The Kleiman–Piene conjecture and node polynomials for plane curves in $\mathbb{P}^3$

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Abstract

For a relative effective divisor $\mathcal{C}$ on a smooth projective family of surfaces $q : \mathcal{S} \rightarrow B$, we consider the locus in B over which the fibres of $\mathcal{C}$ are δ -nodal curves. We prove a conjecture by Kleiman and Piene on the universality of an enumerating cycle on this locus. We propose a bivariant class $\gamma(\mathcal{C}) \in A^*(B)$ motivated by the BPS calculus of Pandharipande and Thomas, and show that it can be expressed universally as a polynomial in classes of the form $q_*(c_1(\mathcal{O}(\mathcal{C}))^a c_1(T_{\mathcal{S}/B})^b c_2(T_{\mathcal{S}/B})^c)$. Under an ampleness assumption, we show that $\gamma(\mathcal{C}) \cap [B]$ is the class of a natural effective cycle with support equal to the closure of the locus of δ -nodal curves. Finally, we apply our method to calculate node polynomials for plane curves intersecting general lines in $\mathbb{P}^3$. We verify our results using nineteenth century geometry of Schubert.

Mathematics Subject Classification Primary 14N10; Secondary 14C20 · 14N35 · 14N15

1 Introduction

1.1 The Kleiman–Piene conjecture

All schemes we consider are separated and of finite type over $\mathbb{C}$. Let B be a base scheme, and let $q : \mathcal{S} \rightarrow B$ be a smooth family of surfaces, i.e. a smooth projective morphism of relative dimension 2. By a *curve*, we mean a proper 1-dimensional scheme, not necessary irreducible or reduced. Let $\mathcal{C}$ be a *relative effective (Cartier) divisor* on the family $\mathcal{S} \rightarrow B$, i.e. an effective Cartier divisor on $\mathcal{S}$, such that the morphism $\mathcal{C} \rightarrow B$ is flat. Fix a non-negative integer δ . We call a curve δ -nodal if it is reduced, has δ nodes and no other singularities. Consider the following counting problem:

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Problem 1 What is, if finite, the number of δ -nodal curves in the family $\mathcal{C} \rightarrow B$?

More generally, consider the locus

$$B(\delta) := \{b \in B \mid \mathcal{C}_b \text{ is a } \delta\text{-nodal curve}\}$$

and write $\overline{B(\delta)}$ for its closure in B .

Problem 2 What is the class $[\overline{B(\delta)}] \in A_*(B)$?

Assume that B is of pure dimension n . In [19], Kleiman and Piene construct a natural effective cycle $U(\delta)$ with support equal to the closure of the locus of δ -nodal curves. For $\delta \leq 8$ they prove that the class $[U(\delta)]$ is given in a rather specific form as a polynomial in the classes

$$\epsilon(a, b, c) := q_*(c_1(\mathcal{O}_{\mathcal{S}}(\mathcal{C}))^a c_1(T_{\mathcal{S}/B})^b c_2(T_{\mathcal{S}/B})^c).$$

Kleiman and Piene work with certain assumptions on the dimensions of equisingular strata in the family. For a curve C , let $\mathbf{D}$ be its equisingularity type. It can be represented by an Enriques diagram, which encodes the numerical invariants of the singularities of C [18]. Conversely, for an equisingularity type (or Enriques diagram) $\mathbf{D}$, we write $B(\mathbf{D}) \subset B$ for the locus of curves in the family $\mathcal{C} \rightarrow B$, with equisingularity type $\mathbf{D}$.

One of the invariants of an equisingularity type is the codimension $\text{cod}(\mathbf{D})$. It is the ‘expected codimension’ in which curves with equisingularity type $\mathbf{D}$ appear in a family. More precisely, in [18], it is characterized as the codimension of the locus of curves with equisingularity type $\mathbf{D}$ in the universal family $\mathcal{C} \rightarrow |L|$ of any sufficiently ample complete linear system. The hypotheses on the family $\mathcal{C} \rightarrow B$ under which the class $U(\delta)$ is constructed and which we will denote by **DIM_{KP}** are the following:

- The locus of non-reduced curves $B(\infty)$ has codimension $> \delta$;
- For each equisingularity type $\mathbf{D}$, the locus $B(\mathbf{D})$ has at least the expected codimension $\text{cod}(\mathbf{D})$, or codimension $> \delta$.

Here we use the convention $\text{codim}(\emptyset) = \infty$. In [19] and [17], the authors prove the following theorem:

Theorem 1.1 (Kleiman–Piene) *Under the above hypotheses **DIM_{KP}**, the locus $B(\delta)$ of δ -nodal curves is either empty, or has pure codimension δ . There is a natural non-negative cycle $U(\delta)$ with support $\overline{B(\delta)}$. For $\delta \leq 8$, the rational equivalence class $[U(\delta)]$ is given by a universal polynomial¹ in the classes $\epsilon(a, b, c)$.*

Moreover, in [19] the following conjecture is made.

Conjecture 1.2 (Kleiman–Piene) *Theorem 1.1 holds for all $\delta \geq 0$.*

¹ In fact, their statement is more precise: The polynomials are of the form $P_\delta(a_1, \dots, a_\delta)/\delta!$, in which P_δ is the δ -th Bell polynomial, and a_i is a linear combination of classes $\epsilon(a, b, c)$ with $a + b + 2c = i + 2$, so that $a_i \in A^i(B)$. Moreover, an algorithm is given that produces these classes.

In this paper we propose a class $\gamma(C) \in A^\delta(B)$, enumerating the δ -nodal curves, inspired by the BPS calculus of Pandharipande and Thomas [26]. We will show that if B is complete, the class $\gamma(C) \cap [B]$ is the rational equivalence class of a natural cycle with support $\overline{B(\delta)}$. For this we work with hypotheses **DIM**, similar to but slightly weaker than **DIM_{KP}**, and an additional ampleness assumption **AMP**. By means of a family version of an algorithm by Ellingsrud, Göttsche and Lehn [8], we show that without assumptions, the class $\gamma(C)$ is a universal polynomial in the classes $\epsilon(a, b, c)$. This will be the content of Theorem A below.

1.2 BPS numbers

Let C be a locally planar, reduced curve of arithmetic genus g . In [26] the authors consider the following transformation of the generating series of topological Euler characteristics of Hilbert schemes $C^{[i]}$ of i points on C , which defines the *BPS numbers* $n_{r,C}$ of C .

$$\sum_{i=0}^{\infty} e(C^{[i]}) q^i = \sum_{r=-\infty}^g n_{r,C} q^{g-r} (1-q)^{2r-2}.$$

They prove the following:

Theorem 1.3 (Pandharipande–Thomas) *The numbers $n_{r,C}$ are zero, unless $g - \delta \leq r \leq g$, where δ is the δ -invariant of C , i.e., $g - \delta$ is the geometric genus of C .*

Shende proves in [30] that the number $n_{g-i,C}$ equals the degree of the subvariety of i -nodal curves in the versal deformation space of C . In particular it is positive for $0 \leq i \leq \delta$.

Let B be a scheme and let $p: \mathcal{C} \rightarrow B$ be a family of (not necessarily reduced) curves, i.e., a projective flat morphism of relative dimension 1. Assume that the fibres are locally planar curves of genus g . Let

$$p^{[i]}: \mathcal{C}_B^{[i]} = \text{Hilb}^i(\mathcal{C}/B) \rightarrow B$$

be the relative Hilbert scheme of i points on the fibres of $\mathcal{C} \rightarrow B$. We define constructible functions $n_r = n_r(\mathcal{C})$ on B by

$$\sum_{i=0}^{\infty} p_*^{[i]}(1_{\mathcal{C}_B^{[i]}}) q^i = \sum_{r=-\infty}^g n_r q^{g-r} (1-q)^{2r-2}. \quad (1)$$

In other words, n_r is the function that assigns the number $n_{r,C_b^{\text{red}}}$ to a point $b \in B$. By Theorem 1.3, the function $n_{g-\delta}$ is supported on the locus of curves that have δ -invariant $\geq \delta$ or are non-reduced. In the same paper it is shown that $n_{g-\delta,C} = 1$ for a δ -nodal curve C .

Let $q: \mathcal{S} \rightarrow B$ be a smooth family of surfaces and let $\mathcal{C} \subset \mathcal{S}$ be a relative effective divisor. We will use the embedding $\mathcal{C} \hookrightarrow \mathcal{S}$ to analogously define classes $n_r^{cl} \in A^*B$.

In fact, $\mathcal{C}_B^{[i]}$ is a subscheme of $\mathcal{S}_B^{[i]} = \text{Hilb}^i(\mathcal{S}/B)$, cut out regularly (in particular, in the expected codimension i) by the tautological bundle $\mathcal{O}(\mathcal{C})_B^{[i]}$ (see Lemma 2.7). The scheme $\mathcal{S}_B^{[i]}$ is smooth over B [1] and the *virtual tangent bundle* $T_{\mathcal{C}_B^{[i]}/B}$, as defined in [10, B.7.6], is given by the class

$$\left[T_{\mathcal{S}_B^{[i]}/B} \Big|_{\mathcal{C}_B^{[i]}} - \mathcal{O}(\mathcal{C})_B^{[i]} \Big|_{\mathcal{C}_B^{[i]}} \right]$$

in the Grothendieck group $K(\mathcal{C}_B^{[i]})$ of vector bundles on $\mathcal{C}_B^{[i]}$. Let

$$c: K \Rightarrow (A^*)^\times$$

be the total Chern class. Then the classes $n_r^{cl} = n_r^{cl}(\mathcal{C}) \in A^*(B)$ are defined by the equation

$$\sum_{i=0}^{\infty} p_*^{[i]} c(T_{\mathcal{C}_B^{[i]}/B}) q^i = \sum_{r=-\infty}^g n_r^{cl} q^{g-r} (1-q)^{2r-2}. \quad (2)$$

Here the homomorphism

$$p_*^{[i]}: A^*(\mathcal{C}_B^{[i]}) \rightarrow A^*(B)$$

denotes the Gysin push-forward as defined in [10, Chapter 17]. We define

$$\gamma(\mathcal{C}) = \left\{ n_{g-\delta}^{cl}(\mathcal{C}) \right\}_\delta \in A^\delta(B)$$

to be equal to the degree- δ part of $n_{g-\delta}^{cl}(\mathcal{C})$. We will show that it reflects some of the properties of $n_{g-\delta}^{cl}(\mathcal{C})$. In fact, in Proposition 3.1, we will compare n_r and n_r^{cl} by means of the Chern–Schwartz–MacPherson class.

Remark 1.4 Göttsche and Shende [12] also mention the CSM-class of the constructible function n_r as an invariant counting nodal curves. Moreover they consider an analogous class, using the virtual tangent bundle of Hilbert schemes of points of the curve. However, the use of the *relative* tangent bundle, which is natural from the point of view of [19], is essential for our results.

1.3 Results

Recall that a line bundle L on a smooth projective surface S is called δ -very ample if for any finite subscheme $Z \subset S$ of length $\delta + 1$, the map $H^0(S, L) \rightarrow H^0(Z, L|_Z)$ is surjective [5]. For a line bundle $\mathcal{L}$ on a smooth family of surfaces $S \rightarrow B$, consider the following ampleness hypotheses, which we denote by **AMP**:

- For every $b \in B$, the line bundle $\mathcal{L}_b = \mathcal{L}|_{\mathcal{S}_b}$ on $\mathcal{S}_b$ is δ -very ample.
- The dimension of the vector spaces $H^0(\mathcal{S}_b, \mathcal{L}_b)$ is locally constant on B .

Now let $\mathcal{C}$ and $q: S \rightarrow B$ be given as above, and let $\mathcal{L} = \mathcal{O}(\mathcal{C})$. Without making any assumptions on the dimensions of the equisingular strata, **AMP** guarantees that the class $\gamma(\mathcal{C}) \cap [B]$ is supported on the locus of curves with δ -invariant $\geq \delta$. If B is equidimensional, it will follow (Proposition 4.4) that $\gamma(\mathcal{C}) \cap [B]$ is the class of a natural effective cycle with support $\overline{B}(\delta)$ if we assume the following hypotheses, which we denote by **DIM**:

- The locus of δ -nodal curves, if non-empty, has codimension δ .
- The loci of curves of the following type have codimension $> \delta$:
 - Curves with δ -invariant $> \delta$;
 - Curves with δ -invariant $= \delta$, but with singularities other than nodes;
 - Non-reduced curves.

As explained in [18], for an equisingularity type $\mathbf{D}$ we have $\text{cod}(\mathbf{D}) \geq \delta(\mathbf{D})$, with equality only for δ -nodal curves. It follows that **DIM** is slightly weaker than **DIM_{KP}**. To summarize, we will prove the following theorem:

Theorem A *Let B be a scheme and fix an integer δ . Let $\mathcal{C}$ be a relative effective divisor on a flat family of smooth surfaces $S \rightarrow B$. Then the class $\gamma(\mathcal{C})$ can be expressed universally as a polynomial of degree δ in classes of the form $\epsilon(a, b, c) = q_*(c_1(\mathcal{O}(\mathcal{C}))^a c_1(T_{S/B})^b c_2(T_{S/B})^c)$. Now assume that B is complete of pure dimension n , that the line bundle $\mathcal{O}_S(\mathcal{C})$ satisfies **AMP**, and moreover assume that $\mathcal{C} \rightarrow B$ satisfies **DIM**. Then the class $\gamma(\mathcal{C}) \cap [B] \in A_{n-\delta}(B)$ is the class of a natural cycle on B with support $\overline{B}(\delta)$.*

Remark 1.5 The natural cycle in the theorem is constructed in Sect. 4. See Proposition 4.4.

The conjecture of Kleiman and Piene is a family version of the Göttsche conjecture [11]. For a sufficiently ample line bundle L on a smooth surface S , the latter asserts that the degree of the Severi locus of δ -nodal curves in the complete linear system $|L|$, is given by a universal polynomial in the numbers L^2 , $(L \cdot K)$, K^2 and $c_2(S)$, for K the canonical divisor on S . Equivalently, the number of δ -nodal curves in a general linear system $\mathbb{P}^\delta \subset |L|$ is given by such a polynomial.

The Göttsche conjecture was first proved using algebraic methods by Tzeng in [31]. Other proofs were given in [16, 24] and [21]. In [23] and [27] the result is generalized to other singularity types.

Our theorem implies the Göttsche conjecture for δ -very ample L , but it is not independent from existing results. In fact, our method can be seen as a family version of the proof in [21]. Moreover, sharper results are known in terms of the required ampleness [20].

1.4 Application to plane curves in $\mathbb{P}^3$

The motivation for the project was the following counting problem. For fixed integers $\delta \geq 0$ and $d > 1$ write

$$n = \frac{d(d+3)}{2} + 3 - \delta.$$

and consider lines $\ell_1, \dots, \ell_n \subset \mathbb{P}^3$. The space of curves of degree d that lie on a plane in $\mathbb{P}^3$ and that intersect the lines $\ell_1, \dots, \ell_n$ has expected dimension δ . We will show that, if we choose the lines $\ell_1, \dots, \ell_n$ sufficiently general, the subspace of δ -nodal curves is finite (and reduced, as a scheme). For $d \geq \delta$, we can use our method to calculate the number $N_{\delta,d}$ of δ -nodal plane curves of degree d intersecting the lines $\ell_1, \dots, \ell_n$.

Let $\mathcal{C} \rightarrow B$ be the universal plane curve of degree d in $\mathbb{P}^3$. We will show in Proposition 6.1 that for $d \geq \delta$, we have $N_{\delta,d} = \gamma(\mathcal{C}) \cap [B_{\ell_1, \dots, \ell_n}]$, in which $B_{\ell_1, \dots, \ell_n} \subset B$ is the closed subvariety of curves intersecting the general lines $\ell_1, \dots, \ell_n$. We use this to prove our second main result:

Theorem B *Let $\delta \geq 0$. The number of planar δ -nodal curves of degree $d \geq \delta$ in $\mathbb{P}^3$ intersecting $n = \frac{d(d+3)}{2} + 3 - \delta$ general lines is given by a polynomial $N_\delta(d)$ in d of degree $\leq 9 + 2\delta$. Moreover, for $\delta \leq 12$ these polynomials are the ones given in Appendix A.*

Remark 1.6 Our computations suggest that $N_\delta(d)$ has degree *exactly* $9 + 2\delta$, with leading coefficient $\frac{3^\delta}{162\delta!}$, but we do not prove this.

2 Preliminaries

2.1 Chern–Schwartz–MacPherson classes

To any constructible function f on a complete scheme X , one can assign a class $c_{SM}(f)$ in the Chow group of X , called the *Chern–Schwartz–MacPherson class* of f . The existence of well-behaved Chern classes for constructible functions was conjectured by Deligne and Grothendieck and proved by MacPherson in [25]. Several other constructions are known. See [2] for an overview and a new construction in a more general set-up.

For a subset V of a scheme X , write $1_V : X \rightarrow \mathbb{Z}$ for the function with constant value 1 on its support V . Recall that a constructible function is a map $f : X \rightarrow \mathbb{Z}$ that can be written as a finite sum

$$f = \sum_{i \in I} \alpha_i 1_{V_i}$$

with $\alpha_i \in \mathbb{Z}$ and $V_i \subset X$ closed. Let $F(X)$ be the group of constructible functions on X . For a proper morphism $g : X \rightarrow Y$, there is a homomorphism $g_* : F(X) \rightarrow F(Y)$ given by

$$g_*(1_V)(y) = e(V \cap g^{-1}(y)), \quad y \in Y,$$

in which $V \subset X$ is closed and e is the topological Euler characteristic. For a scheme X let $A_*(X)$ denote the Chow group of X . In the following theorem, we will view A_* as a (covariant) functor on the category of complete schemes with proper morphisms to the category of abelian groups. Let c denote the total Chern class.

Theorem 2.1 (MacPherson) *There is a unique natural transformation*

$$c_{SM}: F \Rightarrow A_*$$

satisfying $c_{SM}(1_X) = c(T_X)$ for X smooth projective.

Remark 2.2 The uniqueness of such a natural transformation follows from resolution of singularities. MacPherson proved the naturality of the homology class, but in fact his argument gives this stronger result (see [10, 19.1.7]).

Definition 2.3 For a complete scheme X and a constructible function $f \in F(X)$, we call $c_{SM}(f) = c_{SM}(X)(f)$ the *Chern–Schwartz–MacPherson class* of f . We also write

$$c_{SM}(X) = c_{SM}(1_X)$$

and call this class the Chern–Schwartz–MacPherson class of X .

It follows directly from the definitions that for a complete scheme X , we have

$$\int c_{SM}(X) = \pi_*(1_X) = e(X)$$

with $\pi: X \rightarrow \{*\}$ the morphism to a point. At the other extreme, we have the following lemma.

Lemma 2.4 *Let X be a scheme and let $V \subset X$ be a locally closed subset of dimension n and let $\overline{V}$ be its closure (with the reduced scheme structure). Then*

$$c_{SM}(1_V) = [\overline{V}] + \text{cycles of dimension} < n.$$

Proof Let $g: \tilde{V} \rightarrow \overline{V}$ be a birational morphism from a nonsingular projective variety $\tilde{V}$. Let $U \subset \overline{V}$ be a dense open over which g is an isomorphism and let $Z = \overline{V} \setminus U$ be its complement. Write

$$\partial V = \overline{V} \setminus V$$

for the boundary of V . Then we have

$$\begin{aligned} g_* 1_{\tilde{V}} &= 1_{\overline{V}} + f \\ &= 1_V + 1_{\partial V} + f \end{aligned}$$

with f a constructible function on Z . On the other hand, we have

$$c_{SM}(1_{\tilde{V}}) = c(T_{\tilde{V}}) = [\tilde{V}] + \text{cycles of dimension} < n.$$

Since the functions f and $1_{\partial V}$ are supported on closed subsets of dimension $< n$, by naturality the same holds for their Chern–Schwartz–Macpherson classes. It follows that we have

$$\begin{aligned} c_{SM}(1_V) &= g_* c_{SM}(\tilde{V}) - c_{SM}(f) - c_{SM}(1_{\partial V}) \\ &= [\tilde{V}] + \text{cycles of dimension } < n. \end{aligned}$$

□

2.2 Hilbert schemes of points

Let S be a smooth projective surface, and let $S^{[n]}$ be the Hilbert scheme of n points on S . Let $\mathcal{Z}$ be the universal subscheme of length n , with natural morphisms

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{i} & S \\ \downarrow \pi & & \\ S^{[n]} & & \end{array}$$

For a vector bundle F on S , we write $F^{[n]} := \pi_* i^* F$ for the tautological vector bundle on $S^{[n]}$ with fibre

$$F^{[n]} \Big|_{[Z]} = H^0(Z, F|_Z)$$

over a point $[Z] \in S^{[n]}$.

In [8], Ellingsrud, Göttsche and Lehn describe a method to calculate certain tautological integrals on $S^{[n]}$. In fact, they give a constructive proof of the following theorem:

Theorem 2.5 (Ellingsrud, Göttsche, Lehn) *Let $F_1, \dots, F_l$ be vector bundles on S of respective ranks $r_1, \dots, r_l$. Let P be a polynomial in the Chern classes of $T_{S^{[n]}}$ and the Chern classes of the bundles $F_i^{[n]}$. Then there is a universal polynomial Q , depending only on P , in numbers*

$$\int_S p(T_S, F_1, \dots, F_l),$$

in which p is a polynomial in the Chern classes of T_S , the ranks r_i and the Chern classes of the bundles F_i , such that

$$\int_{S^{[n]}} P = Q.$$

Now let B be a base scheme and let $q: S \rightarrow B$ be proper and smooth of relative dimension 2. Let $S_B^{[i]} = \text{Hilb}^i(S/B)$ denote the relative Hilbert scheme of i points on

the fibres of $\mathcal{S} \rightarrow B$, with structure morphism $q^{[i]}: \mathcal{S}_B^{[i]} \rightarrow B$. For a vector bundle $\mathcal{F}$ on $\mathcal{S}$ we can define as above the tautological bundle $\mathcal{F}_B^{[i]} = \pi_* i^* \mathcal{F}$ on $\mathcal{S}_B^{[i]}$, in which π and i are the natural morphisms in

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{i} & \mathcal{S} \\ \downarrow \pi & & \downarrow q \\ \mathcal{S}_B^{[i]} & \xrightarrow{q^{[i]}} & B \end{array}$$

from the universal length n subscheme $\mathcal{Z}$ on the fibres of $\mathcal{S} \rightarrow B$. It restricts to

$$\mathcal{F}_B^{[i]}|_{\mathcal{S}_b^{[i]}} = (\mathcal{F}|_{\mathcal{S}_b})^{[i]}$$

on the fibre $\mathcal{S}_b^{[i]}$ over a point $b \in B$.

The following is proved in [1] and generalizes a well known result by Fogarty:

Lemma 2.6 *The morphism $q^{[i]}: \mathcal{S}_B^{[i]} \rightarrow B$ is smooth of relative dimension $2i$.*

Now let $\mathcal{S} \rightarrow B$ be given as above, and let $\mathcal{C} \subset \mathcal{S}$ be a relative effective divisor.

Lemma 2.7 *The Hilbert scheme $\mathcal{C}_B^{[i]}$ is cut out regularly (in particular, in the expected codimension i) by the canonical section of the vector bundle $\mathcal{O}(\mathcal{C})_B^{[i]}$ on $\mathcal{S}_B^{[i]}$. Moreover, the morphism $\mathcal{C}_B^{[i]} \rightarrow B$ is flat.*

Proof We follow [1]. Let $b \in B$ be an arbitrary point and consider the fibres $\mathcal{C} = \mathcal{C}_b$ and $\mathcal{S} = \mathcal{S}_b$ over b . By [13, §11.3.8], it suffices to check that the Hilbert scheme $\mathcal{C}^{[i]}$ is cut out regularly by the canonical section of the bundle $\mathcal{O}(\mathcal{C})^{[i]}$ on $\mathcal{S}^{[i]}$. Since $\mathcal{S}^{[i]}$ is smooth, we only need to check that for a divisor $\mathcal{C} \subset \mathcal{S}$ on a smooth surface $\mathcal{S}$, the Hilbert scheme $\mathcal{C}^{[i]}$ has the expected dimension i . But this can be seen by inspection of the fibres of the Hilbert–Chow morphism $\mathcal{C}^{[i]} \rightarrow \mathcal{C}^{(i)}$. In fact, by [14], the locus in $\mathcal{S}^{[i]}$ of subschemes of length i supported at a point has dimension $i - 1$. From this it follows directly that the locus in $\mathcal{C}^{(i)}$ over which the fibres of the morphism $\mathcal{C}^{[i]} \rightarrow \mathcal{C}^{(i)}$ have dimension r , has codimension $\geq r$. $\square$

3 The smooth case

Let B be a scheme. Let $\mathcal{S} \rightarrow B$ be smooth projective of relative dimension 2 and let $\mathcal{C} \subset \mathcal{S}$ be a relative effective divisor. For a fixed δ , let

$$n_{g-\delta}^{cl} = n_{g-\delta}^{cl}(\mathcal{C}) \in A^*(B)$$

be the class defined by Eq. (2) in the introduction. The following situation is the model for our results.

Proposition 3.1 *Assume B is projective and that the relative Hilbert schemes of points $\mathcal{C}_B^{[i]}$ for $i = 0, \dots, \delta$ are non-singular. (So in particular B is non-singular). Then we have the following identity in $A_*(B)$:*

$$c(T_B) n_{g-\delta}^{cl} \cap [B] = c_{SM}(n_{g-\delta}). \quad (3)$$

*In particular, the class $n_{g-\delta}^{cl} \cap [B]$ is supported on the locus of curves that have δ -invariant $\geq \delta$ or are non-reduced, i.e. it is the push forward of a class on this locus. If B is of pure dimension n and the family of curves $\mathcal{C} \rightarrow B$ satisfies **DIM**, we find*

$$n_{g-\delta}^{cl} \cap [B] = \left[\overline{B(\delta)} \right] + \beta$$

with β a sum of cycles of dimension $< n - \delta$.

Proof By the defining Eq. (1), the constructible function $n_{g-\delta}$ is a linear combination of the terms $p_*^{[i]}(1_{\mathcal{C}_B^{[i]}})$. It is easy to see that only the terms with $i = 0, \dots, \delta$ are involved. Similarly, $n_{g-\delta}^{cl}$ is a linear combination (with the same coefficients) of the classes $p_*^{[i]}(c(T_{\mathcal{C}_B^{[i]}/B}))$, with $i = 0 \dots \delta$. Therefore it suffices to verify that relation (3) holds for the first $\delta + 1$ terms in the left hand sides of (1) and (2).

Recall that we have defined the class $T_{\mathcal{C}_B^{[i]}/B}$ in the Grothendieck group $K(\mathcal{C}_B^{[i]})$ by

$$T_{\mathcal{C}_B^{[i]}/B} = T_{\mathcal{S}_B^{[i]}/B} \Big|_{\mathcal{C}_B^{[i]}} - \mathcal{O}(\mathcal{C})^{[i]} \Big|_{\mathcal{C}_B^{[i]}}.$$

For $i = 0, \dots, \delta$, we have by Lemma 2.6 and by Lemma 2.7 the following relations in $K(\mathcal{S}_B^{[i]})$ and $K(\mathcal{C}_B^{[i]})$ respectively:

$$\begin{aligned} T_{\mathcal{S}_B^{[i]}} &= T_{\mathcal{S}_B^{[i]}/B} + T_B; \\ T_{\mathcal{S}_B^{[i]}} \Big|_{\mathcal{C}_B^{[i]}} &= T_{\mathcal{C}_B^{[i]}} + \mathcal{O}(\mathcal{C})_B^{[i]} \Big|_{\mathcal{C}_B^{[i]}}. \end{aligned}$$

It follows that

$$T_{\mathcal{C}_B^{[i]}/B} = T_{\mathcal{C}_B^{[i]}} - T_B$$

and hence

$$c(T_{\mathcal{C}_B^{[i]}/B}) = c(T_{\mathcal{C}_B^{[i]}}) c(T_B)^{-1} \in A^*(\mathcal{C}_B^{[i]}).$$

By the defining properties of Chern–Schwartz–MacPherson classes (Theorem 2.1) we obtain

$$c_{SM}(p_*^{[i]}(1_{\mathcal{C}_B^{[i]}})) = p_*^{[i]} c_{SM}(1_{\mathcal{C}_B^{[i]}})$$

$$\begin{aligned}
 &= p_*^{[i]} \left(c(T_{\mathcal{C}_B^{[i]}}) \cap [\mathcal{C}_B^{[i]}] \right) \\
 &= p_*^{[i]} \left(c(T_B) c(T_{\mathcal{C}_B^{[i]}/B}) \cap [\mathcal{C}_B^{[i]}] \right) \\
 &= c(T_B) p_*^{[i]} \left(c(T_{\mathcal{C}_B^{[i]}/B}) \cap [B] \right).
 \end{aligned}$$

By Theorem 1.3 the support of the constructible function $n_{g-\delta}$ lies in the locus in B over which the curves in the family $\mathcal{C} \rightarrow B$ have δ -invariant $\geq \delta$. By the functoriality of the Chern–Schwartz–MacPherson class, the cycle class

$$c(T_B) n_{g-\delta}^{cl} \cap [B]$$

is the push-forward of a cycle class on this locus. Since $c(T_B)$ is invertible, the same holds for $n_{g-\delta}^{cl} \cap [B]$. By [26, Prop. 3.23], the function $n_{g-\delta}$ has constant value 1 on the locus of δ -nodal curves. If $\mathcal{C}$ satisfies **DIM**, it follows that the support of the constructible function $n_{g-\delta} - 1_{B(\delta)}$ lies in a closed subset of codimension $> \delta$. Hence the last assertion follows from Lemma 2.4. $\square$

Example 3.2 In [21] it is shown that both conditions of the proposition are satisfied by the universal curve $\mathcal{C} \rightarrow |L|$ over the linear system of a δ -very ample line bundle L on a smooth surface S , i.e., $\mathcal{C}$ satisfies **DIM**, and the relative Hilbert schemes $\mathcal{C}_{|L|}^{[i]}$ are non-singular for $i \leq \delta$. By Bertini’s theorem it then follows that the same holds for the restriction $\mathcal{C}_{\mathbb{P}^\delta}$ of the universal curve to a general linear system $\mathbb{P}^\delta \subset |L|$. In particular the set $\mathbb{P}^\delta(\delta)$ is finite, and it follows that the degree

$$\int_{\mathbb{P}^\delta} n_{g-\delta}^{cl}(\mathcal{C}_{\mathbb{P}^\delta}) \cap [\mathbb{P}^\delta] = \int_{\mathbb{P}^\delta} [\mathbb{P}^\delta(\delta)]$$

equals the number of δ -nodal curves in the linear system $\mathbb{P}^\delta$.

Remark 3.3 It should be noted that the integrals differ slightly from the ones in [21]. In fact, in loc. cit. the authors consider a linear combination of the integrals

$$\int_{\mathcal{C}_{\mathbb{P}^\delta}^{[i]}} c \left(T_{\mathcal{C}_{\mathbb{P}^\delta}^{[i]}} \right) \tag{4}$$

whereas we use the relative (virtual) tangent bundles

$$T_{\mathcal{C}_{\mathbb{P}^\delta}^{[i]}/\mathbb{P}^\delta} = T_{\mathcal{C}_{|L|}^{[i]}/|L|} \Big|_{\mathcal{C}_{\mathbb{P}^\delta}^{[i]}}$$

and consider the integrals

$$\int_{\mathcal{C}_{\mathbb{P}^\delta}^{[i]}} c \left(T_{\mathcal{C}_{\mathbb{P}^\delta}^{[i]}/\mathbb{P}^\delta} \right) \tag{5}$$

Interestingly, (4) does not equal (5) in general, but after taking the BPS linear combination of the integrals for $i = 0, \dots, \delta$, they both calculate the number of δ nodal curves in the linear system $\mathbb{P}^\delta$.

More generally, let B be a scheme and let $p: \mathcal{S} \rightarrow B$ be smooth projective of relative dimension 2. Let $\mathcal{L}$ be a line bundle on $\mathcal{S}$ and assume $\mathcal{L}$ satisfies **AMP**. Then $p_*\mathcal{L}$ is a vector bundle and the fiber of the projective bundle

$$|\mathcal{L}/B| := \mathbb{P}(p_*\mathcal{L}) = \text{Proj}(\text{Sym}((p_*\mathcal{L})^*)) \rightarrow B$$

over a point $b \in B$ is the δ -very ample complete linear system $|\mathcal{L}_b|$. Consider the family

$$\mathcal{S}_{|\mathcal{L}/B|} = \mathcal{S} \times_B |\mathcal{L}/B| \rightarrow |\mathcal{L}/B|$$

and let $\mathcal{C} \subset \mathcal{S}_{|\mathcal{L}/B|}$ be the universal divisor.

Lemma 3.4 *The family of curves $\mathcal{C} \rightarrow |\mathcal{L}/B|$ satisfies **DIM**. If B is smooth, the Hilbert schemes $\mathcal{C}_{|\mathcal{L}/B|}^{[i]}$ are non-singular for $i \leq \delta$.*

Proof The first statement can be checked fibrewise over B . By Lemma 2.7, the Hilbert schemes of points $\mathcal{C}_{|\mathcal{L}/B|}^{[i]}$ are flat over $|\mathcal{L}/B|$, and hence over B . Hence, if B is smooth, also the second statement can be checked fibrewise over B . Hence, the lemma follows from Example 3.2. $\square$

4 Functoriality and support

Let B be a scheme. Let $\mathcal{C}$ be a relative effective divisor on a smooth family of surfaces $q: \mathcal{S} \rightarrow B$. Let $f: B' \rightarrow B$ be a morphism and consider the relative effective divisor $\mathcal{C}_{B'} = \mathcal{C} \times_B B'$ on the smooth family of surfaces

$$\mathcal{S}_{B'} = \mathcal{S} \times_B B' \rightarrow B'.$$

Then we have the following:

Lemma 4.1 *i) $f^*(n_{g-\delta}^{cl}(\mathcal{C})) = n_{g-\delta}^{cl}(\mathcal{C}_{B'})$ in $A^*(B')$;
ii) $f_*(n_{g-\delta}^{cl}(\mathcal{C}_{B'}) \cap [B']) = n_{g-\delta}^{cl}(\mathcal{C}) \cap f_*[B']$ in $A_*(B)$.*

Proof It suffices to show the relations for the LHS of equation (2). Note that we have Cartesian squares

$$\begin{array}{ccc} \mathcal{C}_{B'}^{[i]} & \xrightarrow{f_i} & \mathcal{C}_B^{[i]} \\ \downarrow p_{B'}^{[i]} & & \downarrow p_B^{[i]} \\ B' & \xrightarrow{f} & B. \end{array}$$

By flatness of the vertical maps, we have

$$\begin{aligned} f^* p_{B*}^{[i]} c(T_{\mathcal{C}_B^{[i]}/B}) &= p_{B'*}^{[i]} f_i^* c(T_{\mathcal{C}_B^{[i]}/B}) \\ &= p_{B'*}^{[i]} c(f_i^* T_{\mathcal{C}_B^{[i]}/B}) \\ &= p_{B'*}^{[i]} c(T_{\mathcal{C}_{B'}^{[i]}/B'}). \end{aligned}$$

The second part follows from the projection formula. $\square$

Let $\mathcal{C}$ be given as above. Write $\mathcal{L} = \mathcal{O}(\mathcal{C})$ and assume $\mathcal{L}$ satisfies **AMP**. As in the previous section, we can form the projective bundle $|\mathcal{L}/B| := \mathbb{P}(q_* \mathcal{L})$ over B , with fibre over b the δ -very ample complete linear system $|\mathcal{L}_b|$. Let $\mathcal{C}'' \subset \mathcal{S}_{|\mathcal{L}/B|}$ denote the universal relative effective divisor on the family $\mathcal{S}_{|\mathcal{L}/B|} \rightarrow |\mathcal{L}/B|$. The bundle $\pi : |\mathcal{L}/B| \rightarrow B$ has a canonical section

$$\begin{array}{c} |\mathcal{L}/B| \\ \pi \downarrow \nearrow s \\ B \end{array} \quad (6)$$

induced by the inclusion $\mathcal{O}(-\mathcal{C}) \hookrightarrow \mathcal{O}$, and the divisor $\mathcal{C}$ is obtained by restricting the divisor $\mathcal{C}''$.

Lemma 4.2 *Assume that B is equidimensional, and let $W \subset \overline{|\mathcal{L}/B|(\delta)}$ be an irreducible component. Then $\pi(W)$ is an irreducible component of B .*

Proof Clearly $\pi(W)$ is closed and irreducible, so it suffices to show that we have $\dim(\pi(W)) = \dim(B)$. Write W° for the open $W \cap |\mathcal{L}/B|(\delta)$ of W . By Lemma 3.4 and the first statement of Theorem 1.1², $|\mathcal{L}/B|(\delta)$ has pure codimension δ . Hence $W^\circ \subset |\mathcal{L}/B|$ has codimension δ . By Example 3.2, a fibre W_b° over $b \in B$ is empty, or has codimension δ in $|\mathcal{L}/B|_b = |\mathcal{L}_b|$. It follows that

$$\begin{aligned} \dim(\pi(W)) &= \dim(\pi(W^\circ)) \\ &= \dim(|\mathcal{L}/B|) - \delta - (\dim(|\mathcal{L}_b|) - \delta) \\ &= \dim(B). \end{aligned}$$

$\square$

Definition 4.3 Define a cycle

$$U(\delta) := \sum_W l(\mathcal{O}_{\pi(W), B}) [W] \in A_*(|\mathcal{L}/B|)$$

in which the sum is taken over the irreducible components of $\overline{|\mathcal{L}/B|(\delta)}$. Here $l(\mathcal{O}_{\pi(W), B})$ denotes the length of the local ring along the subvariety $\pi(W)$ of B .

² Our assumptions are slightly weaker, but the argument given in [19] still holds if we replace **DIM**_{KP} by **DIM**.

As in the introduction, we will use the notation

$$\gamma(\mathcal{C}) = \{n_{g-\delta}^{cl}(\mathcal{C})\}_{\delta} \in A^{\delta}(B).$$

Proposition 4.4 *Let B be a complete base scheme, and $\mathcal{C}$ be a relative effective divisor on a smooth family of surfaces $\mathcal{S} \rightarrow B$. Assume the line bundle $\mathcal{L} = \mathcal{O}(\mathcal{C})$ satisfies **AMP**. Then we have*

$$n_{g-\delta}^{cl}(\mathcal{C}) \in A^{\geq \delta}(B).$$

Now assume that B is equidimensional, and let s be given as in (6). Then we have

$$\gamma(\mathcal{C}) \cap [B] = s^!(U(\delta)).$$

In particular, $\gamma(\mathcal{C}) \cap [B]$ is supported on the locus of curves with δ -invariant $\geq \delta$. Finally, if moreover $\mathcal{C}$ satisfies **DIM**, then $\gamma(\mathcal{C}) \cap [B]$ is the class of a natural cycle with support $\overline{B(\delta)}$.

Proof Assume that B is equidimensional. Without loss of generality, we may assume that B is connected so that $H^0(\mathcal{S}_b, \mathcal{L}_b)$ is constant and $|\mathcal{L}/B|$ equidimensional. By resolution of singularities, there is a proper surjective morphism $f: B' \rightarrow B$ from a smooth projective scheme B' with $f_*[B'] = [B]$. In fact, we consider the union of $l(\mathcal{O}_{Z,B})$ copies of Z for each irreducible component $Z \subset B$. Then we choose for each component a birational morphism from a smooth projective variety.

Let $\mathcal{C}'$, $\mathcal{S}'$ and $\mathcal{L}'$ be the base changes of $\mathcal{C}$, $\mathcal{S}$, and $\mathcal{L}$ along f . Then

$$\mathcal{L}' = f_{\mathcal{S}}^* \mathcal{O}_{\mathcal{S}}(\mathcal{C}) = \mathcal{O}_{\mathcal{S}'}(\mathcal{C}')$$

satisfies **AMP** and we have the following diagram with Cartesian squares:

$$\begin{array}{ccc} \mathcal{C}''' & \longrightarrow & \mathcal{C}'' \\ \downarrow & & \downarrow \\ |\mathcal{L}'/B'| & \xrightarrow{f_{\pi}} & |\mathcal{L}/B| \\ \pi' \downarrow & & \pi \downarrow \wr^s \\ B' & \xrightarrow{f} & B, \end{array}$$

in which $\mathcal{C}'''$ and $\mathcal{C}''$ are the universal divisors on $\mathcal{S}'_{|\mathcal{L}'/B'|}$ and $\mathcal{S}_{|\mathcal{L}/B|}$ in the linear systems $|\mathcal{L}'/B'|$ and $|\mathcal{L}/B|$ respectively.

Note that $(f_{\pi})_*[|\mathcal{L}'/B'|] = [|\mathcal{L}/B|]$. By Lemma 3.4 and Proposition 3.1 we have

$$n_{g-\delta}^{cl}(\mathcal{C}''') \cap |\mathcal{L}'/B'| = \left[\overline{|\mathcal{L}'/B'|(\delta)} \right] + \alpha \in A_*(|\mathcal{L}'/B'|),$$

with α a sum of classes with codimension $> \delta$. We have (set-theoretically)

$$f_{\pi}(\overline{|\mathcal{L}'/B'|(\delta)}) = \overline{|\mathcal{L}/B|(\delta)}.$$

Now let W be an irreducible component of $\overline{|\mathcal{L}/B|(\delta)}$. We will prove that the multiplicity of W in $(f_\pi)_*([\overline{|\mathcal{L}'/B'|(\delta)}])$ is given by $l(\mathcal{O}_{\pi(W), B})$ ($\star$). By Lemma 4.2, $\pi(W)$ (with the reduced scheme structure) is an irreducible component of B . Any irreducible component W' of $|\overline{\mathcal{L}'/B'|(\delta)}$ that maps onto W lies over one of the $l(\mathcal{O}_{\pi(W), B})$ copies of a resolution of singularities

$$\rho: \widetilde{\pi(W)} \rightarrow \pi(W),$$

so W' lies in $W \times_{\pi(W)} \widetilde{\pi(W)}$. Let $U \subset \pi(W)$ be an open over which ρ is an isomorphism. Then

$$W \times_{\pi(W)} \rho^{-1}(U) \rightarrow W \cap \pi^{-1}(U)$$

is an isomorphism. Since W' is irreducible and maps onto W , we have

$$W' = \overline{W \times_{\pi(W)} \rho^{-1}(U)}$$

and the morphism $W' \rightarrow W$ is generically of degree 1. We have now proved $\star$. It follows that

$$(f_\pi)_* \left([\overline{|\mathcal{L}'/B'|(\delta)}] \right) = U(\delta).$$

Hence, by Lemma 4.1 we have

$$\begin{aligned} n_{g-\delta}^{cl}(\mathcal{C}) \cap [B] &= s^*(n_{g-\delta}^{cl}(\mathcal{C}'')) \cap [B] \\ &= s^!(n_{g-\delta}^{cl}(\mathcal{C}'') \cap [|\mathcal{L}/B|]) \\ &= s^!(n_{g-\delta}^{cl}(\mathcal{C}'') \cap (f_\pi)_*[\overline{|\mathcal{L}'/B'|}]) \\ &= s^!(f_\pi)_*(n_{g-\delta}^{cl}(\mathcal{C}''') \cap [|\mathcal{L}'/B'|]) \\ &= s^!(f_\pi)_* \left([\overline{|\mathcal{L}'/B'|(\delta)}] \right) + \beta \\ &= s^!(U(\delta)) + \beta \end{aligned}$$

with $\beta \in A_*(B)$ a sum of classes with codimension $> \delta$. In particular we find

$$\gamma(\mathcal{C}) \cap [B] = s^!(U(\delta)).$$

If $\mathcal{C}$ satisfies **DIM**, it follows that $s(B)$ and $\overline{|\mathcal{L}/B|(\delta)}$ intersect properly in $|\mathcal{L}/B|$, i.e. in dimension $\dim B - \delta$, and we have set theoretically

$$s^{-1} \left(\overline{|\mathcal{L}/B|(\delta)} \right) = \overline{B(\delta)}. \quad (7)$$

To see this, note that since $|\mathcal{L}/B|(\delta)$ has codimension δ in $|\mathcal{L}/B|$, an irreducible component of $s^{-1} \left(\overline{|\mathcal{L}/B|(\delta)} \right)$ has codimension $\leq \delta$. On the other hand, it consists of

curves with δ -invariant $\geq \delta$. So by **DIM**, it has pure codimension δ in B . Moreover, the set

$$s^{-1} \left(\overline{|\mathcal{L}/B|(\delta)} \right) \setminus B(\delta)$$

consists of curves with δ -invariant $\geq \delta$ that are not δ -nodal. By **DIM**, it has codimension $> \delta$. Hence $B(\delta)$ lies dense in $s^{-1} \left(\overline{|\mathcal{L}/B|(\delta)} \right)$, proving equation (7).

It follows that $\gamma(\mathcal{C}) \cap [B]$ is a the class of a natural effective cycle with support equal to $\overline{B(\delta)}$.

Now let B be any complete scheme and let $V \rightarrow B$ be a morphism from an n -dimensional variety V . By the above we have

$$n_{g-\delta}^{cl}(\mathcal{C}) \cap [V] = n_{g-\delta}^{cl}(\mathcal{C}_V) \cap [V] \in A_{\leq n-\delta}(V)$$

and hence we have an equality of bivariant classes

$$n_{g-\delta}^{cl}(\mathcal{C}) = \gamma(\mathcal{C}) + \alpha \in A^*(B)$$

with $\alpha \in A^*(B)$ a sum of classes of degree $> \delta$. □

5 Universality: relative EGL

Let B be a scheme, and let $\mathcal{C}$ be a relative effective divisor on a smooth family of surfaces $q: \mathcal{S} \rightarrow B$. The arithmetic genus of a curve in the family $p: \mathcal{C} \rightarrow B$ is denoted by g , which we view as a locally constant function on B . We consider the transformation of power series (2) and rewrite it as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} p_*^{[n]}(c(T_{\mathcal{C}^{[n]}/B}) q^n) &= \sum_{r=-\infty}^g n_r^{cl} q^{g-r} (1-q)^{2r-2} \\ &= \sum_{i=0}^{\infty} n_{g-i}^{cl} q^i (1-q)^{2(g-i)-2} \\ &= \sum_{i=0}^{\infty} n_{g-i}^{cl} q^i \sum_{j=0}^{\infty} (-q)^j \binom{2(g-i)-2}{j} \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^n n_{g-i}^{cl} (-1)^{n-i} \binom{2(g-i)-2}{n-i} q^n. \end{aligned}$$

It follows that we can write

$$n_{g-\delta}^{cl} = \sum_{i=0}^{\delta} a_i p_*^{[i]}(c(T_{\mathcal{C}_B^{[i]}/B})), \quad (8)$$

in which the a_i are polynomials of degree $\delta - i$ in g , depending only on δ and i . In fact, $a_i = a_{i\delta}$ can be found by inverting the upper triangular matrix with 1's on the diagonal

$$\left[(-1)^{j-i} \binom{2(g-i)-2}{j-i} \right]_{0 \leq i, j \leq \delta}.$$

Proposition 5.1 *The class $\gamma(\mathcal{C})$, can be expressed universally as a polynomial of degree δ in the classes*

$$\epsilon(a, b, c) := q_*(c_1(\mathcal{O}(\mathcal{C}))^a c_1(T_{\mathcal{S}/B})^b c_2(T_{\mathcal{S}/B})^c),$$

in which q_* denotes the Gysin push-forward.

We prove the existence of a universal polynomial in Lemma 5.2. In Lemma 5.4 we compute its degree.

Lemma 5.2 *There exists a polynomial as in the proposition of degree $\leq \delta$ in the classes $\epsilon(a, b, c)$.*

Proof We can view g as an element of $A^0(B)$. In fact we have

$$\begin{aligned} 2g - 2 &= p_*(c_1(T_{\mathcal{C}/B}^\vee)) \\ &= p_*(c_1(\mathcal{O}(\mathcal{C})|_{\mathcal{C}}) - c_1(T_{\mathcal{S}/B}|_{\mathcal{C}})) \\ &= q_*((c_1(\mathcal{O}(\mathcal{C})))^2 - c_1(\mathcal{O}(\mathcal{C}))c_1(T_{\mathcal{S}/B})). \end{aligned}$$

By the Eq. (8), it suffices therefore to prove that the degree- δ parts of the classes $p_*^{[i]}(c(T_{\mathcal{C}_B^{[i]}/B}))$ can be expressed universally as polynomials of degree i in the classes $\epsilon(a, b, c)$. By Lemma 2.7, we have the equality

$$p_*^{[i]}(c(T_{\mathcal{C}_B^{[i]}/B})) = q_*^{[i]} \left(\frac{c(T_{\mathcal{S}_B^{[i]}/B})}{c(\mathcal{O}(\mathcal{C})^{[i]})} c_i(\mathcal{O}(\mathcal{C})^{[i]}) \right), \quad (9)$$

and hence the lemma follows by the following generalisation of Theorem 2.5. $\square$

Let $q: \mathcal{S} \rightarrow B$ proper and smooth relative dimension 2. Write

$$q^{[n]}: \mathcal{S}_B^{[n]} = \text{Hilb}^n(\mathcal{S}/B) \rightarrow B \quad \text{and} \quad q^n: \mathcal{S}_B^n = \mathcal{S} \times_B \cdots \times_B \mathcal{S} \rightarrow B$$

for the structure morphisms.

Theorem 5.3 *Let $\mathcal{F}_1, \dots, \mathcal{F}_l$ be vector bundles on $\mathcal{S}$ of respective ranks $r_1, \dots, r_l$. Let P be a polynomial in the Chern classes of $T_{\mathcal{S}_B^{[n]}/B}$ and the Chern classes of the bundles $\mathcal{F}_i^{[n]}$. Then there is a universal polynomial Q , depending only on P , of degree $\leq n$ in the classes in $A^*(B)$ of the form $q_* p(T_{\mathcal{S}/B}, \mathcal{F}_1, \dots, \mathcal{F}_l)$, with p a polynomial*

in the Chern classes of the bundles in the brackets and the ranks $r_1, \dots, r_l$, such that we have

$$q_*^{[n]} P = Q.$$

Proof The argument given in [8] directly generalises to the relative case. For the case $S = S \times B$, see also [22], Section 4. In fact, Proposition 3.1 in [8] still holds if we replace $S^{[n+1]} \times S^m$ by $S_B^{[n+1]} \times_B S_B^m$, the bundle $T_{S^{[i]}}$ by the relative tangent bundle $T_{S_B^{[i]}/B}$ and the integrals by push-forward to B . It follows that we can find a universal polynomial $\tilde{P}$ in the Chern classes of the sheaves $T_{S/B}$, $\mathcal{O}_\Delta$ and the $\mathcal{F}_i$, pulled back along the several projections

$$pr_i: S_B^n \rightarrow S \quad \text{and} \quad pr_{ij}: S_B^n \rightarrow S \times_B S,$$

such that we have an equation

$$q_*^{[n]} P = q_*^n \tilde{P}.$$

By Grothendieck–Riemann–Roch we have

$$ch(\mathcal{O}_\Delta) = \Delta_*(td(-T_{S/B}))$$

for the projections $pr_i: S \times_B S \rightarrow S$. It follows that $\tilde{P}$ is a polynomial in the Chern classes the bundles $pr_i^* T_{S/B}$ and $pr_i^* \mathcal{F}_i$ and the classes $p_{ij}^*[\Delta]$. By the excess intersection formula, a product of classes of the latter form is a polynomial in Chern classes of the bundles $pr_i^* T_{S/B}$, intersected with a product

$$\Delta^{k_1} \times \dots \times \Delta^{k_m}: S_B^m \hookrightarrow S_B^n$$

of diagonals

$$\Delta^{k_i}: S \hookrightarrow S_B^{k_i}$$

for integers

$$k_1, \dots, k_m \geq 1, \quad k_1 + \dots + k_m = n.$$

It follows that $q_*^n \tilde{P}$ is a sum of classes $q_*^m \tilde{P}_m$ for $m = 1, \dots, n$ and polynomials $\tilde{P}_m$ in the Chern classes of the bundles $pr_i^* T_{S/B}$ and $pr_i^* \mathcal{F}_i$, pulled back along the several projections $S_B^m \rightarrow S$. Now use the fact that for classes $\alpha_1, \dots, \alpha_m \in A^*(S)$ we have

$$q_* \alpha_1 \dots q_* \alpha_m = (q^m)_*(pr_1^* \alpha_1 \dots pr_m^* \alpha_m)$$

in $A^*(B)$. □

Lemma 5.4 *The degree of the polynomial of Lemma 5.2 is δ .*

Proof By Lemma 5.2, the class $\gamma(\mathcal{C})$ can be expressed universally as polynomial γ in classes $\epsilon(a, b, c)$ of degree $\leq \delta$. Now let $\mathcal{C}$ be the universal curve in a complete linear system $|L|$ on a surface S , and let $\mathbb{P}^\delta \subset |L|$ be a general linear system. Let $\omega \in A^1(\mathbb{P}^\delta)$ be the class of a hyperplane. As explained in the proof of [21, Thm. 4.1], the algorithm of [8] applied to the right hand side of (9) for $i = \delta$ produces a term $c_2(S)^\delta / \delta!$ coming from the term

$$c_{2\delta}(T_{S^{[\delta]}})\omega^\delta = c_{2\delta}(T_{S^{[\delta]}/\mathbb{P}^\delta})\omega^\delta.$$

As noted in Remark 3.3, the integrals in [21] differ by a factor $c(T_{\mathbb{P}^\delta})$. However, this does not affect the term $c_{2\delta}(T_{S^{[\delta]}})\omega^\delta$. It follows that γ is a polynomial of degree δ in classes the classes $\epsilon(a, b, c)$. $\square$

Proof of Proposition 5.1 Combine Lemmas 5.2 and 5.4. $\square$

This completes the proof of our first main result:

Proof of Theorem A Combine Propositions 4.4 and 5.1. $\square$

5.1 Multiplicativity

We will check that the class γ has the expected multiplicative behaviour, cf. [19] and [11]. Let B be a base scheme. For $k = 1, 2$, let $\mathcal{S}_k \rightarrow B$ be proper and smooth of relative dimension 2, and let $\mathcal{C}_k$ be a relative effective divisor on $\mathcal{S}_k$, and write $p_k: \mathcal{C}_k \rightarrow B$ for the morphism to B . Let $\mathcal{C}$ be the union

$$\mathcal{C} := \mathcal{C}_1 \amalg \mathcal{C}_2 \subset \mathcal{S}_1 \amalg \mathcal{S}_2 \rightarrow B.$$

We have the following relations:

$$\begin{aligned} \mathcal{C}_B^{[n]} &= \coprod_{i+j=n} (\mathcal{C}_1)_B^{[i]} \times_B (\mathcal{C}_2)_B^{[j]}; \\ p_*^{[i]} c(T_{\mathcal{C}_B^{[n]}/B}) &= \sum_{i+j=n} p_{1*}^{[i]} c(T_{(\mathcal{C}_1)_B^{[i]}/B}) \cdot p_{2*}^{[j]} c(T_{(\mathcal{C}_2)_B^{[j]}/B}). \end{aligned} \quad (10)$$

For $k = 1, 2$, let g_k be the arithmetic genus of a curve in the family $\mathcal{C}_k \rightarrow B$, so we have

$$g - 1 = g_1 - 1 + g_2 - 1,$$

with g the genus of a curve in the family $\mathcal{C} \rightarrow B$. It follows easily from (10) that we have the identity

$$n_{g-\delta}^{cl}(\mathcal{C}) = \sum_{i+j=\delta} n_{g_1-i}^{cl}(\mathcal{C}_1) n_{g_2-j}^{cl}(\mathcal{C}_2). \quad (11)$$

For any $i \geq 0$, let γ_i be the degree- i part of n_{g-i}^{cl} . We record the following lemma.

Lemma 5.5 *Let B be complete and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be given as above. Assume that for $k = 1, 2$, the line bundle $\mathcal{O}(\mathcal{C}_k)$ on $\mathcal{S}_k$ satisfies **AMP**. Then we have the relation*

$$\gamma_\delta(\mathcal{C}) = \sum_{i+j=\delta} \gamma_i(\mathcal{C}_1) \gamma_j(\mathcal{C}_2)$$

Proof By Proposition 4.4, it follows directly from (11), by taking degree- δ parts on both sides of the equation. $\square$

Remark 5.6 As was shown in [11], this gives, in the case of the Göttsche conjecture, the generating series $\gamma = \sum \gamma_\delta q^\delta$ in terms of the Bell polynomials cf. [19]. In this case, there are four power series, corresponding to L^2 , KL , K^2 and $c_2(S)$, that determine the generating series. In the general case, one needs to determine the coefficients of the classes $\epsilon(a, b, c)$ with $a + b + 2c = \delta + 2$ for each δ , rather than just the ones with $b + 2c \leq 2$. Expectedly, it suffices to evaluate γ on a sufficiently large class of examples to obtain the qualitative result.

6 Application: plane curves in $\mathbb{P}^3$

We will apply the results to the problem of counting δ -nodal plane curves of degree d in $\mathbb{P}^3$. As we will see below, the space of such curves has dimension

$$n := \frac{d(d+3)}{2} + 3 - \delta. \quad (12)$$

Let $N_{\delta,d}$ denote the number of δ -nodal plane curves of degree d that intersect general lines $\ell_1, \dots, \ell_n \subset \mathbb{P}^3$. The main result of this section is that for each δ , and $d \geq \delta$, the numbers $N_{\delta,d}$ are given by polynomial of degree $\leq 2\delta + 9$ in d .

Let $\text{Gr} := \text{Gr}(2, \mathbb{P}^3)$ be the Grassmannian of planes in $\mathbb{P}^3$ and let $\mathcal{U}$ be the tautological vector bundle on Gr . Let $\mathcal{O}_{\text{Gr}}(1)$ be the bundle corresponding to the hyperplane class via the identification $\text{Gr} = \mathbb{P}^3$. These two bundles are related via the tautological short exact sequence

$$0 \rightarrow \mathcal{U} \rightarrow \mathbb{C}^4 \otimes \mathcal{O}_{\text{Gr}} \rightarrow \mathcal{O}_{\text{Gr}}(1) \rightarrow 0. \quad (13)$$

Let $q: \mathcal{S} = \mathbb{P}(\mathcal{U}) \rightarrow \text{Gr}$ be the universal plane. As a family of subvarieties of $\mathbb{P}^3$, it comes with a relatively very ample line bundle $\mathcal{O}_{\mathcal{S}}(1)$. Choose an integer $d > 1$ and consider the line bundle $\mathcal{L} := \mathcal{O}_{\mathcal{S}}(d)$. As in Section 3, we can form the projective bundle

$$B := |\mathcal{L}/\text{Gr}| = \mathbb{P}(q_*\mathcal{L}) \xrightarrow{\pi} \text{Gr},$$

which comes with a canonical bundle $\mathcal{O}_B(1)$. The fibre over a point $[V] \in \text{Gr}$ corresponding to a plane $V \subset \mathbb{P}^3$ is the complete linear system $|\mathcal{O}_V(d)|$. In particular π is of relative dimension $r - 1$, with

$$r = \operatorname{rank}(q_*\mathcal{L}) = \frac{(d+1)(d+2)}{2}.$$

Moreover, it follows that B parametrizes planes in $\mathbb{P}^3$, together with a degree d curve on that plane. As a planar curve in $\mathbb{P}^3$ of degree > 1 lies in a unique plane, the variety in fact parametrizes planar curves in $\mathbb{P}^3$.

Let $p: \mathcal{C} \rightarrow B$ be the universal curve. Then $\mathcal{C}$ is a relative effective divisor on the family $\mathcal{S}_B = \mathcal{S} \times_{\operatorname{Gr}} B$ and we have

$$\mathcal{O}_{\mathcal{S}_B}(\mathcal{C}) = \mathcal{L}(1) = \mathcal{L} \otimes \mathcal{O}_B(1).$$

Now let $\delta \leq d$. Note that for $b \in B$, we have

$$\mathcal{S}_b \cong \mathbb{P}^2 \quad \text{and} \quad \mathcal{L}(1)|_{\mathcal{S}_b} \cong \mathcal{O}_{\mathbb{P}^2}(d),$$

so $\mathcal{L}(1)$ satisfies **AMP**, as $\mathcal{O}(d)$ is δ -very ample [5]. By Lemma 3.4, the locus $B(\delta) \subset B$ has codimension δ and we have $\gamma(\mathcal{C}) = \overline{B(\delta)} \subset B$. For lines $\ell_1, \dots, \ell_n$ in $\mathbb{P}^3$, denote the (reduced) locus of curves intersecting the line ℓ_i by $B_{\ell_i} \subset B$, and let

$$B_{\ell_1, \dots, \ell_n} = B_{\ell_1} \cap \dots \cap B_{\ell_n} \subset B$$

be the (scheme theoretic) intersection. We will use the notation

$$\partial B(\delta) = \overline{B(\delta)} \setminus B(\delta).$$

Proposition 6.1 *For*

$$n = \frac{d(d+3)}{2} + 3 - \delta$$

as above and general lines $\ell_1, \dots, \ell_n$, we have

$$B_{\ell_1, \dots, \ell_n} \cap \partial B(\delta) = \emptyset.$$

Moreover, the intersection $B_{\ell_1, \dots, \ell_n} \cap B(\delta)$ is finite and reduced, and its degree is given by

$$N_{\delta, d} = \int_{B_{\ell_1, \dots, \ell_n}} \gamma(\mathcal{C}|_{B_{\ell_1, \dots, \ell_n}}).$$

Remark 6.2 More precisely, in the proof we will construct a non-empty Zariski open subset

$$U \subset \operatorname{Gr}(1, \mathbb{P}^3)^n$$

of the n -fold product of the Grassmannian of lines in $\mathbb{P}^3$, such that the proposition holds for any n -tuple of lines $(\ell_1, \dots, \ell_n) \in U$.

Remark 6.3 It should be noted that for $\delta > 0$, the scheme $B_{\ell_1, \dots, \ell_n}$ is singular. In fact, for every i , a local computation shows that the singular locus of the variety B_{ℓ_i} is the divisor of curves $C \in B_{\ell_i}$ such that ℓ_i lies in the plane spanned by C .

Proof We will prove the first two statements by an argument in the spirit of Lemma 4.7 in [18]. We have $n = r + 2 - \delta = \dim B - \delta$. It follows that, cf. loc. cit., the expected dimension of $B_{\ell_1, \dots, \ell_n} \cap B(\delta)$ is

$$\dim B - n - \delta = 0. \quad (14)$$

Let $\mathrm{Gr}(1, \mathbb{P}^3)$ be the Grassmannian of lines in $\mathbb{P}^3$, and let $\mathbb{L} \rightarrow \mathrm{Gr}(1, \mathbb{P}^3)$ be the universal line. Let P be the limit of the following diagram:

$$\begin{array}{ccc} & & \overline{B(\delta)} \\ & & \downarrow \\ & \mathcal{C}_B^n & \longrightarrow B \\ & \downarrow & \\ \mathbb{L}^n & \longrightarrow & (\mathbb{P}^3)^n \\ \downarrow & & \\ \mathrm{Gr}(1, \mathbb{P}^3)^n, & & \end{array} \quad (15)$$

in which we use the notation

$$\mathcal{C}_B^n = \mathcal{C} \times_B \cdots \times_B \mathcal{C}.$$

Then P parametrises the following data:

- lines $\ell_1, \dots, \ell_n \subset \mathbb{P}^3$;
- points $p_1, \dots, p_n \in \mathbb{P}^3$;
- a plane $V \subset \mathbb{P}^3$;
- a curve $C \in \overline{B(\delta)}$;

subject to the following conditions:

- $C \subset V$;
- $p_i \in \ell_i$ for $i = 1, \dots, n$;
- $p_i \in C$ for $i = 1, \dots, n$.

The horizontal maps in the diagram are flat with relative dimensions $2n$ and n respectively. Since $\overline{B(\delta)}$ has dimension n , it follows that

$$\dim(P) = 4n = \dim(\mathrm{Gr}(1, \mathbb{P}^3)^n).$$

A curve $C \in B(\delta)$ has a dense smooth open subset C° (as it is nodal). Therefore also the universal curve $\mathcal{C} \rightarrow B$ restricted to $\overline{B(\delta)}$ has this property (the latter being reduced). As $P \rightarrow \mathcal{C}_B^n|_{\overline{B(\delta)}}$ is smooth (with fibre $\cong (\mathbb{P}^2)^n$), we see that P is generically

smooth. In particular, the singular locus P^{sing} of P has dimension $< 4n$. Consider the morphism

$$\phi: P \rightarrow \text{Gr}(1, \mathbb{P}^3)^n.$$

Let $U_1 = \text{Gr}(1, \mathbb{P}^3)^n \setminus \phi(P^{\text{sing}})$ be the complement of the image of the singular locus in $\text{Gr}(1, \mathbb{P}^3)^n$. As $\text{Gr}(1, \mathbb{P}^3)^n$ has dimension $4n$, the open $U_1 \subset \text{Gr}(1, \mathbb{P}^3)^n$ is non-empty. Since $\phi^{-1}(U_1)$ is smooth, there is a non-empty open $U_2 \subset U_1$ such that the morphism ϕ is finite and reduced over U_2 . Moreover, the closed subsets

$$Z_i = \{(\vec{\ell}, \vec{p}, V, C) \in P \mid \ell_i \subset V\} \subset P, \quad i = 0, \dots, n$$

and

$$P_{\partial B(\delta)} = \{(\vec{\ell}, \vec{p}, V, C) \in P \mid C \in \partial B(\delta)\} \subset P$$

have positive codimension. Therefore the open

$$U = U_2 \cap \left(\text{Gr}(1, \mathbb{P}^3)^n \setminus \phi \left(\bigcup_{i=1}^n Z_i \cup P_{\partial B(\delta)} \right) \right)$$

is non-empty.

Now let $\underline{\ell} = (\ell_1, \dots, \ell_n) \in U$ be an n -tuple of lines and let

$$P_{\underline{\ell}} = \phi^{-1}(\underline{\ell})$$

be the fibre over $\underline{\ell}$. Consider the morphism

$$\psi: P \rightarrow \overline{B(\delta)} \subset B.$$

For a point $[C] \in \overline{B(\delta)}$, the fibre of $P_{\underline{\ell}}$ over $[C]$ is the scheme

$$P_{\underline{\ell}} \cap \psi^{-1}([C]) = \prod_{i=1}^n (\ell_i \cap C) \subset (\mathbb{P}^3)^n. \quad (16)$$

Let $V \subset \mathbb{P}^3$ be the plane spanned by C . By definition of U , we have $\ell_i \not\subset V$ for the lines $\ell_1, \dots, \ell_n$ in $\underline{\ell}$. It follows that the intersection

$$C \cap \ell_i \subset \ell_i \cap V$$

is a reduced point, if it is non-empty. Hence $P_{\underline{\ell}}$ maps isomorphically to its image in B .

On the other hand, the scheme (16) is non-empty if and only if C intersects the lines ℓ_i . By definition of U , it follows that we have

$$\begin{aligned}\psi(P_{\bar{\ell}}) &= (B_{\ell_1, \dots, \ell_n} \cap \overline{B(\delta)})^{red} \\ &= (B_{\ell_1, \dots, \ell_n} \cap B(\delta))^{red}.\end{aligned}$$

Finally, we will show that in fact

$$\psi(P_{\bar{\ell}}) = B_{\ell_1, \dots, \ell_n} \cap B(\delta).$$

To see this, first note that $\psi(P_{\bar{\ell}})$ lies in the open subset $W \subset B$ consisting of curves C with $\ell_i \not\subset V$ for $i = 1, \dots, n$ and V the plane spanned by C . As noted before, for a curve $C \in W$, the scheme $C \cap \ell_i$ is empty or consists of a single reduced point. It follows that over W , the scheme

$$\ell_i \times_{\mathbb{P}^3} C$$

is mapped isomorphically to its image B_{ℓ_i} in B , since a *scheme theoretic* fibre of

$$\ell_i \times_{\mathbb{P}^3} C_W \rightarrow B_{\ell_i} \cap W$$

is a *reduced* point. We conclude that

$$\begin{aligned}\psi(P_{\bar{\ell}}) &= \psi\left((\ell_1 \times_{\mathbb{P}^3} C) \times_B \cdots \times_B (\ell_n \times_{\mathbb{P}^3} C) \times_B \overline{B(\delta)}\right) \\ &= B_{\ell_1} \cap \cdots \cap B_{\ell_n} \cap \overline{B(\delta)} \\ &= B_{\ell_1, \dots, \ell_n} \cap B(\delta).\end{aligned}$$

As the intersection $B(\delta) \cap B_{\ell_1, \dots, \ell_n}$ is finite and reduced, it is *transverse*, by (14). It follows by Lemma 4.1, Proposition 4.4 and Lemma 3.4 that we have

$$\begin{aligned}\#(B(\delta) \cap B_{\ell_1, \dots, \ell_n}) &= \int_B ([\overline{B(\delta)}] \cdot [B_{\ell_1, \dots, \ell_n}]) \\ &= \int_B \gamma(C) \cap [B_{\ell_1, \dots, \ell_n}] \\ &= \int_{B_{\ell_1, \dots, \ell_n}} \gamma(C|_{B_{\ell_1, \dots, \ell_n}}).\end{aligned}$$

□

The following lemma is essentially [32], Exercise 3.4. See also [10], Example 3.2.22. For completeness, we will include the proof.

Lemma 6.4 *For a line $\ell \subset \mathbb{P}^3$, the closed subvariety $B_{\ell} \subset B$ is a divisor, cut out by a section of the line bundle*

$$\pi^* \mathcal{O}_{\text{Gr}}(d) \otimes \mathcal{O}_B(1).$$

Proof It suffices to construct the section outside the codimension two subvariety $Z \subset \text{Gr}$ of planes $[V] \in \text{Gr}$ containing the line ℓ . Let $U := \text{Gr} - Z$ be the complement. Consider the fibre product $\mathcal{S}_\ell$ of the following diagram:

$$\begin{array}{ccc} & \mathcal{S} & \\ & \downarrow & \\ \ell & \longrightarrow & \mathbb{P}^3. \end{array}$$

Then the morphism $\mathcal{S}_\ell \rightarrow \text{Gr}$ has fibre $V \cap \ell \subset \mathbb{P}^3$ over a point $[V] \in \text{Gr}$. In particular, it restricts to an isomorphism over U . Now consider the fibre product $\mathcal{C}_\ell$ of the diagram

$$\begin{array}{ccc} & \mathcal{C} & \\ & \downarrow & \\ \ell & \longrightarrow & \mathbb{P}^3. \end{array}$$

The morphism $\mathcal{C}_\ell \rightarrow B$ has *scheme theoretic* fibre $C \cap \ell$ over a point $[C] \in B$. It follows that $\mathcal{C}_\ell$ is mapped onto B_ℓ and the morphism restricts to an isomorphism over $B_\ell \times_{\text{Gr}} U$.

As noted before, $\mathcal{C}$ is the zero locus of a canonical section of the line bundle $\mathcal{O}_\mathcal{S}(d) \otimes \mathcal{O}_B(1)$ on $\mathcal{S} \times_{\text{Gr}} B$, in which $\mathcal{O}_\mathcal{S}(d)$ denotes the pull-back of the bundle $\mathcal{O}_{\mathbb{P}^3}(d)$ on $\mathbb{P}^3$ to the universal plane $\mathcal{S}$. It follows that $\mathcal{C}_\ell \subset \mathcal{S}_\ell \times_{\text{Gr}} B$ is cut out by a section of the bundle $\mathcal{O}_\ell(d) \otimes \mathcal{O}_B(1)$, in which $\mathcal{O}_\ell(d)$ is the restriction of $\mathcal{O}_{\mathbb{P}^3}(d)$ to $\mathbb{P}^1 \cong \ell \subset \mathbb{P}^3$. Hence, it suffices to show that $\mathcal{O}_{\mathbb{P}^3}(d)$ equals $\mathcal{O}_{\text{Gr}}(d)$ when pulled back to $\mathcal{S}_\ell \times_{\text{Gr}} B$. But this can be seen easily by noting that in the diagram (with Cartesian square)

$$\begin{array}{ccccc} \mathcal{S}_\ell & \longrightarrow & \mathcal{S} & \longrightarrow & \text{Gr} \\ \downarrow & & \downarrow & & \\ \ell & \longrightarrow & \mathbb{P}^3 & & \end{array}$$

the fibre $(\mathcal{S}_\ell)_x$ over a point $x \in \ell$ is the inverse image of the divisor on Gr of planes in $\mathbb{P}^3$ containing the point x . $\square$

The following corollary of Proposition 6.1 is the first part of Theorem B, our second main result.

Corollary 6.5 *For every $\delta \geq 0$, there is polynomial N_δ of degree $\leq 9 + 2\delta$ such that $N_{\delta,d} = N_\delta(d)$ for $d \geq \delta$.*

Proof By Proposition 6.1, we need to compute the integral

$$\int_{B_{\ell_1, \dots, \ell_n}} \gamma(\mathcal{C}|_{B_{\ell_1, \dots, \ell_n}}) = \int_B \gamma(\mathcal{C}) \cap [B_{\ell_1, \dots, \ell_n}].$$

Let $H = c_1(\mathcal{O}_{\text{Gr}}(1))$ and $\xi = c_1(\mathcal{O}_B(1))$. Then, by the Lemma 6.4, we have for general lines $\ell_1, \dots, \ell_n \subset \mathbb{P}^3$ the equation

$$\begin{aligned} [B_{\ell_1, \dots, \ell_n}] &= (dH + \xi)^n \\ &= \sum_{i=0}^3 \binom{n}{i} (dH)^i \xi^{n-i} \end{aligned}$$

in $A^*(B)$. On the other hand, we know by Theorem A that $\gamma(C)$ is a polynomial of degree δ in classes

$$\epsilon(a, b, c) = (q_B)_*(c_1(\mathcal{O}(C))^a c_1(T_{S/\text{Gr}})^b c_2(T_{S/\text{Gr}})^c).$$

It will follow that $\gamma(C)$ is a polynomial in H, ξ and d . To see this, it suffices to show that the classes $\epsilon(a, b, c)$ are polynomials in H, ξ and d . Let $\eta = c_1(\mathcal{O}_S(1))$. Then we have

$$\begin{aligned} c_1(\mathcal{O}(C))^a c_1(T_{S/\text{Gr}})^b c_2(T_{S/\text{Gr}})^c &= (d\eta + \xi)^a (3\eta + c_1(\mathcal{U}))^b (3\eta^2 + 2\eta c_1(\mathcal{U}) + c_2(\mathcal{U}))^c \\ &= (d\eta + \xi)^a (3\eta - H)^b (3\eta^2 - 2\eta H + H^2)^c, \end{aligned}$$

in which we use the short exact sequences

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{U} \otimes \mathcal{O}_S(1) \rightarrow T_{S/\text{Gr}} \rightarrow 0$$

and (13). The structure of the Chow ring of S_B is given by

$$A^*(S) = A^*(B)[\eta]/(\eta^3 + c_1(\mathcal{U})\eta^2 + c_2(\mathcal{U})\eta + c_3(\mathcal{U}))$$

and the push-forward

$$\epsilon(a, b, c) = (q_B)_*(c_1(\mathcal{O}(C))^a c_1(T_{S/\text{Gr}})^b c_2(T_{S/\text{Gr}})^c)$$

is computed by repeatedly substituting the equation

$$\begin{aligned} \eta^3 &= -c_1(\mathcal{U})\eta^2 - c_2(\mathcal{U})\eta - c_3(\mathcal{U}) \\ &= H\eta^2 - H^2\eta + H^3. \end{aligned}$$

and taking the coefficient of η^2 . Since $H^4 = 0$, the substitution procedure terminates after three steps. For fixed a, b and c , we obtain a polynomial in H, ξ and d . It follows that $\gamma(C)$ can be written as a polynomial in H, ξ and d .

Note that the coefficient of H^i in $\epsilon(a, b, c)$ has degree at most $2+i$ as a polynomial in d . As $\gamma(C)$ is a polynomial of degree δ in classes $\epsilon(a, b, c)$, the coefficient of H^i in this $\gamma(C)$ has degree at most $2\delta + i$, as a polynomial in d .

We consider the class

$$\gamma(\mathcal{C}|_{B_{\ell_1, \dots, \ell_n}}) = \gamma(\mathcal{C}) \cap [B_{\ell_1, \dots, \ell_n}]$$

in $A_0(B)$. We will show that its degree is a polynomial in d . The Chow ring of $B = \mathbb{P}(q_*\mathcal{L})$ is given by

$$\begin{aligned} A_*(B) &= A^*(\mathrm{Gr})[\xi]/(\xi^r + c_1(q_*\mathcal{L})\xi^{r-1} + c_2(q_*\mathcal{L})\xi^{r-2} + c_3(q_*\mathcal{L})\xi^{r-3}) \\ &= \mathbb{Z}[H, \xi]/(H^4, \xi^r + c_1(q_*\mathcal{L})\xi^{r-1} + c_2(q_*\mathcal{L})\xi^{r-2} + c_3(q_*\mathcal{L})\xi^{r-3}). \end{aligned}$$

We have

$$q_*\mathcal{L} = \mathrm{Sym}^d(\mathcal{U}^*),$$

and hence its Chern class is a polynomial in d and H . In fact, we have

$$\begin{aligned} c(q_*\mathcal{L}) &= 1 + \frac{d(d+1)(d+2)}{6} H \\ &\quad + \frac{d(d+1)(d+2)(d+3)(d^2+2)}{72} H^2 \\ &\quad + \frac{d(d+1)(d+2)(d+3)(d^2+2)(d^3+3d^2+2d+12)}{1296} H^3. \end{aligned}$$

Note that coefficient of H^i is a polynomial in d of degree $3i$. We can compute the degree

$$\int_B \gamma(\mathcal{C}|_{B_{\ell_1, \dots, \ell_n}}) = \int_B \gamma(\mathcal{C}) \cap \sum_{i=0}^3 \binom{n}{i} (dH)^i \xi^{n-i}$$

as follows. Note that the class $\gamma(\mathcal{C}|_{B_{\ell_1, \dots, \ell_n}})$ is homogeneous of degree $r+2$ in H and ξ . Using the relations

$$\xi^r = -c_1(q_*\mathcal{L})\xi^{r-1} - c_2(q_*\mathcal{L})\xi^{r-2} - c_3(q_*\mathcal{L})\xi^{r-3} \quad \text{and} \quad H^4 = 0$$

we can rewrite it as

$$uH^3\xi^{r-1}$$

for a polynomial u in d depending only on δ . Now we have

$$\int_B \gamma(\mathcal{C}|_{B_{\ell_1, \dots, \ell_n}}) = u.$$

Recall that we write $\pi : B \rightarrow \mathrm{Gr}$ for the projection. Then the class

$$uH^3 = \pi_*(uH^3\xi^{r-1})$$

is a product of classes of the following types:

- The coefficients of $\gamma(\mathcal{C})$ as a polynomial in ξ ;
- Classes of the form $\binom{n}{i}(dH)^i$;
- Polynomials in the Chern classes of $q_*\mathcal{L}$.

As remarked before, the coefficient of H^i in $\gamma(\mathcal{C})$ has degree $2\delta + i$ in d . In other words, every factor $d^{2\delta+i}$ appearing in the terms of $\gamma(\mathcal{C})$ is accompanied by a factor H^i . Similarly, the coefficients of H^i of classes of the second and third type, are polynomials of degree $3i$ in d , so every factor d^j appearing in the terms of these classes is accompanied by a factor $H^{\lceil j/3 \rceil}$. It follows that uH^3 has degree at most $2\delta + 9$ in d . $\square$

7 Torus localization

As in the previous section, let $\text{Gr} = \text{Gr}(2, \mathbb{P}^3)$ be the Grassmannian of planes in $\mathbb{P}^3$, with universal plane $q: \mathcal{S} = \mathbb{P}(\mathcal{U}) \rightarrow \text{Gr}$, in which $\mathcal{U}$ is the tautological vector bundle on Gr . On $\mathcal{S}$ we have defined the line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{S}}(d)$, which we use to construct the projective bundle $B = \mathbb{P}(q_*\mathcal{L})$ over Gr parametrizing planar curves of degree d in $\mathbb{P}^3$, with universal curve $\mathcal{C} \rightarrow B$. The variety B has dimension $r + 2$, with

$$r = \frac{(d+1)(d+2)}{2}$$

the rank of $q_*\mathcal{L}$. Finally, we define

$$n := r + 2 - \delta = \frac{d(d+3)}{2} + 3 - \delta$$

and write $B_{\ell_1, \dots, \ell_n}$ for the locus of curves intersecting general lines $\ell_1, \dots, \ell_n$.

Rather than using the algorithm of [8], we can use the Bott residue formula to evaluate the integral

$$\int_B \gamma(\mathcal{C}|_{B_{\ell_1, \dots, \ell_n}}) = \int_B \gamma(\mathcal{C}) \cap [B_{\ell_1, \dots, \ell_n}].$$

of Proposition 6.1. By the Lemmas 6.4 and 2.7, we need to compute

$$\begin{aligned} & \int_{\mathcal{C}_B^{[i]}} c(T_{\mathcal{C}_B^{[i]}/B}) \cap [\mathcal{C}_{B_{\ell_1, \dots, \ell_n}}^{[i]}] \\ &= \int_{\mathcal{S}_B^{[i]}} \frac{c(T_{\mathcal{S}_B^{[i]}/B})}{c(\mathcal{O}(\mathcal{C})_B^{[i]})} c_i(\mathcal{O}(\mathcal{C})_B^{[i]}) c_1(\mathcal{O}_{\text{Gr}}(d) \otimes \mathcal{O}_B(1))^n \end{aligned} \quad (17)$$

for $i = 1, \dots, \delta$. We have equations

$$\mathcal{O}_{\mathcal{S}_B}(\mathcal{C})_B^{[i]} = (\mathcal{L} \otimes \mathcal{O}_B(1))_B^{[i]} = \mathcal{L}_{\text{Gr}}^{[i]} \otimes \mathcal{O}_B(1)$$

and

$$T_{S_B^{[i]}/B} = \pi_{S^{[i]}}^* T_{S_{\text{Gr}}^{[i]}/\text{Gr}}.$$

It follows that the class on the right hand side of (17) is a polynomial in classes pulled back from $S_{\text{Gr}}^{[i]}$, and the first Chern class of the line bundle $\mathcal{O}_B(1)$. We will continue to use the notation $\xi = c_1(\mathcal{O}_B(1))$ and $H = c_1(\mathcal{O}_{\text{Gr}}(1))$. We can rewrite the factors involving the bundle $\mathcal{O}_B(1)$ in the integral as follows:

$$\begin{aligned} c_1(\mathcal{O}_{\text{Gr}}(d) \otimes \mathcal{O}_B(1))^n &= (dH + \xi)^n \\ &= \left(\xi^3 + n dH \xi^2 + \binom{n}{2} (dH)^2 \xi + \binom{n}{3} (dH)^3 \right) \xi^{n-3} \\ c_i(\mathcal{L}_{\text{Gr}}^{[i]} \otimes \mathcal{O}_B(1)) &= \sum_{k=0}^i c_k(\mathcal{L}_{\text{Gr}}^{[i]}) \xi^{i-k} \\ \frac{1}{c(\mathcal{L}_{\text{Gr}}^{[i]} \otimes \mathcal{O}_B(1))} &= \sum_{j=0}^{\infty} \left(1 - c(\mathcal{L}_{\text{Gr}}^{[i]} \otimes \mathcal{O}_B(1)) \right)^j \\ &= \sum_{j=0}^{\infty} \left(1 - \sum_{k=0}^i c_k(\mathcal{L}_{\text{Gr}}^{[i]}) (1 + \xi)^{i-k} \right)^j \\ &= \sum_{j=0}^{3+2i+\delta} \left(1 - \sum_{k=0}^i c_k(\mathcal{L}_{\text{Gr}}^{[i]}) (1 + \xi)^{i-k} \right)^j + \alpha. \end{aligned}$$

In the last expression, $\alpha \in A^*(S_B^{[i]})$ is a sum of classes of degree $> 3 + 2i + \delta$. It follows that

$$\alpha \xi^{n-3} = 0,$$

as we have

$$(n - 3) + (3 + 2i + \delta) = r + 2 + 2i = \dim(S_B^{[i]}).$$

Hence we can compute (17) by integrating the class

$$\begin{aligned} q_*^{[i]} \left(c(T_{S_{\text{Gr}}^{[i]}/\text{Gr}}) \times \left(\xi^3 + n dH \xi^2 + \binom{n}{2} (dH)^2 \xi + \binom{n}{3} (dH)^3 \right) \xi^{n-3} \right. \\ \left. \times \sum_{k=0}^i c_k(\mathcal{L}_{\text{Gr}}^{[i]}) \xi^{i-k} \times \sum_{j=0}^{3+2i+\delta} \left(1 - \sum_{k=0}^i c_k(\mathcal{L}_{\text{Gr}}^{[i]}) (1 + \xi)^{i-k} \right)^j \right). \end{aligned}$$

As in the proof of Corollary 6.5, the push-forward along the projection

$$\pi : \mathcal{S}_B^{[i]} = \mathbb{P}(q_*\mathcal{L}) \times_{\text{Gr}} \mathcal{S}_{\text{Gr}}^{[i]} \rightarrow \mathcal{S}_{\text{Gr}}^{[i]}$$

can be computed by substituting the equation

$$\xi^r = -c_1(q_*\mathcal{L})\xi^{r-1} - c_2(q_*\mathcal{L})\xi^{r-2} - c_3(q_*\mathcal{L})\xi^{r-3}$$

and taking the coefficient of ξ^{r-1} . Hence we can rewrite (17) as an integral

$$\int_{\mathcal{S}_{\text{Gr}}^{[i]}} P(T_{\mathcal{S}_{\text{Gr}}^{[i]}/\text{Gr}}, \mathcal{L}_{\text{Gr}}^{[i]}, \mathcal{O}_{\text{Gr}}(1), d), \quad (18)$$

in which P is a polynomial³ in d and the Chern classes of the bundles in the brackets.

We can evaluate this integral using the Bott residue formula. For notation and definitions, see [7]. Recall that for a torus T acting on a smooth variety X , the fixed locus X^T is smooth [15], so a connected component $F \subset X^T$ has normal bundle $N_F X$ of rank equal to the codimension d_F of F in X .)

Theorem 7.1 (Bott residue formula ([7])) *Let $E_1, \dots, E_r$ be T -equivariant vector bundles on a complete, smooth n -dimensional variety X with a torus action by T . Let $p(E)$ be a polynomial in the Chern classes of the bundles E_i . Then*

$$\int_X p(E) \cap [X] = \sum_{F \subset X^T} \int_F \left(\frac{p^T(E|_F) \cap [F]_T}{c_{d_F}^T(N_F X)} \right),$$

in which we sum over the connected components F of the fixed locus X^T of the torus action.

Consider the natural action of the torus $T = (\mathbb{G}_m)^4$ on $\mathbb{P}^3$ given by

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \cdot (a_0 : a_1 : a_2 : a_3) = (\lambda_0 a_0 : \lambda_1 a_1 : \lambda_2 a_2 : \lambda_3 a_3).$$

It induces a dual action of T on the Grassmanian $\text{Gr} = \check{\mathbb{P}}^3$ which lifts to an equivariant structure on $\mathcal{O}_{\text{Gr}}(1)$. The action on $\check{\mathbb{P}}^3 \times \mathbb{P}^3$ restricts to an action on $\mathcal{S}$ (which is simply the incidence variety). This action, in turn, lifts to an equivariant structure on the line bundle $\mathcal{O}_{\mathcal{S}}(1)$. Moreover, we obtain actions on the Hilbert schemes $\mathcal{S}^{[i]}$ and induced equivariant structures on the bundles $T_{\mathcal{S}^{[i]}/\text{Gr}}$ and $\mathcal{L}^{[i]} = \mathcal{O}_{\mathcal{S}}(d)^{[i]}$.

Lemma 7.2 *The fixed locus of the action of T on $\mathcal{S}_{\text{Gr}}^{[i]}$ is finite and reduced.*

³ The fact that the expression is polynomial in d is not important for the computation. However, it does give another proof of the polynomiality of $N_{\delta, d}$ for $d \geq \delta$, that does not depend on the algorithm of [8].

Proof As the variety $\mathcal{S}_{\text{Gr}}^{[i]}$ is smooth, so is the fixed locus, as remarked above. Hence it suffices to show that the underlying set is finite. The fixed points of Gr are the four planes

$$V_k = \mathcal{Z}(x_i) \subset \mathbb{P}^3 = \text{Proj } \mathbb{C}[x_0, x_1, x_2, x_3]$$

for $k = 0, \dots, 3$, given by the vanishing of a coordinate. Note that the morphism $\mathcal{S}_{\text{Gr}}^{[i]} \rightarrow \text{Gr}$ is equivariant for the T -action. It follows that T acts on the fibres $V_k^{[i]}$. Now we follow [9], Section 4. Let $Z \subset V_0$ be a subscheme of length i , fixed under the action of T . Then Z is supported on the T -invariant locus $\{P_1, P_2, P_3\} \subset V_0$, with

$$P_1 = (0 : 1 : 0 : 0), \quad P_2 = (0 : 0 : 1 : 0), \quad P_3 = (0 : 0 : 0 : 1).$$

For $k = 1, 2, 3$, let Z_k the component of Z , supported on $\{P_k\}$, and let i_k the length of Z_k . On an open neighbourhood of P_1 , we have coordinates $u = x_2/x_1$ and $v = x_3/x_1$ on the plane V_0 . Now T acts on the coordinate ring $\mathbb{C}[u, v]$ by

$$\lambda \cdot u = \frac{\lambda_1}{\lambda_2} u \quad \text{and} \quad \lambda \cdot v = \frac{\lambda_1}{\lambda_3} v.$$

As Z is invariant, so is the ideal $\mathcal{I}(Z_1)$ of Z_1 in $\mathbb{C}[u, v]$. It follows that $\mathcal{I}(Z_1)$ is generated by monomials in u and v . The coordinate ring $\mathbb{C}[u, v]/\mathcal{I}(Z_1)$ is spanned by the i_1 monomials $u^k v^l$ not contained in $\mathcal{I}(Z_1)$. For every $k \geq 0$, define

$$d_k = \max\{l \mid u^k v^l \notin \mathcal{I}(Z_1)\}.$$

It is easy to see that the d_k define a partition

$$P_{Z_1} = (d_0 \geq \dots \geq d_n)$$

of length l_1 . Similarly, we get partitions P_{Z_2} and P_{Z_3} . Conversely, any *tripartition* (P_1, P_2, P_3) of length i , consisting of three partitions P_1, P_2, P_3 with $|P_1| + |P_2| + |P_3| = i$, corresponds to a T -invariant length- i subscheme of V_0 . It is clear that there are finitely many such tripartitions. By repeating this argument for the other planes V_k , the result follows. $\square$

We will apply Theorem 7.1 to the integral (18). By Lemma 7.2, the fixed locus consists of isolated points. Hence, for a fixed point $[Z] \in \mathcal{S}_{\text{Gr}}^{[i]}$, the normal bundle $N_{[Z]}\mathcal{S}_{\text{Gr}}^{[i]}$ is just the restriction of the tangent bundle of $\mathcal{S}_{\text{Gr}}^{[i]}$. The restrictions of the bundles $T_{\text{Gr}}, T_{\mathcal{S}_{\text{Gr}}^{[i]}/\text{Gr}}, \mathcal{L}^{[i]}$ and $\mathcal{O}_{\text{Gr}}(1)$ to $[Z]$ are T -representations. Lemma 3 of [7] gives the equivariant Chern classes explicitly as polynomials in the characters of the torus. The details are similar to the computation in [9].

Proof Theorem B, second part We have performed the calculation using Maple. We have computed the polynomials N_δ up to $\delta = 12$. The results are printed in Appendix A. $\square$

Remark 7.3 Up to $\delta = 7$, our answers are in agreement with polynomials communicated by Ritwik Mukherjee, which he calculated using the methods from [3,4] and [33] and verified by means of the algorithm of [19].

Remark 7.4 This method of calculating node polynomials seems to be quite efficient. For example, consider the integral

$$\int_B \gamma(\mathcal{C}) c_1(\mathcal{O}_{\text{Gr}}(1))^3 \cap [B_{\ell_1, \dots, \ell_{n-3}}].$$

It is easy to see that it computes the number of δ -nodal curves of degree d intersecting $n - 3$ general points in a *fixed* plane $\mathbb{P}^2$. By an minor adaptation of our code, we were able to compute the node polynomials up to $\delta = 15$, finding agreement with the polynomials up to $\delta = 14$, published by Block in [6]. However, Göttsche has computed the polynomials up to $\delta \leq 28$ [11]. The polynomial for $\delta = 15$ is given in Appendix B.

8 Low degree checks

Let $\delta, d \geq 1$. We want to determine the contribution of reducible curves to the number $N_{\delta,d}$ of planar δ -nodal curves of degree d in $\mathbb{P}^3$ intersecting general lines $\ell_1, \dots, \ell_n \subset \mathbb{P}^3$. For certain δ and d , *all* curves contributing to $N_{\delta,d}$ are reducible, thereby giving consistency checks of our formulae. If in addition the irreducible components of these reducible curves are smooth or 1-nodal, these can be calculated by classical methods.

Let C be a curve in our counting problem, and assume we can write C as the union of irreducible curves $C = C_1 \cup \dots \cup C_r$. The curves C_i are necessarily nodal (if singular), and intersect transversely. For $i = 1, \dots, r$, let δ_i be the number of nodes of C_i , and d_i its degree. As C lies in a plane, two curves C_i and C_j with $1 \leq i < j \leq r$ intersect in $d_i d_j$ points. We have

$$\begin{aligned} d &= \sum_{i=1}^r d_i, \\ \delta &= \sum_{1 \leq i < j \leq r} d_i d_j + \sum_{i=1}^r \delta_i. \end{aligned} \tag{19}$$

Moreover, there is a partition

$$\{\ell_1, \dots, \ell_n\} = \bigsqcup_{i=1}^r \Sigma_i \tag{20}$$

such that C_i intersects the lines in $\Sigma_i \subset \{\ell_1, \dots, \ell_n\}$.

Conversely, choose a partition as in (20) and integers d_i and δ_i for $i = 1, \dots, r$ such that the equations (19) hold. We will determine the number of curves contributing to $N_{\delta,d}$ that decompose as described above, with these fixed data.

For $i \in \{1, \dots, r\}$, let $B_i := \mathbb{P}(\mathrm{Sym}^{d_i}(\mathcal{U}^*)) \xrightarrow{\pi_i} \mathrm{Gr} = \mathrm{Gr}(2, \mathbb{P}^3)$ be the projective bundle parametrizing planar curves of degree d_i . Consider the locus

$$W_i := W(d_i, \delta_i, \Sigma_i) \subset B_i$$

of *irreducible* δ_i -nodal curves that intersect the lines in Σ_i .

We have the following lemma.

Lemma 8.1 *For general $\ell_1, \dots, \ell_n$ the number of curves that contribute to $N_{\delta, d}$ that decompose with data fixed above, is given by*

$$\int_{\mathrm{Gr}} \prod_{i=1}^r \pi_{i*}[\overline{W}_i],$$

in which $[\overline{W}_i]$ denotes the class in $A_* B_i$ of the closure of W_i .

Remark 8.2 More precisely, in the proof we will construct a non-empty Zariski open

$$U \subset \mathrm{Gr}(1, \mathbb{P}^3)^n$$

such that for an n -tuple $(\ell_1, \dots, \ell_n) \in U$, the statement of the lemma holds.

Proof The argument is similar to the proof of Proposition 6.1, so we will not give all the details. For $i = 1, \dots, r$, let $n_i = \#\Sigma_i$ and form the limit P_i as in the diagram (15). We have natural morphisms $\phi_i: P_i \rightarrow \mathrm{Gr}(1, \mathbb{P}^3)^{n_i}$. Consider the morphism

$$\phi = (\phi_1, \dots, \phi_r): P_1 \times_{\mathrm{Gr}} \cdots \times_{\mathrm{Gr}} P_r \rightarrow \prod_{i=1}^r \mathrm{Gr}(1, \mathbb{P}^3)^{n_i} = \mathrm{Gr}(1, \mathbb{P}^3)^n.$$

As in Proposition 6.1, ϕ is finite and smooth over a non-empty open

$$U_0 \subset \mathrm{Gr}(1, \mathbb{P}^3)^n.$$

Here we use the fact due to Severi, that for the line bundles $\mathcal{O}(d)$ on $\mathbb{P}^2$, the locus of irreducible δ -nodal curves in $|\mathcal{O}(d)|$, if non-empty, has codimension δ [29]. There is a non-empty open $U_1 \subset U_0$, such that for a point $\Sigma = (\Sigma_1, \dots, \Sigma_r) \in U_1$, the fibre over Σ is

$$\begin{aligned} \phi^{-1}(\Sigma) &= \overline{W}_1 \times_{\mathrm{Gr}} \cdots \times_{\mathrm{Gr}} \overline{W}_r \\ &= W_1 \times_{\mathrm{Gr}} \cdots \times_{\mathrm{Gr}} W_r. \end{aligned}$$

Finally, there is an non-empty open $U_2 \subset U_1$, such that for $\Sigma \in U_2$, and any point $(C_1, \dots, C_r) \in \phi^{-1}(\Sigma)$, the curves $C_1, \dots, C_r$ intersect transversely, i.e.

$$C = C_1 \cup \cdots \cup C_r \subset V$$

is a nodal curve, in which the union is taken in the plane $V \subset \mathbb{P}^3$ corresponding to the image of $(C_1, \dots, C_r)$ in Gr . By a count of dimensions, the sets $\pi_i(W_i)$ intersect properly in Gr . It follows that the contribution to $N_{\delta,d}$ by curves of this type is given by

$$\begin{aligned} \#\phi^{-1}(\Sigma) &= \int_{P_1 \times_{\text{Gr}} \cdots \times_{\text{Gr}} P_r} [\overline{W}_1 \times_{\text{Gr}} \cdots \times_{\text{Gr}} \overline{W}_r] \\ &= \int_{\text{Gr}} \prod_{i=1}^r \pi_{i*}[\overline{W}_i]. \end{aligned}$$

□

Notation 8.3 Let $\ell_1, \dots, \ell_n$ general lines in $\mathbb{P}^3$, and let $\Sigma_i \subset \{\ell_1, \dots, \ell_n\}$ be a subset with $\#\Sigma_i = n_i$. Let $W_i = W(d_i, \delta_i, \Sigma_i)$ be the locus in $B_i = \mathbb{P}(\text{Sym}^{d_i}(\mathcal{U}^*)) \xrightarrow{\pi} \text{Gr} = \text{Gr}(2, \mathbb{P}^3)$ of irreducible δ_i -nodal degree d_i plane curves in $\mathbb{P}^3$ intersecting the lines in Σ_i . We will write ν_{d_i, δ_i, n_i} for the class $\pi_*[\overline{W}_i] \in A_*(\text{Gr})$.

We formulate the conclusion of the discussion above in the following proposition.

Proposition 8.4 *The number of δ -nodal plane curves in $\mathbb{P}^3$ of degree d , intersecting $n = \frac{d(d+3)}{2} + 3 - \delta$ general lines $\Sigma = \{\ell_1, \dots, \ell_n\}$, is given by*

$$N_{\delta,d} = \sum_{r=1}^{\infty} \sum_{\Theta} \mu(\Theta) \nu_{d_1, \delta_1, n_1} \cdots \nu_{d_r, \delta_r, n_r}$$

in which the second sum is taken over unordered r -tuples (or multisets) of triples

$$\Theta = \{(d_i, \delta_i, n_i)\}_{i=1, \dots, r}$$

of integers $d_i \geq 1$, $\delta_i \geq 0$, $n_i \geq 0$ satisfying (19) and

$$n = n_1 + \cdots + n_r,$$

and in which the multiplicity $\mu(\Theta)$ is the number of unordered partitions of the set Σ in sets of lengths $n_1, \dots, n_r$, labeled by d_i and δ_i . In fact, this number is given by

$$\mu(\Theta) = \frac{1}{\#\text{Stab}_{S_r}(d, \delta, n)} \binom{n}{n_1, \dots, n_r}$$

in which the denominator is the order of the stabilizer subgroup of

$$\overline{(d, \delta, n)} = ((d_i, \delta_i, n_i))_{i=1, \dots, r} \in (\mathbb{Z}^3)^r$$

for the action of the symmetric group S_r on $(\mathbb{Z}^3)^r$.

Remark 8.5 The generality condition in the proposition means that the lines have to be general in the sense of Lemma 8.1, for every triple $(\bar{d}, \bar{\delta}, \bar{n})$ appearing in the second sum.

Using the proposition, we will compute the numbers $N_{\delta,d}$, for $0 \leq \delta \leq 6$ and $\delta = 8$ and certain low d . We will compare results with the numbers $N_{\delta}(d)$, with N_{δ} the node polynomial as computed in the previous section, and given in the appendix for $\delta \leq 12$. In these cases, we can choose d in such a way that the irreducible components of the curves are smooth or 1-nodal.

Lemma 8.6 *For the following δ and d , the irreducible components of δ -nodal plane curves of degree d are lines $[(\delta, d) = (1, 2), (3, 3), (6, 4)]$.*

In the following cases, a δ -nodal plane curve of degree d has only linear components besides one smooth conic component $[(\delta, d) = (0, 2), (2, 3), (5, 4)]$.

Finally, δ -nodal plane curves of degree d of the following types have only linear components besides two smooth conic components, or a nodal cubic component $[(\delta, d) = (4, 4), (8, 5)]$.

Remark 8.7 See Tables 1, 2 and 3. Since we can apply Theorem A only under the assumption $d \geq \delta$, we have not proved that the value of the polynomial $N_{\delta}(d)$ equals the curve count $N_{\delta,d}$ in the cases indicated with *. However, as we will prove below, in these cases the polynomials give the right numbers. In general, the *Göttsche threshold* $d \geq \lceil \delta/2 \rceil + 1$ for nodal curves in $\mathbb{P}^2$, determined in [20], seems to hold also in our case, i.e. that the node polynomials N_{δ} have value $N_{\delta,d}$ in these d .

Table 1 Nodal plane curves consisting of lines in $\mathbb{P}^3$

δ	1	3	6
d	2	3	4
$N_{\delta}(d)$	140	7280	261,800*

Table 2 Nodal plane curves with a smooth conic component

δ	0	2	5
d	2	3	4
$N_{\delta}(d)$	92	15,660	1,303,500*

Table 3 Nodal plane curves with two conics or a nodal cubic

δ	4	8
d	4	5
$N_{\delta}(d)$	3,071,796	385,022,820*

Proof For an integral curve C , and its normalisation $\tilde{C}$, we have

$$g(C) - \delta(C) = g(\tilde{C}) \geq 0.$$

It follows that the number of nodes of an irreducible plane curve of degree d is bounded by its arithmetic genus $\frac{(d-1)(d-2)}{2}$. Now use the equations (19). $\square$

We have the following elementary lemma.

Lemma 8.8 *Let H be the hyperplane class in $\text{Gr} \cong \check{\mathbb{P}}^3$. Then we have*

$$\begin{aligned} v_{1,0,2} &= 1 & v_{2,0,5} &= 1 & v_{3,1,8} &= 12 \\ v_{1,0,3} &= 2H & v_{2,0,6} &= 8H & v_{3,1,9} &= 216H \\ v_{1,0,4} &= 2H^2 & v_{2,0,7} &= 34H^2 & v_{3,1,10} &= 2040H^2 \\ v_{1,0,5} &= 0 & v_{2,0,8} &= 92H^3 & v_{3,1,11} &= 12,960H^3 \end{aligned}$$

Proof Let $n := \frac{d(d+3)}{2} + 3 - \delta$. First note that for $d \geq \delta + 2$, all δ -nodal curves of degree d are irreducible, so we have by Lemma 3.4, Proposition 4.4 and a slightly adapted version of Proposition 6.1 the identity

$$v(d, \delta, n - i) = \pi_*(\gamma(\mathcal{C}|_{B_{\ell_1, \dots, \ell_{n-i}}})) \in A_i(\text{Gr}).$$

Hence we can compute the classes by the methods of the previous sections. In the case that $\delta = 0, 1$, however, the classes can be computed by elementary means. Let $\delta = 0$. The locus of curves in $B = \mathbb{P}(\text{Sym}^d(\mathcal{U}^*))$ intersecting a line, is cut out by a section of $\mathcal{O}_{\text{Gr}}(d) \otimes \mathcal{O}_B(1)$. Note that a general such curve is smooth. It follows that we have the equation

$$v_{d,0,n-i} = \pi_*(c_1(\mathcal{O}_{\text{Gr}}(d) \otimes \mathcal{O}_B(1))^{n-i}),$$

the right hand side of which can easily be calculated.

Now let $\delta = 1$. For a curve $C \subset \mathbb{P}^2$, given by a degree d polynomial f , the singular locus is given by the equations

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = 0, \quad f = 0.$$

The rules $f \mapsto df$ and $f \mapsto f$ define homomorphisms

$$\mathcal{O}_B(-1) \rightarrow \Omega_{S/\text{Gr}} \otimes \mathcal{O}_S(d) \quad \text{and} \quad \mathcal{O}_B(-1) \rightarrow \mathcal{O}_S(d)$$

of bundles on $\mathcal{S} \times_{\text{Gr}} B$, in which $\mathcal{S} = \mathbb{P}(\mathcal{U}) \rightarrow \text{Gr}$ is the universal $\mathbb{P}^2$ -bundle over the Grassmannian. The homomorphisms simultaneously vanish on the singular locus of the fibres of the universal curve $\mathcal{C} \rightarrow B$. As curves with one node lie dense in this

locus, it follows that the class of the closure of the locus of 1-nodal curves in B is given by

$$\alpha = (pr_B)_*(c_3((\Omega_{S/\text{Gr}} \oplus \mathcal{O}_{S \times_{\text{Gr}} B}) \otimes \mathcal{O}_S(d) \otimes \mathcal{O}_B(1)) \in A^1(B).$$

We conclude that

$$v_{d,1,n-i} = \pi_*(c_1(\mathcal{O}_{\text{Gr}}(d) \otimes \mathcal{O}_B(1))^{n-i} \cap \alpha).$$

Again, by a straight-forward computation, we obtain the numbers in the third column. $\square$

For $n = \frac{d(d+3)}{2} + 3 - \delta$, the number

$$\int v_{d,\delta,n-i} H^i$$

has the following interpretation: it is the number of planar curves C in $\mathbb{P}^3$ of degree d , with δ nodes, intersecting general lines $\ell_1, \dots, \ell_{n-i} \subset \mathbb{P}^3$, such that the plane of the curve contains general points $P_1, \dots, P_i \in \mathbb{P}^3$. For certain cases, this enumerative problem has already been studied by Schubert using his calculus introduced in [28]. E.g. he treats conics, planar and twisted cubics and planar quartic curves in $\mathbb{P}^N$ that intersect points, lines and planes. The curves are allowed to have nodal singularities, or a cusp in the case of the planar cubic. The degrees of the classes in the second and third column of Lemma 8.8 can be found in §20 and §24 of loc. cit. respectively. By Proposition 8.4 it follows that the numbers in Tables 1, 2 and 3 can be computed by 19th century geometry and some elementary combinatorics.

Curves with only linear components

- $\delta = 1, d = 2$

$$N_{1,2} = \binom{7}{3,4} \times 2^2 = 140 = N_1(2).$$

- $\delta = 3, d = 3$

$$N_{3,3} = \binom{9}{4,3,2} \times 2^2 + \frac{1}{3!} \times \binom{9}{3,3,3} \times 2^3 = 7280 = N_3(3).$$

- $\delta = 6, d = 4$

$$N_{6,4} = \frac{1}{2!} \times \binom{11}{4,3,2,2} \times 2^2 + \frac{1}{3!} \times \binom{11}{3,3,3,2} \times 2^3 = 261,800 = N_6(4).$$

Curves with one conic component

- $\delta = 0, d = 2$

$$N_{0,2} = \nu_{2,0,8} = 92 = N_2(0).$$

- $\delta = 2, d = 3$

$$N_{2,3} = \binom{10}{8} \times 92 + \binom{10}{7} \times 34 \times 2 + \binom{10}{6} \times 8 \times 2 = 15,660 = N_2(3).$$

- $\delta = 5, d = 4$

$$\begin{aligned} N_{5,4} &= \frac{1}{2!} \times \binom{12}{8, 2, 2} \times 92 + \binom{12}{7, 3, 2} \times 34 \times 2 + \binom{12}{6, 4, 2} \times 8 \times 2 + \\ &\quad \frac{1}{2!} \times \binom{12}{6, 3, 3} \times 8 \times 2^2 + \binom{12}{5, 4, 3} \times 2^2 = 1,303,500 = N_5(4). \end{aligned}$$

Curves with a nodal cubic or two conic components

- $\delta = 4, d = 4$ The contribution of curves with two conic components is

$$\binom{13}{8} \times 92 + \binom{13}{7} \times 34 \times 8 = 585,156.$$

The contribution of curves consisting of a nodal cubic and a line is

$$\binom{13}{11} \times 12,960 + \binom{13}{10} \times 2040 \times 2 + \binom{13}{9} \times 216 \times 2 = 2,486,640.$$

It total we have:

$$N_{4,4} = 585,156 + 2,486,640 = 3,071,796 = N_4(4).$$

- $\delta = 8, d = 5$ Two conics and a line:

$$\begin{aligned} &\binom{15}{8, 5, 2} \times 92 + \binom{15}{7, 6, 2} \times 34 \times 8 + \binom{15}{7, 5, 3} \times 34 \times 2 \\ &+ \frac{1}{2!} \times \binom{15}{6, 6, 3} \times 8 \times 8 \times 2 + \binom{15}{6, 5, 4} \times 8 \times 2 = 122,942,820. \end{aligned}$$

A nodal cubic, and two lines:

$$\begin{aligned} & \frac{1}{2!} \times \binom{15}{11, 2, 2} \times 12,960 + \binom{15}{10, 3, 2} \times 2040 \times 2 + \binom{15}{9, 4, 2} \times 216 \times 2 \\ & + \frac{1}{2!} \times \binom{15}{9, 3, 3} \times 216 \times 2 \times 2 + \binom{15}{8, 4, 3} \times 12 \times 2 \times 2 = 262,080,000. \end{aligned}$$

Total:

$$N_{8,5} = 122,942,820 + 262,080,000 = 385,022,820 = N_8(5).$$

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Appendix A. Node polynomials for $\delta = 0, \dots, 12$.

In order to keep the denominators under control, we will print the node polynomials for curves with *ordered* nodes, i.e. $N_\delta^o = \delta! N_\delta$.

$$\begin{aligned} N_0^o &= \frac{1}{324} d(d-1)(d+2)(d+1)(d^2+4d+6)(2d^3+6d^2+13d+3) \\ N_1^o &= \frac{1}{108} d(d+3)(d+2)(2d^4+4d^3+d^2-10d-6)(d-1)^2(d+1)^2 \\ N_2^o &= \frac{1}{108} d(d-1)(d-2)(d+2)(d+1)(6d^8+30d^7-25d^6-255d^5-142d^4 \\ &\quad + 333d^3+629d^2+18d+198) \\ N_3^o &= \frac{1}{108} d(d-1)(d-2)(18d^{12}+108d^{11}-315d^{10}-2664d^9+470d^8 \\ &\quad + 21,919d^7+19,103d^6-58,136d^5-106,948d^4+7039d^3+129,360d^2 \\ &\quad - 165,798d+110,700) \\ N_4^o &= \frac{1}{36} (d-1)(d-3)(18d^{15}+90d^{14}-747d^{13}-3843d^{12}+11,660d^{11} \\ &\quad + 63,140d^{10}-75,352d^9-486,678d^8+73,143d^7+1,773,729d^6+1,150,606d^5 \\ &\quad - 4,123,550d^4-3,282,032d^3+12,893,256d^2-11,795,040d+3,404,160) \\ N_5^o &= \frac{1}{36} (d-1)(54d^{18}-4545d^{16}+1152d^{15}+159,342d^{14}-67,218d^{13} \\ &\quad - 2,985,967d^{12}+1,450,512d^{11}+32,041,927d^{10}-129,36,036d^9-198,254,910d^8 \\ &\quad + 9,946,932d^7+752,976,733d^6+563,804,514d^5-2,869,526,338d^4 \\ &\quad - 1,811,459,616d^3+11,267,964,504d^2-12,007,211,040d+4,224,182,400) \\ N_6^o &= \frac{1}{36} (162d^{21}-486d^{20}-17,901d^{19}+56,781d^{18}+836,361d^{17}-2,772,558d^{16} \end{aligned}$$

$$\begin{aligned}
& -21,438,711 d^{15} + 73,412,631 d^{14} + 327,808,568 d^{13} - 1,138,677,007 d^{12} \\
& -3,072,121,759 d^{11} + 10,302,259,428 d^{10} + 18,632,510,223 d^9 - 50,159,288,793 d^8 \\
& -99,732,049,025 d^7 + 130,703,793,592 d^6 + 629,801,216,266 d^5 \\
& -777,706,339,956 d^4 - 2,089,991,213,304 d^3 + 5,446,674,186,768 d^2 \\
& -4,582,360,442,880 d + 1,359,752,313,600)
\end{aligned}$$

$$\begin{aligned}
N_7^o = \frac{1}{12} & (162 d^{23} - 810 d^{22} - 22,275 d^{21} + 117,045 d^{20} + 1,315,044 d^{19} - 7,305,633 d^{18} \\
& -43,435,062 d^{17} + 257,593,851 d^{16} + 875,704,283 d^{15} - 56,206,23,440 d^{14} \\
& -11,055,698,265 d^{13} + 77,840,061,643 d^{12} + 89,179,790,228 d^{11} \\
& -672,462,975,543 d^{10} - 563,743,329,044 d^9 + 3,506,892,852,821 d^8 \\
& +4,693,983,485,919 d^7 - 13,574,568,995,962 d^6 - 37,376,320,692,374 d^5 \\
& +84,863,008,074,540 d^4 + 101,290,677,876,264 d^3 - 419,002,213,496,112 d^2 \\
& +415,086,981,865,920 d - 136,551,736,742,400)
\end{aligned}$$

$$\begin{aligned}
N_8^o = \frac{1}{12} & (486 d^{25} - 3402 d^{24} - 80,271 d^{23} + 603,369 d^{22} + 5,736,609 d^{21} \\
& -47,210,985 d^{20} - 230,681,484 d^{19} + 2,141,947,278 d^{18} + 5,649,412,578 d^{17} \\
& -62,197,110,162 d^{16} - 84,069,436,618 d^{15} + 1,201,119,124,190 d^{14} \\
& +695,539,180,710 d^{13} - 15,500,834,280,650 d^{12} - 2,727,660,315,107 d^{11} \\
& +131,722,402,261,845 d^{10} + 25,466,213,716,945 d^9 - 750,756,824,927,669 d^8 \\
& -664,023,356,945,796 d^7 + 3,782,983,980,383,618 d^6 + 6,489,582,893,159,132 d^5 \\
& -24,182,782,626,411,432 d^4 - 11,635,999,979,827,824 d^3 + 98,923,354,020,446,400 d^2 \\
& -113,846,941,521,653,760 d + 40,910,206,904,985,600)
\end{aligned}$$

$$\begin{aligned}
N_9^o = \frac{1}{12} & (1458 d^{27} - 13,122 d^{26} - 282,123 d^{25} + 2,810,295 d^{24} + 23,620,086 d^{23} \\
& -269,654,670 d^{22} - 1,102,117,023 d^{21} + 15,284,004,291 d^{20} + 30,114,876,816 d^{19} \\
& -567,444,295,476 d^{18} - 422,910,483,264 d^{17} + 14,442,428,462,976 d^{16} \\
& -173,655,449,080 d^{15} - 256,024,966,449,048 d^{14} + 117,007,607,498,013 d^{13} \\
& +3,154,205,513,887,891 d^{12} - 1,917,888,357,827,630 d^{11} \\
& -26,939,284,058,835,262 d^{10} + 9,381,328,237,342,969 d^9 \\
& +170,575,538,835,999,315 d^8 + 81,091,482,477,623,574 d^7 \\
& -1,043,320,220,595,663,742 d^6 - 1,048,469,186,302,651,972 d^5 \\
& +6,715,930,311,282,223,672 d^4 + 37,331,737,163,479,536 d^3 \\
& -24,266,279,644,066,088,640 d^2 + 32,202,247,356,878,376,960 d \\
& -12,516,744,443,551,488,000)
\end{aligned}$$

$$\begin{aligned}
N_{10}^o = \frac{1}{4} & (1458 d^{29} - 16,038 d^{28} - 324,405 d^{27} + 4,083,129 d^{26} + 30,991,005 d^{25} \\
& -471,257,676 d^{24} - 1,603,277,307 d^{23} + 32,578,597,143 d^{22} + 43,377,665,589 d^{21} \\
& -1,500,353,595,792 d^{20} - 174,393,237,924 d^{19} + 48,387,112,634,196 d^{18} \\
& -31,729,963,605,856 d^{17} - 1,117,426,679,453,368 d^{16} + 1,278,946,167,008,861 d^{15}
\end{aligned}$$

$$\begin{aligned}
& + 18,584,197,566,356,041 d^{14} - 25,929,838,100,941,987 d^{13} \\
& - 222,292,527,680,013,236 d^{12} + 301,840,036,425,933,217 d^{11} \\
& + 1,945,082,392,976,187,963 d^{10} - 1,702,396,861,961,183,657 d^9 \\
& - 13,840,054,897,129,551,538 d^8 - 869,477,353,690,603,586 d^7 \\
& + 97,969,270,160,191,718,168 d^6 + 43,400,439,640,826,602,848 d^5 \\
& - 629,779,784,557,096,225,952 d^4 + 234,491,922,967,527,070,944 d^3 \\
& + 2,074,544,035,031,110,584,960 d^2 - 3,163,830,549,287,964,595,200 d \\
& + 1,320,649,590,395,681,280,000) \\
N_{11}^o &= \frac{1}{4} (4374 d^{31} - 56,862 d^{30} - 1,102,977 d^{29} + 16,973,307 d^{28} + 117,608,112 d^{27} \\
& - 2,318,176,989 d^{26} - 6,425,985,798 d^{25} + 191,682,452,511 d^{24} \\
& + 133,119,473,328 d^{23} - 10,693,744,028,253 d^{22} + 5,846,987,132,217 d^{21} \\
& + 424,349,459,112,114 d^{20} - 577,264,977,532,776 d^{19} - 12,296,681,379,527,388 d^{18} \\
& + 23,489,692,414,637,363 d^{17} + 263,030,758,384,442,289 d^{16} \\
& - 593,247,075,529,299,782 d^{15} - 4,162,564,652,290,610,993 d^{14} \\
& + 9,903,092,880,496,934,734 d^{13} + 49,018,479,445,283,034,499 d^{12} \\
& - 106,385,448,246,367,215,702 d^{11} - 447,930,561,908,076,256,091 d^{10} \\
& + 645,540,608,889,693,443,477 d^9 + 3,606,790,056,461,753,863,832 d^8 \\
& - 1,341,780,384,161,439,521,626 d^7 - 28,540,272,186,090,313,415,704 d^6 \\
& + 1,205,795,435,057,498,651,584 d^5 + 182,406,168,443,172,371,488,448 d^4 \\
& - 128,952,276,571,759,318,016,928 d^3 - 557,203,918,390,573,072,878,720 d^2 \\
& + 976,485,597,554,969,367,782,400 d - 435,294,406,202,292,274,176,000) \\
N_{12}^o &= \frac{1}{4} (13,122 d^{33} - 196,830 d^{32} - 3,706,965 d^{31} + 68,070,375 d^{30} + 432,815,319 d^{29} \\
& - 10,850,751,657 d^{28} - 23,535,697,932 d^{27} + 1,055,997,538,326 d^{26} \\
& + 50,357,440,881 d^{25} - 70,021,319,228,739 d^{24} + 873,74,064,448,161 d^{23} \\
& + 3,341,431,959,709,527 d^{22} - 7,111,742,962,317,408 d^{21} \\
& - 118,120,713,345,379,188 d^{20} + 326,384,557,105,326,777 d^{19} \\
& + 3,137,002,724,874,226,941 d^{18} - 10,106,444,734,270,420,903 d^{17} \\
& - 62,918,353,809,936,417,707 d^{16} + 220,730,277,300,344,083,152 d^{15} \\
& + 956,623,940,326,083,332,050 d^{14} - 3,393,890,378,620,526,954,445 d^{13} \\
& - 11,241,179,417,671,261,959,041 d^{12} + 35,363,022,169,384,877,426,927 d^{11} \\
& + 109,549,515,125,017,919,639,429 d^{10} - 229,128,080,595,756,761,453,742 d^9 \\
& - 1,002,037,783,274,877,109,543,198 d^8 + 840,677,011,541,967,726,001,824 d^7 \\
& + 8,641,146,394,045,844,077,954,112 d^6 - 4,223,694,280,033,586,640,137,824 d^5 \\
& - 547,79,602,892,548,858,064,166,240 d^4 + 56,217,660,837,944,500,164,819,456 d^3 \\
& + 156,589,791,424,366,871,478,896,640 d^2 - 316,225,057,234,071,161,731,737,600 d \\
& + 149,867,365,795,069,610,096,640,000)
\end{aligned}$$

Appendix B. A node polynomial for curves in $\mathbb{P}^2$

The number of 15-nodal curves degree $d \geq 9$ (by [20]) in $\mathbb{P}^2$ containing $\frac{d(d+1)}{2} - 15$ points in general position is given by the following polynomial.

$$\begin{aligned} & \frac{1}{15!} (14,348,907 d^{30} - 430,467,210 d^{29} - 789,189,885 d^{28} + 144,134,770,815 d^{27} \\ & - 800,302,316,310 d^{26} - 21,505,566,260,997 d^{25} + 206,046,709,321,635 d^{24} \\ & + 1,830,389,081,571,180 d^{23} - 25,973,085,837,797,631 d^{22} - 90,805,122,781,323,093 d^{21} \\ & + 2,106,764,580,151,475,244 d^{20} + 1,842,311,595,032,520,885 d^{19} \\ & - 120,731,061,785,804,511,795 d^{18} + 83,105,496,803,044,790,514 d^{17} \\ & + 5,106,565,375,968,131,056,197 d^{16} - 8,800,802,481,614,659,877,511 d^{15} \\ & - 162,890,506,083,253,675,564,674 d^{14} + 397,425,775,424,906,515,333,221 d^{13} \\ & + 3,952,008,654,242,554,161,166,365 d^{12} - 11,546,375,323,786,656,779,457,252 d^{11} \\ & - 72,858,625,897,371,563,437,077,825 d^{10} + 232,182,939,704,411,137,229,570,133 d^9 \\ & + 1,010,825,449,711,157,998,476,650,988 d^8 - 3,241,105,115,881,805,786,551,102,893 d^7 \\ & - 10,336,040,203,392,280,930,456,480,032 d^6 + 30,163,840,992,557,581,783,875,044,832 d^5 \\ & + 74,721,661,229,894,928,962,601,063,456 d^4 - 168,817,217,722,446,315,040,796,818,224 d^3 \\ & - 347,671,495,806,428,829,919,633,280,640 d^2 + 429,634,898,369,604,339,129,576,633,600 d \\ & + 794,015,010,296,634,348,660,582,144,000) \end{aligned}$$

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