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The algebraisation of higher Deligne–Lusztig representations

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Abstract In this paper we study higher Deligne–Lusztig representations of reductive groups over finite quotients of discrete valuation rings. At even levels, we show that these geometrically constructed representations, defined by Lusztig, coincide with certain explicit induced representations defined by Gérardin, thus giving a solution to a problem raised by Lusztig. In particular, we determine the dimensions of these representations. As an immediate application we verify a conjecture of Letellier for GL_2 and GL_3.

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1 Introduction

In [13] Lusztig proposed a geometric (cohomological) construction of representations of reductive groups over finite rings $\mathcal{O}_r = \mathcal{O}/\pi^r$, where $\mathcal{O}$ is the ring of integers in a non-archimedean local field with residue field $\mathbb{F}_q$, π a uniformiser, and $r \geq 1$

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a positive integer (the asserted fundamental properties were later proved in [15] for function fields and in [17] in general). This generalises the construction of Deligne and Lusztig [4] corresponding to the case $r = 1$, which is the only known way to realise all irreducible representations of a general connected reductive group over a finite field. This generalised Deligne–Lusztig theory is a unified way to deal with all $r \geq 1$. However, for $r > 1$, besides the geometric construction, there is also a Clifford theoretic algebraic construction of representations of these groups. This algebraic method depends on the parity of r, and can be traced back to Shintani (see [16]), and then Gérardin (see [6] and [7]), who use this construction to study the representations of split p-adic groups.

Let $\mathbb{G}$ be a reductive group scheme over $\mathcal{O}_r$. For $r > 1$, the geometrically constructed representations and the algebraically constructed representations of $\mathbb{G}(\mathcal{O}_r)$ share the same set of parameters, namely, the pairs consisting of a maximal torus in $\mathbb{G}$, and a character of the $\mathcal{O}_r$-points of the torus satisfying some regularity conditions (see Definition 2.2 and 2.3). So a natural question, suggested by Lusztig in [15, Section 1], is whether the two representations coincide. In Sect. 4 we give a positive answer to this question when r is even.

In the special case of GL_n over $\mathbb{F}_q[[\pi]]/\pi^r$, note that (e.g. by Weil restriction) the group $\mathrm{GL}_n(\overline{\mathbb{F}}_q[[\pi]]/\pi^r)$ admits a natural algebraic group structure over $\overline{\mathbb{F}}_q$, together with a Frobenius endomorphism F such that

$$\mathrm{GL}_n(\overline{\mathbb{F}}_q[[\pi]]/\pi^r)^F = \mathrm{GL}_n(\mathbb{F}_q[[\pi]]/\pi^r).$$

Moreover, we can talk about the reduction morphisms of these algebraic groups (on points they are the natural reduction maps $\mathrm{GL}_n(\overline{\mathbb{F}}_q[[\pi]]/\pi^r) \to \mathrm{GL}_n(\overline{\mathbb{F}}_q[[\pi]]/\pi^i)$, where $i \in \{1, \ldots, r\}$), whose kernels are closed subgroups.

By applying the Greenberg functor technique, the above description goes through for a general reductive group $\mathbb{G}$ over a general $\mathcal{O}_r$ (where $\mathcal{O}$ can be of any characteristic), namely, there exists an algebraic group G over $\overline{\mathbb{F}}_q$, together with a Frobenius endomorphism F, such that

$$G^F \cong \mathbb{G}(\mathcal{O}_r)$$

as finite groups. Moreover, we can talk about reduction maps $G \to G_i$ and the corresponding kernels G^i; see Sect. 2 for more details. For a maximal torus of $\mathbb{G}$, there exists a closed F-stable subgroup T of G with the same property regarding its F-fixed points.

We now describe our main result. Assume that r is even and write $r = 2l$. Let θ be a character of T^F; it admits a pull-back $\widetilde{\theta}$ to $(TG^l)^F$ (see Sect. 3). Our main results (Theorem 4.1 and Corollary 4.7) say that, under either Gérardin's conditions (see Remark 3.4) or the genericity condition (see Definition 3.5), one has an isomorphism between irreducible representations

$$R_T^\theta \cong \mathrm{Ind}_{(TG^l)^F}^{G^F} \widetilde{\theta}, \tag{1}$$

where R_T^θ is the higher Deligne–Lusztig representation (see Definition 2.1). As a consequence, under the conditions of Gérardin's construction, the higher Deligne–

Lusztig representations coincide with Gérardin's representations for r even, hence we get an affirmative answer to Lusztig's question. Another consequence is that, from the above isomorphism, we obtain a dimension formula for R_T^θ. As far as we know, this dimension formula was not known earlier for $r > 1$, except in the principal series case where R_T^θ is Harish-Chandra induced.

The strategy of the proof is to first realise $\mathrm{Ind}_{(TG^l)^F}^{G^F}\widetilde{\theta}$ as the cohomology of the Lang pre-image of a unipotent algebraic group (see Proposition 3.3), and then show that the inner product of these two representations equals 1 (this is the most difficult part). The argument for the computation of inner product is generalised from the GL_n case in [1].

We remark that: a) In the principal series case the above isomorphism (1) follows easily from the Mackey intertwining formula, b) the isomorphism (1) can fail when θ is not regular (as can be seen from the example computed by Lusztig in [15, Section 3]), and c) we expect that a similar result holds for odd r, but this case requires further considerations and is work in progress.

Finally, we use our main result to deduce some consequences for the invariant characters of Lie algebras over finite fields. Let $\mathfrak{g}$ be the Lie algebra of the reductive group G_1. For $\mathcal{O}_r = \mathbb{F}_q[[\pi]]/\pi^2$, by restricting the higher Deligne–Lusztig characters to the kernel $(G^1)^F \cong \mathfrak{g}^F$ one obtains invariant characters of finite Lie algebras. This was studied by Letellier in [12], where he proposed several conjectures. One of them says roughly that any irreducible invariant character of $\mathfrak{g}^F$ appears in the restriction of some Deligne–Lusztig character. We verify this conjecture for GL_2 and GL_3 in Sect. 5. Previously this was only known for GL_2 under the condition that $|\mathbb{F}_q| > 3$.

During a summer school in Jul–Aug 2015, when we communicated with Lusztig about our methods and results, he told us that at the time when he stated the expected relation between the algebraic and the geometric constructions, he had found a proof in the type A_n case with $r = 2$ (unpublished), by a method very different from ours.

2 Higher Deligne–Lusztig theory

Here we recall the main results developed in [15], and [17].

Throughout this paper we fix an arbitrary positive integer $r \geq 1$. Let $\mathcal{O}^{\mathrm{ur}}$ be the ring of integers in the maximal unramified extension of the field of fractions of $\mathcal{O}$, and put $\mathcal{O}_r^{\mathrm{ur}} = \mathcal{O}^{\mathrm{ur}}/\pi^r$. Denote the residue field of $\mathcal{O}^{\mathrm{ur}}$ by $k = \overline{\mathbb{F}}_q$. For $\mathbf{H}$ a smooth affine group scheme over $\mathcal{O}_r^{\mathrm{ur}}$, we have an associated algebraic group $H = H_r = \mathcal{F}\mathbf{H}$ over $k = \overline{\mathbb{F}}_q$, where $\mathcal{F}$ is the Greenberg functor; see [8,9,17], and [18] for its further properties. This H is an affine smooth algebraic group over k such that $H(k) \cong \mathbf{H}(\mathcal{O}_r^{\mathrm{ur}})$.

From now on, let $\mathbb{G}$ be a reductive group scheme over $\mathcal{O}_r$ (in other words, $\mathbb{G}$ is an affine smooth group scheme whose geometric fibre $\mathbb{G}_k$ is a connected reductive algebraic group in the classical sense; see e.g. [3, XIX 2.7]). Let $\mathbf{G}$ be the base change of $\mathbb{G}$ to $\mathcal{O}_r^{\mathrm{ur}}$, then

$$G = G_r := \mathcal{F}(\mathbf{G})$$

is a smooth affine algebraic group over k such that $G(k) \cong \mathbf{G}(\mathcal{O}_r^{\mathrm{ur}})$. Let $F\colon G \to G$ be a surjective algebraic group endomorphism such that the fixed points G^F form a finite group; we call such a morphism a *Frobenius endomorphism*. A closed subgroup $H \subseteq G$ is said to be F-rational (or rational when F is fixed), if $F(H) \subseteq H$. In this paper we will only be concerned with the following typical situation: The Frobenius element F in $\mathrm{Gal}(k/\mathbb{F}_q)$ extends to an automorphism of $\mathcal{O}_r^{\mathrm{ur}}$, and by the Greenberg functor this gives a rational structure on G over $\mathbb{F}_q$; we denote the associated geometric Frobenius endomorphism again by F. In this case we have an isomorphism of finite groups $G^F \cong \mathbb{G}(\mathcal{O}_r)$. We write $L\colon g \mapsto g^{-1}F(g)$ for the Lang map associated to F.

For any integer i such that $r \geq i \geq 1$, let $\rho_{r,i}\colon G \to G_i$ be the reduction map modulo π^i. Note that this is a surjective algebraic group morphism; denote the kernel by $G^i = G_r^i$. We also set $G^0 = G$ (this is not the identity component G°). Similar notation applies to closed subgroups of G. Note that if $\mathcal{O} = \mathbb{F}_q[[\pi]]$, then there is a natural semi-direct product $G \cong G_1 \ltimes G^1$; however, if $\mathrm{char}(\mathcal{O}) = 0$, this product does not hold in general: For example, if $\mathcal{O} = \mathbb{Z}_p$, then $\mathcal{O}_r = W_r(\mathbb{F}_p)$ is the truncated Witt vector ring and $\mathbf{G}(\mathcal{O}_r^{\mathrm{ur}}) = \mathbf{G}(W_r(k))$ (this is why $\mathbf{G}(\mathcal{O}_r^{\mathrm{ur}})$ admits an algebraic group structure over k in this case), but in general there is no group embedding from $\mathbf{G}(k)$ to $\mathbf{G}(W_r(k))$.

Let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus such that $T = \mathcal{F}\mathbf{T}$ is F-rational, and let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. Consider the Levi decomposition $\mathbf{B} = \mathbf{UT}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. The functor $\mathcal{F}$ gives a semi-direct product $B = \mathcal{F}\mathbf{B} = UT$ of closed subgroups of G, where $U = \mathcal{F}\mathbf{U}$. Let $\ell \neq p := \mathrm{char}(\mathbb{F}_q)$ be a fixed prime number. We are interested in the *higher level Deligne–Lusztig variety* associated to T and U

$$S_{T,U} := \{g \in G \mid g^{-1}F(g) \in FU\} = L^{-1}(FU),$$

where here, and in what follows, we often write FU for $F(U)$. Note that $G^F \times T^F$ acts on $S_{T,U}$ by $(g,t)\colon x \mapsto gxt$, which induces an action on the compactly supported ℓ-adic cohomology groups $H_c^i(S_{T,U}) := H_c^i(S_{T,U}, \overline{\mathbb{Q}}_\ell)$.

For any $\theta \in \widehat{T^F} = \mathrm{Hom}(T^F, \overline{\mathbb{Q}}_\ell^\times)$, we denote by $H_c^i(S_{T,U})_\theta$ the θ-isotypical part of $H_c^i(S_{T,U})$. This is a G^F-submodule of $H_c^i(S_{T,U})$. We use the notation $H_c^*(-)$ for the alternating sum

$$H_c^*(-) = \sum_{i \geq 0} (-1)^i H_c^i(-, \overline{\mathbb{Q}}_\ell).$$

Definition 2.1 The *higher Deligne–Lusztig representation* of G^F associated to $\theta \in \widehat{T^F}$ is the virtual representation

$$R_{T,U}^\theta = \sum_{i \geq 0} (-1)^i H_c^i(S_{T,U})_\theta.$$

In the situation we are interested in (see Theorem 2.4), $R_{T,U}^\theta$ is independent of the choice of U, and when this is the case we denote $R_{T,U}^\theta$ by R_T^θ.

The higher Deligne–Lusztig representations considered in this paper are the irreducible ones, or more precisely, the ones associated to the characters of T^F which are regular and in general position. We explain these notions.

For any root $\alpha \in \Phi = \Phi(\mathbf{G}, \mathbf{T})$ of $\mathbf{T}$, denote by $\mathbf{T}^\alpha$ the image of the coroot $\check{\alpha}$, and let $T^\alpha = \mathcal{F}\mathbf{T}^\alpha$. We write $\mathbf{U}_\alpha$ for the root subgroup of α, and write U_α for its

Greenberg functor image. For simplicity, we write $\mathcal{T}^\alpha$ for $(T^\alpha)^{r-1}$, the kernel of T^α along $\rho_{r,r-1}$. Note that $\mathbf{B}$ determines a subset of negative roots $\Phi^- \subseteq \Phi$ of $\mathbf{T}$ by the condition $-\alpha \in \Phi^-$ iff $\mathbf{U}_\alpha \subseteq \mathbf{B}$. From now on we fix an arbitrary total order on Φ^-.

Definition 2.2 Let a be a fixed positive integer such that $F^a(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$ for every root $\alpha \in \Phi$ of $\mathbf{T}$. Consider the norm map $N_F^{F^a}(t) := t \cdot F(t) \cdots F^{a-1}(t)$ on T^{F^a}. Then $\theta \in \widehat{T^F}$ is called *regular* if it is non-trivial on $N_F^{F^a}((\mathcal{T}^\alpha)^{F^a})$ for every root $\alpha \in \Phi$. One knows that a regular character is regular with respect to any such a; see [17, 2.8].

Since $\mathcal{O}_r^{\mathrm{ur}}$ is a strictly henselian local ring, the reductive group scheme $\mathbf{G}$ is split with respect to every maximal torus (see [17, 2.1]), therefore we can identify the Weyl group $W(T) := N(T)/T \cong W(T_1) := N(T_1)/T_1$; see [3, XXII 3.4].

Definition 2.3 $\theta \in \widehat{T^F}$ is said to be in *general position* if no non-trivial element in $W(T)^F = N(T)^F/T^F$ stabilises θ.

The following is one of the main results of [15] (in the function field case) and [17] (in the general case).

Theorem 2.4 *If $\theta \in \widehat{T^F}$ is regular, then $R_{T,U}^\theta = R_T^\theta$ is independent of the choice of U, and if moreover θ is in general position, then R_T^θ is an irreducible representation up to sign.*

Proof See [15] for the function fields and [17] for the general situation. □

3 The algebraic construction

From now on we assume $r = 2l$ is even (note that l is not the fixed prime ℓ). Let $B_0 = T_0 U_0$ (resp. T_0, U_0) be the Greenberg functor image of a Borel subgroup $\mathbf{B}_0$ (resp. maximal torus $\mathbf{T}_0$, unipotent radical $\mathbf{U}_0$) of $\mathbf{G}$, such that B_0 is F-rational. Let $\lambda \in G$ be such that $B = \lambda B_0 \lambda^{-1}$ and $T = \lambda T_0 \lambda^{-1}$. Note that $\lambda^{-1}F(\lambda) = \hat{w} \in N(T_0)$ is a lift of some Weyl element $w \in W(T_0)$.

Definition 3.1 Along with the above notation, we denote by $U^\pm$ the commutative unipotent group $(U^-)^l U^l$, and call it the *arithmetic radical* associated to T.

Note that $T = \mathcal{F}\mathbf{T}$ is usually not a torus, but we sometimes still call it a torus. For convenience, we similarly say "Borel subgroup" for $B = \mathcal{F}\mathbf{B}$.

Lemma 3.2 *$U^\pm$ is normalised by $N(T)$, and it is F-rational.*

This easy result follows from the fact that both $N(T)$ and F act on the root subgroups U_α, hence they permute the groups U_α^l and preserve the group $U^\pm$.

The variety $L^{-1}(U^\pm)$ admits a left G^F-action and a right T^F-action, so $H_c^*(L^{-1}(U^\pm))$ is a $G^F \times T^F$-module.

Proposition 3.3 *For every $\theta \in \widehat{T^F}$ we have $H_c^*(L^{-1}(U^\pm))_\theta \cong \mathrm{Ind}_{(TU^\pm)^F}^{G^F} \widetilde{\theta}$, where $\widetilde{\theta}$ is the trivial lift of θ from T^F to $(TU^\pm)^F$ (that is, $\widetilde{\theta}$ is the pull-back of $(TU^\pm)^F \to T^F$).*

Proof This is an argument analogous to the last paragraph in [5, p. 81]. Consider the natural morphism $L^{-1}(U^{\pm}) \to G/U^{\pm}$ given by $g \mapsto gU^{\pm}$. Note that $F(gU^{\pm}) = gL(g)U^{\pm} = gU^{\pm}$, so its image is $(G/U^{\pm})^F \cong G^F/(U^{\pm})^F$. Note that its fibres are isomorphic to an affine space ($\cong U^{\pm}$), therefore $H_c^*(L^{-1}(U^{\pm})) \cong \overline{\mathbb{Q}}_\ell[G^F/(U^{\pm})^F]$ by basic properties of ℓ-adic cohomology. Finally, $\overline{\mathbb{Q}}_\ell[G^F/(U^{\pm})^F] \otimes_{\overline{\mathbb{Q}}_\ell[T^F]} \theta \cong \overline{\mathbb{Q}}_\ell[G^F] \otimes_{\overline{\mathbb{Q}}_\ell[(TU^{\pm})^F]} \widetilde{\theta}$ as $\overline{\mathbb{Q}}_\ell[G^F]$-modules, thus $H_c^*(L^{-1}(U^{\pm}))_\theta \cong \mathrm{Ind}_{(TU^{\pm})^F}^{G^F} \widetilde{\theta}$.

Remark 3.4 The representations $\mathrm{Ind}_{(TU^{\pm})^F}^{G^F} \widetilde{\theta}$ have been considered by Gérardin [7] in a more restrictive situation. To be more precise, he assumed $\mathbb{G}(\mathcal{O}_r)$ is the $\mathcal{O}_r$-points of a split reductive group over the field of fractions of $\mathcal{O}$, whose derived subgroup is assumed to be simply connected, and he assumed the maximal tori to be "special" in the sense of [7, 3.3.9]; see [7, 4.1.1]. Under these conditions, Gérardin proved that $\mathrm{Ind}_{(TU^{\pm})^F}^{G^F} \widetilde{\theta}$ is irreducible if θ is regular and in general position; see [7, 4.4.1]. Note that Gérardin denoted $\mathrm{Ind}_{(TU^{\pm})^F}^{G^F} \widetilde{\theta}$ by κ_θ, and defined the regularity of θ in the language of conductor of Galois orbits (see [7, 4.2.2 and 4.2.3]).

We formulate a similar irreducibility condition for a general $\mathbb{G}$. First, note that one has $(TU^{\pm})^F \subseteq \mathrm{Stab}_{G^F}(\widetilde{\theta}|_{(G^l)^F})$, and by Clifford theory, if equality holds, then $\mathrm{Ind}_{(TU^{\pm})^F}^{G^F} \widetilde{\theta}$ is irreducible. In the following definition, we consider a condition on $\mathrm{Stab}_{G^F}(\widetilde{\theta}|_{(G^l)^F})$ which is weaker than equality, but still implies irreducibility (see the proof of Corollary 4.7).

Definition 3.5 A character $\theta \in \widehat{T^F}$ is *generic* if it is regular, in general position, and $\mathrm{Stab}_{G^F}(\widetilde{\theta}|_{(G^l)^F}) = (TU^{\pm})^F \cdot \mathrm{Stab}_{N_G(T)^F}(\widetilde{\theta}|_{(G^l)^F})$.

Remark 3.6 We explain how the genericity in the above definition appears in a natural way. Let $\psi : \mathcal{O}_l \to \overline{\mathbb{Q}}_\ell^\times$ be a character which is not trivial on $\pi^{l-1}\mathcal{O}_l$, and let $\mathfrak{g}$ be the $\mathcal{O}_l^{\mathrm{ur}}$-points of the Lie algebra of $\mathbb{G}$. Identify $\mathfrak{g}^F$ with $(G^l)^F$ via $x \mapsto 1 + \pi^l x$. For $\beta \in \mathrm{Hom}_{\mathcal{O}_l}(\mathfrak{g}^F, \mathcal{O}_l)$ we have a character ψ_β of $(G^l)^F$, defined by

$$\psi_\beta(1 + \pi^l x) = \psi(\beta(x)).$$

Any character of $(G^l)^F$ is of the form ψ_β for a unique β and $\mathrm{Stab}_{G^F}(\psi_\beta) = \rho_{r,l}^{-1}(C_{G_l^F}(\beta))$, where G^F acts via the co-adjoint action. In many situations, for example when $\mathbb{G} = \mathrm{GL}_n$ or p is a very good prime for $\mathbb{G}$, there exists a G^F-equivariant bijection $\mathfrak{g}^F \cong \mathrm{Hom}_{\mathcal{O}_r}(\mathfrak{g}^F, \mathcal{O}_r)$, and then β can be taken in the Lie algebra rather than in the dual. Let β be such that $\widetilde{\theta}|_{(G^l)^F} = \psi_\beta$. Then, by taking quotients modulo $(G^l)^F$, we see that the stabiliser equality in Definition 3.5 is equivalent to

$$C_{G_l^F}(\beta) = C_{N_{G_l}(T_l)^F}(\beta).$$

Note that, analogously, regularity of a semisimple element β in $\mathfrak{g}(\overline{\mathbb{F}}_q)$ (or in the reductive group G_1) is equivalent to the equality $C_{G_1}(\beta) = C_{N_{G_1}(T_1)}(\beta)$, for some maximal torus T_1.

It seems that in some cases, regularity and general position together imply genericity. It is an interesting problem to determine exactly when this is the case. Moreover, in some situations the equality $\mathrm{Stab}_{G^F}(\widetilde{\theta}|_{(G^l)^F}) = (TU^{\pm})^F$ is equivalent to regularity of θ, and implies the general position condition. In the following result, we verify this for the Coxeter torus in a general linear group.

Proposition 3.7 *For $\mathbb{G} = \mathrm{GL}_n$ over $\mathcal{O}_r$, let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus corresponding to the Coxeter element $w = (1, 2, \ldots, n)$. Then for $\theta \in \widehat{T^F}$, The following two conditions are equivalent:*

(i) $(TU^{\pm})^F = \mathrm{Stab}_{G^F}(\widetilde{\theta}|_{(G^l)^F})$,
(ii) θ *is regular.*

Furthermore, under these conditions, θ is in general position.

Proof We have $(G^l)^F \cong \mathrm{M}_n(\mathcal{O}_l)$, and as in the above remark, its irreducible characters are of the form $\psi_\beta(1 + \pi^l x) = \psi(\mathrm{Tr}(\beta x))$, where $\beta \in \mathrm{M}_n(\mathcal{O}_l)$. Let β be such that $\widetilde{\theta}|_{(G^l)^F} = \psi_\beta$. Then, by taking quotients modulo $(G^l)^F$, the condition $(TU^{\pm})^F = \mathrm{Stab}_{G^F}(\widetilde{\theta}|_{(G^l)^F})$ is equivalent to $C_{G_l^F}(\beta) = T_l^F$.

Since $\widetilde{\theta}$ is trivial on $(U^{\pm})^F$ and since $(G^l)^F \cong (T^l)^F \times (U^{\pm})^F$, we have $\beta \in T_l^F$, so

$$\beta_0 = \lambda^{-1}\beta\lambda = \mathrm{diag}(\beta_1, \ldots, \beta_n) \in \mathrm{M}_n(\mathcal{O}_l^{\mathrm{ur}})$$

(here the image of λ modulo π^l is again denoted by λ); here we can write $\beta_1 = \beta' \in (\mathcal{O}_l^{\mathrm{ur}})^{F^n}$ and $\beta_i = F^{i-1}(\beta')$ (for $i \in \{1, \ldots, n\}$) since w is the Coxeter element $(1, \ldots, n)$. As we are concerned with the general linear groups, we can assume $\lambda^{-1}F(\lambda) = \hat{w} \in N(T_0)$, a lift of w, is the standard monomial matrix. Denote by v the image of $\hat{w}$ in G_l, and still view it as the monomial matrix.

With the above notation, the condition $C_{G_l^F}(\beta) = T_l^F$ is equivalent to $C_{\lambda^{-1}G_l^F\lambda}(\beta_0) = \lambda^{-1}T_l^F\lambda$. However, as $\lambda^{-1}T_l^F\lambda$ is a group consisting of some diagonal matrices, this happens if and only if $\beta_i - \beta_j$ is invertible for all $i \neq j \in \{1, \ldots, n\}$: Indeed, note that $\beta_i - \beta_j = F^{i-1}(\beta' - F^{j-i}(\beta'))$ is invertible if and only if it is non-zero modulo π; now, if $\beta_i - \beta_j \mod \pi$ is zero for some i, j, then $\beta' \mod \pi \in \mathbb{F}_{q^{n'}}$ for some $n' < n$ satisfying $n' \mid n$, and so the non-diagonal matrix $I + v^{n'}\pi^{l-1}$ (if $l = 1$, replace $I + v^{n'}\pi^{l-1}$ by $v^{n'}$) stabilises β_0 (note that $v \in \lambda^{-1}G_l^F\lambda$), a contradiction; the other direction is immediate. In particular, in this situation θ is in general position.

For any $t \in T_l$, we have $F(t) = \lambda v F(\lambda^{-1}t\lambda)v^{-1}\lambda^{-1}$. Denote by F' the endomorphism $F'(g) = vF(g)v^{-1}$, then for any root α, and any positive integer m such that $F^m(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$, we have

$$N_F^{F^m}(t) = \lambda t_0 F'(t_0) \cdots F'^{m-1}(t_0)\lambda^{-1},$$

where $t \in (\mathcal{T}^\alpha)^{F^m}$ and $t_0 = \lambda^{-1}t\lambda$. Thus, since $\psi(\mathrm{Tr}(\beta N_F^{F^m}(t))) = \psi(\mathrm{Tr}(\beta_0 N_{F'}^{F'^m}(t_0)))$, the regularity of θ is equivalent to: For each given root α and integer m,

$$\psi(\mathrm{Tr}(\beta_0 N_{F'}^{F'^m}(t_0))) \neq 1$$

for some $t_0 \in (\lambda^{-1}\mathcal{T}^\alpha\lambda)^{F'^m}$.

Note that for any $g \in G$ such that $gTg^{-1} = T_0$, we have $gT^\alpha g^{-1} = T_0^{\alpha_0}$, for some root α_0 corresponding to the torus $\mathbf{T}_0$. Hence we can write

$$t_0 = \mathrm{diag}(0, \dots, 0, s, 0, \dots, 0, -s, 0, \dots, 0) \in \mathrm{M}_n(\mathcal{O}_l^{\mathrm{ur}})$$

for any $t_0 \in \lambda^{-1}\mathcal{T}^\alpha\lambda$, where $s \in \pi^{l-1}\mathcal{O}_l^{\mathrm{ur}} \cong k$ is at position (a, a) and $-s$ is at position (b, b). As v is a Coxeter element, we can take $m = n$, and thus

$$\mathrm{Tr}(\beta_0 N_{F'}^{F'^m}(t_0)) = \sum_{d=0}^{n-1}(\beta_{v^d(a)} - \beta_{v^d(b)})F^d(s) = \sum_{d=0}^{n-1} F^d(F^a(\beta' - F^{b-a}(\beta'))s)$$

(here v acts on $a, b \in \{1, \dots, n\}$ by permutation). Therefore the regularity of θ is equivalent to that, for any $b - a \in [1, \dots, n-1]$, the element $\beta' - F^{b-a}(\beta') \in \mathcal{O}_l^{\mathrm{ur}}$ is invertible, that is, $\beta_i - \beta_j$ is invertible for all $i \neq j \in \{1, \dots, n\}$, and we see from the above this is equivalent to the stabiliser condition. $\square$

4 The main result

As before, $\mathbb{G}$ is a reductive group scheme over $\mathcal{O}_r$, F is the corresponding Frobenius on G and $\mathbf{T}$ is a maximal torus in $\mathbf{G}$ such that T is F-rational. Moreover, U is the Greenberg functor image of the unipotent radical of a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{T}$. For any $v \in W(T)$, we fix a lift $\hat{v} \in N(T)$. Recall that (see Lemma 3.2) $F(U^\pm) = U^\pm$ and $\hat{v}U^\pm\hat{v}^{-1} = U^\pm$. Given two elements x and y in a group, we sometimes use the shorthand notation $x^y := y^{-1}xy$ and ${}^y x := yxy^{-1}$ for conjugations.

We now present our main result. We start with the computation of inner products of Deligne–Lusztig representations and the representations produced from the arithmetic radicals.

Theorem 4.1 *Suppose that $r = 2l$ is even and $\theta \in \widehat{T^F}$ is regular, then*

$$\langle \mathrm{Ind}_{(TU^\pm)^F}^{G^F} \widetilde{\theta}, R_T^\theta \rangle_{G^F} = \#\mathrm{Stab}_{W(T)^F}(\theta).$$

In particular, if θ is moreover in general position, then $\langle \mathrm{Ind}_{(TU^\pm)^F}^{G^F} \widetilde{\theta}, R_T^\theta \rangle_{G^F} = 1$.

Proof We want to compare the cohomology of $S_{T,U} = L^{-1}(FU)$ with the cohomology of the Lang pre-image $L^{-1}(FU^\pm)$ of the arithmetic radical (see Proposition 3.3). One has

$$\langle H_c^*(L^{-1}(FU^\pm))_\theta, R_T^\theta \rangle_{G^F} = \dim H_c^*(\Sigma)_{\theta^{-1},\theta},$$

where

$$\Sigma := \{(x, x', y) \in U^\pm \times FU \times G \mid xF(y) = yx'\}.$$

This follows from the $T^F \times T^F$-equivariant isomorphism

$$G^F\backslash L^{-1}(U^{\pm}) \times L^{-1}(FU) \cong \Sigma, \quad (g, g') \mapsto (g^{-1}F(g), g'^{-1}F(g'), g^{-1}g')$$

and the Künneth formula; here $T^F \times T^F$ acts on Σ by $(t, t')\colon (x, x', y) \mapsto (x^t, (x')^{t'}, t^{-1}yt')$.

In the following we will compute the cohomology following a general argument of Lusztig (for the orthogonality of Deligne–Lusztig representations) by first decomposing Σ into pieces according to the Bruhat decomposition, and then computing the cohomology of each piece.

The Bruhat decomposition $G_1 = \coprod_{v\in W(T)} B_1\hat{v}B_1$ of $G_1 = \mathbf{G}(k)$ gives the finite stratification (see, e.g. the proof of [17, Lemma 2.3]) $G = \coprod_{v\in W(T)} G_v$, where

$$G_v := (U \cap \hat{v}U^-\hat{v}^{-1})(\hat{v}(U^-)^1\hat{v}^{-1})\hat{v}TU,$$

and hence a finite partition into disjoint locally closed subvarieties

$$\Sigma = \coprod_{v\in W(T)} \Sigma_v,$$

where

$$\Sigma_v := \{(x, x', y) \in U^{\pm} \times FU \times G_v \mid xF(y) = yx'\}.$$

For each v, consider the variety

$$\mathcal{Z}_v := (U \cap \hat{v}U^-\hat{v}^{-1}) \times \hat{v}(U^-)^1\hat{v}^{-1};$$

this allows us to consider

$$\widehat{\Sigma}_v := \{(x, x', u', u^-, \tau, u) \in U^{\pm}\times FU\times\mathcal{Z}_v\times T\times U \mid xF(u'u^-\hat{v}\tau u) = u'u^-\hat{v}\tau ux'\}.$$

This is a locally trivial fibration $\widehat{\Sigma}_v \to \Sigma_v$ by an affine space ($\cong U \cap \hat{v}(U^-)^1\hat{v}^{-1}$), on which $T^F \times T^F$ acts as

$$(t, t')\colon (x, x', u', u^-, \tau, u) \longmapsto (t^{-1}xt, t'^{-1}x't', t^{-1}u't, t^{-1}u^-t, (t^{\hat{v}})^{-1}\tau t', t'^{-1}ut').$$

By the change of variable $x'F(u)^{-1} \mapsto x'$ we can rewrite $\widehat{\Sigma}_v$ as

$$\widehat{\Sigma}_v = \{(x, x', u', u^-, \tau, u) \in U^{\pm}\times FU\times\mathcal{Z}_v\times T\times U \mid xF(u'u^-\hat{v}\tau) = u'u^-\hat{v}\tau ux'\},$$

on which the $T^F \times T^F$-action does not change (therefore $H_c^*(\widehat{\Sigma}_v)$ and $H_c^*(\Sigma_v)$ afford the same virtual $T^F \times T^F$-representations).

For $i = 0, 1, \ldots, r-1$ let $\mathcal{Z}_v(i)$ be the pre-image of $(\hat{v}U^-\hat{v}^{-1})^i = \hat{v}(U^-)^i\hat{v}^{-1}$ under the product morphism

$$\mathcal{Z}_v = (U \cap \hat{v}U^-\hat{v}^{-1}) \times \hat{v}(U^-)^1\hat{v}^{-1} \longrightarrow \hat{v}U^-\hat{v}^{-1}.$$

Recall that for $i = 0$ we always let $G^0 = G$ for an algebraic group G. For each v consider the partition $\widehat{\Sigma}_v = \Sigma'_v \sqcup \Sigma''_v$ of locally closed subvarieties, where

$$\Sigma'_v := \{(x, x', u', u^-, \tau, u) \in \widehat{\Sigma}_v \mid (u', u^-) \in \mathcal{Z}_v \setminus \mathcal{Z}_v(l)\}$$

and

$$\Sigma''_v := \{(x, x', u', u^-, \tau, u) \in \widehat{\Sigma}_v \mid (u', u^-) \in \mathcal{Z}_v(l)\}.$$

In order to compute the inner product, an Euler characteristic, our goal is to compute $\dim H^*_c(\Sigma''_v)_{\theta^{-1},\theta}$ and $\dim H^*_c(\Sigma'_v)_{\theta^{-1},\theta}$ explicitly, for all v.

For the first one, we have the following lemma:

Lemma 4.2 *One has* $\dim H^*_c(\Sigma''_v)_{\theta^{-1},\theta} = \begin{cases} 1, & \textit{if } v \in \mathrm{Stab}_{W(T)^F}(\theta) \\ 0, & \textit{otherwise}. \end{cases}$

As one can see from its proof, this lemma is true for any θ, regular or not.

For the second one, we have the following lemma:

Lemma 4.3 *One has* $\dim H^*_c(\Sigma'_v)_{\theta^{-1},\theta} = 0$*, for all* v.

It is in the proof of this second lemma that the regularity of θ is required.

By the above two lemmas, $\dim H^*_c(\Sigma)_{\theta^{-1},\theta} = \sum_{v\in \mathrm{Stab}_{W(T)^F}(\theta)} 1 = \#\mathrm{Stab}_{W(T)^F}(\theta)$. Thus we get the desired result.

It remains to prove Lemmas 4.2 and 4.3.

Proof of Lemma 4.2 Note that for any $(x, x', u', u^-, \tau, u) \in \Sigma''_v$ we have

$$u'u^- \in \hat{v}(U^-)^l\hat{v}^{-1} \subseteq U^{\pm} = FU^{\pm},$$

so we can apply the changes of variables $(u'u^-)^{-1}x \mapsto x$, and then $xF(u'u^-) \mapsto x$. This allows us to rewrite Σ''_v as

$$\widetilde{\Sigma}''_v := \{(x, x', u', u^-, \tau, u) \in U^{\pm} \times FU \times \mathcal{Z}_v(l) \times T \times U \mid xF(\hat{v}\tau) = \hat{v}\tau u x'\},$$

on which $T^F \times T^F$ acts in the same way as before.

Consider the algebraic group

$$H = \{(t, t') \in T_1 \times T_1 \mid tF(t^{-1}) = F(\hat{v})t'F(t')^{-1}F(\hat{v}^{-1})\}.$$

Note that the action of $T_1^F \times T_1^F$ on $\widetilde{\Sigma}''_v$ extends to an action of H (the torus T_1 is always a subgroup of T) in a natural way. The identity component H° is a torus acting on $\widetilde{\Sigma}''_v$, and thus by basic properties of ℓ-adic cohomology (see e.g. [5, 10.15]) we have

$$\dim H^*_c(\widetilde{\Sigma}''_v)_{\theta^{-1},\theta} = \dim H^*_c((\widetilde{\Sigma}''_v)^{H^\circ})_{\theta^{-1},\theta}.$$

The Lang–Steinberg theorem implies that both the first and the second projections of H° to T_1 are surjective. Therefore $(x, x', u', u^-, \tau, u) \in (\widetilde{\Sigma}''_v)^{H^\circ}$ only if $x = x' = u' = u^- = u = 1$. Thus $(\widetilde{\Sigma}''_v)^{H^\circ} = \{(1, 1, 1, 1, \tau, 1) \mid F(\hat{v}\tau) = \hat{v}\tau\}^{H^\circ}$. The set $(\hat{v}T)^F$ is empty unless $\hat{v}^{-1}F(\hat{v}) \in T$ (i.e. unless $v \in W(T)^F$), in which case $\{(1, 1, 1, 1, \tau, 1) \mid F(\hat{v}\tau) = \hat{v}\tau\}$ is actually stable under the action of H, so it is also stable under the action of H°. We only need to treat the non-empty case. As a finite set $(\hat{v}T)^F$ admits only the trivial action of the connected non-trivial group H°, thus

$$(\widetilde{\Sigma}''_v)^{H^\circ} = \{(1, 1, 1, 1, \tau, 1) \mid F(\hat{v}\tau) = \hat{v}\tau\}^{H^\circ} \cong (\hat{v}T)^F.$$

Therefore $H_c^*(\widetilde{\Sigma}''_v) = \overline{\mathbb{Q}}_\ell[(\hat{v}T)^F]$, on which $T^F \times T^F$ acts via $(t, t')\colon \hat{v}\tau \mapsto \hat{v}(t^{\hat{v}})^{-1}\tau t'$; note that this is the regular representation of both the left T^F and the right T^F in $T^F \times T^F$. In particular, the irreducible constituents of $H_c^*(\widetilde{\Sigma}''_v)$ are of the form $H_c^*(\widetilde{\Sigma}''_v)_{(\phi^{\hat{v}})^{-1},\phi}$, where ϕ runs over $\widehat{T^F}$. Hence $H_c^*(\widetilde{\Sigma}''_v)_{\theta^{-1},\theta}$ is non-zero if and only if $\theta^{\hat{v}} = \theta$, in other words, if and only if $v \in \mathrm{Stab}_{W(T)^F}(\theta)$. Now, for $v \in \mathrm{Stab}_{W(T)^F}(\theta)$, we have $\dim H_c^*(\widetilde{\Sigma}''_v)_{(\theta^{\hat{v}})^{-1},\theta} = \dim H_c^*(\widetilde{\Sigma}''_1)_{\theta^{-1},\theta} = 1$ for any $\theta \in \widehat{T^F}$, since $|\widehat{T^F}| = |T^F| = |(\hat{v}T)^F|$. This proves the lemma. □

The proof of Lemma 4.3 is more difficult than that of Lemma 4.2, and we need two extra inputs; the first input is a general homotopy result from [4]:

Lemma 4.4 *Let H be a connected algebraic group over k, and Y a separated scheme of finite type over k. Suppose there is a morphism $f\colon H \times Y \to Y$ such that $f(1, -)$ is the identity map and $(h, y) \mapsto (h, f(h, y))$ is an automorphism on $H \times Y$. Then for any $h \in H$, the induced endomorphism of $f(h, -)$ on $H_c^i(Y, \overline{\mathbb{Q}}_\ell)$ is the identity map.*

Proof The same argument as in [4, p. 136] works here.

The second input is a variant of [15, Lemma 1.7]. For general linear groups this can be done in an ad hoc way explicitly (see [1]); for general reductive groups we will prove the following lemma. We first fix several pieces of notation:

Definition 4.5 Let Φ^+ and Φ^- be a choice of positive and negative roots of $\mathbf{T}$, respectively. For $\beta \in \Phi^-$, let $\mathrm{ht}(\beta)$ be the largest integer n such that $\beta = \beta_1 + \cdots + \beta_n$, for $\beta_i \in \Phi^-$ (note that this is the negative of the height function defined with respect to the positive roots Φ^+).

(1) Suppose Φ^- is equipped with a total order refining the natural order given by $\mathrm{ht}(-)$. For $z \in U^-$ and $\beta \in \Phi^-$, define $x_\beta^z \in U_\beta = \mathcal{F}\mathbf{U}_\beta$ by the decomposition $z = \prod_{\beta \in \Phi^-} x_\beta^z$, where the product is with respect to the following order: If $\mathrm{ht}(\beta) < \mathrm{ht}(\beta')$, then x_β^z is to the left of $x_{\beta'}^z$; and if $\mathrm{ht}(\beta) = \mathrm{ht}(\beta')$ and $\beta < \beta'$, then x_β^z is to the left of $x_{\beta'}^z$.
(2) For a fixed $\alpha \in \Phi^+$ and $i \in \{0, \ldots, l-1\}$, denote by $Z^\alpha(i) \subseteq U^-$ the subvariety consisting of all z such that:
   i. $z \in (U^-)^i \setminus (U^-)^{i+1}$;
   ii. $x_{-\alpha}^z \neq 1$;
   iii. $x_\beta^z = 1$ for $\forall \beta \in \Phi^-$ such that $\beta < -\alpha$.

Recall that $\mathcal{T}^\alpha := (\mathcal{F}\mathbf{T}^\alpha)^{r-1}$ is a 1-dimensional affine space.

Lemma 4.6 *Suppose $\alpha \in \Phi^+$ and $i \in \{0, \dots, l-1\}$. Then for $z \in Z^\alpha(i)$ and $\xi \in U_\alpha^{r-i-1}$, one has*

$$[\xi, z] := \xi z \xi^{-1} z^{-1} = \tau_{\xi,z}\omega_{\xi,z},$$

where $\tau_{\xi,z} \in \mathcal{T}^\alpha$ and $\omega_{\xi,z} \in (U^-)^{r-1}$ are uniquely determined. Moreover,

$$U_\alpha^{r-i-1} \longrightarrow \mathcal{T}^\alpha, \qquad \xi \longmapsto \tau_{\xi,z}$$

is a surjective morphism admitting a section Ψ_z^α such that $\Psi_z^\alpha(1) = 1$ and such that the map

$$Z^\alpha(i) \times \mathcal{T}^\alpha \longrightarrow U_\alpha^{r-i-1}, \qquad (z, \tau) \longmapsto \Psi_z^\alpha(\tau)$$

is a morphism.

Proof Write $z = x_{-\alpha}^z z'$, then

$$[\xi, z] = \xi x_{-\alpha}^z z' \xi^{-1} z'^{-1} (x_{-\alpha}^z)^{-1} = [\xi, x_{-\alpha}^z] \cdot {}^{x_{-\alpha}^z}[\xi, z']. \tag{2}$$

We need to determine $[\xi, x_{-\alpha}^z]$ and ${}^{x_{-\alpha}^z}[\xi, z']$ separately.

Following the notation in [3, XX] we write $p_\beta\colon (\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}} \cong \mathbf{U}_\beta$ for every $\beta \in \Phi$ (and we use the same notation for the isomorphism $\mathcal{F}(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}} \cong U_\beta$ induced by p_β via the Greenberg functor). Then there exists $a \in \mathbb{G}_m(\mathcal{O}_r^{\mathrm{ur}})$ such that, for all $x, y \in \mathbb{G}_a(\mathcal{O}_r^{\mathrm{ur}})$, we have

$$p_{-\alpha}(y)p_\alpha(x) = p_\alpha(\frac{x}{1+axy})\check{\alpha}((1+axy)^{-1})p_{-\alpha}(\frac{y}{1+axy}); \tag{3}$$

see [3, XX 2.2]. Let x, y be such that $p_\alpha(x) = \xi$ and $p_{-\alpha}(y) = x_{-\alpha}^z$ (note that in our case $x^2 = 0$, so that $(1+axy)^{-1} = 1 - axy$). By applying (3) to $p_{-\alpha}(y)p_\alpha(-x)$, we see that

$$\begin{aligned}
[\xi, x_{-\alpha}^z] &= p_\alpha(x)p_{-\alpha}(y)p_\alpha(-x)p_{-\alpha}(-y) \\
&= p_\alpha(x)p_\alpha(\frac{-x}{1-axy})\check{\alpha}(1+axy)p_{-\alpha}(\frac{y}{1-axy})p_{-\alpha}(-y) \\
&= \check{\alpha}(1+axy)p_{-\alpha}(axy^2).
\end{aligned} \tag{4}$$

Note that since $\xi \in G^{r-i-1}$ and $x_{-\alpha}^z \in G^i$ (in other words, $\pi^{r-i-1} \mid x$ and $\pi^i \mid y$), we have $p_{-\alpha}(axy^2) \in U_{-\alpha}^{r-1}$. We will see below that $\check{\alpha}(1+axy)$ is the required $\tau_{\xi,z}$.

Now turn to $[\xi, z']$; we want to show that $[\xi, z'] \in (U^-)^{r-1}$. First, the relation $[G^i, G^j] \subseteq G^{i+j}$ implies that $[\xi, z'] \in G^{r-1}$, so we only need to show that $[\xi, z'] = ({}^\xi z')z'^{-1} \in U^-$, or equivalently, that ${}^\xi z' \in U^-$. Write

$$z' = \prod_{\beta \in \Phi^-} x_\beta^{z'},$$

according to Definition 4.5, and let $y_\beta \in \mathbb{G}_a(\mathcal{O}_r^{\mathrm{ur}})$ be such that $p_\beta(y_\beta) = x_\beta^{z'}$. By the Chevalley commutator formula (see [2, 3.3.4.1]), we have

$$[\xi, x_\beta^{z'}] = \prod_{\substack{j,j\geq 1\\ j\beta+j'\alpha\in\Phi}} p_{j\beta+j'\alpha}(a_{j,j'}y_\beta^j x^{j'}) \in \prod_{\substack{j,j\geq 1\\ j\beta+j'\alpha\in\Phi}} U_{j\beta+j'\alpha}^{r-1},$$

for some $a_{j,j'} \in \mathbb{G}_a(\mathcal{O}_r^{\mathrm{ur}})$. Since $z \in Z^\alpha(i)$, we have $\mathrm{ht}(\beta) \geq \mathrm{ht}(-\alpha)$. In the above formula, if $y_\beta^j x^{j'} \neq 0$ and $j\beta + j'\alpha \in \Phi^+$ with $j, j' \geq 1$, then $j' = 1$ (as $x^2 = 0$) and $j\,\mathrm{ht}(\beta) < \mathrm{ht}(-\alpha)$, so $j < 1$; contradiction. Hence, if $y_\beta^j x^{j'} \neq 0$, then $j\beta + j'\alpha \in \Phi^-$. Thus $[\xi, x_\beta^{z'}] \in (U^-)^{r-1}$, and hence ${}^\xi(x_\beta^{z'}) \in U^-$, for every β. Therefore ${}^\xi z' = \prod_\beta {}^\xi(x_\beta^{z'}) \in U^-$, as required.

By (2) and (4) we have

$$[\xi, z] = [\xi, x_{-\alpha}^z] \cdot {}^{x_{-\alpha}^z}[\xi, z'] = \check{\alpha}(1 + axy) \cdot p_{-\alpha}(axy^2) \cdot {}^{x_{-\alpha}^z}[\xi, z'].$$

From this expression, put

$$\tau_{\xi,z} = \check{\alpha}(1 + axy)$$

and

$$\omega_{\xi,z} = p_{-\alpha}(axy^2) \cdot {}^{x_{-\alpha}^z}[\xi, z'].$$

Note that $\tau_{\xi,z} \in \mathcal{T}^\alpha$ and $\omega_{\xi,z} \in (U^-)^{r-1}$ (since $[\xi, z'] \in (U^-)^{r-1}$). The elements $\tau_{\xi,z}$ and $\omega_{\xi,z}$ are uniquely determined because of the Iwahori decomposition.

Now, as $\tau_{\xi,z}$ is defined to be $\check{\alpha}(1 + ap_\alpha^{-1}(\xi)p_{-\alpha}^{-1}(x_{-\alpha}^z))$, the map $\xi \mapsto \tau_{\xi,z}$, whose target is a connected 1-dimensional algebraic group, is a surjective algebraic group morphism (note that $z \mapsto x_{-\alpha}^z$ is a projection, hence a morphism). The section morphism Ψ_z^α can be defined in the following way: The isomorphism of additive groups

$$(\pi^i) \cong \mathcal{O}_{r-i}^{\mathrm{ur}}, \quad \pi^i a + (\pi^r) \longmapsto a + (\pi^{r-i})$$

induces an isomorphism of affine spaces (by the Greenberg functor)

$$\mu_i\colon (\mathcal{F}(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}})^i \longrightarrow (\mathcal{F}(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}})_{r-i}.$$

Note that this isomorphism depends on the choice of π. Meanwhile, let

$$\mu^i\colon (\mathcal{F}(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}})_{r-i} \cong \mathcal{F}(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}}/(\mathcal{F}(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}})^{r-i} \longrightarrow \mathcal{F}(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}}$$

be a section morphism to the quotient morphism such that $\mu^i(0) = 0$ (μ^i exists because $\mathcal{F}(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}}$ is an affine space). For $\tau \in \mathcal{T}^\alpha$ we put

$$\Psi_z^\alpha(\tau) := p_\alpha\left(a^{-1} \cdot \mu^i\left(\mu_i\left(\check{\alpha}^{-1}(\tau) - 1\right) \cdot \mu_i\left(p_{-\alpha}^{-1}(x_{-\alpha}^z)\right)^{-1}\right)\right).$$

Here $\check{\alpha}^{-1}$ is defined on $\mathcal{T}^\alpha = (\mathcal{F}\mathbf{T}^\alpha)^{r-1} \cong (\mathcal{F}(\mathbb{G}_m)_{\mathcal{O}_r^{\mathrm{ur}}})^{r-1}$ as the inverse to $\check{\alpha}$, and we view $\check{\alpha}^{-1}(\tau)$ as an element in $\mathcal{F}(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}}$ by the natural open immersion $(\mathbb{G}_m)_{\mathcal{O}_r^{\mathrm{ur}}} \to (\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}}$, so the minus operation $\check{\alpha}^{-1}(\tau) - 1$ is well-defined. On the other hand, by our assumption on z (see Definition 4.5 (2) i), $\mu_i\left(p_{-\alpha}^{-1}(x_{-\alpha}^z)\right)$ is an element in $\mathcal{F}(\mathbb{G}_m)_{\mathcal{O}_{r-i}^{\mathrm{ur}}}$, so its multiplicative inverse exists. Moreover, the product operation "$\cdot$" is by viewing $(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}}$ (resp. $\mathcal{F}(\mathbb{G}_a)_{\mathcal{O}_r^{\mathrm{ur}}}$) as a ring scheme (resp. k-ring variety). Thus Ψ_z^α is well-defined as a morphism.

Finally, by the definition of μ_i and μ^i, for $\tau \in \mathcal{T}^\alpha(k)$ we have

$$\begin{aligned}
\tau_{\Psi_z^\alpha(\tau),z} &= \check{\alpha}\left(1 + a p_{-\alpha}^{-1}(\Psi_z^\alpha(\tau)) p_{-\alpha}^{-1}(x_{-\alpha}^z)\right) \\
&= \check{\alpha}\left(1 + \mu^i\left(\mu_i\left(\check{\alpha}^{-1}(\tau) - 1\right) \cdot \mu_i\left(p_{-\alpha}^{-1}(x_{-\alpha}^z)\right)^{-1}\right) \cdot p_{-\alpha}^{-1}(x_{-\alpha}^z)\right) \\
&= \check{\alpha}\left(1 + \pi^i \cdot \mu^i \mu_i(\check{\alpha}^{-1}(\tau) - 1)\right) = \tau
\end{aligned}$$

(for the last equality, note that $\check{\alpha}^{-1}(\tau)$ is of the form $1 + s\pi^{r-1}$ for some $s \in \mathcal{O}_r^{\mathrm{ur}}$, as an element in $\mathbb{G}_m(\mathcal{O}_r^{\mathrm{ur}})$), thus $\tau \mapsto \Psi_z^\alpha(\tau) \mapsto \tau_{\Psi_z^\alpha(\tau),z}$ is the identity map on the k-points $\mathcal{T}^\alpha(k)$ of the 1-dimensional affine space $\mathcal{T}^\alpha \cong \mathbb{A}_k^1$, hence it is the identity morphism. So Ψ_z^α is a section to $\xi \mapsto \tau_{\xi,z}$, and the other assertions in the lemma follow from its definition. $\square$

Now we proceed to prove Lemma 4.3 itself.

Proof of Lemma 4.3 By the changes of variables $\hat{v}\tau\hat{v}^{-1} \mapsto \tau$, $\tau^{-1}u^-\tau \mapsto u^-$, and $\tau^{-1}u'\tau \mapsto u'$, we can rewrite Σ_v' as

$$\begin{aligned}
\widetilde{\Sigma}_v' := \{&(x, x', u', u^-, \tau, u) \in U^\pm \times FU \times \mathcal{Z}_v \setminus \mathcal{Z}_v(l) \times T \\
&\times U \mid xF(\tau u' u^- \hat{v}) = \tau u' u^- \hat{v} u x'\},
\end{aligned}$$

on which $(t, t') \in T^F \times T^F$ acts by sending $(x, x', u', u^-, \tau, u)$ to

$$(t^{-1}xt, t'^{-1}x't', (t'^{\hat{v}})^{-1}u'(t')^{\hat{v}}, (t'^{\hat{v}})^{-1}u^-(t')^{\hat{v}}, t^{-1}\tau(t')^{\hat{v}}, t'^{-1}ut').$$

To show $\dim H_c^*(\widetilde{\Sigma}_v')_{\theta^{-1},\theta} = 0$, it suffices to show

$$\dim H_c^*(\widetilde{\Sigma}_v')_{\theta^{-1}|_{(T^{r-1})^F}} = 0,$$

for the subgroup $(T^{r-1})^F = (T^{r-1})^F \times 1 \subseteq T^F \times T^F$. Note that the $(T^{r-1})^F$-action on $\widetilde{\Sigma}_v'$ is given by

$$t\colon (x, x', u', u^-, \tau, u) \mapsto (x, x', u', u^-, t^{-1}\tau, u).$$

Recall that we fixed an order on Φ^-. For $\beta \in \Phi^-$, let $F(\beta) \in \Phi$ be the root defined by $F(U)_{F(\beta)} = F(U_\beta)$, then the order on Φ^- produces an order on $F(\Phi^-)$; similarly we can define F on Φ^+, and hence get a bijection on $\Phi = \Phi^- \sqcup \Phi^+ = F(\Phi^-) \sqcup F(\Phi^+)$, and then a bijection on $\{U_\beta\}_{\beta\in\Phi}$; it is clear that $F(-\alpha) = -F(\alpha)$ for any $\alpha \in \Phi$. Following the notation in Definition 4.5, let $\mathcal{Z}_v^\beta(i)$ be the subvariety of $\mathcal{Z}_v(i) \setminus \mathcal{Z}_v(i+1)$ consisting of (u', u^-) such that, in the decomposition $F(z) := F(\hat{v}^{-1}u'u^-\hat{v}) = \prod_{\beta'\in F(\Phi^-)} x_{\beta'}^{F(z)}$ the following conditions hold: ($\mathrm{ht}(-)$ is defined on $F(\Phi^-)$)

(1) $x_{\beta'}^{F(z)} = 1$ whenever $\mathrm{ht}(\beta') < \mathrm{ht}(F(\beta))$,
(2) $x_{\beta'}^{F(z)} = 1$ whenever $\mathrm{ht}(\beta') = \mathrm{ht}(F(\beta))$ and $\beta' < F(\beta)$,
(3) $x_{F(\beta)}^{F(z)} \neq 1$.

(This means that

$$\mathcal{Z}_v^\beta(i) = \{(u', u^-) \in \mathcal{Z}_v(i) \setminus \mathcal{Z}_v(i+1) \mid F((u'u^-)^{\hat{v}}) \in Z^{-F(\beta)}(i)\}$$

according to the notation in Definition 4.5 (2), after formally replacing α by $-F(\beta)$ and Φ^- by $F(\Phi^-)$). We then obtain a finite partition

$$\mathcal{Z}_v \setminus \mathcal{Z}_v(l) = \coprod_{i=0}^{l-1} \coprod_{\beta\in\Phi^-} \mathcal{Z}_v^\beta(i).$$

And hence a partition of $\widetilde{\Sigma}'_v$ into locally closed subvarieties

$$\widetilde{\Sigma}'_v = \coprod_{i=0}^{l-1} \coprod_{\beta\in\Phi^-} \Sigma_v^\beta(i),$$

where

$$\begin{aligned}\Sigma_v^\beta(i) := \{(x, x', u', u^-, \tau, u) \in U^\pm \times FU \times \mathcal{Z}_v^\beta(i) \times T \\ \times U \mid xF(\tau u'u^-\hat{v}) = \tau u'u^-\hat{v}ux'\}.\end{aligned}$$

Each subvariety $\Sigma_v^\beta(i)$ inherits the $(T^{r-1})^F$-action:

$$t\colon (x, x', u', u^-, \tau, u) \mapsto (x, x', u', u^-, t^{-1}\tau, u),$$

so it suffices to show:

$$H_c^*(\Sigma_v^\beta(i))_{\theta^{-1}|_{(T^{r-1})^F}} = 0$$

for every $i \in \{0, \ldots, l-1\}$ and every $\beta \in \Phi^-$.

From now on we fix an $\alpha \in \Phi^+$. Consider the closed subgroup

$$H := \{t \in T^{r-1} \mid F(\hat{v})^{-1}F(t)t^{-1}F(\hat{v}) \in \mathcal{T}^{F(\alpha)}\}$$

of T^{r-1}. For any $t \in H$, define $g_t\colon FU \to FU$ by

$$g_t\colon x' \mapsto x' \cdot \Psi^{F(\alpha)}_{F(z)}\left(F(\hat{v})^{-1}F(t^{-1})tF(\hat{v})\right)^{-1}$$

with the parameter $z := \hat{v}^{-1}u'u^-\hat{v}$, where $(u', u^-) \in \mathcal{Z}_v^{-\alpha}(i)$. This is well-defined because $F(z)$ satisfies the conditions in Lemma 4.6, with respect to $F(U^-)$ and $F(\Phi^-)$. Note that if $F(t) = t$, then $g_t(x') = x'$.

Moreover, for any $t \in H$, define the morphism $f_t\colon U^{\pm} \to U^{\pm}$ by

$$f_t\colon x \mapsto x \cdot {}^{F(\tau)}\left(t^{-1} \cdot {}^{F(\hat{v}z)}\left(x'^{-1}g_t(x')\right)F(t)\right),$$

with the parameters $x' \in FU$, $\tau \in T$, and $z = \hat{v}^{-1}u'u^-\hat{v}$ (where $(u', u^-) \in \mathcal{Z}_v^{-\alpha}(i)$, as for g_t). To see this is well-defined one needs to check the right hand side is in $U^{\pm}$: By the definition of $\Psi^{F(\alpha)}_{F(z)}$ and the first assertion of Lemma 4.6 we see

$$F(z)x'^{-1}g_t(x')F(z^{-1}) = \Psi^{F(\alpha)}_{F(z)}\left(F(\hat{v})^{-1}F(t^{-1})tF(\hat{v})\right)^{-1} \cdot F(\hat{v})^{-1}F(t^{-1})tF(\hat{v}) \cdot \omega$$

for some $\omega \in U^{r-1}$. Hence by definition of f_t we get

$$(x^{-1}f_t(x))^{F(\tau)} = ({}^{F(\hat{v})}\Psi)^t \cdot ({}^{F(\hat{v})}\omega)^{F(t)} \in \prod_{\beta\in\Phi} U_\beta^{r-i-1} \subseteq U^{\pm},$$

where $\Psi := \Psi^{F(\alpha)}_{F(z)}(F(\hat{v})^{-1}F(t^{-1})tF(\hat{v}))^{-1}$. Thus $x^{-1}f_t(x) \in U^{\pm}$, and f_t is therefore well-defined. Moreover, if $F(t) = t$, then $f_t(x) = x$.

For any $t \in H$, the above preparations on f_t and g_t allow us to define the following automorphism of $\Sigma_v^{-\alpha}(i)$:

$$h_t\colon (x, x', u', u^-, \tau, u) \mapsto (f_t(x), g_t(x'), u', u^-, t^{-1}\tau, u),$$

where the involved parameter z is $\hat{v}^{-1}u'u^-\hat{v}$. To see this is well-defined, one needs to show the right hand side satisfies the defining equation of $\Sigma_v^{-\alpha}(i)$, in other words, satisfies

$$f_t(x)F(t^{-1}\tau u'u^-\hat{v}) = t^{-1}\tau u'u^-\hat{v}ug_t(x');$$

this can be seen by just expanding the definition of f_t: (note that $t \in T^{r-1}$ commutes with $x \in U^{\pm}$, and $xF(\tau u'u^-\hat{v}) = \tau u'u^-\hat{v}ux'$)

$$\begin{aligned} f_t(x)F(t^{-1}\tau u'u^-\hat{v}) &= x \cdot {}^{F(\tau)}\left(t^{-1} \cdot {}^{F(\hat{v}z)}\left(x'^{-1}g_t(x')\right)F(t)\right) \cdot F(t^{-1}\tau u'u^-\hat{v}) \\ &= t^{-1}xF(\tau u'u^-\hat{v})x'^{-1}g_t(x') \\ &= t^{-1}\tau u'u^-\hat{v}ug_t(x'). \end{aligned}$$

Moreover, it is clear that in the case $F(t) = t$, the automorphism h_t coincides with the $(T^{r-1})^F$-action, so by Lemma 4.4, the induced endomorphism of h_t on $H_c^*(\Sigma_v^{-\alpha}(i))$ is the identity map for any t in the identity component H° of H.

Let $a \geq 1$ be an integer such that $F^a(F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1}) = F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1}$, then the image of the norm map $N_F^{F^a}(t) = t \cdot F(t) \cdots F^{a-1}(t)$ on $F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1}$ is a connected subgroup of H, hence contained in H°. Moreover $N_F^{F^a}((F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1})^{F^a}) \subseteq (T^{r-1})^F \cap H^\circ$. Thus, as θ is regular,

$$H_c^*(\Sigma_v^\beta(i))_{\theta^{-1}}\Big|_{N_F^{F^a}\left(\left(F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1}\right)^{F^a}\right)} = 0.$$

Therefore $H_c^*(\Sigma_v^\beta(i))_{\theta^{-1}}|_{(T^{r-1})^F} = 0$.

This completes the whole proof of the theorem. ◻

Theorem 4.1 leads to an affirmative answer to Lusztig's question mentioned in the introduction, for r even:

Corollary 4.7 *Let $r = 2l$ and suppose $\theta \in \widehat{T^F}$ is regular and in general position. Denote by $\widetilde{\theta}$ the trivial lift of θ to $(TU^\pm)^F = (TG^l)^F$, then*

$$R_T^\theta \cong \mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta}$$

if at least one of the following two conditions is satisfied:

(i) θ is generic;
(ii) G^F and T^F satisfy Gérardin's conditions (see Remark 3.4).

In particular, in these situations R_T^θ has dimension $|G_l^F|/|T_l^F|$.

Proof If (i) is satisfied, then

$$\mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta} = \mathrm{Ind}_{(TU^\pm)^F\cdot\mathrm{Stab}_{N(T)^F}(\widetilde{\theta}|_{(G^l)^F})}^{G^F}\mathrm{Ind}_{(TU^\pm)^F}^{(TU^\pm)^F\cdot\mathrm{Stab}_{N(T)^F}(\widetilde{\theta}|_{(G^l)^F})}\widetilde{\theta}$$

is irreducible by Clifford theory (note that $\mathrm{Ind}_{(TU^\pm)^F}^{(TU^\pm)^F\cdot\mathrm{Stab}_{N(T)^F}(\widetilde{\theta}|_{(G^l)^F})}\widetilde{\theta}$ is irreducible by the Mackey intertwining formula and the assumption that θ is in general position). If (ii) is satisfied, then $\mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta}$ is irreducible according to Remark 3.4. Now the result follows from Theorem 4.1.

In the case of $\mathbb{G} = \mathrm{GL}_n$, the above result implies that, if θ satisfies $\mathrm{Stab}_{G^F}(\widetilde{\theta}|_{(G^l)^F}) = (TU^\pm)^F$, then the Deligne–Lusztig representations are regular semisimple in the sense of Hill [10] (note that, following the notation in the proof of Proposition 3.7, β is semisimple, and the genericity of θ implies β is regular; see [10, 3.6]).

5 An application to finite Lie algebras

In this last section we assume $\mathcal{O} = \mathbb{F}_q[[\pi]]$ and $r = 2$. Note that the kernel group G^1 is isomorphic to the additive group of the Lie algebra $\mathfrak{g}$ of G_1, and the adjoint action of G_1^F on $\mathfrak{g}^F$ is the conjugation action under this isomorphism. Since $T^F \cong T_1^F \times (T^1)^F$, any character θ^1 of $\mathfrak{t}^F \cong (T^1)^F$ extends (trivially) to a character θ of T^F. Thus, viewing $R_{T,U}^{\theta}$ as a $\mathfrak{g}^F \cong (G^1)^F$-module by restriction, we can view $R_{\mathfrak{t},\mathfrak{u}}^{\theta^1} := R_{T,U}^{\theta}$ as a Deligne–Lusztig theory for the finite Lie algebra $\mathfrak{g}^F$ (here $\mathfrak{u}$ is the Lie algebra of U_1).

An *invariant character* of $\mathfrak{g}^F$ is a $\overline{\mathbb{Q}}_\ell$-character of the finite abelian group $\mathfrak{g}^F$ that is invariant under the adjoint action of G_1^F, and it is said to be irreducible if it is not the sum of two non-zero invariant characters (these functions have interesting relations with character sheaves; see e.g. [14] and [11]). Letellier studied this construction in [12], where he compared it with a different construction he considered earlier in [11], and made a conjecture that every irreducible invariant character Ψ of $\mathfrak{g}^F$ "appear" in some $R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}$ in the sense that

$$(\Psi, R_{\mathfrak{t},\mathfrak{u}}^{\theta^1})_{\mathfrak{g}^F} := \frac{1}{|G_1^F|} \sum_{g\in\mathfrak{g}^F} \Psi(g) R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}(-g) \neq 0$$

(note that the bracket $(,)$ is different from the usual inner product $\langle , \rangle$ because of the denominator G_1^F). Letellier showed that this conjecture is true for GL_2 with the assumption that $|\mathbb{F}_q| > 3$. Here, as a simple application of our main result, we prove it for GL_2 and GL_3, without assumptions on the residue field.

Proposition 5.1 *Along with the above notation, if $\mathbb{G} = \mathrm{GL}_2$ or GL_3, then for any irreducible invariant character Ψ of $\mathfrak{g}^F$, we have*

$$(\Psi, R_{\mathfrak{t},\mathfrak{u}}^{\theta^1})_{\mathfrak{g}^F} \neq 0,$$

for some $R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}$.

Proof Firstly note that $(\Psi, R_{\mathfrak{t},\mathfrak{u}}^{\theta^1})_{\mathfrak{g}^F} \neq 0$ if and only if $\langle \Psi, R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}\rangle_{(G^1)^F} \neq 0$. Also note that a $\mathfrak{g}^F$-representation is invariant if and only if it is G^F-invariant as a $(G^1)^F$-representation, so we can focus on characters of the group $(G^1)^F$. Suppose χ is an irreducible character of $(G^1)^F$, then

$$\chi^O := \sum_{s\in G^F/\mathrm{Stab}_{G^F}(\chi)} \chi^s$$

is an invariant character of $(G^1)^F$, and any invariant character containing χ contains χ^O (so χ^O is the unique irreducible invariant character containing χ). On the other hand, any G^F-module is an invariant $(G^1)^F$-module, thus we only need to show that any irreducible character χ of $(G^1)^F$ is "contained" in some $R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}$ in the sense that $\langle \chi, R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}\rangle_{(G^1)^F} \neq 0$.

For $\mathbb{G} = \mathrm{GL}_2$ (resp. GL_3), the irreducible characters of $\mathfrak{g}^F$ are of the form $\chi = \psi_\beta(-) = \psi(\mathrm{Tr}(\beta \cdot (-)))$, where ψ is some fixed non-trivial $\overline{\mathbb{Q}}_\ell$-character of $\mathbb{F}_q$ and $\beta \in \mathrm{M}_2(\mathbb{F}_q)$ (resp. $\beta \in \mathrm{M}_3(\mathbb{F}_q)$). The conjugacy classes of $\beta \in \mathrm{M}_2(\mathbb{F}_q)$ are of the following two types:

(1) $\begin{bmatrix} a & * \\ 0 & b \end{bmatrix}$, where $*$ is 0 or 1;

(2) $\begin{bmatrix} 0 & 1 \\ -\Delta & s \end{bmatrix}$, where $x^2 - sx + \Delta$ is irreducible over $\mathbb{F}_q$.

And the conjugacy classes of $\beta \in \mathrm{M}_3(\mathbb{F}_q)$ are of the following three types:

(1') $\begin{bmatrix} a & *_1 & 0 \\ 0 & b & *_2 \\ 0 & 0 & c \end{bmatrix}$, where $*_1$ and $*_2$ are 0 or 1;

(2') $\begin{bmatrix} 0 & 1 & 0 \\ -\Delta & s & 0 \\ 0 & 0 & a \end{bmatrix}$, where $x^2 - sx + \Delta$ is irreducible over $\mathbb{F}_q$;

(2") N, where $\det(x \cdot I - N)$ is irreducible over $\mathbb{F}_q$.

For types (1) and (1'), the corresponding $\chi = \psi_\beta$ is trivial on the rational points of the Lie algebra of the unipotent radical $\mathbf{U}_0$ of some rational Borel subgroup $\mathbf{B}_0$. Let $\mathbf{T} = \mathbf{T}_0$ be a rational maximal torus contained in $\mathbf{B}_0$, and following the previous notation we denote by θ^1 the restriction of χ to $\mathfrak{t}^F = (T^1)^F$. Then we have

$$\left\langle \mathrm{Res}_{\mathfrak{g}^F}^{G^F} \mathrm{Ind}_{B_0^F}^{G^F} \widetilde{\theta}, \chi \right\rangle_{(G^1)^F} = \sum_{s \in B_0^F \backslash G^F / \mathfrak{g}^F} \left\langle \mathrm{Ind}_{({}^s(B_0)^1)^F}^{\mathfrak{g}^F} \left(\widetilde{\theta}^{s^{-1}}|_{({}^s(B_0)^1)^F} \right), \chi \right\rangle_{(G^1)^F}$$

by the Mackey intertwining formula. Note that

$$\left\langle \mathrm{Ind}_{({}^s B_0^1)^F}^{\mathfrak{g}^F} \left(\widetilde{\theta}^{s^{-1}}|_{({}^s B_0^1)^F} \right), \chi \right\rangle_{(G^1)^F} = \left\langle \left(\widetilde{\theta}^{s^{-1}}|_{({}^s B_0^1)^F} \right), \chi|_{({}^s(B_0)^1)^F} \right\rangle_{({}^s(B_0)^1)^F}$$

by the Frobenius reciprocity, which is non-zero in the case $s = 1$. Therefore χ appears in $\mathrm{Ind}_{B_0^F}^{G^F} \widetilde{\theta} = R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}$.

For type (2) (resp. types (2'), and (2")), the β is a semisimple regular element in $\mathrm{M}_2(\mathbb{F}_q)$ (resp. $\mathrm{M}_3(\mathbb{F}_q)$), in particular the corresponding θ is in general position and $\mathrm{Stab}_{G^F}(\theta|_{(G^l)^F}) = (TU^{\pm})^F$. For GL_2 (resp. GL_3) conjugate β to be a diagonal matrix in $\mathrm{M}_2(k)$ (resp. $\mathrm{M}_3(k)$), and view T^1 as the set of diagonal matrices in $\mathrm{M}_2(k)$ (resp. $\mathrm{M}_3(k)$) with Frobenius endomorphism being the canonical one conjugating by an element in the Weyl group, then the same argument of Proposition 3.7 shows θ is regular. So thanks to Corollary 4.7 we only need to show $\chi = \psi_\beta$ appears in $\mathrm{Ind}_{(TU^{\pm})^F}^{G^F} \widetilde{\theta}$. Actually, again by the Mackey intertwining formula we have

$$\left\langle \mathrm{Res}^{G^F}_{\mathfrak{g}^F} \mathrm{Ind}^{G^F}_{(TU^{\pm})^F} \widetilde{\theta}, \chi \right\rangle_{(G^1)^F} = \sum_{s \in (TU^{\pm})^F \backslash G^F / \mathfrak{g}^F} \left\langle \widetilde{\theta}^{s^{-1}}|_{\mathfrak{g}^F}, \chi \right\rangle_{(G^1)^F},$$

which is non-zero (take $s = 1$). □

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