. With this convention, if (I, X, τ) is a Satake diagram and $i \in I \setminus X$ then $(\{i, \tau(i)\} \cup X, X, \tau|_{\{i, \tau(i)\} \cup X})$ is also a Satake diagram.

Graphically, the ingredients of a Satake diagram are recorded in the Dynkin diagram of $\mathfrak{g}$. The nodes labelled by X are colored black and a double sided arrow is used to indicate the diagram automorphism. See [1, p. 32/33] for a complete list of Satake diagrams for simple $\mathfrak{g}$.

We can now construct an involution θ corresponding to the Satake diagram (X, τ) as in [12, 2.4]. Let $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ be the Chevalley involution given by

$$\omega(e_i) = -f_i, \quad \omega(h_i) = -h_i, \quad \omega(f_i) = -e_i. \quad (2.1)$$

The diagram automorphism τ can be lifted to a Lie algebra automorphism $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$ denoted by the same symbol. Recall from [28, Lemma 56] that the action of the Weyl group W on $\mathfrak{h}$ can be lifted to an action of the corresponding braid group $Br(W)$ on $\mathfrak{g}$. We denote the action of $w \in Br(W)$ on $\mathfrak{g}$ by $\text{Ad}(w)$. Let $s : I \rightarrow \mathbb{C}^\times$ be a function such that

$$s(i) = 1 \quad \text{if } i \in X \text{ or } \tau(i) = i, \quad (2.2)$$

$$\frac{s(i)}{s(\tau(i))} = (-1)^{\alpha_i(2\rho_X^\vee)} \quad \text{if } i \notin X \text{ and } \tau(i) \neq i. \quad (2.3)$$

This map extends to a group homomorphism $s_Q : Q \rightarrow \mathbb{C}^\times$ such that $s_Q(\alpha_i) = s(i)$ for each simple root α_i . Let $\text{Ad}(s) : \mathfrak{g} \rightarrow \mathfrak{g}$ denote the Lie algebra automorphism such that the restriction to any root space $\mathfrak{g}_\alpha$ is given by multiplication by $s_Q(\alpha)$.

Given a Satake diagram (X, τ) we define a Lie algebra automorphism $\theta = \theta(X, \tau)$ of $\mathfrak{g}$ by

$$\theta(X, \tau) = \text{Ad}(s) \circ \text{Ad}(w_X) \circ \tau \circ \omega : \mathfrak{g} \rightarrow \mathfrak{g}. \quad (2.4)$$

Here, the longest element $w_X \in W_X$ is considered as an element in the braid group $Br(W)$. The Lie algebra automorphism $\theta = \theta(X, \tau)$ is involutive, that is $\theta^2 = \text{id}$. Any involutive Lie algebra automorphism of $\mathfrak{g}$ is $\text{Aut}(\mathfrak{g})$ -conjugate to an automorphism of the form $\theta(X, \tau)$, see for example [12, Theorems 2.5, 2.7].

2.2 The subgroup W^Θ

For any Satake diagram (X, τ) , the automorphism $\theta = \theta(X, \tau)$ satisfies $\theta(\mathfrak{h}) = \mathfrak{h}$. More explicitly, the definition (2.4) implies that $\theta|_{\mathfrak{h}} = -w_X \circ \tau$. The dual map $\Theta : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ is given by the same expression

$$\Theta = -w_X \circ \tau \quad (2.5)$$

where now w_X and τ act on $\mathfrak{h}^*$. We obtain a group automorphism

$$\Theta_W : W \rightarrow W, \quad w \mapsto \Theta \circ w \circ \Theta. \quad (2.6)$$

Let $W^\Theta = \{w \in W \mid \Theta_W(w) = w\}$ denote the subgroup of elements fixed by Θ_W . In this section we recall the structure of the subgroup W^Θ following [8] and [22]. For any $i \in I$ one has

$$\Theta_W(\sigma_i) = (-w_X \circ \tau) \circ \sigma_i \circ (-w_X \circ \tau) = w_X \sigma_{\tau(i)} w_X. \quad (2.7)$$

This formula and property (2) in Definition 2.1 imply that W_X is a subgroup of W^Θ . Recall that for any subset $J \subset I$ we write w_J to denote the longest element in the parabolic subgroup W_J . We use this notation in particular for $J = \{i, \tau(i)\} \cup X$. For all $i \in I \setminus X$ define

$$\tilde{\sigma}_i = w_{\{i, \tau(i)\} \cup X} w_X^{-1}. \quad (2.8)$$

Recall that there exists a diagram automorphism $\tau_0 : I \rightarrow I$ such that the longest element $w_0 \in W$ satisfies

$$w_0(\alpha_i) = -\alpha_{\tau_0(i)} \quad (2.9)$$

for all $i \in I$. By inspection of the list of Satake diagrams in [1, p. 32/33] one sees that the set X is τ_0 -invariant.

By Remark 2.2 the triple $(\{i, \tau(i)\} \cup X, X, \tau|_{\{i, \tau(i)\} \cup X})$ is a Satake diagram for any $i \in I \setminus X$. Hence, for any $i \in I \setminus X$ there exists a diagram automorphism $\tau_{0,i} : \{i, \tau(i)\} \cup X \rightarrow \{i, \tau(i)\} \cup X$ such that X is $\tau_{0,i}$ -invariant and

$$w_{\{i, \tau(i)\} \cup X}(\alpha_j) = -\alpha_{\tau_{0,i}(j)} \quad (2.10)$$

for all $j \in \{i, \tau(i)\} \cup X$. With this notation we get

$$w_{\{i, \tau(i)\} \cup X} \sigma_j = \sigma_{\tau_{0,i}(j)} w_{\{i, \tau(i)\} \cup X} \quad (2.11)$$

and hence

$$w_{\{i, \tau(i)\} \cup X} w_X = \tau_{0,i}(w_X) w_{\{i, \tau(i)\} \cup X} = w_X w_{\{i, \tau(i)\} \cup X}.$$

This proves that w_X and $w_{\{i, \tau(i)\} \cup X}$ commute in W for any $i \in I \setminus X$. Hence Eq. (2.7) implies that

$$\tilde{\sigma}_i \in W^\Theta \quad \text{for all } i \in I \setminus X. \quad (2.12)$$

Let $\tilde{W} \subset W^\Theta$ denote the subgroup of W generated by all $\tilde{\sigma}_i$ for $i \in I \setminus X$. Let $l : W \rightarrow \mathbb{N}_0$ denote the length function with respect to W . Let

$$W^X = \{w \in W \mid l(\sigma_i w) > l(w) \text{ for all } i \in X\} \quad (2.13)$$

be the set of minimal length left coset representatives of W_X . By [24, 25.1] the subset

$$\mathcal{W} = \{w \in W^X \mid w W_X = W_X w\} \quad (2.14)$$

is a subgroup of W . By (2.8) and (2.11) we have $\tilde{\sigma}_i \in \mathcal{W}$ for all $i \in I \setminus X$. Let $\mathcal{W}^\tau = \{w \in \mathcal{W} \mid \tau(w) = w\}$. As $\tau(X) = X$ the definition of $\tilde{\sigma}_i$ implies that $\tilde{W} \subseteq \mathcal{W}^\tau$. The following lemma is a generalisation of [24, A.1(a)] and [8, Lemma 2] in the spirit of [8, Remark 8].

Lemma 2.3 *Any $w \in \mathcal{W}^\tau$ may be written as $w = \tilde{\sigma}_{i_1} \cdots \tilde{\sigma}_{i_k}$ such that $\tilde{\sigma}_{i_1}, \dots, \tilde{\sigma}_{i_k} \in \tilde{W}$ and $l(w) = l(\tilde{\sigma}_{i_1}) + \cdots + l(\tilde{\sigma}_{i_k})$.*

Lemma 2.3 implies that $\tilde{W} = \mathcal{W}^\tau$. Set $\tilde{S} = \{\tilde{\sigma}_i \mid i \in I \setminus X\}$. The following theorem is stated in [24, 25.1] and proved in [22, Theorem 5.9(i)]. A proof using only Weyl group combinatorics is indicated in [8, Remark 8].

Theorem 2.4 *The pair $(\tilde{W}, \tilde{S})$ is a Coxeter system.*

Let $\lambda : \tilde{W} \rightarrow \mathbb{N}_0$ denote the length function with respect to $\tilde{W}$. The following proposition, proved in [22, Theorem 5.9(iii)] and indicated in [8, Corollary 6] implies that reduced expressions in $\tilde{W}$ are also reduced in W .

Proposition 2.5 *Let $w, w' \in \tilde{W}$. Then $l(w w') = l(w) + l(w')$ if and only if $\lambda(w w') = \lambda(w) + \lambda(w')$.*

Remark 2.6 By (2.11) the group $\tilde{W}$ acts on W_X by conjugation. Moreover $W_X \cap \tilde{W} = \{\text{id}\}$. One can show that $W^\Theta = W_X \rtimes \tilde{W}$.

2.3 The restricted root system

The group $\tilde{W}$ has an interpretation as the Weyl group of the restricted root system of the symmetric Lie algebra $(\mathfrak{g}, \theta)$. This fact is implicit in [22] but we feel that it is beneficial to explain this connection in some detail. As $\theta(\mathfrak{h}) = \mathfrak{h}$, we obtain a direct sum decomposition

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{a} \quad (2.15)$$

where $\mathfrak{h}_1 = \{x \in \mathfrak{h} \mid \theta(x) = x\}$ and $\mathfrak{a} = \{x \in \mathfrak{h} \mid \theta(x) = -x\}$. The restricted root system $\Sigma \subset \mathfrak{a}^*$ is obtained by restriction of all roots in Φ to $\mathfrak{a}$, that is

$$\Sigma = \Phi|_{\mathfrak{a}} \setminus \{0\}. \quad (2.16)$$

As $\Theta(\Phi) = \Phi$ we have $\Theta(V) = V$. Moreover, the inner product $(-, -)$ is Θ -invariant. Hence we obtain an orthogonal direct sum decomposition

$$V = V_{+1} \oplus V_{-1} \quad (2.17)$$

where $V_\lambda = \{\alpha \in V \mid \Theta(\alpha) = \lambda\alpha\}$ for $\lambda = \pm 1$. Indeed, any $\alpha \in V$ can be written as

$$\alpha = \frac{\alpha + \Theta(\alpha)}{2} + \frac{\alpha - \Theta(\alpha)}{2} \quad (2.18)$$

where $(\alpha + \lambda\Theta(\alpha))/2 \in V_\lambda$ for $\lambda = \pm 1$. For any $\beta \in V_{-1}$ and $h \in \mathfrak{h}_1$, we have $\beta(h) = 0$ as $\beta(h) = \beta(\Theta(h)) = \Theta(\beta)(h) = -\beta(h)$. Hence we may consider V_{-1} as a subspace of $\mathfrak{a}^*$ and $V_{-1} = \mathbb{R}\Sigma$. For any $\beta \in V$ we define

$$\tilde{\beta} = \frac{\beta - \Theta(\beta)}{2}. \quad (2.19)$$

Equation (2.18) implies that $\Sigma = \{\tilde{\beta} \mid \beta \in \Phi, \tilde{\beta} \neq 0\}$. We write $\tilde{\Pi} = \{\tilde{\alpha}_i \mid i \in I \setminus X\}$ and define $Q(\Sigma) = \mathbb{Z}\Sigma = \mathbb{Z}\tilde{\Pi}$ and $Q^+(\Sigma) = \mathbb{N}_0\tilde{\Pi}$. For any $w \in W^\Theta$, we have

$$\begin{aligned} w(\tilde{\beta}) &= w\left(\frac{\beta - \Theta(\beta)}{2}\right) = \frac{w(\beta) - w(\Theta(\beta))}{2} \\ &= \frac{w(\beta) - \Theta(w(\beta))}{2} = \widetilde{w(\beta)}. \end{aligned} \quad (2.20)$$

Hence the group W^Θ acts on Σ .

We have an inner product on V_{-1} by restriction of the inner product on V . As the inner product on V is W -invariant and V_{-1} is a W^Θ -invariant subspace, the restriction

of the inner product on V_{-1} is W^Θ -invariant. For any $i \in X$, we have $\alpha_i \in V_{+1}$. Hence the direct sum decomposition (2.17) implies that $\sigma_i(\tilde{\beta}) = \tilde{\beta}$ for all $\tilde{\beta} \in \Sigma$, $i \in X$. The group $\tilde{W}$ on the other hand can be interpreted in terms of the restricted root system Σ . We include a pedestrian proof, avoiding the more sophisticated setting in [22].

Proposition 2.7 (1) *The reflections at the hyperplanes perpendicular to elements of Σ generate a finite reflection group $W(\Sigma)$.*

(2) *There is an isomorphism of groups $\rho : \tilde{W} \rightarrow W(\Sigma)$ which sends $\tilde{\sigma}_i$ to the reflection at the hyperplane orthogonal to $\tilde{\alpha}_i$ for any $i \in I \setminus X$.*

Proof For any $i \in I \setminus X$ we have

$$\begin{aligned}\tilde{\sigma}_i(\tilde{\alpha}_i) &= (w_X^{-1} w_{\{i, \tau(i)\} \cup X})(\tilde{\alpha}_i) \\ &= w_{\{i, \tau(i)\} \cup X}(\tilde{\alpha}_i) \\ &= (w_{\{i, \tau(i)\} \cup X}(\alpha_i))|_{\mathfrak{a}}.\end{aligned}$$

It hence follows from (2.10) that

$$\tilde{\sigma}_i(\tilde{\alpha}_i) = -\tilde{\alpha}_i. \quad (2.21)$$

Now suppose $\tilde{\beta} \in V_{-1}$ such that $(\tilde{\beta}, \tilde{\alpha}_i) = 0$. Using the W^Θ -invariance of the bilinear form on V_{-1} , we obtain

$$\begin{aligned}(\tilde{\sigma}_i(\tilde{\beta}), \tilde{\alpha}_i) &= (\tilde{\beta}, \tilde{\sigma}_i(\tilde{\alpha}_i)) \\ &= -(\tilde{\beta}, \tilde{\alpha}_i) \\ &= 0.\end{aligned}$$

On the other hand, by the definition of $\tilde{\sigma}_i$ we have

$$\tilde{\sigma}_i(\beta) = \beta + n_i \alpha_i + n_{\tau(i)} \alpha_{\tau(i)} + \sum_{j \in X} n_j \alpha_j$$

for some $n_j \in \mathbb{Q}$. From this we obtain

$$\tilde{\sigma}_i(\tilde{\beta}) = \tilde{\beta} + m_i \tilde{\alpha}_i$$

where $m_i = n_i + n_{\tau(i)}$. Hence,

$$\begin{aligned}0 &= (\tilde{\sigma}_i(\tilde{\beta}), \tilde{\alpha}_i) = (\tilde{\beta} + m_i \tilde{\alpha}_i, \tilde{\alpha}_i) \\ &= (\tilde{\beta}, \tilde{\alpha}_i) + m_i (\tilde{\alpha}_i, \tilde{\alpha}_i) \\ &= m_i (\tilde{\alpha}_i, \tilde{\alpha}_i).\end{aligned}$$

The inner product is positive definite so it follows that $m_i = 0$. Hence $\tilde{\sigma}_i(\tilde{\beta}) = \tilde{\beta}$. This together with (2.21) implies that $\tilde{\sigma}_i$ is the reflection at the hyperplane orthogonal

to $\tilde{\alpha}_i$. As seen above, the action of $\tilde{W}$ on V_{-1} gives a group homomorphism

$$\rho : \tilde{W} \rightarrow W(\Sigma).$$

Adapting the proof of [10, Theorem 1.5] one shows that ρ is surjective which implies part 1. To prove part 2 it remains to show that ρ is injective. This is a consequence of Lemma 2.8 below. $\square$

Lemma 2.8 *The action of $\tilde{W}$ on Σ is faithful.*

Proof Assume that there exists $w \in \tilde{W}$ such that $w \neq 1_{\tilde{W}}$ and

$$w(\tilde{\alpha}_i) = \tilde{\alpha}_i \quad \text{for all } i \in I.$$

We can rewrite this formula as

$$w(\alpha_i) - w(\Theta(\alpha_i)) = \alpha_i - \Theta(\alpha_i). \quad (2.22)$$

For all $i \in X$ we have $w(\alpha_i) > 0$ as $l(w\sigma_i) = l(w) + 1$. Hence there exists $i \in I \setminus X$ such that $w(\alpha_i) < 0$. In this case also $w(\alpha_{\tau(i)}) < 0$ since elements of $\tilde{W}$ are fixed under τ . Consider Eq. (2.22) for this i : The right hand side lies in Q^+ and is of the form

$$\alpha_i + \alpha_{\tau(i)} + \sum_{j \in X} n_j \alpha_j \quad (2.23)$$

where $n_j \in \mathbb{N}_0$ for each $j \in X$. We can write the left hand side as

$$w(\alpha_i) - w(\Theta(\alpha_i)) = w(\alpha_i) + w(\alpha_{\tau(i)}) + \sum_{j \in X} m_j w(\alpha_j) \quad (2.24)$$

where $m_j \in \mathbb{N}_0$ for each $j \in X$. Hence inserting (2.23) and (2.24) into (2.22), we get

$$w(\alpha_i) + w(\alpha_{\tau(i)}) + \sum_{j \in X} m_j w(\alpha_j) = \alpha_i + \alpha_{\tau(i)} + \sum_{j \in X} n_j \alpha_j.$$

Now we apply the tilde map to the above equation. The terms involving α_j for $j \in X$ vanish, because the tilde map is zero on Q_X and w commutes with Θ . We get

$$\widetilde{w(\alpha_i)} + \widetilde{w(\alpha_{\tau(i)})} = \tilde{\alpha}_i + \tilde{\alpha}_{\tau(i)}.$$

The right hand side lies in $Q^+(\Sigma)$. The left hand side lies in $-Q^+(\Sigma)$ because $w(\alpha_i)$ and $w(\alpha_{\tau(i)})$ lie in $-Q^+$. Hence both sides of the equation must vanish. However, this is not possible, in particular for the right hand side which is $2\tilde{\alpha}_i$. We have a contradiction. $\square$

Proposition 2.7 has the following consequence which we note for later reference.

Corollary 2.9 *For any $i \in I \setminus X$ and $\mu \in Q(\Sigma)$ the relation*

$$\tilde{\sigma}_i(\mu) = \mu - 2 \frac{(\mu, \tilde{\alpha}_i)}{(\tilde{\alpha}_i, \tilde{\alpha}_i)} \tilde{\alpha}_i$$

$$\text{holds and } 2 \frac{(\mu, \tilde{\alpha}_i)}{(\tilde{\alpha}_i, \tilde{\alpha}_i)} \in \mathbb{Z}.$$

3 Factorisation of quasi K -matrices

As explained in the introduction, the quasi R -matrix for $U_q(\mathfrak{g})$ [23, Chapter 4] has a deep connection to the Weyl group W . This was first observed by Levendorskiĭ and Soibelman [21], and independently by Kirillov and Reshetikhin [14]. In this section, we establish a similar connection between the quasi K -matrix $\mathfrak{K}$ for a quantum symmetric pair and the restricted Weyl group $\tilde{W}$. In particular, we will see in many cases that the quasi K -matrix $\mathfrak{K}$ factorises into a product of quasi K -matrices for Satake diagrams of rank one.

We first fix notation for quantised enveloping algebras and quantum symmetric pairs in Sects. 3.1 and 3.2. Recall that the construction of quantum symmetric pairs depends on an additional choice of parameters $\mathbf{c} \in \mathcal{C}$ and $\mathbf{s} \in \mathcal{S}$, see Definition 3.1. In Sects. 3.3 and 3.4 we find explicit formulas for $\mathfrak{K}$ in the case $\mathbf{s} = (0, \dots, 0)$. Section 3.5 then deals with the case of general parameters $\mathbf{s}$.

3.1 Quantised enveloping algebras

In this section we fix notation for quantum groups mostly following the conventions in [23] and [11]. Let q be an indeterminate and $\mathbb{K}$ a field of characteristic zero. Denote by $\mathbb{K}(q)$ the field of rational functions in q with coefficients in $\mathbb{K}$. The quantised enveloping algebra $U_q(\mathfrak{g})$ is the associative $\mathbb{K}(q)$ -algebra generated by elements $E_i, F_i, K_i^{\pm 1}$ for all $i \in I$ satisfying the relations given in [23, 3.1.1] or [11, 4.3]. The algebra $U_q(\mathfrak{g})$ has the structure of a Hopf algebra with coproduct Δ , counit ε and antipode S given by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \varepsilon(E_i) = 0, \quad S(E_i) = -K_i^{-1} E_i, \quad (3.1)$$

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \varepsilon(F_i) = 0, \quad S(F_i) = -F_i K_i, \quad (3.2)$$

$$\Delta(K_i) = K_i \otimes K_i, \quad \varepsilon(K_i) = 1, \quad S(K_i) = K_i^{-1}, \quad (3.3)$$

for all $i \in I$. Let U^+, U^- and U^0 denote the subalgebras of $U_q(\mathfrak{g})$ generated by $\{E_i \mid i \in I\}$, $\{F_i \mid i \in I\}$ and $\{K_i^{\pm 1} \mid i \in I\}$, respectively. For $\lambda = \sum_{i \in I} n_i \alpha_i \in Q$ we also write

$$K_\lambda = \prod_{i \in I} K_i^{n_i}. \quad (3.4)$$

The elements K_λ for $\lambda \in Q$ form a vector space basis of U^0 . For any U^0 -module M and any $\mu \in Q$ let

$$M_\mu = \{m \in M \mid K_i m = q^{(\mu, \alpha_i)} m \text{ for all } i \in I\} \quad (3.5)$$

denote the corresponding weight space. In particular, both U^+ and U^- are U^0 -modules via the left adjoint action so we may apply the above notation. This gives algebra gradings

$$U^+ = \bigoplus_{\mu \in Q^+} U_\mu^+, \quad U^- = \bigoplus_{\mu \in Q^+} U_{-\mu}^-. \quad (3.6)$$

By [23, 39.4.3] the braid group $Br(W)$ acts on $U_q(\mathfrak{g})$ by algebra automorphisms analogously to the action of $Br(W)$ on $\mathfrak{g}$. For any $i \in I$ let T_i be the algebra automorphism of $U_q(\mathfrak{g})$ denoted by $T''_{i,1}$ in [23, 37.1]. The automorphisms T_i for $i \in I$ satisfy braid relations

$$\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}} \quad (3.7)$$

where m_{ij} denotes the order of $\sigma_i \sigma_j \in W$. Hence for any $w \in W$, there is a well-defined algebra automorphism $T_w : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ defined by

$$T_w = T_{i_1} T_{i_2} \cdots T_{i_k} \quad (3.8)$$

where $w = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$ is a reduced expression.

In Sects. 3.2 and 3.4 it is necessary to consider a completion $\mathcal{U}$ of $U_q(\mathfrak{g})$. We recall the construction of $\mathcal{U}$ following [3, 3.1]. Let $\mathcal{O}_{int}$ denote the category of finite dimensional $U_q(\mathfrak{g})$ -modules of type 1, and let $\mathcal{Vect}$ denote the category of vector spaces over $\mathbb{K}(q)$. Both categories are tensor categories and the forgetful functor $\mathcal{For} : \mathcal{O}_{int} \rightarrow \mathcal{Vect}$ is a tensor functor. We let $\mathcal{U} = \text{End}(\mathcal{For})$ be the $\mathbb{K}(q)$ -algebra of all natural transformations from the functor $\mathcal{For}$ to itself. Observe that $U_q(\mathfrak{g})$ and $\widehat{U}^+ = \prod_{\mu \in Q^+} U_\mu^+$ may be considered as subalgebras of $\mathcal{U}$, see [3, 3.1]. We usually write elements in $\widehat{U}^+$ additively as infinite sums $\sum_{\mu \in Q^+} u_\mu$ with $u_\mu \in U_\mu^+$.

By [23, 1.2.13] for any $i \in I$ there exist uniquely determined linear maps ${}_i r : U^+ \rightarrow U^+$ satisfying

$${}_i r(E_j) = \delta_{ij}, \quad {}_i r(xy) = {}_i r(x)y + q^{(\alpha_i, \mu)} x {}_i r(y) \quad (3.9)$$

$${}_i r(E_j) = \delta_{ij}, \quad {}_i r(xy) = q^{(\alpha_i, \nu)} r_i(x)y + x r_i(y), \quad (3.10)$$

for all $j \in I$, $x \in U_\mu^+$ and $y \in U_\nu^+$ where $\mu, \nu \in Q^+$. We may extend the skew derivation ${}_i r : U^+ \rightarrow U^+$ to a linear map

$${}_i r : \widehat{U}^+ \rightarrow \widehat{U}^+, \quad \sum_{\mu \in Q^+} u_\mu \mapsto \sum_{\mu \in Q^+} {}_i r(u_\mu) \quad (3.11)$$

where ${}_i r(u_\mu)$ is the component in $U_{\mu-\alpha_i}^+$ for all $\mu \in Q^+$ with $\mu \geq \alpha_i$. Similarly we may extend the skew derivation $r_i : U^+ \rightarrow U^+$ to a linear map $r_i : \widehat{U^+} \rightarrow \widehat{U^+}$.

Finally, recall that the bar involution for $U_q(\mathfrak{g})$ is the $\mathbb{K}$ -algebra automorphism

$$-^U : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}), \quad u \mapsto \overline{u}^U \quad (3.12)$$

defined by $\overline{q}^U = q^{-1}$ and $\overline{E_i}^U = E_i$, $\overline{F_i}^U = F_i$, $\overline{K_i}^U = K_i^{-1}$ for all $i \in I$.

3.2 Quantum symmetric pairs

We recall the definition of quantum symmetric pair coideal subalgebras $B_{\mathbf{c},\mathbf{s}}$ as introduced by Letzter [16]. Here we follow the conventions in [12]. Let (X, τ) be a Satake diagram and let $s : I \rightarrow \mathbb{K}^\times$ be a function satisfying equations (2.2) and (2.3). Let $\mathcal{M}_X = U_q(\mathfrak{g}_X)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i, F_i, K_i^{\pm 1} \mid i \in X\}$, and let U_Θ^0 be the subalgebra of U^0 generated by $\{K_\lambda \mid \lambda \in Q, \Theta(\lambda) = \lambda\}$. Quantum symmetric pair coideal subalgebras depend on a choice of parameters $\mathbf{c} = (c_i)_{i \in I \setminus X} \in (\mathbb{K}(q)^\times)^{I \setminus X}$ and $\mathbf{s} = (s_i)_{i \in I \setminus X} \in (\mathbb{K}(q)^\times)^{I \setminus X}$ with added constraints. Define

$$I_{ns} = \{i \in I \setminus X \mid \tau(i) = i \text{ and } a_{ij} = 0 \text{ for all } j \in X\} \quad (3.13)$$

where $a_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ for $i, j \in I$ are the entries of the Cartan matrix of $\mathfrak{g}$. Define the parameter sets

$$\mathcal{C} = \{\mathbf{c} \in (\mathbb{K}(q)^\times)^{I \setminus X} \mid c_i = c_{\tau(i)} \text{ if } \tau(i) \neq i \text{ and } (\alpha_i, \Theta(\alpha_i)) = 0\}, \quad (3.14)$$

$$\mathcal{S} = \{\mathbf{s} \in (\mathbb{K}(q)^\times)^{I \setminus X} \mid s_j \neq 0 \Rightarrow (j \in I_{ns} \text{ and } a_{ij} \in -2\mathbb{N}_0 \forall i \in I_{ns} \setminus \{j\})\}. \quad (3.15)$$

Definition 3.1 Let (X, τ) be a Satake diagram, $\mathbf{c} = (c_i)_{i \in I \setminus X} \in \mathcal{C}$ and $\mathbf{s} = (s_i)_{i \in I \setminus X} \in \mathcal{S}$. The quantum symmetric pair coideal subalgebra $B_{\mathbf{c},\mathbf{s}} = B_{\mathbf{c},\mathbf{s}}(X, \tau)$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $\mathcal{M}_X$, U_Θ^0 and the elements

$$B_i = F_i - c_i s(\tau(i)) T_{w_X}(E_{\tau(i)}) K_i^{-1} + s_i K_i^{-1} \quad (3.16)$$

for all $i \in I \setminus X$.

By [12, Proposition 5.2] the algebra $B_{\mathbf{c},\mathbf{s}}$ is a right coideal subalgebra of $U_q(\mathfrak{g})$, that is

$$\Delta(B_{\mathbf{c},\mathbf{s}}) \subseteq B_{\mathbf{c},\mathbf{s}} \otimes U_q(\mathfrak{g}). \quad (3.17)$$

All through this paper we assume that the parameters $\mathbf{c} = (c_i)_{i \in I \setminus X} \in \mathcal{C}$ and $\mathbf{s} = (s_i)_{i \in I \setminus X} \in \mathcal{S}$ satisfy the additional relations

$$c_{\tau(i)} = q^{(\alpha_i, \Theta(\alpha_i) - 2\rho_X)} \overline{c_i}^U, \quad (3.18)$$

$$\overline{s_i}^U = s_i \quad (3.19)$$

for all $i \in I \setminus X$. By [2, Theorem 3.11], if condition (3.18) holds, then there exists a $\mathbb{K}$ -algebra automorphism

$$^{-B} : B_{\mathbf{c}, \mathbf{s}} \rightarrow B_{\mathbf{c}, \mathbf{s}}, \quad b \mapsto \overline{b}^B \quad (3.20)$$

such that

$$^{-B}|_{\mathcal{M}_X U_\Theta^0} = {}^{-U}|_{\mathcal{M}_X U_\Theta^0}, \quad \overline{B_i}^B = B_i \quad \text{for all } i \in I \setminus X. \quad (3.21)$$

The map $^{-B}$ is called the bar involution for $B_{\mathbf{c}, \mathbf{s}}$ and plays a similar role as the bar involution (3.12) for $U_q(\mathfrak{g})$. In particular there exists a quasi K -matrix $\mathfrak{X} \in \mathcal{U}$ which resembles the quasi R -matrix for $U_q(\mathfrak{g})$. More explicitly, following a program outlined by H. Bao and W. Wang in [4], it was proved in [3, Theorem 6.10] that there exists a uniquely determined element $\mathfrak{X} = \sum_{\mu \in Q^+} \mathfrak{X}_\mu \in \prod_{\mu \in Q^+} U_\mu^+$ with $\mathfrak{X}_0 = 1$ and $\mathfrak{X}_\mu \in U_\mu^+$ such that the equation

$$\overline{b}^B \mathfrak{X} = \mathfrak{X} \overline{b}^U \quad (3.22)$$

holds in $\mathcal{U}$ for all $b \in B_{\mathbf{c}, \mathbf{s}}$. For symmetric pairs of type $AIII/IV$ with $X = \emptyset$ the existence of the quasi K -matrix satisfying (3.22) was first observed in [4]. The quasi K -matrix is an essential building block for the construction of the universal K -matrix in [3]. To unify notation we define $c_i = s_i = 0$ and $B_i = F_i$ for $i \in X$. Moreover, we write

$$X_i = -s(\tau(i))T_{w_X}(E_{\tau(i)}) \quad \text{for all } i \in I \setminus X, \quad (3.23)$$

$$X_j = 0 \quad \text{for all } j \in X. \quad (3.24)$$

By [3, Proposition 6.1] the quasi K -matrix $\mathfrak{X} = \sum_{\mu \in Q^+} \mathfrak{X}_\mu$ is the unique solution to the recursive system of equations

$${}_i r(\mathfrak{X}_\mu) = -(q_i - q_i^{-1})(q^{-(\Theta(\alpha_i), \alpha_i)} c_i X_i \mathfrak{X}_{\mu + \Theta(\alpha_i) - \alpha_i} + s_i \mathfrak{X}_{\mu - \alpha_i}) \quad \text{for all } i \in I \quad (3.25)$$

with the normalisation $\mathfrak{X}_0 = 1$. Using the extension of the skew derivation ${}_i r$ to $\widehat{U^+}$ given by (3.11) we can rewrite (3.25) in the compact form

$${}_i r(\mathfrak{X}) = -(q_i - q_i^{-1})(q^{-(\Theta(\alpha_i), \alpha_i)} c_i X_i \mathfrak{X} + s_i \mathfrak{X}) \quad \text{for all } i \in I. \quad (3.26)$$

In Sect. 3.3 and in Appendix A we will use the above formula to perform uniform calculations with $\mathfrak{X}$. Equation (3.25) implies in particular that

$${}_j r(\mathfrak{X}_\mu) = 0 \quad \text{for all } j \in X \quad (3.27)$$

as $c_j = s_j = 0$ for all $j \in X$. This property has the following consequence which was already observed in [5, Proposition 4.15]. Recall that $w_0 \in W$ denotes the longest element. Moreover, for any $w \in W$ recall the definition of the subalgebra $U^+[w]$ of U^+ given in [11, 8.21, 8.24].

Lemma 3.2 *For any $\mu \in Q^+$ the relation $\mathfrak{X}_\mu \in U^+[w_X w_0]$ holds.*

Proof In view of (3.27), Equation (4) of [11, 8.26] implies that $\mathfrak{X}_\mu \in U^+[\sigma_j w_0]$ for all $j \in X$. By [9, Theorem 7.3] we have

$$\bigcap_{j \in X} U^+[\sigma_j w_0] = U^+[w_X w_0]$$

which completes the proof of the Lemma. $\square$

We write $\mathfrak{X}_{\mathbf{c}, \mathbf{s}}$ for $\mathfrak{X}$ if we need to specify the dependence on the parameters. Any diagram automorphism $\eta : I \rightarrow I$ induces a map $\eta : \mathbb{K}(q)^{I \setminus X} \rightarrow \mathbb{K}(q)^{I \setminus X}$ by

$$\eta(\mathbf{c}) = \mathbf{d} \quad \text{with } d_i = c_{\eta^{-1}(i)} \text{ where } \mathbf{c} = (c_i)_{i \in I \setminus X} \text{ and } \mathbf{d} = (d_i)_{i \in I \setminus X}. \quad (3.28)$$

This notation allows us to record the effect of diagram automorphisms on the quasi K -matrix $\mathfrak{X}$.

Lemma 3.3 *Let $\eta : I \rightarrow I$ be a diagram automorphism and $\mathbf{c} \in \mathcal{C}$, $\mathbf{s} \in \mathcal{S}$. Then $\eta(\mathbf{c}) \in \mathcal{C}$, $\eta(\mathbf{s}) \in \mathcal{S}$ and*

$$\eta(\mathfrak{X}_{\mathbf{c}, \mathbf{s}}) = \mathfrak{X}_{\eta(\mathbf{c}), \eta(\mathbf{s})}. \quad (3.29)$$

Proof The relations $\eta(\mathbf{c}) \in \mathcal{C}$, $\eta(\mathbf{s}) \in \mathcal{S}$ follow from the definitions (3.14) and (3.15) of $\mathcal{C}$ and $\mathcal{S}$. By [3, Proposition 6.1], property (3.22) is equivalent to the relation

$$B_i^{\mathbf{c}, \mathbf{s}} \mathfrak{X}_{\mathbf{c}, \mathbf{s}} = \mathfrak{X}_{\mathbf{c}, \mathbf{s}} \overline{B_i^{\mathbf{c}, \mathbf{s}}}^U \quad \text{for all } i \in I \quad (3.30)$$

where we write $B_i^{\mathbf{c}, \mathbf{s}}$ instead of B_i to denote the dependence on $\mathbf{c}$ and $\mathbf{s}$.

By construction, $\eta(B_i^{\mathbf{c}, \mathbf{s}}) = B_{\eta(i)}^{\eta(\mathbf{c}), \eta(\mathbf{s})}$ and $\eta : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ commutes with the bar involution ${}^{-U}$ on $U_q(\mathfrak{g})$. Hence applying η to relation (3.30) we obtain

$$B_{\eta(i)}^{\eta(\mathbf{c}), \eta(\mathbf{s})} \eta(\mathfrak{X}_{\mathbf{c}, \mathbf{s}}) = \eta(\mathfrak{X}_{\mathbf{c}, \mathbf{s}}) \overline{B_{\eta(i)}^{\eta(\mathbf{c}), \eta(\mathbf{s})}}^U \quad \text{for all } i \in I$$

which proves (3.29). $\square$

3.3 Quasi K -matrices for Satake diagrams of rank one

For the remainder of this paper, following Remark 2.2, we denote Satake diagrams as triples (I, X, τ) to also indicate the underlying Lie algebra.

Definition 3.4 A subdiagram of a Satake diagram (I, X, τ) is a triple $(J, X \cap J, \tau|_J)$ such that $J \subset I$ and $(J, X \cap J, \tau|_J)$ is a Satake diagram for the subdiagram of the Dynkin diagram of $\mathfrak{g}$ indexed by J .

Adding a connected component consisting entirely of black dots to a Satake diagram does not change the corresponding quasi K -matrix. For this reason we only consider subdiagrams $(J, X \cap J, \tau|_J)$ with the property that any connected component of $X \cap J$ is connected to a white node of J . Let $\tilde{I}$ be the set of τ -orbits of $I \setminus X$. There is a projection map

$$\pi: I \setminus X \longrightarrow \tilde{I} \quad (3.31)$$

that takes any white node to the τ -orbit it belongs to.

Definition 3.5 The rank of a Satake diagram (I, X, τ) is defined by $\text{rank}(I, X, \tau) = |\pi(I \setminus X)|$.

In other words, a Satake diagram has rank n if there are n distinct orbits of white nodes. By Proposition 2.7 the rank of a Satake diagram coincides with the rank of the corresponding restricted root system Σ .

Given a Satake diagram (I, X, τ) , any $i \in I \setminus X$ determines a subdiagram $(\{i, \tau(i)\} \cup X, X, \tau|_{\{i, \tau(i)\} \cup X})$ of rank one. Let $\mathfrak{X}_i$ be the quasi K -matrix corresponding to this rank one subdiagram. For any $w \in W$ we define $\widehat{U^+[w]} = \prod_{\mu \in Q^+} U^+[w]_\mu$. As $U[w]^+$ is a subalgebra of U^+ we obtain that $\widehat{U^+[w]}$ is a subalgebra of $\widehat{U^+}$ and hence of $\mathcal{U}$. Formulating Lemma 3.2 in the present setting we obtain

$$\mathfrak{X}_i \in \widehat{U^+[\tilde{\sigma}_i]}. \quad (3.32)$$

In the following lemma we consider the case $\tau(i) = i$ and make the dependence of $\mathfrak{X}_i$ on the parameter c_i more explicit.

Lemma 3.6 Assume that $\mathbf{s} = (0, \dots, 0)$ and $i \in I \setminus X$ satisfies $\tau(i) = i$. Then

$$\mathfrak{X}_i = \sum_{n \in \mathbb{N}_0} c_i^n \mathcal{E}_{n(\alpha_i - \Theta(\alpha_i))} \quad (3.33)$$

where $\mathcal{E}_{n(\alpha_i - \Theta(\alpha_i))} \in U_{n(\alpha_i - \Theta(\alpha_i))}^+$ is independent of $\mathbf{c}$.

Proof It follows from the recursion (3.25) and the assumption $s_i = 0$ that $\mathfrak{X}_i = \sum_{n \in \mathbb{N}_0} \mathfrak{X}_{n(\alpha_i - \Theta(\alpha_i))}$ with $\mathfrak{X}_{n(\alpha_i - \Theta(\alpha_i))} \in U_{n(\alpha_i - \Theta(\alpha_i))}^+$. Again by (3.25), the elements $\mathcal{E}_{n(\alpha_i - \Theta(\alpha_i))} = c_i^{-n} \mathfrak{X}_{n(\alpha_i - \Theta(\alpha_i))}$ for $n \in \mathbb{N}$ satisfy the relations

$${}_i r(\mathcal{E}_{n(\alpha_i - \Theta(\alpha_i))}) = -(q - q^{-1})q^{-(\Theta(\alpha_i), \alpha_i)} X_i \mathcal{E}_{(n-1)(\alpha_i - \Theta(\alpha_i))} \quad (3.34)$$

and

$${}_j r(\mathcal{E}_{n(\alpha_i - \Theta(\alpha_i))}) = 0 \quad \text{for } j \in X. \quad (3.35)$$

Table 1 Satake diagrams of symmetric pairs of rank one

AI_1		$BII, n \geq 2$
AII_3		$CII, n \geq 3$
$AIII_{11}$		$DII, n \geq 4$
$AIV, n \geq 2$		FII

The relations (3.34) and (3.35) are independent of $\mathbf{c}$ and determine $\mathcal{E}_{n(\alpha_i - \Theta(\alpha_i))}$ uniquely if we additionally impose $\mathcal{E}_0 = 1$. $\square$

The quasi K -matrices of rank one are the building blocks for quasi K -matrices of higher rank. In the following we give explicit formulas for rank one quasi K -matrices of type A shown on the left hand side of Table 1 in the case $\mathbf{s} = (0, \dots, 0)$. The additional rank one Satake diagrams on the right hand side of Table 1 are not tackled in this paper. However, the rank one quasi K -matrices of type BII and DII have been determined in [6, 5.3].

Recall the definition of the q -number

$$[n]_{q_i} = [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} \quad \text{for any } n \in \mathbb{Z} \text{ and } i \in I \quad (3.36)$$

and of the q -factorial $[n]_i! = [1]_i[2]_i \dots [n]_i$ for $n \geq 0$, see for example [11, Chapter 0]. If all roots $\alpha \in \Phi$ are of the same length, then we simply write $[n]$ and $[n]!$.

Throughout the following calculations in the present section and in Appendix A, we use a modified form of the q -number $[n]_i$. For $n \geq 1$ define

$$\{n\}_i = q_i^{n-1} [n]_i = 1 + q_i^2 + \dots + q_i^{2(n-1)}. \quad (3.37)$$

Define $\{n\}_i! = \prod_{k=1}^n \{k\}_i$, and let $\{n\}_i!!$ be the double factorial of $\{n\}_i$ defined by

$$\{n\}_i!! = \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} \{n - 2k\}_i. \quad (3.38)$$

For $n = 0$ we set $\{0\}_i! = 1$ and $\{0\}_i!! = 1$. Again, we omit the index i if all roots are of the same length. Further we use the following conventions. For any $x, y \in U_q(\mathfrak{g})$, $a \in \mathbb{K}(q)$ we denote by $[x, y]_a$ the element $xy - ayx$. For any $i, j \in I$ we write $T_{ij} = T_i \circ T_j : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ and we extend this definition recursively.

3.3.1 Type AI_1

Consider the Satake diagram of type AI_1 .

Lemma 3.7 *The quasi K -matrix $\mathfrak{X}$ of type AI_1 is given by*

$$\mathfrak{X} = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n E_1^{2n}. \quad (3.39)$$

Proof By Eq. (3.26), we need to show that

$${}_1r(\mathfrak{X}) = (q - q^{-1})(q^2 c_1) E_1 \mathfrak{X}.$$

Using the recursive formula (3.9) for ${}_1r$, we see that

$${}_1r(E_1^n) = \{n\} E_1^{n-1} \quad \text{for all } n \in \mathbb{N}. \quad (3.40)$$

Hence

$$\begin{aligned} {}_1r(\mathfrak{X}) &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n {}_1r(E_1^{2n}) \\ &= \sum_{n \geq 1} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n \{2n\} E_1^{2n-1} \\ &= \sum_{n \geq 0} \frac{(q - q^{-1})^{n+1}}{\{2n\}!!} (q^2 c_1)^{n+1} E_1^{2n+1} \\ &= (q - q^{-1})(q^2 c_1) E_1 \mathfrak{X} \end{aligned}$$

as required. $\square$

3.3.2 Type AI_3

Consider the Satake diagram of type AI_3 .

Lemma 3.8 *The quasi K -matrix $\mathfrak{X}$ of type AI_3 is given by*

$$\mathfrak{X} = \sum_{n \geq 0} \frac{(qc_2)^n}{\{n\}!} [E_2, T_{13}(E_2)]_{q^{-2}}^n. \quad (3.41)$$

Proof Since $T_{13}(E_2) = [E_1, T_3(E_2)]_{q^{-1}}$, property (3.9) of the skew derivative ${}_1r$ implies that ${}_1r(T_{13}(E_2)) = (1 - q^{-2})T_3(E_2)$. Again by property (3.9), it follows that ${}_1r([E_2, T_{13}(E_2)]_{q^{-2}}) = 0$. Hence ${}_1r(\mathfrak{X}) = 0$. By symmetry, we also have ${}_3r(\mathfrak{X}) = 0$.

We want to show that

$${}_2r(\mathfrak{X}) = (q - q^{-1})c_2T_{13}(E_2)\mathfrak{X}.$$

Since ${}_2r(T_{13}(E_2)) = 0$ by [11, 8.26, (4)], the relation

$${}_2r([E_2, T_{13}(E_2)]_{q^{-1}}) = (1 - q^{-2})T_{13}(E_2)$$

holds in $U_q(\mathfrak{sl}_4)$.

Moreover, the element $T_{13}(E_2)$ commutes with the element $[E_2, T_{13}(E_2)]_{q^{-2}}$. Indeed, this follows from the fact that E_2 commutes with $[T_{213}(E_2), E_2]_{q^{-2}}$ by applying the automorphism T_{13} . This implies that the relation

$${}_2r([E_2, T_{13}(E_2)]_{q^{-2}}^n) = (1 - q^{-2})\{n\}T_{13}(E_2)[E_2, T_{13}(E_2)]_{q^{-2}}^{n-1}$$

holds in $U_q(\mathfrak{sl}_4)$. Using this, we obtain

$$\begin{aligned} {}_2r(\mathfrak{X}) &= \sum_{n \geq 0} \frac{(qc_2)^n}{\{n\}!} {}_2r([E_2, T_{13}(E_2)]_{q^{-2}}^n) \\ &= (1 - q^{-2})T_{13}(E_2) \sum_{n \geq 1} \frac{(qc_2)^n}{\{n-1\}!} [E_2, T_{13}(E_2)]_{q^{-2}}^{n-1} \\ &= (q - q^{-1})c_2T_{13}(E_2)\mathfrak{X} \end{aligned}$$

as required. $\square$

3.3.3 Type $AIII_{11}$

Consider the Satake diagram of type $AIII_{11}$.

Note that $s(1) = s(2) = 1$ by (2.3) and $c_1 = c_2$ by (3.14) and (3.18).

Lemma 3.9 *The quasi K -matrix $\mathfrak{X}$ of type $AIII_{11}$ is given by*

$$\mathfrak{X} = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{n\}!} c_1^n (E_1 E_2)^n. \quad (3.42)$$

Proof By symmetry, we only need to show that

$${}_1r(\mathfrak{X}) = (q - q^{-1})c_1E_2\mathfrak{X}.$$

By Eq. (3.40), we have

$$\begin{aligned} {}_1r(\mathfrak{X}) &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{n\}!} c_1^n {}_1r((E_1E_2)^n) \\ &= \sum_{n \geq 1} \frac{(q - q^{-1})^n}{\{n\}!} \{n\} c_1^n E_1^{n-1} E_2^n \\ &= (q - q^{-1})c_1E_2\mathfrak{X} \end{aligned}$$

as required. $\square$

3.3.4 Type AIV for $n \geq 2$

Consider the Satake diagram of type AIV for $n \geq 2$.

By (2.3), we have $s(1) = -s(n)$ and by (3.18), we have $c_1 = q^{-2}\overline{c_n}$.

Lemma 3.10 *The quasi K -matrix $\mathfrak{X}$ of type AIV is given by*

$$\mathfrak{X} = \left(\sum_{k \geq 0} \frac{(c_1s(n))^k}{\{k\}!} T_1T_{w_X}(E_n)^k \right) \left(\sum_{k \geq 0} \frac{(c_ns(1))^k}{\{k\}!} T_nT_{w_X}(E_1)^k \right). \quad (3.43)$$

Proof We have ${}_ir(\mathfrak{X}) = 0$ for $i \in X$. Hence by symmetry we only need to show that

$${}_1r(\mathfrak{X}) = (q - q^{-1})q^{-1}c_1s(n)T_{w_X}(E_n)\mathfrak{X}$$

since $T_1T_{w_X}(E_n)$ and $T_nT_{w_X}(E_1)$ commute. We have

$$\begin{aligned} {}_1r(T_nT_{w_X}(E_1)^k) &= 0, \\ {}_1r(T_1T_{w_X}(E_n)^k) &= q^{-1}(q - q^{-1})\{k\}T_{w_X}(E_n)T_1T_{w_X}(E_n)^{k-1}. \end{aligned}$$

Using this, we obtain

$$\begin{aligned} {}_1r(\mathfrak{X}) &= \left(\sum_{k \geq 0} \frac{(c_1s(n))^k}{\{k\}!} {}_1r(T_1T_{w_X}(E_n)^k) \right) \left(\sum_{k \geq 0} \frac{(c_ns(1))^k}{\{k\}!} T_nT_{w_X}(E_1)^k \right) \\ &= q^{-1}(q - q^{-1})c_1s(n)T_{w_X}(E_n)\mathfrak{X} \end{aligned}$$

as required. $\square$

Remark 3.11 Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ and let $\mathcal{A}U^+$ be the $\mathcal{A}$ -subalgebra of U^+ generated by $E_i^{(n)} = \frac{E_i^n}{[n]!}$ for all $n \in \mathbb{N}_0, i \in I$. Set $\widehat{\mathcal{A}U^+} = \prod_{\mu \in Q^+} \mathcal{A}U_\mu^+$ where $\mathcal{A}U_\mu^+ = \mathcal{A}U^+ \cap U_\mu^+$ for all $\mu \in Q^+$. By [5, Theorem 5.3] we have $\mathfrak{X} \in \widehat{\mathcal{A}U^+}$ if $c_{is}(\tau(i)) \in \pm q^{\mathbb{Z}}$ for all $i \in I \setminus X$. This integrality property is crucial for the theory of canonical bases of quantum symmetric pairs developed in [5].

We observe that the integrality of the quasi K -matrix in rank one can in some cases be read off from the explicit formulas given in this section. Indeed, Lemma 3.7, 3.9 and 3.10 imply that $\mathfrak{X} \in \widehat{\mathcal{A}U^+}$ in the rank one cases of type $AI, AIII$ and AIV . The rank one case AII_3 is more complicated, and Lemma 3.8 does not give an obvious way to see that $\mathfrak{X} \in \widehat{\mathcal{A}U^+}$. Nevertheless, $\mathfrak{X}$ is also integral in this case, as shown in [5, A.5]. Based on the present remark, the integrality of $\mathfrak{X}$ in higher rank is discussed in Remark 3.23.

3.4 Partial quasi K -matrices

All through this section we make the assumption that $\mathbf{s} = \mathbf{0} = (0, 0, \dots, 0) \in \mathcal{S}$. In Sect. 3.5 we discuss the case of general parameters $\mathbf{s} \in \mathcal{S}$. Recall that the Lusztig automorphisms T_i of $U_q(\mathfrak{g})$ for all $i \in I$ give rise to a representation of $Br(W)$ on $U_q(\mathfrak{g})$. Since $\widetilde{W}$ is a subgroup of W , we obtain algebra automorphisms of $U_q(\mathfrak{g})$ defined by

$$\widetilde{T}_i := T_{\widetilde{\sigma}_i} \quad \text{for each } i \in I \setminus X.$$

By Theorem 2.4 and Proposition 2.5 the algebra automorphisms $\widetilde{T}_i$ give rise to a representation of $Br(\widetilde{W})$ on $U_q(\mathfrak{g})$.

Recall from Sect. 2.3 that $\widetilde{\Pi} = \{\widetilde{\alpha}_i \mid i \in I \setminus X\}$ and define $Q(2\Sigma) = 2\mathbb{Z}\widetilde{\Pi}$ and $Q^+(2\Sigma) = 2\mathbb{N}_0\widetilde{\Pi}$. By (3.25) and the assumption $\mathbf{s} = \mathbf{0}$ we have

$$ir(\mathfrak{X}_\mu) = -(q - q^{-1})q^{-(\Theta(\alpha_i), \alpha_i)} c_i X_i \mathfrak{X}_{\mu - 2\widetilde{\alpha}_i} \quad \text{for any } \mu \in Q^+.$$

Hence we may consider the quasi K -matrix $\mathfrak{X}$ as an element in $\prod_{\mu \in Q^+(2\Sigma)} U_\mu^+ \subset \mathcal{U}$. For any $w \in W$ define

$$\widetilde{U}^+[w] = \bigoplus_{\mu \in Q^+(2\Sigma)} U^+[w]_\mu$$

and set $\widetilde{U}^+ = \bigoplus_{\mu \in Q^+(2\Sigma)} U_\mu^+$. Then $\widetilde{U}^+$ and $\widetilde{U}^+[w]$ are $\mathbb{K}(q)$ -subalgebras of U^+ and $U^+[w]$, respectively. In particular by Eq. (3.32) we have

$$\mathfrak{X}_i \in \widetilde{U}^+[\widetilde{\sigma}_i] = \prod_{\mu \in Q^+(2\Sigma)} \widetilde{U}^+[\widetilde{\sigma}_i]_\mu \quad \text{for any } i \in I \setminus X.$$

Let $\mathbb{K}'$ be a field extension of $\mathbb{K}(q)$ which contains $q^{1/2}$ and elements $\widetilde{c}_i$ such that

$$\widetilde{c}_i^2 = c_i c_{\tau(i)} s(i) s(\tau(i)) \quad \text{for all } i \in I \setminus X. \quad (3.44)$$

We extend $\tilde{U}^+$ and $\tilde{U}^+[w]$ for $w \in W$ to $\mathbb{K}'$ -algebras $\tilde{U}_{1/2}^+ = \mathbb{K}' \otimes_{\mathbb{K}(q)} \tilde{U}^+$ and $\tilde{U}_{1/2}^+[w] = \mathbb{K}' \otimes_{\mathbb{K}(q)} \tilde{U}^+[w]$. Define an algebra automorphism $\Psi : \tilde{U}_{1/2}^+ \rightarrow \tilde{U}_{1/2}^+$ by

$$\Psi(E_{2\tilde{\alpha}_i}) = q^{(\tilde{\alpha}_i, \tilde{\alpha}_i)} \tilde{c}_i E_{2\tilde{\alpha}_i} \quad \text{for all } E_{2\tilde{\alpha}_i} \in U_{2\tilde{\alpha}_i}^+. \quad (3.45)$$

For each $i \in I \setminus X$ define an algebra homomorphism

$$\Omega_i = \Psi \circ \tilde{T}_i \circ \Psi^{-1} : \tilde{U}_{1/2}^+[\tilde{\sigma}_i w_0] \rightarrow \tilde{U}_{1/2}^+.$$

The role of the conjugation by Ψ in the definition of Ω_i is discussed in Remark 3.16 below. We consider the restriction of the algebra homomorphism Ω_i to the subalgebra $\tilde{U}^+[\tilde{\sigma}_i w_0]$, and we denote this restriction also by Ω_i . Crucially, by the following proposition, the image of the restriction Ω_i belongs to $\tilde{U}^+$ and does not involve any of the adjoined square roots.

Proposition 3.12 *For every $i \in I \setminus X$ the map $\Omega_i : \tilde{U}^+[\tilde{\sigma}_i w_0] \rightarrow \tilde{U}^+$ is a well defined algebra homomorphism.*

Proof It remains to show that the image of Ω_i is contained in $\tilde{U}^+$. Observe that $\tilde{T}_i(\tilde{U}_\mu^+) \subseteq \tilde{U}_{\tilde{\sigma}_i(\mu)}^+$ for all $\mu \in Q^+(2\Sigma)$. By Corollary 2.9 we have

$$\tilde{\sigma}_i(\mu) = \mu - \frac{2(\mu, \tilde{\alpha}_i)}{(\tilde{\alpha}_i, \tilde{\alpha}_i)} \tilde{\alpha}_i \quad \text{for all } \mu \in Q^+(2\Sigma).$$

Hence Eq. (3.45) implies that

$$\Omega_i|_{\tilde{U}_\mu^+} = q^{-(\mu, \tilde{\alpha}_i)} \tilde{c}_i^{-(\mu, \tilde{\alpha}_i)/(\tilde{\alpha}_i, \tilde{\alpha}_i)} \tilde{T}_i|_{\tilde{U}_\mu^+}.$$

Since $\mu \in Q^+(2\Sigma)$ it follows that the exponent $-(\mu, \tilde{\alpha}_i)$ is an integer. Moreover, Corollary 2.9 implies that the exponent $-(\mu, \tilde{\alpha}_i)/(\tilde{\alpha}_i, \tilde{\alpha}_i)$ is an integer.

If $i = \tau(i)$ then Eq. (3.44) and condition (2.2) imply that $\tilde{c}_i = \pm c_i$. This implies that the image of Ω_i is contained in $\tilde{U}^+$ in this case.

Suppose instead that $i \in I \setminus X$ satisfies $i \neq \tau(i)$. If additionally $(\alpha_i, \Theta(\alpha_i)) = 0$, then (3.14) implies that $c_i = c_{\tau(i)}$. Moreover in this case $\Theta(\alpha_i) = -\alpha_{\tau(i)}$ and hence $s(i) = s(\tau(i))$ by (2.3). Hence we get $\tilde{c}_i = \pm c_i s(i)$ in the case $i \neq \tau(i)$, $(\alpha_i, \Theta(\alpha_i)) = 0$ which implies that the image of Ω_i is contained in $\tilde{U}^+$ in this case.

Finally, we consider the case that $i \neq \tau(i)$ and $(\alpha_i, \Theta(\alpha_i)) \neq 0$. We are then in Case 3 in [20, p. 17] and hence the restricted root system Σ is of type $(BC)_n$ for $n \geq 1$ and $(\tilde{\alpha}_i, \tilde{\alpha}_i) = \frac{1}{4}(\alpha_i, \alpha_i)$. Since $\mu \in Q^+(2\Sigma) \subset Q$ we have

$$\frac{(\mu, \tilde{\alpha}_i)}{(\tilde{\alpha}_i, \tilde{\alpha}_i)} = 4 \frac{(\mu, \alpha_i)}{(\alpha_i, \alpha_i)} \in 2\mathbb{Z}.$$

Hence the image of Ω_i is contained in $\tilde{U}^+$ in all cases as required. $\square$

Consider $\tilde{w} \in \tilde{W}$ and let $\tilde{w} = \tilde{\sigma}_{i_1} \tilde{\sigma}_{i_2} \cdots \tilde{\sigma}_{i_t}$ be a reduced expression. For $k = 1, \dots, t$ let

$$\mathfrak{X}_{\tilde{w}}^{[k]} = \Omega_{i_1} \Omega_{i_2} \cdots \Omega_{i_{k-1}}(\mathfrak{X}_{i_k}) = \Psi \circ \tilde{T}_{i_1} \cdots \tilde{T}_{i_{k-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_k}). \quad (3.46)$$

By Proposition 2.5 we have $U^+[\tilde{\sigma}_{i_k}] \subset U^+[\tilde{\sigma}_{i_{k-1}} w_0]$ for $k = 2, \dots, t$, and

$$\tilde{T}_{i_l} \cdots \tilde{T}_{i_{k-1}}(U^+[\tilde{\sigma}_{i_k}]) \subset U^+[\tilde{\sigma}_{i_{l-1}} w_0] \quad \text{for } l = 2, \dots, k-1$$

and hence the elements $\mathfrak{X}_{\tilde{w}}^{[k]}$ are well-defined. Moreover, by Proposition 3.12 we have

$$\mathfrak{X}_{\tilde{w}}^{[k]} \in \widehat{U^+[\tilde{w}]} = \prod_{\mu \in Q^+(2\Sigma)} U^+[\tilde{w}]_{\mu} \quad \text{for } k = 1, \dots, t.$$

When clear, we omit the subscript $\tilde{w}$ and write $\mathfrak{X}^{[k]}$ instead of $\mathfrak{X}_{\tilde{w}}^{[k]}$.

Definition 3.13 Let $\tilde{w} \in \tilde{W}$ and let $\tilde{w} = \tilde{\sigma}_{i_1} \tilde{\sigma}_{i_2} \cdots \tilde{\sigma}_{i_t}$ be a reduced expression. The partial quasi K -matrix $\mathfrak{X}_{\tilde{w}}$ associated to $\tilde{w}$ and the given reduced expression is defined by

$$\mathfrak{X}_{\tilde{w}} = \mathfrak{X}^{[k]} \mathfrak{X}^{[k-1]} \cdots \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}. \quad (3.47)$$

We expect that the partial quasi K -matrix $\mathfrak{X}_{\tilde{w}}$ only depends on $\tilde{w} \in \tilde{W}$ and not on the chosen reduced expression. As we will see in Theorem 3.17 it suffices to check the independence of the reduced expression in rank two. If the Satake diagram is of rank two then the restricted Weyl group $\tilde{W}$ is of one of the types $A_1 \times A_1$, A_2 , B_2 or G_2 . In each case, only the longest word for $\tilde{W}$ has distinct reduced expressions.

Conjecture 3.14 Assume that (I, X, τ) is a Satake diagram of rank two. Then the element $\mathfrak{X}_{\tilde{w}} \in \mathcal{U}$ defined by (3.47) depends only on $\tilde{w} \in \tilde{W}$ and not on the chosen reduced expression.

In Appendix A, we prove the following Theorem which confirms Conjecture 3.14 in many cases. The proof is performed by showing that for both reduced expressions of the longest word in $\tilde{W}$ the resulting elements $\mathfrak{X}_{\tilde{w}}$ satisfy the relations (3.26).

Theorem 3.15 Assume that $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ or $X = \emptyset$. Then Conjecture 3.14 holds.

Remark 3.16 The Hopf algebra automorphism Ψ in the definition of Ω_i turns out to be necessary for the rank two calculations in Appendix A which prove Theorem 3.15. The conjugation by Ψ affects the coefficients in the partial quasi K -matrix associated to a reduced expression of an element $\tilde{w} \in \tilde{W}$. In rank two the two partial quasi K -matrices associated to the longest word $\tilde{w}_0 \in \tilde{W}$ coincide only after this change of coefficients. The effect of the conjugation by Ψ can be seen in particular in Sects. A.3 and A.4 of the appendix which treat type $AIII_n$ for $n \geq 3$.

Once the rank two case is established, we can generalise to higher rank cases.

Theorem 3.17 Suppose that (I, X, τ) is a Satake diagram such that all subdiagrams $(J, X \cap J, \tau|_J)$ of rank two satisfy Conjecture 3.14. Then the element $\mathfrak{X}_{\tilde{w}} \in \mathcal{U}$ depends on $\tilde{w} \in \tilde{W}$ and not on the chosen reduced expression.

Proof Let $\tilde{w}$ and $\tilde{w}'$ be reduced expressions which represent the same element in $\tilde{W}$. Assume that $\tilde{w}$ and $\tilde{w}'$ differ by a single braid relation. The following are the possible braid relations:

$$\begin{aligned}\tilde{\sigma}_p \tilde{\sigma}_r &= \tilde{\sigma}_r \tilde{\sigma}_p, \\ \tilde{\sigma}_p \tilde{\sigma}_r \tilde{\sigma}_p &= \tilde{\sigma}_r \tilde{\sigma}_p \tilde{\sigma}_r, \\ (\tilde{\sigma}_p \tilde{\sigma}_r)^2 &= (\tilde{\sigma}_r \tilde{\sigma}_p)^2, \\ (\tilde{\sigma}_p \tilde{\sigma}_r)^3 &= (\tilde{\sigma}_r \tilde{\sigma}_p)^3.\end{aligned}\tag{3.48}$$

The argument for each relation is the same, so we only consider the second case. Assume that $\tilde{w}$ and $\tilde{w}'$ differ by relation (3.48), that is

$$\begin{aligned}\tilde{w} &= \tilde{\sigma}_{i_1} \cdots \tilde{\sigma}_{i_{k-1}} (\tilde{\sigma}_p \tilde{\sigma}_r \tilde{\sigma}_p) \tilde{\sigma}_{i_{k+3}} \cdots \tilde{\sigma}_{i_t}, \\ \tilde{w}' &= \tilde{\sigma}_{i_1} \cdots \tilde{\sigma}_{i_{k-1}} (\tilde{\sigma}_r \tilde{\sigma}_p \tilde{\sigma}_r) \tilde{\sigma}_{i_{k+3}} \cdots \tilde{\sigma}_{i_t}\end{aligned}$$

for some $k = 1, \dots, t-2$. For $l = 1, \dots, k-1$, we have

$$\mathfrak{X}_{\tilde{w}}^{[l]} = \Psi \circ \tilde{T}_{i_1} \cdots \tilde{T}_{i_{l-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_l}) = \mathfrak{X}_{\tilde{w}'}^{[l]}.$$

Since the algebra automorphisms $\tilde{T}_i$ satisfy braid relations, we have

$$\begin{aligned}\mathfrak{X}_{\tilde{w}}^{[l]} &= \Psi \circ \tilde{T}_{i_1} \cdots \tilde{T}_{i_{k-1}} (\tilde{T}_p \tilde{T}_r \tilde{T}_p) \tilde{T}_{i_{k+3}} \cdots \tilde{T}_{i_{l-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_l}) \\ &= \Psi \circ \tilde{T}_{i_1} \cdots \tilde{T}_{i_{k-1}} (\tilde{T}_r \tilde{T}_p \tilde{T}_r) \tilde{T}_{i_{k+3}} \cdots \tilde{T}_{i_{l-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_l}) = \mathfrak{X}_{\tilde{w}'}^{[l]}\end{aligned}$$

for $l = k+3, \dots, t$. Finally, consider the rank two subdiagram $(J, X \cap J, \tau|_J)$ obtained by taking $J = J_1 \cup J_2$, where $J_1 = \{r, p, \tau(r), \tau(p)\}$ and $J_2 \subset X$ is the union of connected components of X which are connected to a node of J_1 . By assumption,

$$\begin{aligned}\mathfrak{X}_{\tilde{\sigma}_p \tilde{\sigma}_r \tilde{\sigma}_p} &= \mathfrak{X}_p \cdot \Psi \tilde{T}_p \Psi^{-1}(\mathfrak{X}_r) \cdot \Psi \tilde{T}_p \tilde{T}_r \Psi^{-1}(\mathfrak{X}_p) \\ &= \mathfrak{X}_r \cdot \Psi \tilde{T}_r \Psi^{-1}(\mathfrak{X}_p) \cdot \Psi \tilde{T}_r \tilde{T}_p \Psi^{-1}(\mathfrak{X}_r) = \mathfrak{X}_{\tilde{\sigma}_r \tilde{\sigma}_p \tilde{\sigma}_r}.\end{aligned}$$

It follows from this that

$$\begin{aligned}\mathfrak{X}_{\tilde{w}}^{[k]} \mathfrak{X}_{\tilde{w}}^{[k+1]} \mathfrak{X}_{\tilde{w}}^{[k+2]} &= \Psi \tilde{T}_{i_1} \cdots \tilde{T}_{i_{k-1}} \Psi^{-1}(\mathfrak{X}_{\tilde{\sigma}_p \tilde{\sigma}_r \tilde{\sigma}_p}) \\ &= \Psi \tilde{T}_{i_1} \cdots \tilde{T}_{i_{k-1}} \Psi^{-1}(\mathfrak{X}_{\tilde{\sigma}_r \tilde{\sigma}_p \tilde{\sigma}_r}) \\ &= \mathfrak{X}_{\tilde{w}'}^{[k]} \mathfrak{X}_{\tilde{w}'}^{[k+1]} \mathfrak{X}_{\tilde{w}'}^{[k+2]}\end{aligned}$$

Hence we have $\mathfrak{X}_{\tilde{w}} = \mathfrak{X}_{\tilde{w}'}$ as required.

If $\tilde{w}$ and $\tilde{w}'$ differ by more than a single relation, then we can find a sequence of reduced expressions

$$\tilde{w} = \tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_n = \tilde{w}'$$

such that for each $i = 1, \dots, n-1$, the expressions $\tilde{w}_i$ and $\tilde{w}_{i+1}$ differ by a single relation. We repeat the above argument at each step and obtain $\mathfrak{X}_{\tilde{w}} = \mathfrak{X}_{\tilde{w}'}$. $\square$

Recall from (2.9) that there exists a diagram automorphism $\tau_0 : I \rightarrow I$ such that the longest element $w_0 \in W$ satisfies

$$w_0(\alpha_i) = -\alpha_{\tau_0(i)} \quad (3.49)$$

for all $i \in I$.

Proposition 3.18 *Let $\tilde{w}_0 \in \tilde{W}$ be the longest element with reduced expression $\tilde{w}_0 = \tilde{\sigma}_{i_1} \cdots \tilde{\sigma}_{i_t}$. Then*

$$\mathfrak{X}_{\tilde{w}_0}^{[t]} = \mathfrak{X}_{\tau_0(i_t)}. \quad (3.50)$$

Proof To simplify notation we write $i_t = i$. By construction we have

$$\begin{aligned} \tilde{w}_0 w_X &= w_0, \\ w_X \tilde{\sigma}_i &= w_{\{i, \tau(i)\} \cup X} \quad \text{for all } i \in I \setminus X. \end{aligned}$$

As w_X and $w_{\{i, \tau(i)\} \cup X}$ commute, we get

$$\begin{aligned} \mathfrak{X}_{\tilde{w}_0}^{[t]} &= \Psi \circ T_{\tilde{w}_0} T_{\tilde{\sigma}_i}^{-1} \circ \Psi^{-1}(\mathfrak{X}_i) \\ &= \Psi \circ T_{w_0} T_{w_X}^{-1} T_{w_X} T_{w_{\{i, \tau(i)\} \cup X}}^{-1} \circ \Psi^{-1}(\mathfrak{X}_i) \\ &= \Psi \circ T_{w_0} T_{w_{\{i, \tau(i)\} \cup X}}^{-1} \circ \Psi^{-1}(\mathfrak{X}_i). \end{aligned}$$

Recall that

$$T_{w_0} = \text{tw}^{-1} \circ \tau_0 \quad (3.51)$$

where $\text{tw} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ is the algebra automorphism defined by

$$\text{tw}(E_i) = -K_i^{-1} F_i, \quad \text{tw}(F_i) = -E_i K_i, \quad \text{tw}(K_i) = K_i^{-1}$$

for $i \in I$, see [3, 7.1]. Analogously we have on $U_q(\mathfrak{g}_{\{i, \tau(i)\} \cup X})$ the relation

$$T_{w_{\{i, \tau(i)\} \cup X}} = \text{tw}^{-1} \circ \tau_{0,i} = \tau_{0,i} \circ \text{tw}^{-1} \quad (3.52)$$

where $\tau_{0,i} : \{i, \tau(i)\} \cup X \rightarrow \{i, \tau(i)\} \cup X$ is the diagram automorphism satisfying (2.10). We obtain

$$\begin{aligned}\mathfrak{X}_{\tilde{w}_0}^{[t]} &= \Psi \circ \text{tw}^{-1} \circ \tau_0 \circ \text{tw} \circ \tau_{0,i} \circ \Psi^{-1}(\mathfrak{X}_i) \\ &= \Psi \circ \tau_0 \tau_{0,i} \circ \Psi^{-1}(\mathfrak{X}_i).\end{aligned}\quad (3.53)$$

Case 1 $\tau(i) = i$.

In this case Lemma 3.3 implies that

$$\tau_{0,i}(\mathfrak{X}_i) = \mathfrak{X}_i. \quad (3.54)$$

Moreover $s(\tau(i)) = s(i) = 1$ and hence by Lemma 3.6 and by definition of Ψ we have

$$\begin{aligned}\mathfrak{X}_{\tilde{w}_0}^{[t]} &= \sum_{n \in \mathbb{N}_0} \Psi \circ \tau_0 \tau_{0,i} \circ \Psi^{-1}(c_i^n \mathcal{E}_{n(\alpha_i - \Theta(\alpha_i))}) \\ &= \sum_{n \in \mathbb{N}_0} q^{-n/2(\alpha_i - \Theta(\alpha_i), \alpha_i)} \Psi \circ \tau_0 \tau_{0,i}(\mathcal{E}_{n(\alpha_i - \Theta(\alpha_i))}) \\ &\stackrel{(3.54)}{=} \sum_{n \in \mathbb{N}_0} q^{-n/2(\alpha_i - \Theta(\alpha_i), \alpha_i)} \Psi \circ \tau_0(\mathcal{E}_{n(\alpha_i - \Theta(\alpha_i))}) \\ &= \sum_{n \in \mathbb{N}_0} q^{-n/2(\alpha_i - \Theta(\alpha_i), \alpha_i)} \Psi(\mathcal{E}_{n(\alpha_{\tau_0(i)} - \Theta(\alpha_{\tau_0(i)}))})\end{aligned}$$

where we use the notation from Lemma 3.6 also for $\mathfrak{X}_{\tau_0(i)}$.

As $(\alpha_{\tau_0(i)} - \Theta(\alpha_{\tau_0(i)}), \alpha_{\tau_0(i)}) = (\alpha_i - \Theta(\alpha_i), \alpha_i)$ and $s(\tau_0(i)) = 1$, formula (3.45) gives us

$$\mathfrak{X}_{\tilde{w}_0}^{[t]} = \sum_{n \in \mathbb{N}_0} c_{\tau_0(i)}^n \mathcal{E}_{n(\alpha_{\tau_0(i)} - \Theta(\alpha_{\tau_0(i)}))} = \mathfrak{X}_{\tau_0(i)} \quad (3.55)$$

which proves the Lemma in this case.

Case 2 $\tau(i) \neq i$.

In this case the rank one Satake subdiagram is either of type $AIII_{11}$ or of type AIV for $n \geq 2$ as in Table 1.

If the rank one Satake subdiagram is of type AIV for $n \geq 2$ then $\tau = \tau_0$ and $\tau_{0,i}$ coincide on $\{i, \tau(i)\} \cup X$ and hence (3.53) implies that

$$\mathfrak{X}_{\tilde{w}_0}^{[t]} = \mathfrak{X}_i = \mathfrak{X}_{\tau \tau_0(i)} = \mathfrak{X}_{\tau_0(i)}. \quad (3.56)$$

If the rank one subdiagram is of type $AIII_{11}$ then $\tau_{0,i}(i) = i$. If additionally $\tau_0(i) = i$ then $\tau_0(\tau(i)) = \tau(i)$ and hence (3.53) implies that $\mathfrak{X}_{\tilde{w}_0}^{[t]} = \mathfrak{X}_i = \mathfrak{X}_{\tau_0(i)}$ in this case.

If on the other hand $\tau_0(i) \neq i$ then $\tau_0 = \tau$ and we invoke the fact that

$$s(i) = s(\tau(i)), \quad c_i = c_{\tau(i)} \quad (3.57)$$

which hold by (2.3) and (3.14). Relation (3.57) and $\tau = \tau_0$ imply that $\tau_0 \circ \Psi^{-1}(\mathfrak{X}_i) = \Psi^{-1} \circ \tau_0(\mathfrak{X}_i)$. Hence Eq. (3.53) implies that $\mathfrak{X}_{\tilde{w}_0}^{[t]} = \mathfrak{X}_{\tau_0(i)}$ also in this case. $\square$

Lemma 3.19 *Let $\tilde{w}_0 = \tilde{\sigma}_{i_1} \cdots \tilde{\sigma}_{i_t}$ be a reduced expression for the longest word in $\tilde{W}$. Then $\mathfrak{X}_{\tilde{w}_0}^{[i]} \in U^+[\widehat{\tilde{\sigma}_k \tilde{w}_0}]$ for $i = 1, \dots, t-1$ and $k = \tau_0(i_t)$.*

Proof We have

$$\tilde{\sigma}_k \tilde{w}_0 = \tilde{\sigma}_k w_0 w_X = w_0 \tilde{\sigma}_{\tau_0(k)} w_X = w_0 w_X \tilde{\sigma}_{\tau_0(k)} = \tilde{w}_0 \tilde{\sigma}_{i_t} = \tilde{\sigma}_{i_1} \cdots \tilde{\sigma}_{i_{t-1}}.$$

By definition of $U^+[w]$ for each $w \in W$ and Proposition 2.5 we have

$$\tilde{T}_{i_1} \cdots \tilde{T}_{i_{j-1}}(U^+[\tilde{\sigma}_{i_j}]) \subseteq U^+[\tilde{\sigma}_k \tilde{w}_0]$$

for $j = 1, \dots, t-1$. Now the claim of the lemma follows from Eq. (3.32), Proposition 3.12 and the fact that

$$\mathfrak{X}_{\tilde{w}_0}^{[j]} = \Psi \circ \tilde{T}_{i_1} \cdots \tilde{T}_{i_{j-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_j})$$

for $j = 1, \dots, t-1$. $\square$

With the above preparations we are ready to prove the main result of the paper.

Theorem 3.20 *Suppose that (I, X, τ) is a Satake diagram such that all subdiagrams $(J, X \cap J, \tau|_J)$ of rank two satisfy Conjecture 3.14. Then $\mathfrak{X}_{\tilde{w}_0}$ coincides with the quasi K -matrix $\mathfrak{X}$.*

Proof It suffices to show that

$${}_i r(\mathfrak{X}_{\tilde{w}_0}) = (q - q^{-1})q^{-(\Theta(\alpha_i), \alpha_i)} c_i s(\tau(i)) T_{w_X}(E_{\tau(i)}) \mathfrak{X}_{\tilde{w}_0} \quad (3.58)$$

for all $i \in I \setminus X$. By Theorem 3.17 the element $\mathfrak{X}_{\tilde{w}_0}$ is independent of the chosen reduced expression for $\tilde{w}_0$. Fix $i \in I \setminus X$ and choose a reduced expression $\tilde{w}_0 = \tilde{\sigma}_{i_1} \cdots \tilde{\sigma}_{i_t}$ such that $i_t = \tau_0(i)$. Proposition 3.18 implies that

$$\mathfrak{X}_{\tilde{w}_0} = \mathfrak{X}_i \mathfrak{X}^{[t-1]} \cdots \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}.$$

By the previous lemma we have $\mathfrak{X}^{[j]} \in U^+[\widehat{\tilde{\sigma}_i \tilde{w}_0}]$ for $j = 1, \dots, t-1$. By [11, 8.26, (4)] this implies that ${}_i r(\mathfrak{X}^{[j]}) = 0$ for $j = 1, \dots, t-1$. By Eq. (3.26) and the skew derivative property (3.9) we obtain

$${}_i r(\mathfrak{X}_{\tilde{w}_0}) = {}_i r(\mathfrak{X}_i) \mathfrak{X}^{[t-1]} \cdots \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}$$

$$= (q - q^{-1})q^{-(\Theta(\alpha_i), \alpha_i)} c_i s(\tau(i)) T_{w_X}(E_{\tau(i)}) \mathfrak{X}_i \mathfrak{X}^{[t-1]} \dots \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}$$

which proves (3.58). $\square$

Combining Theorems 3.15 and 3.20 we obtain the following result.

Corollary 3.21 *Let $\mathfrak{g}$ be of type A_n or $X = \emptyset$. Then the quasi K -matrix $\mathfrak{X}$ is given by $\mathfrak{X} = \mathfrak{X}_{\tilde{w}_0}$ for any reduced expression of the longest word $\tilde{w}_0 \in \tilde{W}$.*

Conjecture 3.22 *The statement of Corollary 3.21 holds for any Satake diagram of finite type.*

Remark 3.23 We continue the discussion of the integrality of the quasi K -matrix $\mathfrak{X}$ from Remark 3.11 under the assumption that $c_i s(\tau(i)) \in \pm q^{\mathbb{Z}}$ for all $i \in I \setminus X$. In this case $\mathfrak{X}_{\tilde{w}}^{[k]} \in \widehat{\mathcal{U}^+}$ for $k = 1, \dots, t$ if $\tilde{w} \in \tilde{W}$ has a reduced expression $\tilde{w} = \tilde{\sigma}_{i_1} \tilde{\sigma}_{i_2} \dots \tilde{\sigma}_{i_t}$. Indeed, the discussion in the proof of Proposition 3.12 shows that $\mathfrak{X}_{\tilde{w}}^{[k]}$ differs from $\tilde{T}_{i_1} \dots \tilde{T}_{i_t}(\mathfrak{X}_{i_k})$ by a factor in $\pm q^{\mathbb{Z}}$. Hence we obtain $\mathfrak{X}_{\tilde{w}} \in \widehat{\mathcal{U}^+}$ for all $\tilde{w} \in \tilde{W}$. By Corollary 3.21, choosing $\tilde{w} = \tilde{w}_0$, we obtain $\mathfrak{X} \in \widehat{\mathcal{U}^+}$ whenever $\mathfrak{g}$ is of type A_n or $X = \emptyset$. In these cases we have hence reproduced [5, Theorem 5.3] for $\mathbf{s} = \mathbf{0}$ without the use of canonical bases. The case of general Satake diagrams hinges on Conjecture 3.22 and the integrality in rank one from [5, Appendix A].

3.5 Quasi K -matrices for general parameters

We now give a description of the quasi K -matrix $\mathfrak{X}$ for general parameters $\mathbf{s} \in \mathcal{S}$. Recall that we denote the generators B_i by $B_i^{\mathbf{c}, \mathbf{s}}$ if we need to specify the dependence on the parameters. By [18, Theorem 7.1], [12, Theorem 7.1] the algebra $B_{\mathbf{c}, \mathbf{s}}$ can be given in terms of the generators B_i for $i \in I \setminus X$ and $\mathcal{M}_X U_{\Theta}^0$ by relations which are independent of $\mathbf{s}$. Hence there exists an algebra isomorphism $\varphi_{\mathbf{s}} : B_{\mathbf{c}, \mathbf{0}} \rightarrow B_{\mathbf{c}, \mathbf{s}}$ given by

$$\varphi_{\mathbf{s}}(B_i^{\mathbf{c}, \mathbf{0}}) = B_i^{\mathbf{c}, \mathbf{s}}, \quad \varphi_{\mathbf{s}}(b) = b \quad \text{for all } i \in I \setminus X, b \in \mathcal{M}_X U_{\Theta}^0.$$

This algebra isomorphism allows us to define a one dimensional representation $\chi_{\mathbf{s}} : B_{\mathbf{c}, \mathbf{0}} \rightarrow \mathbb{K}(q)$ by $\chi_{\mathbf{s}} = \varepsilon \circ \varphi_{\mathbf{s}}$. By definition we have

$$\chi_{\mathbf{s}}(B_i^{\mathbf{c}, \mathbf{0}}) = s_i \quad \text{for all } i \in I \setminus X, \quad \chi_{\mathbf{s}}|_{\mathcal{M}_X U_{\Theta}^0} = \varepsilon|_{\mathcal{M}_X U_{\Theta}^0}.$$

By [12, (5.5)] we have

$$\Delta(B_i) - B_i \otimes K_i^{-1} \in \mathcal{M}_X U_{\Theta}^0 \otimes U_q(\mathfrak{g}) \quad (3.59)$$

which implies that

$$\varphi_{\mathbf{s}} = (\chi_{\mathbf{s}} \otimes \text{id}) \circ \Delta \quad (3.60)$$

on $B_{\mathbf{c}, \mathbf{0}}$. For later use we observe the following compatibility with the bar involution.

Lemma 3.24 *For all $b \in B_{\mathbf{c},0}$ we have*

$$(\chi_s \otimes \text{id}) \circ (-^{B_{\mathbf{c},0}} \otimes -^U) \circ \Delta(b) = \overline{\varphi_s(b)}^U. \quad (3.61)$$

Proof As $(\chi_s \otimes \text{id}) \circ (-^{B_{\mathbf{c},0}} \otimes -^U) \circ \Delta$ and $-^U \circ \varphi_s$ are $\mathbb{K}$ -algebra homomorphisms, it suffices to check equation (3.61) on the generators $B_i^{\mathbf{c},0}$ for $i \in I \setminus X$ and on $\mathcal{M}_X U_\Theta^0$. If $b \in \mathcal{M}_X U_\Theta^0$ then both sides of (3.61) coincide with $\overline{b}^U$. If $b = B_i^{\mathbf{c},0}$ for some $i \notin \{j \in I_{ns} \mid a_{jk} \in -2\mathbb{N}_0 \text{ for all } k \in I_{ns} \setminus \{j\}\}$ then $s_i = 0$ by the definition of $\mathcal{S}$ in (3.15) and hence $B_i^{\mathbf{c},0} = B_i^{\mathbf{c},s}$. Using the membership property (3.59) we get

$$((\chi_s \circ -^{B_{\mathbf{c},0}}) \otimes -^U) \circ \Delta(B_i^{\mathbf{c},0}) = ((-^U \circ \varepsilon) \otimes -^U) \circ \Delta(B_i^{\mathbf{c},0}) = \overline{B_i^{\mathbf{c},0}}^U = \overline{\varphi_s(b)}^U$$

which proves (3.61) in this case. Finally, if $i \in \{j \in I_{ns} \mid a_{jk} \in -2\mathbb{N}_0 \text{ for all } k \in I_{ns} \setminus \{j\}\}$, then the definition (3.13) of I_{ns} implies that

$$B_i^{\mathbf{c},s} = F_i - c_i E_i K_i^{-1} + s_i K_i^{-1}.$$

Hence, using $s_i = \overline{s_i}^U$ from (3.19), we get

$$\begin{aligned} ((\chi_s \circ -^{B_{\mathbf{c},0}}) \otimes -^U) \circ \Delta(B_i^{\mathbf{c},0}) &= ((\chi_s \circ -^{B_{\mathbf{c},0}}) \otimes -^U)(B_i^{\mathbf{c},0} \otimes K_i^{-1} + 1 \otimes B_i^{\mathbf{c},0}) \\ &= s_i K_i + \overline{B_i^{\mathbf{c},0}}^U \\ &= \overline{B_i^{\mathbf{c},s}}^U \\ &= \overline{\varphi_s(B_i^{\mathbf{c},0})}^U \end{aligned}$$

which completes the proof of the lemma. $\square$

As in [3, 3.2] we consider the algebra

$$\mathcal{W}_0^{(2)} = \text{End}(\mathcal{F} \circ \otimes : \mathcal{O}_{int} \times \mathcal{O}_{int} \rightarrow \text{Vect})$$

and observe that $\prod_{\mu \in Q^+} U_\mu^- \otimes U_\mu^+$ is a subalgebra of $\mathcal{W}_0^{(2)}$. Let $R \in \prod_{\mu \in Q^+} U_\mu^- \otimes U_\mu^+$ be the quasi R -matrix for $U_q(\mathfrak{g})$, see [23, Theorem 4.1.2]. Following [4, 3.1] we define an element

$$R^\theta = \Delta(\mathfrak{X}) \cdot R \cdot (\mathfrak{X}^{-1} \otimes 1) \in \mathcal{W}_0^{(2)}, \quad (3.62)$$

see also [13, 3.4]. In [4] the element R^θ is called the quasi R -matrix for $B_{\mathbf{c},s}$. By [4, Proposition 3.2] it satisfies the following intertwiner property

$$\Delta(\overline{b}^B) \cdot R^\theta = R^\theta \cdot (-^B \otimes -^U) \circ \Delta(b) \quad \text{for all } b \in B = B_{\mathbf{c},s} \quad (3.63)$$

in $\mathcal{W}_0^{(2)}$. Moreover, by [4, Proposition 3.5], [13, Proposition 3.10] we can write R^θ as an infinite sum

$$R^\theta = \sum_{\mu \in Q^+} R_\mu^\theta \quad \text{with } R_\mu^\theta \in B_{\mathbf{c},\mathbf{s}} \otimes U_\mu^+. \quad (3.64)$$

Similarly to the notation $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ introduced at the end of Sect. 3.2, we write $R_{\mathbf{c},\mathbf{s}}^\theta$ if we need to specify the dependence on the parameters. Observe that once we have an explicit formula for $\mathfrak{X}_{\mathbf{c},\mathbf{0}}$, Eq. (3.62) provides us with an explicit formula for $R_{\mathbf{c},\mathbf{0}}^\theta$. This in turn provides a formula for the quasi K -matrix $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ for general parameters $\mathbf{s} \in \mathcal{S}$. Indeed, by Eq. (3.64) we can apply the character $\chi_{\mathbf{s}}$ to the first tensor factor of $R_{\mathbf{c},\mathbf{0}}^\theta$ to obtain an element $\mathfrak{X}' = (\chi_{\mathbf{s}} \otimes \text{id})(R_{\mathbf{c},\mathbf{0}}^\theta)$ which can be written as

$$\mathfrak{X}' = \sum_{\mu \in Q^+} \mathfrak{X}'_\mu \quad \text{with } \mathfrak{X}'_\mu \in U_\mu^+.$$

Moreover, Eq. (3.64) implies that $\mathfrak{X}'_0 = 1$. By the following proposition the element $\mathfrak{X}' \in \mathcal{U}$ is the quasi K -matrix for $B_{\mathbf{c},\mathbf{s}}$.

Proposition 3.25 *For any $\mathbf{c} \in \mathcal{C}$, $\mathbf{s} \in \mathcal{S}$ we have $\mathfrak{X}_{\mathbf{c},\mathbf{s}} = (\chi_{\mathbf{s}} \otimes \text{id})(R_{\mathbf{c},\mathbf{0}}^\theta)$.*

Proof We keep the notation $\mathfrak{X}' = (\chi_{\mathbf{s}} \otimes \text{id})(R_{\mathbf{c},\mathbf{0}}^\theta)$ from above. By Eq. (3.63) we have

$$\Delta(\bar{b}^{B_{\mathbf{c},\mathbf{0}}}) \cdot R_{\mathbf{c},\mathbf{0}}^\theta = R_{\mathbf{c},\mathbf{0}}^\theta \cdot (-^{B_{\mathbf{c},\mathbf{0}}} \otimes -^U) \circ \Delta(b) \quad \text{for all } b \in B_{\mathbf{c},\mathbf{0}}.$$

Applying $\chi_{\mathbf{s}} \otimes \text{id}$ to both sides of this relation, we obtain in view of Eq. (3.60) the relation

$$\varphi_{\mathbf{s}}(\bar{b}^{B_{\mathbf{c},\mathbf{0}}}) \cdot \mathfrak{X}' = \mathfrak{X}'(\chi_{\mathbf{s}} \otimes \text{id}) \circ (-^{B_{\mathbf{c},\mathbf{0}}} \otimes -^U) \circ \Delta(b) \quad \text{for all } b \in B_{\mathbf{c},\mathbf{0}}.$$

By Lemma 3.24 the above relation implies that

$$\varphi_{\mathbf{s}}(\bar{b}^{B_{\mathbf{c},\mathbf{0}}}) \mathfrak{X}' = \mathfrak{X}' \overline{\varphi_{\mathbf{s}}(b)}^U \quad \text{for all } b \in B_{\mathbf{c},\mathbf{0}}.$$

This gives in particular $B_i^{\mathbf{c},\mathbf{s}} \mathfrak{X}' = \mathfrak{X}' \overline{B_i^{\mathbf{c},\mathbf{s}}}$ for all $i \in I$ and $b \mathfrak{X}' = \mathfrak{X}' b$ for all $b \in \mathcal{M}_X U_0^\theta$. This means that $\mathfrak{X}'$ satisfies the defining relation (3.22) of $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ and hence, in view of the normalisation $\mathfrak{X}'_0 = 1$ observed above, we get $\mathfrak{X}' = \mathfrak{X}_{\mathbf{c},\mathbf{s}}$. $\square$

Remark 3.26 The existence of the quasi K -matrix $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ was established in [3, Theorem 6.10] by fairly involved calculations. It was noted in [3, Remark 6.9] that these calculations simplify significantly if one restricts to the case $\mathbf{s} = \mathbf{0}$. Proposition 3.25 now shows that in the presence of (3.64) the existence of $\mathfrak{X}_{\mathbf{c},\mathbf{0}}$ implies the existence of $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ for any $\mathbf{s} \in \mathcal{S}$ satisfying (3.19). Relation (3.64) was established in [13].

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Appendix A: Rank two calculations

In rank two, there are two distinct reduced expressions for the longest word $\tilde{w}_0 \in \tilde{W}$. All irreducible rank two Satake diagrams for simple $\mathfrak{g}$ are shown in Table 2. Using the explicit formulas from Sect. 3.3, we confirm that the partial quasi K -matrices for the two reduced expressions of $\tilde{w}_0$ coincide with the quasi K -matrix if the rank two Satake diagram is of type A_n or B_2 . We also state this result in type G_2 , but we refer to [6, 6.6] for the proof, which involves significantly longer calculations. The calculations in this appendix and [6] prove Theorem 3.15.

Table 2 Irreducible Satake diagrams of rank two for simple $\mathfrak{g}$

AI_2		CII_4
AII_5		$DI_n, n \geq 5$
$AIII_3$		$DIII_4$
$AIII_n, n \geq 4$		$DIII_5$
$BI_n, n \geq 3$		$EIII$
CI_2		EIV
$CII_n, n \geq 5$		G

A.1 Type AI_2

Consider the Satake diagram of type AI_2 .

Since $\Theta = -\text{id}$ the restricted Weyl group $\tilde{W}$ coincides with the Weyl group W . The longest word of the Weyl group has two reduced expressions given by

$$w_0 = \sigma_1 \sigma_2 \sigma_1, \quad w'_0 = \sigma_2 \sigma_1 \sigma_2.$$

Proposition A.1 *In this case, the partial quasi K -matrices $\mathfrak{X}_{w_0}$ and $\mathfrak{X}_{w'_0}$ coincide with the quasi K -matrix $\mathfrak{X}$. Hence $\mathfrak{X}_{w_0} = \mathfrak{X}_{w'_0}$.*

Before we prove this, we need to know how the Lusztig skew derivations ${}_1r$ and ${}_2r$ act on certain elements and their powers, and also some commutation relations. These are given in the following two lemmas, whose proofs are obtained by straightforward computation.

Lemma A.2 *For any $n \in \mathbb{N}$, the relations*

$$\begin{aligned} E_2^n E_1 &= q^n E_1 E_2^n - q\{n\} E_2^{n-1} T_1(E_2), \\ E_1^n E_2 &= q^n E_2 E_1^n - q\{n\} E_1^{n-1} T_2(E_1), \\ T_1(E_2)^n E_1 &= q^{-n} E_1 T_1(E_2)^n, \\ T_2(E_1)^n E_2 &= q^{-n} E_2 T_2(E_1)^n \end{aligned}$$

hold in $U_q(\mathfrak{sl}_3)$.

Lemma A.3 *For any $n \in \mathbb{N}$, the relations*

$$\begin{aligned} {}_1r(E_2^n) &= {}_1r(T_2(E_1)^n) = {}_2r(E_1^n) = {}_2r(T_1(E_2)^n) = 0, \\ {}_1r(E_1^n) &= \{n\} E_1^{n-1}, \\ {}_2r(E_2^n) &= \{n\} E_2^{n-1}, \\ {}_1r(T_1(E_2)^n) &= (1 - q^{-2})\{n\} E_2 T_1(E_2)^{n-1}, \\ {}_2r(T_2(E_1)^n) &= (1 - q^{-2})\{n\} E_1 T_2(E_1)^{n-1} \end{aligned}$$

hold in $U_q(\mathfrak{sl}_3)$.

Proof of Proposition A.1 Consider first the element $\mathfrak{X}_{w_0}$. Using (3.47) and Lemma 3.7, we write

$$\mathfrak{X}_{w_0} = \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \quad (\text{A.1})$$

where

$$\begin{aligned} \mathfrak{X}^{[3]} &\stackrel{(3.50)}{=} \mathfrak{X}_2 = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_2)^n E_2^{2n}, \\ \mathfrak{X}^{[2]} &= \Psi \circ T_1 \circ \Psi^{-1}(\mathfrak{X}_2) = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n (q^2 c_2)^n T_1(E_2)^{2n}, \end{aligned}$$

$$\mathfrak{X}^{[1]} = \mathfrak{X}_1 = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n E_1^{2n}.$$

For $i = 1, 2, 3$, let $\mathfrak{X}_{[i]} = K_1 \mathfrak{X}^{[i]} K_1^{-1}$. The difference between $\mathfrak{X}_{[i]}$ and $\mathfrak{X}^{[i]}$ is the occurrence of a q -power in each summand of the infinite series. By Eq. (3.26), to show that $\mathfrak{X}_{w_0}$ coincides with the quasi K -matrix $\mathfrak{X}$ we show that

$$\begin{aligned} {}_1r(\mathfrak{X}_{w_0}) &= (q - q^{-1})(q^2 c_1) E_1 \mathfrak{X}_{w_0}, \\ {}_2r(\mathfrak{X}_{w_0}) &= (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{w_0}. \end{aligned}$$

By Lemmas A.3 and 3.7, we see that

$$\begin{aligned} {}_2r(\mathfrak{X}_{w_0}) &= {}_2r(\mathfrak{X}^{[3]}) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\ &= (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\ &= (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{w_0}. \end{aligned}$$

By the property (3.9) of the skew derivative ${}_1r$, we have

$${}_1r(\mathfrak{X}_{w_0}) = \mathfrak{X}_{[3]} {}_1r(\mathfrak{X}^{[2]}) \mathfrak{X}^{[1]} + \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} {}_1r(\mathfrak{X}^{[1]}) \quad (\text{A.2})$$

Using Lemma A.3, we have

$$\begin{aligned} {}_1r(\mathfrak{X}^{[2]}) &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n (q^2 c_2)^n {}_1r(T_1(E_2)^{2n}) \\ &= q^{-1} (q - q^{-1}) E_2 \sum_{n \geq 1} \frac{(q - q^{-1})^n}{\{2n - 2\}!!} (q^2 c_1)^n (q^2 c_2)^n T_1(E_2)^{2n-1} \\ &= (q - q^{-1})^2 q^{-1} (q^2 c_1) (q^2 c_2) E_2 T_1(E_2) \mathfrak{X}^{[2]}, \\ {}_1r(\mathfrak{X}^{[1]}) &= (q - q^{-1})(q^2 c_1) E_1 \mathfrak{X}^{[1]}. \end{aligned}$$

The second summand of Eq. (A.2) is of the form

$$(q - q^{-1})(q^2 c_1) \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} E_1 \mathfrak{X}^{[1]}.$$

We use Lemma A.2 to bring the E_1 in the above summand to the front. We have

$$\begin{aligned} \mathfrak{X}_{[2]} E_1 &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n (q^2 c_2)^n q^{2n} T_1(E_2)^{2n} E_1 \\ &= E_1 \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n (q^2 c_2)^n T_1(E_2)^{2n} \\ &= E_1 \mathfrak{X}^{[2]}, \end{aligned}$$

$$\begin{aligned}
\mathfrak{X}_{[3]}E_1 &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_2)^n q^{-2n} E_2^{2n} E_1 \\
&= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_2)^n q^{-2n} (q^{2n} E_1 E_2^{2n} - q \{2n\} E_2^{2n-1} T_1(E_2)) \\
&= E_1 \mathfrak{X}^{[3]} - q \sum_{n \geq 1} \frac{(q - q^{-1})^n}{\{2n - 2\}!!} (q^2 c_2)^n q^{-2n} E_2^{2n-1} T_1(E_2) \\
&= E_1 \mathfrak{X}^{[3]} - (q - q^{-1}) q^{-1} (q^2 c_2) \mathfrak{X}_{[3]} E_2 T_1(E_2).
\end{aligned}$$

Hence,

$$\begin{aligned}
(q - q^{-1})(q^2 c_1) \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} E_1 \mathfrak{X}^{[1]} &= (q - q^{-1})(q^2 c_1) \mathfrak{X}_{[3]} E_1 \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\
&= (q - q^{-1})(q^2 c_1) E_1 \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\
&\quad - (q - q^{-1})^2 q^{-1} (q^2 c_1)(q^2 c_2) \mathfrak{X}_{[3]} E_2 T_1(E_2) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\
&= (q - q^{-1})(q^2 c_1) E_1 \mathfrak{X}_{w_0} - \mathfrak{X}_{[3]} 1r(\mathfrak{X}^{[2]}) \mathfrak{X}^{[1]}.
\end{aligned}$$

It follows from (A.2) that $1r(\mathfrak{X}_{w_0}) = (q - q^{-1})(q^2 c_1) E_1 \mathfrak{X}_{w_0}$, so $\mathfrak{X}_{w_0}$ coincides with the quasi K -matrix $\mathfrak{X}$. Instead of repeating the same calculation for $\mathfrak{X}_{w'_0}$, we use the underlying symmetry in type AI_2 , which implies that $\mathfrak{X}_{w'_0}$ also coincides with the quasi K -matrix $\mathfrak{X}$. $\square$

A.2 Type AI_5

Consider the Satake diagram of type AI_5 .

In this case the involutive automorphism $\Theta : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ is given by

$$\Theta = -\sigma_1 \sigma_3 \sigma_5.$$

There are two τ -orbits of white nodes given by the sets $\{2\}$ and $\{4\}$. The restricted root system is of type A_2 since the restricted roots

$$\tilde{\alpha}_2 = \frac{\alpha_1 + 2\alpha_2 + \alpha_3}{2}, \quad \tilde{\alpha}_4 = \frac{\alpha_3 + 2\alpha_4 + \alpha_5}{2}$$

have the same length. The subgroup $\tilde{W} \subset W^\Theta$ is generated by the elements

$$\tilde{\sigma}_2 = \sigma_2 \sigma_1 \sigma_3 \sigma_2, \quad \tilde{\sigma}_4 = \sigma_4 \sigma_3 \sigma_5 \sigma_4.$$

The longest word of the restricted Weyl group has two reduced expressions given by

$$\tilde{w}_0 = \tilde{\sigma}_2 \tilde{\sigma}_4 \tilde{\sigma}_2, \quad \tilde{w}'_0 = \tilde{\sigma}_4 \tilde{\sigma}_2 \tilde{\sigma}_4.$$

By Lemma 3.8 we have

$$\begin{aligned}\mathfrak{X}_2 &= \sum_{n \geq 0} \frac{(qc_2)^n}{\{n\}!} [E_2, T_{13}(E_2)]_{q^{-2}}^n, \\ \mathfrak{X}_4 &= \sum_{n \geq 0} \frac{(qc_4)^n}{\{n\}!} [E_4, T_{35}(E_4)]_{q^{-2}}^n.\end{aligned}$$

Proposition A.4 *The partial quasi K -matrices $\mathfrak{X}_{\tilde{w}_0}$ and $\mathfrak{X}_{\tilde{w}'_0}$ coincide with the quasi K -matrix $\mathfrak{X}$.*

We have the following relations needed for the proof of Proposition A.4. These are proved by induction.

Lemma A.5 *For any $n \in \mathbb{N}$ the relations*

$$\begin{aligned}[E_4, T_{35}(E_4)]_{q^{-2}}^n T_{13}(E_2) &= q^n T_{13}(E_2) [E_4, T_{35}(E_4)]_{q^{-2}}^n \\ &\quad - q \{n\} [E_4, T_{35}(E_4)]_{q^{-2}}^{n-1} [T_3(E_4), T_{1235}(E_4)]_{q^{-2}},\end{aligned}\tag{A.3}$$

$$[T_{23}(E_4), T_{1235}(E_4)]_{q^{-2}}^n T_{13}(E_2) = q^{-n} T_{13}(E_2) [T_{23}(E_4), T_{1235}(E_4)]_{q^{-2}}^n\tag{A.4}$$

hold in $U_q(\mathfrak{sl}_6)$.

Lemma A.6 *For any $n \in \mathbb{N}$ the relation*

$$\begin{aligned}{}_2r([T_{23}(E_4), T_{1235}(E_4)]_{q^{-2}}^n) \\ = q^{-1}(q - q^{-1})\{n\}[T_3(E_4), T_{1235}(E_4)]_{q^{-2}}[T_{23}(E_4), T_{1235}(E_4)]_{q^{-2}}^{n-1}\end{aligned}\tag{A.5}$$

holds in $U_q(\mathfrak{sl}_6)$.

Proof of Proposition A.4 We only confirm that $\mathfrak{X}_{\tilde{w}_0}$ coincides with the quasi K -matrix $\mathfrak{X}$. By the underlying symmetry in type AI_5 , the calculation for $\mathfrak{X}_{\tilde{w}'_0}$ is the same up to a change of indices. By definition, we have

$$\mathfrak{X}_{\tilde{w}_0} = \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}$$

where

$$\begin{aligned}\mathfrak{X}^{[3]} &\stackrel{(3.50)}{=} \mathfrak{X}_4, \\ \mathfrak{X}^{[2]} &= \Psi \circ T_{2132} \circ \Psi^{-1}(\mathfrak{X}_4) = \sum_{n \geq 0} \frac{(q^2 c_2 c_4)^n}{\{n\}!} [T_{23}(E_4), T_{1235}(E_4)]_{q^{-2}}^n, \\ \mathfrak{X}^{[1]} &= \mathfrak{X}_2.\end{aligned}$$

Using Lemma 3.8 and [11, 8.26, (4)] we see that

$${}_4r(\mathfrak{X}_{\tilde{w}_0}) = {}_4r(\mathfrak{X}_4) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}$$

$$\begin{aligned}
&= (q - q^{-1})c_4T_{35}(E_4)\mathfrak{X}_4\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\
&= (q - q^{-1})c_4T_{35}(E_4)\mathfrak{X}_{\tilde{w}_0}.
\end{aligned}$$

We want to show that

$${}_2r(\mathfrak{X}_{\tilde{w}_0}) = (q - q^{-1})c_2T_{13}(E_2)\mathfrak{X}_{\tilde{w}_0}.$$

For $i = 1, 2, 3$, let $\mathfrak{X}_{[i]} = K_2\mathfrak{X}^{[i]}K_2^{-1}$. By property (3.9) of the skew derivative ${}_2r$ and [11, 8.26, (4)] we have

$${}_2r(\mathfrak{X}_{\tilde{w}_0}) = \mathfrak{X}_{[3]}{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_2r(\mathfrak{X}^{[1]}). \quad (\text{A.6})$$

By Lemma 3.8 we have

$${}_2r(\mathfrak{X}^{[1]}) = (q - q^{-1})c_2T_{13}(E_2)\mathfrak{X}^{[1]}.$$

By Lemma A.6 we have

$$\begin{aligned}
{}_2r(\mathfrak{X}^{[2]}) &= \sum_{n \geq 1} \frac{(q^2c_2c_4)^n}{\{n\}!} {}_2r([T_{23}(E_4), T_{1235}(E_4)]_{q^{-2}}^n) \\
&\stackrel{(\text{A.5})}{=} q(q - q^{-1})c_2c_4[T_3(E_4), T_{1235}(E_4)]_{q^{-2}}\mathfrak{X}^{[2]}.
\end{aligned}$$

The second summand of Eq. (A.6) is of the form $(q - q^{-1})c_2\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}T_{13}(E_2)\mathfrak{X}^{[1]}$. Using Lemma A.5, we bring the $T_{13}(E_2)$ term in this expression to the front. We have

$$\mathfrak{X}_{[2]}T_{13}(E_2) \stackrel{(\text{A.4})}{=} T_{13}(E_2)\mathfrak{X}^{[2]}, \quad (\text{A.7})$$

$$\mathfrak{X}_{[3]}T_{13}(E_2) \stackrel{(\text{A.3})}{=} T_{13}(E_2)\mathfrak{X}^{[3]} - qc_4\mathfrak{X}_{[3]}[T_3(E_4), T_{1235}(E_4)]_{q^{-2}}. \quad (\text{A.8})$$

Substituting (A.7) and (A.8) into $(q - q^{-1})c_2\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}T_{13}(E_2)\mathfrak{X}^{[1]}$, we obtain

$$\begin{aligned}
(q - q^{-1})c_2\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}T_{13}(E_2)\mathfrak{X}^{[1]} &\stackrel{(\text{A.7})}{=} (q - q^{-1})c_2\mathfrak{X}_{[3]}T_{13}(E_2)\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\
&\stackrel{(\text{A.8})}{=} (q - q^{-1})c_2T_{13}(E_2)\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\
&\quad - (q - q^{-1})qc_2c_4\mathfrak{X}^{[3]}[T_3(E_4), T_{1235}(E_4)]_{q^{-2}}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\
&= (q - q^{-1})c_2T_{13}(E_2)\mathfrak{X}_{\tilde{w}_0} - \mathfrak{X}_{[3]}{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]}.
\end{aligned}$$

It follows from (A.6) that ${}_2r(\mathfrak{X}_{\tilde{w}_0}) = (q - q^{-1})c_2T_{13}(E_2)\mathfrak{X}_{\tilde{w}_0}$ as required. $\square$

A.3 Type AI/III_3

We consider the diagram of type AI/III_3 with non-trivial diagram automorphism τ and no black dots.

Here, we see that there are 2 nodes in the restricted Dynkin diagram, corresponding to the restricted roots

$$\tilde{\alpha}_1 = \frac{\alpha_1 + \alpha_3}{2}, \quad \tilde{\alpha}_2 = \alpha_2.$$

A quick check confirms that $1/2(\alpha_1 + \alpha_3)$ is the short root, and hence the restricted root system is of type B_2 . The subgroup $\tilde{W}$ is generated by the elements

$$\tilde{\sigma}_1 = \sigma_1\sigma_3, \quad \tilde{\sigma}_2 = \sigma_2.$$

The longest word of the restricted Weyl group has two reduced expressions given by

$$\tilde{w}_0 = \tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1\tilde{\sigma}_2, \quad \tilde{w}'_0 = \tilde{\sigma}_2\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1.$$

The definition (3.14) and condition (3.18) imply that $c_1 = c_3 = \overline{c_1}$. By Lemmas 3.7 and 3.9 we have

$$\mathfrak{X}_1 = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{n\}!} c_1^n (E_1 E_3)^n, \quad (\text{A.9})$$

$$\mathfrak{X}_2 = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_2)^n E_2^{2n}. \quad (\text{A.10})$$

Proposition A.7 *The partial quasi K -matrix $\mathfrak{X}_{\tilde{w}_0}$ coincides with the quasi K -matrix $\mathfrak{X}$.*

The following relations are needed for the proof of Proposition A.7. They are checked by induction.

Lemma A.8 *For any $n \in \mathbb{N}$, the relations*

$$T_{13}(E_2)^n E_3 = q^{-n} E_3 T_{13}(E_2)^n, \quad (\text{A.11})$$

$$(T_1(E_2)T_3(E_2))^n E_3 = E_3 (T_1(E_2)T_3(E_2))^n \\ - q\{n\} (T_1(E_2)T_3(E_2))^{n-1} T_3(E_2)T_{13}(E_2), \quad (\text{A.12})$$

$$E_2^n E_3 = q^n E_3 E_2^n - q\{n\} E_2^{n-1} T_3(E_2) \quad (\text{A.13})$$

hold in $U_q(\mathfrak{sl}_4)$.

Lemma A.9 *For any $n \in \mathbb{N}$, the relations*

$$1^r(T_{13}(E_2)^n) = q^{-1}(q - q^{-1})\{n\}T_3(E_2)T_{13}(E_2)^{n-1}, \quad (\text{A.14})$$

$${}_1r((T_1(E_2)T_3(E_2))^n) = q^{-1}(q - q^{-1})\{n\}E_2T_3(E_2)(T_1(E_2)T_3(E_2))^{n-1} \quad (\text{A.15})$$

hold in $U_q(\mathfrak{sl}_4)$.

Proof of Proposition A.7 Take $\tilde{w}_0 = \tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1\tilde{\sigma}_1$. Then we have

$$\mathfrak{X}_{\tilde{w}_0} = \mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}$$

where

$$\mathfrak{X}^{[4]} \stackrel{(3.50)}{=} \mathfrak{X}_{\tilde{w}_0(2)} = \mathfrak{X}_2,$$

$$\mathfrak{X}^{[3]} = \Psi \circ \tilde{T}_1\tilde{T}_2 \circ \Psi^{-1}(\mathfrak{X}_1) = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{n\}!} (q^2c_1c_2)^n (T_3(E_2)T_1(E_2))^n,$$

$$\mathfrak{X}^{[2]} = \Psi \circ \tilde{T}_1 \circ \Psi^{-1}(\mathfrak{X}_2) = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^4c_1^2c_2)^n T_{13}(E_2)^{2n},$$

$$\mathfrak{X}^{[1]} = \mathfrak{X}_1.$$

By Lemma 3.7, property (3.9) of the skew derivative ${}_2r$ and [11, 8.26, (4)], we see that ${}_2r(\mathfrak{X}_{\tilde{w}_0}) = (q - q^{-1})q^2c_2E_2\mathfrak{X}_{\tilde{w}_0}$. Due to the underlying symmetry in this case, we only need to show that

$${}_1r(\mathfrak{X}_{\tilde{w}_0}) = (q - q^{-1})c_1E_3\mathfrak{X}_{\tilde{w}_0}.$$

For each $i = 1, 2, 3, 4$, let $\mathfrak{X}_{[i]} = K_1\mathfrak{X}^{[i]}K_1^{-1}$. Then by the property (3.9) of the skew derivation ${}_1r$, we have

$${}_1r(\mathfrak{X}_w) = \mathfrak{X}_{[4]}{}_1r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]}).$$

Using Lemmas 3.9 and A.9, it follows that

$${}_1r(\mathfrak{X}^{[3]}) \stackrel{(A.15)}{=} q^{-1}(q - q^{-1})^2(q^2c_1c_2)E_2T_3(E_2)\mathfrak{X}^{[3]}, \quad (\text{A.16})$$

$${}_1r(\mathfrak{X}^{[2]}) \stackrel{(A.14)}{=} q^{-1}(q - q^{-1})^2(q^4c_1^2c_2)T_3(E_2)T_{13}(E_2)\mathfrak{X}^{[2]}, \quad (\text{A.17})$$

$${}_1r(\mathfrak{X}^{[1]}) \stackrel{(3.42)}{=} (q - q^{-1})E_3\mathfrak{X}^{[1]}. \quad (\text{A.18})$$

Using Lemma A.8, we look at the term $\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]})$ in more detail. We have

$$\mathfrak{X}_{[2]}E_3 \stackrel{(A.11)}{=} E_3\mathfrak{X}^{[2]}, \quad (\text{A.19})$$

$$\mathfrak{X}_{[3]}E_3 \stackrel{(A.12)}{=} E_3\mathfrak{X}^{[3]} - q(q - q^{-1})(q^2c_1c_2)\mathfrak{X}_{[3]}T_3(E_2)T_{13}(E_2), \quad (\text{A.20})$$

$$\mathfrak{X}_{[4]}E_3 \stackrel{(A.13)}{=} E_3\mathfrak{X}^{[4]} - q^{-1}(q - q^{-1})(q^2c_2)\mathfrak{X}_{[4]}E_2T_3(E_2). \quad (\text{A.21})$$

It follows that

$$\begin{aligned}
 & \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]1r}(\mathfrak{X}^{[1]}) \\
 & \stackrel{(A.19)}{=} (q - q^{-1})c_1\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}E_3\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\
 & \stackrel{(A.20)}{=} (q - q^{-1})c_1\mathfrak{X}_{[4]}(E_3\mathfrak{X}^{[3]} - q(q - q^{-1})(q^2c_1c_2)\mathfrak{X}_{[3]}T_3(E_2)T_{13}(E_2))\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\
 & \stackrel{(A.21)}{=} (q - q^{-1})c_1(E_3\mathfrak{X}^{[4]} - q^{-1}(q - q^{-1})(q^2c_2)\mathfrak{X}_{[4]}E_2T_3(E_2))\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\
 & \quad - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]1r}(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} \\
 & = (q - q^{-1})c_1E_3\mathfrak{X}_{\tilde{w}_0} - \mathfrak{X}_{[4]1r}(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]1r}(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]}
 \end{aligned}$$

by Eqs. (A.16), (A.17) and (A.18). Hence, it follows that $1r(\mathfrak{X}_{\tilde{w}_0}) = (q - q^{-1})c_1E_3\mathfrak{X}_{\tilde{w}_0}$, as required. $\square$

Proposition A.10 *The partial quasi K -matrix $\mathfrak{X}_{\tilde{w}'_0}$ coincides with the quasi K -matrix $\mathfrak{X}$.*

The following relations are needed for the proof of Proposition A.10. They are checked by induction.

Lemma A.11 *For any $n \in \mathbb{N}$, the relations*

$$(T_2(E_3)T_2(E_1))^n E_2 = q^{-2n} E_2 (T_2(E_3)T_2(E_1))^n, \quad (A.22)$$

$$T_{213}(E_2)^n E_2 = E_2 T_{213}^n(E_2) - (q - q^{-1})\{n\}T_{213}(E_2)^{n-1}T_2(E_3)T_2(E_1), \quad (A.23)$$

$$\begin{aligned}
 (E_1 E_3)^n E_2 &= q^{2n} E_2 (E_1 E_3)^n - q\{n\}(E_1 E_3)^{n-1}(E_3 T_2(E_1) + E_1 T_2(E_3)) \\
 &\quad - q^2\{n\}^2(E_1 E_3)^{n-1}T_{213}(E_2)
 \end{aligned} \quad (A.24)$$

hold in $U_q(\mathfrak{sl}_4)$.

Lemma A.12 *For any $n \in \mathbb{N}$, the relations*

$$2r(T_{213}(E_2)^n) = q^{-2}(q - q^{-1})^2\{n\}E_1 E_3 T_{213}(E_2)^{n-1}, \quad (A.25)$$

$$\begin{aligned}
 2r(T_2(E_3)^n T_2(E_1)^n) &= q^{-1}(q - q^{-1})\{n\}E_3 T_2(E_3)^{n-1}T_2(E_1)^n \\
 &\quad + q^{-1}(q - q^{-1})\{n\}E_1 T_2(E_3)^n T_2(E_1)^{n-1} \\
 &\quad + (q - q^{-1})\{n\}^2 T_{213}(E_2)(T_2(E_1)T_2(E_3))^{n-1}
 \end{aligned} \quad (A.26)$$

hold in $U_q(\mathfrak{sl}_4)$.

Proof of Proposition A.10 For $\tilde{w}'_0 = \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1$ we have

$$\mathfrak{X}_{\tilde{w}'_0} = \mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]},$$

where

$$\mathfrak{X}^{[4]} \stackrel{(3.50)}{=} \mathfrak{X}_{\tau_0(1)} = \mathfrak{X}_1,$$

$$\begin{aligned}\mathfrak{X}^{[3]} &= \Psi \circ \tilde{T}_2 \tilde{T}_1 \circ \Psi^{-1}(\mathfrak{X}_2) = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^4 c_1^2 c_2)^n T_{213}(E_2)^{2n}, \\ \mathfrak{X}^{[2]} &= \Psi \circ \tilde{T}_2 \circ \Psi^{-1}(\mathfrak{X}_1) = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{n\}!} (q^2 c_1 c_2)^n T_2(E_1)^n T_2(E_3)^n, \\ \mathfrak{X}^{[1]} &= \mathfrak{X}_2.\end{aligned}$$

By Lemma 3.9, property (3.9) of the skew derivative ${}_1r$ and [11, 8.26, (4)] we have ${}_1r(\mathfrak{X}_{\tilde{w}'_0}) = (q - q^{-1})c_1 E_3 \mathfrak{X}_{\tilde{w}'_0}$. We want to show that

$${}_2r(\mathfrak{X}_{\tilde{w}'_0}) = (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{\tilde{w}'_0}.$$

For $i = 1, 2, 3, 4$, we let $\mathfrak{X}_{[i]} = K_2 \mathfrak{X}^{[i]} K_2^{-1}$. Note that we have $\mathfrak{X}_{[3]} = \mathfrak{X}^{[3]}$. By the property (3.9) of the skew derivative ${}_2r$, we have

$${}_2r(\mathfrak{X}_{\tilde{w}'_0}) = \mathfrak{X}_{[4]} {}_2r(\mathfrak{X}^{[3]}) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} + \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} {}_2r(\mathfrak{X}^{[2]}) \mathfrak{X}^{[1]} + \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} {}_2r(\mathfrak{X}^{[1]}).$$

Using Lemma A.12, we have

$$\begin{aligned}{}_2r(\mathfrak{X}^{[3]}) &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^4 c_1^2 c_2)^n {}_2r(T_{213}(E_2)^{2n}) \\ &\stackrel{(A.25)}{=} q^{-2} (q - q^{-1})^2 \sum_{n \geq 1} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^4 c_1^2 c_2)^n \{2n\} E_1 E_3 T_{213}(E_2)^{2n-1} \\ &= q^{-2} (q - q^{-1})^2 E_1 E_3 \sum_{n \geq 0} \frac{(q - q^{-1})^{n+1}}{\{2n\}!!} (q^4 c_1^2 c_2)^{n+1} T_{213}(E_2)^{2n+1} \\ &= (q - q^{-1})^3 (q^2 c_1^2 c_2) E_1 E_3 T_{213}(E_2) \mathfrak{X}^{[3]}.\end{aligned}\tag{A.27}$$

Similarly, we have

$$\begin{aligned}{}_2r(\mathfrak{X}^{[2]}) &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{n\}!} (q^2 c_1 c_2)^n {}_2r(T_2(E_1)^n T_2(E_3)^n) \\ &\stackrel{(A.26)}{=} q^{-1} (q - q^{-1})^2 (q^2 c_1 c_2) (E_3 T_2(E_1) + E_1 T_2(E_3)) \mathfrak{X}^{[2]} \\ &\quad + (q - q^{-1}) T_{213}(E_2) \sum_{n \geq 1} \frac{(q - q^{-1})^n}{\{n\}!} (q^2 c_1 c_2)^n \{n\}^2 T_2(E_1)^{n-1} T_2(E_3)^{n-1} \\ &= (q - q^{-1})^2 (q c_1 c_2) (E_3 T_2(E_1) + E_1 T_2(E_3)) \mathfrak{X}^{[2]} \\ &\quad + (q - q^{-1})^2 (q^2 c_1 c_2) T_{213}(E_2) \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{n\}!} (q^2 c_1 c_2)^n \{n+1\} T_2(E_1)^n T_2(E_3)^n.\end{aligned}$$

We want to write the last summand in terms of $\mathfrak{X}^{[2]}$. To do this, we use the fact that $\{n+1\} = 1 + q^2 \{n\}$ for $n \geq 1$. This is a useful fact that will be used again in future

calculations. Using this, we have

$$\begin{aligned}
 & \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{n\}!} (q^2 c_1 c_2)^n \{n + 1\} T_2(E_1)^n T_2(E_3)^n \\
 &= 1 + \sum_{n \geq 1} \frac{(q - q^{-1})^n}{\{n\}!} (q^2 c_1 c_2)^n (1 + q^2 \{n\}) T_2(E_1)^n T_2(E_3)^n \\
 &= \mathfrak{X}^{[2]} + q^2 \sum_{n \geq 1} \frac{(q - q^{-1})^n}{\{n - 1\}!} (q^2 c_1 c_2)^n T_2(E_1)^n T_2(E_3)^n \\
 &= \mathfrak{X}^{[2]} + q^2 \sum_{n \geq 0} \frac{(q - q^{-1})^{n+1}}{\{n\}!} (q^2 c_1 c_2)^{n+1} T_2(E_1)^{n+1} T_2(E_3)^{n+1} \\
 &= \mathfrak{X}^{[2]} + (q - q^{-1})(q^4 c_1 c_2) T_2(E_1) T_2(E_3) \mathfrak{X}^{[2]}.
 \end{aligned}$$

Inserting this equation into the expression for ${}_{2r}(\mathfrak{X}^{[2]})$, we obtain

$$\begin{aligned}
 {}_{2r}(\mathfrak{X}^{[2]}) &= (q - q^{-1})^2 (q c_1 c_2) \left(E_3 T_2(E_1) + E_1 T_2(E_3) \right) \mathfrak{X}^{[2]} \\
 &\quad + (q - q^{-1})^2 (q^2 c_1 c_2) T_{213}(E_2) \mathfrak{X}^{[2]} \\
 &\quad + (q - q^{-1})^3 (q^6 c_1^2 c_2^2) T_{213}(E_2) T_2(E_1) T_2(E_3) \mathfrak{X}^{[2]}. \tag{A.28}
 \end{aligned}$$

Finally, we have

$${}_{2r}(\mathfrak{X}^{[1]}) \stackrel{(3.42)}{=} (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}^{[1]}. \tag{A.29}$$

Now, we use Lemma A.11 to rewrite the term $\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_{2r}(\mathfrak{X}^{[1]})$. By calculations similar to those leading to (A.27) and (A.28), we obtain

$$\mathfrak{X}_{[2]}E_2 = E_2\mathfrak{X}^{[2]}, \tag{A.30}$$

$$\mathfrak{X}_{[3]}E_2 = E_2\mathfrak{X}^{[3]} - (q - q^{-1})^2 (q^4 c_1^2 c_2) \mathfrak{X}_{[3]}T_{213}(E_2)T_2(E_1)T_2(E_3), \tag{A.31}$$

$$\begin{aligned}
 \mathfrak{X}_{[4]}E_2 &= E_2\mathfrak{X}^{[4]} - q^{-1}(q - q^{-1})c_1\mathfrak{X}_{[4]}(E_1T_2(E_3) + E_3T_2(E_1)) \\
 &\quad - (q - q^{-1})c_1(\mathfrak{X}_{[4]} + (q - q^{-1})c_1\mathfrak{X}_{[4]}E_1E_3)T_{213}(E_2). \tag{A.32}
 \end{aligned}$$

We use these to obtain the following.

$$\begin{aligned}
 & \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_{2r}(\mathfrak{X}^{[1]}) \\
 & \stackrel{(A.29)}{=} (q - q^{-1})(q^2 c_2) \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}E_2\mathfrak{X}^{[1]} \\
 & \stackrel{(A.30)}{=} (q - q^{-1})(q^2 c_2) \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}E_2\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\
 & \stackrel{(A.31)}{=} (q - q^{-1})(q^2 c_2) \mathfrak{X}_{[4]}E_2\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\
 & \quad - (q - q^{-1})^3 (q^6 c_1^2 c_2^2) \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}T_{213}(E_2)T_2(E_1)T_2(E_3)\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}
 \end{aligned}$$

$$\begin{aligned}
& \stackrel{(A.32)}{=} (q - q^{-1})(q^2 c_2) \left(E_2 \mathfrak{X}^{[4]} - q^{-1}(q - q^{-1})c_1 \mathfrak{X}_{[4]}(E_1 T_2(E_3) + E_3 T_2(E_1)) \right. \\
& \quad \left. - (q - q^{-1})c_1(\mathfrak{X}_{[4]} + (q - q^{-1})c_1 \mathfrak{X}_{[4]} E_1 E_3) T_{213}(E_2) \right) \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\
& \quad - (q - q^{-1})^3 (q^6 c_1^2 c_2^2) \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} T_{213}(E_2) T_2(E_1) T_2(E_3) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}.
\end{aligned}$$

We gather terms now and get

$$\begin{aligned}
& \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} 2r(\mathfrak{X}^{[1]}) \\
& = (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{\tilde{w}'_0} \\
& \quad - (q - q^{-1})^2 (q c_1 c_2) \mathfrak{X}_{[4]} (E_1 T_2(E_3) + E_3 T_2(E_1)) \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\
& \quad - (q - q^{-1})^2 (q^2 c_1 c_2) \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} T_{213}(E_2) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\
& \quad + (q - q^{-1})^3 (q^6 c_1^2 c_2^2) \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} T_{213}(E_2) T_2(E_1) T_2(E_3) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\
& \quad - (q - q^{-1})^3 (q^2 c_1^2 c_2) \mathfrak{X}_{[4]} E_1 E_3 T_{213}(E_2) \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\
& = (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{\tilde{w}'_0} - \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} 2r(\mathfrak{X}^{[2]}) \mathfrak{X}^{[1]} - \mathfrak{X}_{[4]} 2r(\mathfrak{X}^{[3]}) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}
\end{aligned}$$

where we use the fact that $E_1 T_2(E_3)$ and $E_3 T_2(E_1)$ both commute with $T_{213}(E_2)$, and hence with $\mathfrak{X}^{[3]}$. It follows that

$$\begin{aligned}
2r(\mathfrak{X}_{\tilde{w}'_0}) & = \mathfrak{X}_{[4]} 2r(\mathfrak{X}^{[3]}) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} + \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} 2r(\mathfrak{X}^{[2]}) \mathfrak{X}^{[1]} + \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} 2r(\mathfrak{X}^{[1]}) \\
& = (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{\tilde{w}'_0}
\end{aligned}$$

as required. $\square$

A.4 Type $AIII_n$ for $n \geq 4$

Consider the Satake diagram of type $AIII_n$ for $n \geq 4$.

In this case the restricted root system is of type B_2 with

$$\tilde{\alpha}_1 = \frac{\alpha_1 + \alpha_n}{2}, \quad \tilde{\alpha}_2 = \frac{\alpha_2 + \alpha_3 + \cdots + \alpha_{n-1}}{2}.$$

The subgroup $\tilde{W} \subset W^\Theta$ is generated by the elements

$$\tilde{\sigma}_1 = \sigma_1 \sigma_n, \quad \tilde{\sigma}_2 = w_X w_{\{2, n-1\} \cup X} = \sigma_2 \sigma_3 \cdots \sigma_{n-1} \cdots \sigma_3 \sigma_2.$$

The longest word of the restricted Weyl group has two reduced expressions given by

$$\tilde{w}_0 = \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2, \quad \tilde{w}'_0 = \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1.$$

The definition (3.14) and condition (3.18) imply that $c_1 = c_n = \bar{c}_1$. By Lemmas 3.9 and 3.10 we have

$$\begin{aligned} \mathfrak{X}_1 &= \sum_{k \geq 0} \frac{(q - q^{-1})^k}{\{k\}!} c_1^k (E_1 E_n)^k, \\ \mathfrak{X}_2 &= \left(\sum_{k \geq 0} \frac{(c_2 s(n-1))^k}{\{k\}!} T_2 T_{w_X} (E_{n-1})^k \right) \left(\sum_{k \geq 0} \frac{(c_{n-1} s(2))^k}{\{k\}!} T_{n-1} T_{w_X} (E_2)^k \right). \end{aligned}$$

Proposition A.13 *The partial quasi K -matrix $\mathfrak{X}_{\tilde{w}_0}$ coincides with the quasi K -matrix $\mathfrak{X}$.*

We have the following relations needed in the proof of Proposition A.13, proved by induction.

Lemma A.14 *For any $k \in \mathbb{N}$ the relations*

$$\begin{aligned} T_{1n(n-1)} T_{w_X} (E_2)^k E_n &= q^{-k} E_n T_{1n(n-1)} T_{w_X} (E_2)^k, \\ T_{12n} T_{w_X} (E_{n-1})^k E_n &= q^{-k} E_n T_{12n} T_{w_X} (E_{n-1})^k, \\ T_{n \dots 3} (E_2)^k E_n &= q^{-k} E_n T_{n \dots 3} (E_2)^k, \\ T_{1 \dots (n-2)} (E_{n-1})^k E_n &= q^k E_n T_{1 \dots (n-2)} (E_{n-1})^k \\ &\quad - q \{k\} T_{1 \dots (n-2)} (E_{n-1})^{k-1} T_{12n} T_{w_X} (E_{n-1}), \\ T_{n-1} T_{w_X} (E_2)^k E_n &= q^k E_n T_{n-1} T_{w_X} (E_2)^k \\ &\quad - q \{k\} T_{n-1} T_{w_X} (E_2)^{k-1} T_{n \dots 3} (E_2), \\ T_2 T_{w_X} (E_{n-1})^k E_n &= q^k E_n T_2 T_{w_X} (E_{n-1})^k \\ &\quad - q \{k\} T_2 T_{w_X} (E_{n-1})^{k-1} T_{2n} T_{w_X} (E_{n-1}), \\ T_{2n} T_{w_X} (E_{n-1}) T_{n-1} T_{w_X} (E_2)^k &= q^{-k} T_{n-1} T_{w_X} (E_2)^k T_{2n} T_{w_X} (E_{n-1}) \\ &\quad + q^{1-k} (q - q^{-1}) \{k\} T_{n-1} T_{w_X} (E_2)^{k-1} T_2 T_{w_X} (E_{n-1}) T_{n \dots 3} (E_2), \end{aligned} \quad (\text{A.33})$$

$$\begin{aligned} T_{12n} T_{w_X} (E_{n-1})^k T_{n \dots 3} (E_2) \\ = (q^{-k} + (q - q^{-1}) q^{1-k} \{k\}) T_{n \dots 3} (E_2) T_{12n} T_{w_X} (E_{n-1})^k \end{aligned} \quad (\text{A.34})$$

hold in $U_q(\mathfrak{sl}_{n+1})$.

Lemma A.15 *For any $k \in \mathbb{N}$ the relations*

$$\begin{aligned} {}_1 r(T_{1 \dots (n-2)} (E_{n-1})^k) \\ = q^{-1} (q - q^{-1}) \{k\} T_{2 \dots (n-2)} (E_{n-1}) T_{1 \dots (n-2)} (E_{n-1})^{k-1}, \end{aligned} \quad (\text{A.35})$$

$$\begin{aligned}
& {}_1r(T_{12n}T_{w_X}(E_{n-1})^k) \\
& = q^{-1}(q - q^{-1})\{k\}T_{2n}T_{w_X}(E_{n-1})T_{12n}T_{w_X}(E_{n-1})^{k-1},
\end{aligned} \tag{A.36}$$

$$\begin{aligned}
& {}_1r(T_{1n(n-1)}T_{w_X}(E_2)^k) \\
& = q^{-1}(q - q^{-1})\{k\}T_{n(n-1)}T_{w_X}(E_2)T_{1n(n-1)}T_{w_X}(E_2)^{k-1}
\end{aligned} \tag{A.37}$$

hold in $U_q(\mathfrak{sl}_{n+1})$.

One can show that the above Lemmas still hold if we consider the case where we have no black dots. In this situation, the calculations differ slightly but the results still hold.

Proof of Proposition A.13 We write

$$\mathfrak{X}_{\tilde{w}_0} = \mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}, \tag{A.38}$$

where, recalling the notation $\tilde{c}_2^2 = c_2c_{n-1}s(n-1)s(2)$, we have

$$\begin{aligned}
\mathfrak{X}^{[4]} & \stackrel{(3.50)}{=} \mathfrak{X}_2, \\
\mathfrak{X}^{[3]} & = \sum_{k \geq 0} \frac{(q - q^{-1})^k}{\{k\}!} (qc_1\tilde{c}_2^2)^k T_{n \dots 3}(E_2)^k T_{1 \dots (n-2)}(E_{n-1})^k, \\
\mathfrak{X}^{[2]} & = \left(\sum_{k \geq 0} \frac{(qc_1c_2s(n-1))^k}{\{k\}!} T_{12n}T_{w_X}(E_{n-1})^k \right) \\
& \quad \left(\sum_{k \geq 0} \frac{(qc_1c_{n-1}s(2))^k}{\{k\}!} T_{1n(n-1)}T_{w_X}(E_2)^k \right), \\
\mathfrak{X}^{[1]} & = \mathfrak{X}_1.
\end{aligned}$$

When necessary, since $\mathfrak{X}^{[4]}$ and $\mathfrak{X}^{[2]}$ are a product of two infinite sums, we write $\mathfrak{X}^{[4]} = \mathfrak{X}^{[4;1]}\mathfrak{X}^{[4;2]}$ and $\mathfrak{X}^{[2]} = \mathfrak{X}^{[2;1]}\mathfrak{X}^{[2;2]}$.

For each $i = 1, 2, 3, 4$, let $\mathfrak{X}_{[i]} = K_1\mathfrak{X}^{[i]}K_1^{-1}$. We write $\mathfrak{X}_{[4]} = \mathfrak{X}_{[4;1]}\mathfrak{X}_{[4;2]}$ and $\mathfrak{X}_{[2]} = \mathfrak{X}_{[2;1]}\mathfrak{X}_{[2;2]}$. Now, by the rank one case for $\mathfrak{X}_2$ given in Lemma 3.10 and [11, 8.26, (4)], we obtain

$${}_2r(\mathfrak{X}_{\tilde{w}_0}) = (q - q^{-1})c_2s(n-1)T_{w_X}(E_{n-1})\mathfrak{X}_{\tilde{w}_0}. \tag{A.39}$$

The underlying symmetry implies that we only need to show that

$${}_1r(\mathfrak{X}_{\tilde{w}_0}) = (q - q^{-1})c_1E_n\mathfrak{X}_{\tilde{w}_0}.$$

By property (3.9) of the skew derivative ${}_1r$ and [11, 8.26, (4)] we have

$$\begin{aligned}
{}_1r(\mathfrak{X}_{\tilde{w}_0}) & = \mathfrak{X}_{[4]}{}_1r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}{}_1r(\mathfrak{X})^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]}).
\end{aligned} \tag{A.40}$$

Using Lemma A.15 and Lemma 3.9 we find that

$${}_1r(\mathfrak{X}^{[3]}) \stackrel{(A.35)}{=} (q - q^{-1})^2 c_1 \tilde{c}_2^2 T_{n \dots 3}(E_2) T_{2 \dots (n-2)}(E_{n-1}) \mathfrak{X}^{[3]}, \quad (A.41)$$

$${}_1r(\mathfrak{X}^{[1]}) \stackrel{(3.42)}{=} (q - q^{-1}) c_1 E_n \mathfrak{X}^{[1]}. \quad (A.42)$$

To write an expression for ${}_1r(\mathfrak{X}^{[2]})$, we use the splitting of $\mathfrak{X}^{[2]}$ into a product of two infinite sums. By Eqs. (A.36) and (A.37) of Lemma A.15 we have

$$\begin{aligned} {}_1r(\mathfrak{X}^{[2]}) &= {}_1r(\mathfrak{X}^{[2;1]}) \mathfrak{X}^{[2;2]} + \mathfrak{X}_{[2;1]} {}_1r(\mathfrak{X}^{[2;2]}) \\ &= (q - q^{-1}) c_1 c_2 s(n-1) T_{2n} T_{w_X}(E_{n-1}) \mathfrak{X}^{[2]} \\ &\quad + (q - q^{-1}) c_1 c_{n-1} s(2) \mathfrak{X}_{[2;1]} T_{n(n-1)} T_{w_X}(E_2) \mathfrak{X}^{[2;2]}. \end{aligned} \quad (A.43)$$

We would like the last summand of this expression to be in terms of $\mathfrak{X}^{[2]}$. To this end, we use Eq. (A.34) in Lemma A.14 to obtain

$$\begin{aligned} &\mathfrak{X}_{[2;1]} T_{n(n-1)} T_{w_X}(E_2) \\ &= \left(1 + \sum_{k \geq 1} \frac{(q c_1 c_2 s(n-1))^k}{\{k\}!} q^k T_{12n} T_{w_X}(E_{n-1})^k \right) T_{n(n-1)} T_{w_X}(E_2) \\ &\stackrel{(A.34)}{=} T_{n(n-1)} T_{w_X}(E_2) \mathfrak{X}^{[2;1]} \\ &\quad + q^2 (q - q^{-1}) c_1 c_2 s(n-1) T_{n(n-1)} T_{w_X}(E_2) T_{12n} T_{w_X}(E_{n-1}) \mathfrak{X}^{[2;1]}. \end{aligned}$$

Substituting this into (A.43) it follows that

$$\begin{aligned} {}_1r(\mathfrak{X}^{[2]}) &= (q - q^{-1}) c_1 c_2 s(n-1) T_{2n} T_{w_X}(E_{n-1}) \mathfrak{X}^{[2]} \\ &\quad + (q - q^{-1}) c_1 c_{n-1} s(2) T_{n(n-1)} T_{w_X}(E_2) \mathfrak{X}^{[2]} \\ &\quad + q^2 (q - q^{-1})^2 c_1^2 \tilde{c}_2^2 T_{n(n-1)} T_{w_X}(E_2) T_{12n} T_{w_X}(E_{n-1}) \mathfrak{X}^{[2]}. \end{aligned} \quad (A.44)$$

When we calculate ${}_1r(\mathfrak{X}_{\tilde{w}_0})$, we obtain a component of the form $\mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} {}_1r(\mathfrak{X}^{[1]})$. From Eq. (A.42), we see that we obtain an E_n term that we want to pass to the front of this expression. We use Lemma A.14 to do this. We have

$$\mathfrak{X}_{[2]} E_n = E_n \mathfrak{X}^{[2]}, \quad (A.45)$$

$$\mathfrak{X}_{[3]} E_n = E_n \mathfrak{X}_{[3]} - q^2 (q - q^{-1}) c_1 \tilde{c}_2^2 T_{n \dots 3}(E_2) \mathfrak{X}^{[3]} T_{12n} T_{w_X}(E_{n-1}), \quad (A.46)$$

$$\mathfrak{X}_{[4;2]} E_n = E_n \mathfrak{X}^{[4;2]} - c_1 c_{n-1} s(2) \mathfrak{X}_{[4;2]} T_{n \dots 3}(E_2), \quad (A.47)$$

$$\mathfrak{X}_{[4;1]} E_n = E_n \mathfrak{X}^{[4;1]} - c_1 c_2 s(n-1) \mathfrak{X}_{[4;1]} T_{2n} T_{w_X}(E_{n-1}). \quad (A.48)$$

Now, using Eq. (A.33), we obtain the following expression for $\mathfrak{X}_{[4]} E_n$.

$$\mathfrak{X}_{[4]} E_n \stackrel{(A.47)}{=} \mathfrak{X}_{[4;1]} (E_n \mathfrak{X}^{[4;2]} - c_1 c_{n-1} s(2) \mathfrak{X}_{[4;2]} T_{n \dots 3}(E_2))$$

$$\begin{aligned}
&\stackrel{(A.48)}{=} (E_n \mathfrak{X}^{[4;1]} - c_1 c_2 s(n-1) \mathfrak{X}_{[4;1]} T_{2n} T_{w_X}(E_{n-1})) \mathfrak{X}^{[4;2]} \\
&\quad - c_1 c_{n-1} s(2) \mathfrak{X}_{[4]} T_{n \dots 3}(E_2) \\
&\stackrel{(A.33)}{=} E_n \mathfrak{X}^{[4]} - c_1 c_{n-1} s(2) \mathfrak{X}_{[4]} T_{n \dots 3}(E_2) \\
&\quad - c_1 c_2 s(n-1) \mathfrak{X}_{[4;1]} \mathfrak{X}_{[4;2]} T_{2n} T_{w_X}(E_{n-1}) \\
&\quad + (q - q^{-1}) c_1^2 c_2 c_{n-1} s(2) s(n-1) \mathfrak{X}_{[4;1]} \mathfrak{X}_{[4;2]} T_2 T_{w_X}(E_{n-1}) T_{n \dots 3}(E_2) \\
&= E_n \mathfrak{X}^{[4]} - c_1 c_{n-1} s(2) \mathfrak{X}_{[4]} T_{n \dots 3}(E_2) - c_1 c_2 s(n-1) \mathfrak{X}_{[4]} T_{2n} T_{w_X}(E_{n-1}) \\
&\quad - (q - q^{-1}) c_1^2 \tilde{c}_2^2 \mathfrak{X}_{[4]} T_2 T_{w_X}(E_{n-1}) T_{n \dots 3}(E_2). \tag{A.49}
\end{aligned}$$

Using Eqs. (A.45), (A.46) and (A.49), and comparing with Eqs. (A.41), (A.42) and (A.44), one finds that

$$\begin{aligned}
\mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]1} r(\mathfrak{X}^{[1]}) &= (q - q^{-1}) \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} E_n \mathfrak{X}^{[1]} \\
&= (q - q^{-1}) E_n \mathfrak{X}_{\tilde{w}_0} - \mathfrak{X}_{[4]1} r(\mathfrak{X}^{[3]}) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\
&\quad - \mathfrak{X}_{[4]} \mathfrak{X}_{[3]1} r(\mathfrak{X}^{[2]}) \mathfrak{X}^{[1]},
\end{aligned}$$

and hence it follows that

$${}_1 r(\mathfrak{X}_{\tilde{w}_0}) = (q - q^{-1}) E_n \mathfrak{X}_{\tilde{w}_0},$$

as required. This completes the proof. $\square$

Proposition A.16 *The partial quasi K -matrix $\mathfrak{X}_{\tilde{w}_0'}$ coincides with the quasi K -matrix $\mathfrak{X}$.*

We have the following relations needed in the proof of Proposition A.16, proved by induction.

Lemma A.17 *For any $k \in \mathbb{N}$ the relations*

$$\begin{aligned}
T_{2 \dots (n-1)}(E_n)^k T_{w_X}(E_{n-1}) &= T_{w_X}(E_{n-1}) T_{2 \dots (n-1)}(E_n)^k, \\
T_{(n-1) \dots 2}(E_1)^k T_{w_X}(E_{n-1}) &= q^{-k} T_{w_X}(E_{n-1}) T_{(n-1) \dots 2}(E_1)^k, \\
T_{2 \dots (n-1)} T_{12n} T_{w_X}(E_{n-1})^k T_{w_X}(E_{n-1}) &= T_{w_X}(E_{n-1}) T_{2 \dots (n-1)} T_{12n} T_{w_X}(E_{n-1})^k, \\
T_{(n-1) \dots 2} T_{1n(n-1)} T_{w_X}(E_2)^k T_{w_X}(E_{n-1}) &= T_{w_X}(E_{n-1}) T_{(n-1) \dots 2} T_{1n(n-1)} T_{w_X}(E_2)^k \\
&\quad - (q - q^{-1}) \{k\} T_{(n-1) \dots 2} T_{1n(n-1)} T_{w_X}(E_2)^{k-1} T_{3 \dots (n-1)}(E_n) T_{(n-1) \dots 2}(E_1), \\
E_1^k T_{w_X}(E_{n-1}) &= T_{w_X}(E_{n-1}) E_1^k, \\
E_n^k T_{w_X}(E_{n-1}) &= q^k T_{w_X}(E_{n-1}) E_n^k - q \{k\} E_n^{k-1} T_{w_X} T_{n-1}(E_n)
\end{aligned}$$

hold in $U_q(\mathfrak{sl}_{n+1})$.

Lemma A.18 *For any $k \in \mathbb{N}$ the relations*

$${}_2 r(T_{2 \dots (n-1)}(E_n)^k) = q^{-1} (q - q^{-1}) \{k\} T_{3 \dots (n-1)}(E_n) T_{2 \dots (n-1)}(E_n)^{k-1},$$

$$\begin{aligned}
& {}_2r(T_2 \dots (n-1) T_{12n} T_{w_X}(E_{n-1})^k) \\
&= q^{-2} (q - q^{-1})^2 \{k\} E_1 T_{3 \dots (n-1)}(E_n) T_2 \dots (n-1) T_{12n} T_{w_X}(E_{n-1})^{k-1}
\end{aligned}$$

hold in $U_q(\mathfrak{sl}_{n+1})$.

Proof of Proposition A.16 We write

$$\mathfrak{X}_{\tilde{w}'_0} = \mathfrak{X}^{[4]} \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]},$$

where we have

$$\begin{aligned}
\mathfrak{X}^{[4]} &\stackrel{(3.50)}{=} \mathfrak{X}_1, \\
\mathfrak{X}^{[3]} &= \left(\sum_{k_1 \geq 0} \frac{(q c_1 c_2 s(n-1))^{k_1}}{\{k_1\}!} T_2 \dots (n-1) T_{12n} T_{w_X}(E_{n-1})^{k_1} \right) \\
&\quad \left(\sum_{k_2 \geq 0} \frac{(q c_1 c_{n-1} s(2))^{k_2}}{\{k_2\}!} T_{(n-1) \dots 2} T_{1n(n-1)} T_{w_X}(E_2)^{k_2} \right), \\
\mathfrak{X}^{[2]} &= \sum_{k \geq 0} \frac{(q - q^{-1})^k}{\{k\}!} (q c_1 \tilde{c}_2^2)^k T_{(n-1) \dots 2} (E_1)^k T_2 \dots (n-1) (E_n)^k, \\
\mathfrak{X}^{[1]} &= \mathfrak{X}_2.
\end{aligned}$$

For $i = 1, 2, 3, 4$ let $\mathfrak{X}_{[i]} = K_2 \mathfrak{X}^{[i]} K_2^{-1}$. By Lemma 3.9 and [11, 8.26, (4)] it follows that

$$\begin{aligned}
{}_1r(\mathfrak{X}_{\tilde{w}'_0}) &= {}_1r(\mathfrak{X}^{[4]} \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}) \\
&= (q - q^{-1}) E_n \mathfrak{X}_{\tilde{w}'_0}.
\end{aligned}$$

Hence, we only need to check that

$${}_2r(\mathfrak{X}_{\tilde{w}'_0}) = q^{-1} (q - q^{-1}) c_2 s(n-1) T_{w_X}(E_{n-1}) \mathfrak{X}_{\tilde{w}'_0}.$$

By [11, 8.26, (4)] and property (3.9) of the skew derivative ${}_2r$ we have

$${}_2r(\mathfrak{X}_{\tilde{w}'_0}) = \mathfrak{X}_{[4]} {}_2r(\mathfrak{X}^{[3]}) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} + \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} {}_2r(\mathfrak{X}^{[2]}) \mathfrak{X}^{[1]} + \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} {}_2r(\mathfrak{X}^{[1]}).$$

Using Lemmas 3.10 and A.18, we have

$$\begin{aligned}
{}_2r(\mathfrak{X}^{[3]}) &= q^{-1} (q - q^{-1})^2 c_1 c_2 s(n-1) E_1 T_{3 \dots (n-1)}(E_n) \mathfrak{X}^{[3]}, \\
{}_2r(\mathfrak{X}^{[2]}) &= (q - q^{-1})^2 c_1 \tilde{c}_2^2 T_{3 \dots (n-1)}(E_n) T_{(n-1) \dots 2}(E_1) \mathfrak{X}^{[2]}, \\
{}_2r(\mathfrak{X}^{[1]}) &= q^{-1} (q - q^{-1}) c_2 s(n-1) T_{w_X}(E_{n-1}) \mathfrak{X}^{[1]}.
\end{aligned}$$

By Lemma A.17, we have

$$\mathfrak{X}_{[2]} T_{w_X}(E_{n-1}) = T_{w_X}(E_{n-1}) \mathfrak{X}^{[2]},$$

$$\begin{aligned}
\mathfrak{X}_{[3]}T_{w_X}(E_{n-1}) &= T_{w_X}(E_{n-1})\mathfrak{X}^{[3]} \\
&\quad - q(q - q^{-1})c_1c_{n-1}s(2)\mathfrak{X}_{[3]}T_{3\dots(n-1)}(E_n)T_{(n-1)\dots 2}(E_1), \\
\mathfrak{X}_{[4]}T_{w_X}(E_{n-1}) &= T_{w_X}(E_{n-1})\mathfrak{X}^{[4]} - (q - q^{-1})c_1\mathfrak{X}_{[4]}E_1T_{3\dots(n-1)}(E_n).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]2r}(\mathfrak{X}^{[1]}) &= q^{-1}(q - q^{-1})c_2s(n-1)T_{w_X}(E_{n-1})\mathfrak{X}_{w'} \\
&\quad - \mathfrak{X}_{[4]2r}(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]2r}(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]}
\end{aligned}$$

and therefore we obtain

$$2r(\mathfrak{X}_{\tilde{w}'_0}) = q^{-1}(q - q^{-1})c_2s(n-1)T_{w_X}(E_{n-1})\mathfrak{X}_{\tilde{w}'_0}$$

as required. $\square$

A.5 Type C_2

Consider the Satake diagram of type C_2 .

Since $\Theta = -\text{id}$ the subgroup $\tilde{W}$ coincides with W . The longest word of the Weyl group has two reduced expressions given by

$$w_0 = \sigma_1\sigma_2\sigma_1\sigma_2, \quad w'_0 = \sigma_2\sigma_1\sigma_2\sigma_1.$$

By Lemma 3.7 we have

$$\begin{aligned}
\mathfrak{X}_1 &= \sum_{n \geq 0} \frac{(q^2 - q^{-2})^n}{\{2n\}_1!!} (q^4 c_1)^n E_1^{2n}, \\
\mathfrak{X}_2 &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}_2!!} (q^2 c_2)^n E_2^{2n}.
\end{aligned}$$

Proposition A.19 *The partial quasi K -matrix $\mathfrak{X}_{w_0}$ coincides with the quasi K -matrix $\mathfrak{X}$.*

The following relations are necessary for the proof of Proposition A.19, proved by induction.

Lemma A.20 *For any $n \in \mathbb{N}$ the relations*

$$E_2^n E_1 = q^{2n} E_1 E_2^n - q^2 \{n\}_2 E_2^{n-1} T_1(E_2) - q^3 \{n\}_2 \{n-1\}_2 E_2^{n-1} T_{12}(E_1), \quad (\text{A.50})$$

$$T_1(E_2)^n E_1 = q^{-2n} E_1 T_1(E_2)^n, \quad (\text{A.51})$$

$$T_{12}(E_1)^n E_1 = E_1 T_{12}(E_1)^n - \frac{(q^2 - 1)}{[2]_2} \{n\}_1 T_{12}(E_1)^{n-1} T_1(E_2)^2 \quad (\text{A.52})$$

hold in $U_q(\mathfrak{so}_5)$.

Lemma A.21 For any $n \in \mathbb{N}$ the relations

$$\begin{aligned} {}_1r(T_1(E_2)^{n+1}) &= q^{-2}(q^2 - q^{-2})\{n+1\}_2 E_2 T_1(E_2)^n \\ &\quad + q^{-1}(q^2 - q^{-2})\{n+1\}_2 \{n\}_2 T_{12}(E_1) T_1(E_2)^{n-1}, \end{aligned} \quad (\text{A.53})$$

$${}_1r(T_{12}(E_1)^n) = q^{-3}(q - q^{-1})^2 \{n\}_1 E_2^2 T_{12}(E_1)^{n-1} \quad (\text{A.54})$$

hold in $U_q(\mathfrak{so}_5)$.

Proof of Proposition A.19 We have

$$\mathfrak{X}_{w_0} = \mathfrak{X}^{[4]} \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}$$

where

$$\mathfrak{X}^{[4]} \stackrel{(3.50)}{=} \mathfrak{X}_2,$$

$$\mathfrak{X}^{[3]} = \Psi \circ T_{12} \circ \Psi^{-1}(\mathfrak{X}_1) = \sum_{n \geq 0} \frac{(q^2 - q^{-2})^n}{\{2n\}_1!!} (q^4 c_1)^n (q^2 c_2)^{2n} T_{12}(E_1)^{2n},$$

$$\mathfrak{X}^{[2]} = \Psi \circ T_1 \circ \Psi^{-1}(\mathfrak{X}_2) = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}_2!!} (q^4 c_1)^n (q^2 c_2)^n T_1(E_2)^{2n},$$

$$\mathfrak{X}^{[1]} = \mathfrak{X}_1.$$

By Lemma 3.7 and [11, 8.26, (4)] we have

$$\begin{aligned} {}_2r(\mathfrak{X}_{w_0}) &= {}_2r(\mathfrak{X}_2) \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\ &= (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_2 \mathfrak{X}^{[3]} \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} \\ &= (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{w_0}. \end{aligned}$$

We want to show that

$${}_1r(\mathfrak{X}_{w_0}) = (q^2 - q^{-2})(q^4 c_1) E_1 \mathfrak{X}_{w_0}.$$

For each $i = 1, 2, 3, 4$ let $\mathfrak{X}_{[i]} = K_1 \mathfrak{X}^{[i]} K_1^{-1}$. Note that $\mathfrak{X}_{[3]} = \mathfrak{X}^{[3]}$. By property (3.9) of the skew derivative ${}_1r$ we see that

$${}_1r(\mathfrak{X}_{w_0}) = \mathfrak{X}_{[4]} {}_1r(\mathfrak{X}^{[3]}) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} + \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} {}_1r(\mathfrak{X}^{[2]}) \mathfrak{X}^{[1]} + \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} {}_1r(\mathfrak{X}^{[1]}). \quad (\text{A.55})$$

Lemma A.21 gives

$${}_1r(\mathfrak{X}^{[3]}) = q^{-3}(q - q^{-1})^2(q^2 - q^{-2})(q^4c_1)(q^2c_2)^2E_2^2T_{12}(E_1)\mathfrak{X}^{[3]}, \quad (\text{A.56})$$

$${}_1r(\mathfrak{X}^{[1]}) = (q^2 - q^{-2})(q^4c_1)E_1\mathfrak{X}^{[1]}. \quad (\text{A.57})$$

By Eq. (A.53) we have

$$\begin{aligned} {}_1r(\mathfrak{X}^{[2]}) &= q^{-2}(q - q^{-1})(q^2 - q^{-2})(q^4c_1)(q^2c_2)E_2T_1(E_2)\mathfrak{X}^{[2]} \\ &\quad + q^{-1}(q - q^{-1})(q^2 - q^{-2})(q^6c_1c_2)T_{12}(E_1)\widehat{\mathfrak{X}}^{[2]} \end{aligned} \quad (\text{A.58})$$

where

$$\widehat{\mathfrak{X}}^{[2]} = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}_2!!} (q^6c_1c_2)^n \{2n + 1\}_2 T_1(E_2)^{2n}.$$

For any $n \geq 1$ we have

$$\{2n + 1\}_2 = 1 + q^2\{2n\}_2. \quad (\text{A.59})$$

It hence follows that

$$\begin{aligned} \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}_2!!} (q^4c_1)^n (q^2c_2)^n \{2n + 1\}_2 T_1(E_2)^{2n} \\ = \mathfrak{X}^{[2]} + q^2(q - q^{-1})(q^4c_1)(q^2c_2)T_1(E_2)^2\mathfrak{X}^{[2]}. \end{aligned}$$

Substituting this into Eq. (A.58) we see that

$$\begin{aligned} {}_1r(\mathfrak{X}^{[2]}) &= q^{-2}(q - q^{-1})(q^2 - q^{-2})(q^4c_1)(q^2c_2)E_2T_1(E_2)\mathfrak{X}^{[2]} \\ &\quad + q^{-1}(q - q^{-1})(q^2 - q^{-2})(q^4c_1)(q^2c_2)T_{12}(E_1)\mathfrak{X}^{[2]} \\ &\quad + q(q - q^{-1})^2(q^2 - q^{-2})(q^4c_1)^2(q^2c_2)^2T_{12}(E_1)T_1(E_2)^2\mathfrak{X}^{[2]} \end{aligned} \quad (\text{A.60})$$

When we calculate ${}_1r(\mathfrak{X}_{w_0})$, we obtain a component of the form $\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]})$. From Eq. (A.57), this contains an E_1 term that we pass to the front using Lemma A.20. We have

$$\mathfrak{X}_{[2]}E_1 \stackrel{(\text{A.51})}{=} E_1\mathfrak{X}^{[2]}, \quad (\text{A.61})$$

$$\mathfrak{X}_{[3]}E_1 \stackrel{(\text{A.52})}{=} E_1\mathfrak{X}^{[3]} - q(q - q^{-1})^2(q^4c_1)(q^2c_2)^2\mathfrak{X}_{[3]}T_{12}(E_1)T_1(E_2)^2, \quad (\text{A.62})$$

$$\begin{aligned} \mathfrak{X}_{[4]}E_1 &\stackrel{(\text{A.50})}{=} E_1\mathfrak{X}^{[4]} - q^{-2}(q - q^{-1})(q^2c_2)E_2\mathfrak{X}_{[4]}T_1(E_2) \\ &\quad - q^{-1}(q - q^{-1})(q^2c_2)\mathfrak{X}_{[4]}T_{12}(E_1) \\ &\quad - q^{-3}(q - q^{-1})^2(q^2c_2)^2\mathfrak{X}_{[4]}E_2^2T_{12}(E_1) \end{aligned} \quad (\text{A.63})$$

where we also use (A.59) to obtain (A.63). Note that $\mathfrak{X}^{[3]}$ commutes with $E_2 T_1(E_2)$. Using (A.61), (A.62) and (A.63), and comparing with (A.56) and (A.60) we obtain

$$\begin{aligned} & \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]1}r(\mathfrak{X}^{[1]}) \\ &= (q^2 - q^{-2})(q^4 c_1)E_1 \mathfrak{X}_{w_0} - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]1}r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} - \mathfrak{X}_{[4]1}r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}. \end{aligned}$$

Hence by (A.55) we have

$${}_1r(\mathfrak{X}_{w_0}) = (q^2 - q^{-2})(q^4 c_1)E_1 \mathfrak{X}_{w_0}$$

as required. $\square$

We now consider the reduced expression $w'_0 = \sigma_2 \sigma_1 \sigma_2 \sigma_1$.

Proposition A.22 *The partial quasi K -matrix $\mathfrak{X}_{w'_0}$ coincides with the quasi K -matrix $\mathfrak{X}$.*

The following relations are needed for the proof and are obtained by induction.

Lemma A.23 *For any $n \in \mathbb{N}$ the relations*

$$E_1^n E_2 = q^{2n} E_2 E_1^n - q^2 \{n\}_1 E_1^{n-1} T_{21}(E_2), \quad (\text{A.64})$$

$$T_2(E_1)^n E_2 = q^{-2n} E_2 T_2(E_1)^n, \quad (\text{A.65})$$

$$T_{21}(E_2)^n E_2 = E_2 T_{21}(E_2)^n - [2]_2 \{n\}_2 T_{21}(E_2)^{n-1} T_2(E_1) \quad (\text{A.66})$$

hold in $U_q(\mathfrak{so}_5)$.

Lemma A.24 *For any $n \in \mathbb{N}$ the relations*

$${}_2r(T_2(E_1)^n) = (q - q^{-1})\{n\}_1 T_{21}(E_2) T_2(E_1)^{n-1}, \quad (\text{A.67})$$

$${}_2r(T_{21}(E_2)^n) = q^{-2}(q^2 - q^{-2})\{n\}_2 E_1 T_{21}(E_2)^{n-1} \quad (\text{A.68})$$

hold in $U_q(\mathfrak{so}_5)$.

Proof of Proposition A.22 We have

$$\mathfrak{X}_{w'_0} = \mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}$$

where

$$\begin{aligned} \mathfrak{X}^{[4]} &\stackrel{(3.50)}{=} \mathfrak{X}_1, \\ \mathfrak{X}^{[3]} &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}_2!!} (q^4 c_1)^n (q^2 c_2)^n T_{21}(E_2)^{2n}, \\ \mathfrak{X}^{[2]} &= \sum_{n \geq 0} \frac{(q^2 - q^{-2})^n}{\{2n\}_1!!} (q^4 c_1)^n (q^2 c_2)^{2n} T_2(E_1)^{2n}, \end{aligned}$$

$$\mathfrak{X}^{[1]} = \mathfrak{X}_2.$$

By Lemma 3.7 and [11, 8.26, (4)] we have ${}_1r(\mathfrak{X}_{w'_0}) = (q^2 - q^{-2})(q^4 c_1) E_1 \mathfrak{X}_{w'_0}$. We want to show that

$${}_2r(\mathfrak{X}_{w'_0}) = (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{w'_0}.$$

For each $i = 1, 2, 3, 4$ let $\mathfrak{X}_{[i]} = K_2 \mathfrak{X}^{[i]} K_2^{-1}$. By property (3.9) of the skew derivative ${}_2r$ we have

$${}_2r(\mathfrak{X}_{w'_0}) = \mathfrak{X}_{[4]} {}_2r(\mathfrak{X}^{[3]}) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} + \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} {}_2r(\mathfrak{X}^{[2]}) \mathfrak{X}^{[1]} + \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} {}_2r(\mathfrak{X}^{[1]}). \quad (\text{A.69})$$

Using Lemma A.24 we obtain

$${}_2r(\mathfrak{X}^{[3]}) \stackrel{(\text{A.68})}{=} q^{-2}(q - q^{-1})(q^2 - q^{-2})(q^4 c_1)(q^2 c_2) E_1 T_{21}(E_2) \mathfrak{X}^{[3]}, \quad (\text{A.70})$$

$${}_2r(\mathfrak{X}^{[2]}) \stackrel{(\text{A.67})}{=} (q - q^{-1})(q^2 - q^{-2})(q^4 c_1)(q^2 c_2)^2 T_{21}(E_2) T_2(E_1) \mathfrak{X}^{[2]}, \quad (\text{A.71})$$

$${}_2r(\mathfrak{X}^{[1]}) \stackrel{(\text{3.39})}{=} (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}^{[1]}. \quad (\text{A.72})$$

By Lemma A.23 we have

$$\mathfrak{X}_{[2]} E_2 \stackrel{(\text{A.65})}{=} E_2 \mathfrak{X}^{[2]}, \quad (\text{A.73})$$

$$\mathfrak{X}_{[3]} E_2 \stackrel{(\text{A.66})}{=} E_2 \mathfrak{X}^{[3]} - (q^2 - q^{-2})(q^4 c_1)(q^2 c_2) \mathfrak{X}_{[3]} T_{21}(E_2) T_2(E_1), \quad (\text{A.74})$$

$$\mathfrak{X}_{[4]} E_2 \stackrel{(\text{A.64})}{=} E_2 \mathfrak{X}^{[4]} - q^{-2}(q^2 - q^{-2})(q^4 c_1) \mathfrak{X}_{[4]} E_1 T_{21}(E_2). \quad (\text{A.75})$$

Using Eqs. (A.73), (A.74) and (A.75), and comparing with Eqs. (A.70) and (A.71) we can rewrite the term $\mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} {}_2r(\mathfrak{X}^{[1]})$ as

$$\begin{aligned} & \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} {}_2r(\mathfrak{X}^{[1]}) \\ & \stackrel{(\text{A.72})}{=} (q - q^{-1})(q^2 c_2) \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} \mathfrak{X}_{[2]} E_2 \mathfrak{X}^{[1]} \\ & = (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{w'_0} - \mathfrak{X}_{[4]} {}_2r(\mathfrak{X}^{[3]}) \mathfrak{X}^{[2]} \mathfrak{X}^{[1]} - \mathfrak{X}_{[4]} \mathfrak{X}_{[3]} {}_2r(\mathfrak{X}^{[2]}) \mathfrak{X}^{[1]}. \end{aligned}$$

It hence follows from (A.69) that ${}_2r(\mathfrak{X}_{w'_0}) = (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{w'_0}$ as required. $\square$

A.6 Type G_2

Consider the Satake diagram of type G_2 .

Here the subgroup $\widetilde{W}$ coincides with W and the longest element of the Weyl group has two reduced expressions given by

$$w_0 = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2, \quad w'_0 = \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1.$$

By Lemma 3.7 we have

$$\begin{aligned} \mathfrak{X}_1 &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}_1!!} (q^2 c_1)^n E_1^{2n}, \\ \mathfrak{X}_2 &= \sum_{n \geq 0} \frac{(q^3 - q^{-3})^n}{\{2n\}_2!!} (q^6 c_2)^n E_2^{2n}. \end{aligned}$$

The following proposition is proved in [6, 6.6].

Proposition A.25 *The partial quasi K -matrices $\mathfrak{X}_{w_0}$ and $\mathfrak{X}_{w'_0}$ coincide with the quasi K -matrix $\mathfrak{X}$.*

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