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Asymptotic behavior of solutions to the extension problem for the fractional Laplacian on noncompact symmetric spaces

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Abstract. This work deals with the extension problem for the fractional Laplacian on Riemannian symmetric spaces G/K of noncompact type and of general rank, which gives rise to a family of convolution operators, including the Poisson operator. More precisely, motivated by Euclidean results for the Poisson semigroup, we study the long-time asymptotic behavior of solutions to the extension problem for L^1 initial data. In the case of the Laplace–Beltrami operator, we show that if the initial data are bi-K-invariant, then the solution to the extension problem behaves asymptotically as the mass times the fundamental solution, but this convergence may break down in the non-bi-K-invariant case. In the second part, we investigate the long-time asymptotic behavior of the extension problem associated with the so-called distinguished Laplacian on G/K. In this case, we observe phenomena which are similar to the Euclidean setting for the Poisson semigroup, such as L^1 asymptotic convergence without the assumption of bi-K-invariance.

Contents

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1. Introduction

Let $\mathcal{M}$ be a complete, noncompact Riemannian manifold and Δ be its Laplace–Beltrami operator. It is well understood that the long-time behavior of solutions to the heat equation

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x), & t > 0, \quad x \in \mathcal{M}, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

is strongly related to the global geometry of $\mathcal{M}$. This applies also to the heat kernel $h_t\,(x, y)$, that is, the minimal positive fundamental solution of the heat equation or, equivalently, the integral kernel of the heat semigroup $\exp\,(t\Delta)$ (see, for instance, [18]).

The connection between the long-time behavior of the solution $u(t, x)$ of (1.1) for initial data $u_0 \in L^1(\mathcal{M}, \mu)$ (where μ is the Riemannian measure on $\mathcal{M}$) and that of the heat kernel $h_t(x, y)$ has recently been the subject of extensive studies, see, for example, [7,19,30] or see [1,2,7,23] for variants and related questions. Denote by $M = \int_{\mathcal{M}} \mathrm{d}\mu(x)\, u_0(x)$ the mass of the initial data. In the case when $\mathcal{M} = \mathbb{R}^n$ with the Euclidean metric, the heat kernel is given by

$$h_t(x, y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$$

and the solution to (1.1) satisfies as $t \to +\infty$

$$\|u(t, .), , M\, h_t(., x_0)\|_{L^1(\mathbb{R}^n)} \longrightarrow 0 \tag{1.2}$$

and

$$t^{\frac{n}{2}} \|u(t, .) - M\, h_t(., x_0)\|_{L^\infty(\mathbb{R}^n)} \longrightarrow 0. \tag{1.3}$$

By interpolation, a similar convergence holds with respect to any L^p norm when $1 < p < \infty$:

$$t^{\frac{n}{2p'}} \|u(t, .) - M\, h_t(., x_0)\|_{L^p(\mathbb{R}^n)} \longrightarrow 0$$

where p' is the Hölder conjugate of p.

The situation is drastically different in hyperbolic spaces: it was shown by Vázquez [30] that (1.2) fails for a general initial function $f \in L^1$ but is still true if f is spherically symmetric around x_0. Similar results were obtained in [7] in a more general setting of symmetric spaces of noncompact type by using tools of harmonic analysis. In [19], it was shown that (1.3) fails on connected sums $\mathbb{R}^n \# \mathbb{R}^n$, $n \geq 3$.

The *fractional* Laplacian is the operator $(-\Delta)^\sigma$, $\sigma \in (0, 1)$, defined as the spectral σ-th power of the Laplace–Beltrami operator, with $\mathrm{Dom}(-\Delta) \subset \mathrm{Dom}((-\Delta)^\sigma)$. It is connected to *anomalous* diffusion, which accounts for much of the interest in modeling with fractional equations (quasi-geostrophic flows, turbulence and water waves, molecular dynamics, and relativistic quantum mechanics of stars). It also has various applications in probability and finance. One can obtain the fractional Laplacian through a Dirichlet-to-Neumann map extension problem, introduced by Caffarelli and Silvestre [13] on $\mathbb{R}^n$. This extension problem was considered for fractional powers of more general self-adjoint operators by Stinga and Torrea [27], where the authors adopted a rather general spectral/semigroup approach; also, in the same work [27], a Poisson/subordination formula was given, as well as conditions for the existence of a fundamental solution. On certain "good" noncompact Riemannian manifolds $\mathcal{M}$ (e.g., Cartan–Hadamard manifolds or manifolds with nonnegative Ricci curvature), the problem was studied in [9]. More precisely, let $H^\sigma(\mathcal{M})$ denote the usual Sobolev space on $\mathcal{M}$. Then, for any given $v_0 \in H^\sigma(\mathcal{M})$ there exists a unique solution of the extension problem

$$\Delta v + \frac{(1-2\sigma)}{t}\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial t^2} = 0, \quad 0 < \sigma < 1, \tag{1.4}$$

with $v(0, x) = v_0(x)$ where $t > 0$, $x \in \mathcal{M}$, and the fractional Laplacian, can be recovered through

$$(-\Delta)^\sigma v_0(x) = -2^{2\sigma-1}\frac{\Gamma(\sigma)}{\Gamma(1-\sigma)} \lim_{t\to 0^+} t^{1-2\sigma}\frac{\partial v}{\partial t}(x,t).$$

The extension problem has drawn much attention. Since the associated literature is enormous, we shall refer indicatively to [3,9,10,12,16,24,25,27] and the references therein. From a probabilistic point of view, the extension problem corresponds to the property that all symmetric stable processes can be obtained as traces of degenerate Bessel diffusion processes, see [26].

Observe that for $\sigma = 1/2$ we get the Poisson semigroup $e^{-t\sqrt{-\Delta}}$. In the Euclidean case $\mathcal{M} = \mathbb{R}^n$, the Poisson kernel is given by

$$\mathcal{Q}_t(x, y) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{t}{(t^2 + |x-y|^2)^{\frac{n+1}{2}}}, \quad x, y \in \mathbb{R}^n, \quad t > 0. \tag{1.5}$$

Then, for $v_0 \in L^1(\mathbb{R}^n)$ as $t \to +\infty$, it holds [28]

$$\|e^{-t\sqrt{-\Delta}}v_0 - M\,\mathcal{Q}_t(.,x_0)\|_{L^1(\mathbb{R}^n)} \longrightarrow 0 \tag{1.6}$$

and

$$t^n \,\|e^{-t\sqrt{-\Delta}}v_0 \,-\, M\,\mathcal{Q}_t(.,x_0)\|_{L^\infty(\mathbb{R}^n)} \,\longrightarrow\, 0. \tag{1.7}$$

Motivated by this, we examine the long-time behavior of solutions to the extension problem with L^1 initial data on noncompact symmetric spaces $\mathbb{X} = G/K$, for all $0 < \sigma < 1$. More precisely, let Q_t^σ be the fundamental solution to the extension problem (1.4), which will be called *the fractional Poisson kernel* from now on. Then our main result is the following.

Theorem 1.1. *Let $v_0 \in L^1(\mathbb{X})$ be bi-K-invariant and consider the solution v to the extension problem* (1.4) *with initial data v_0. Set $M = \int_{\mathbb{X}} v_0$. Then*

$$\|v(t,\,\cdot\,) - M\,Q_t^\sigma\|_{L^1(\mathbb{X})} \,\longrightarrow\, 0 \quad \text{as} \quad t \to +\infty. \tag{1.8}$$

Moreover, this convergence fails in general without the bi-K-invariance assumption.

It is worth mentioning that for the Euclidean case, in fact for manifolds of non-negative Ricci curvature and certain generalizations of them, the problem was treated recently in [24]: the convergence there is true for all absolutely integrable data, without any further symmetry assumptions. To the best of our knowledge, the present work is the first approach to examine this property on (essentially) negatively curved manifolds. The result is new even for the case of real hyperbolic space.

Remark 1.2. If the bi-K-invariant initial data are in addition compactly supported, we obtain the better estimate

$$\|v(t,\,\cdot\,) - M\,Q_t^\sigma\|_{L^1(\mathbb{X})} \,\le\, C\,t^{-\mu\varepsilon}, \quad \mu = \min\{\sigma, 1/2\} \quad \forall\, t \ge 1,$$

where $C > 0$ is a constant and ε is any positive constant such that $\varepsilon < 2/(\nu + 2\sigma)$, see Sects. 4.1 and 4.2. Here, ν denotes the so-called dimension at infinity of $\mathbb{X}$, see Sect. 2.

Remark 1.3. We also provide the following sup norm (for which no bi-K-invariance is needed) and L^p $(1 < p < \infty)$ norm estimates:

$$\|v(t,\,\cdot\,) - \,M\,Q_t^\sigma\|_{L^\infty(\mathbb{X})} \,=\, \mathrm{O}\left(t^{-\left(\frac{\nu}{2}+\frac{1}{2}-\sigma\right)}e^{-|\rho|t}\right) \tag{1.9}$$

$$\|v(t,\,\cdot\,) - \,M\,Q_t^\sigma\|_{L^p(\mathbb{X})} \,=\, \mathrm{o}\left(t^{-\frac{1}{p'}\left(\frac{\nu}{2}+\frac{1}{2}-\sigma\right)}e^{-\frac{|\rho|t}{p'}}\right) \tag{1.10}$$

as $t \to +\infty$. Here, p' denotes the dual exponent of p, defined by the formula $\frac{1}{p}+\frac{1}{p'} = 1$. However, the sup norm estimate (1.9) in the present context is relatively weaker compared to (1.7) in the Euclidean setting, while the L^p norm estimate is similar. Here, ρ is the half sum of positive roots with multiplicities, see Sect. 2. This is reminiscent of the weak L^∞ convergence for the heat equation on $\mathbb{X}$, observed first on the three-dimensional real hyperbolic space [30] and generalized to arbitrary rank noncompact symmetric spaces [7].

Let $S = N(\exp\mathfrak{a}) = (\exp\mathfrak{a})N$ be the solvable group occurring in the Iwasawa decomposition $G = N(\exp\mathfrak{a})K$. Then S is identifiable, as a manifold, with the symmetric space $\mathbb{X} = G/K$. Our second main contribution is to study the asymptotic convergence for solutions to the extension associated with the so-called distinguished Laplacian $\widetilde{\Delta}$ on S.

In order to state the results, let us introduce some indispensable notation, which will be clarified in Sects. 2 and 5. Denote by φ_0 the ground spherical function, by $\widetilde{\delta}$ the modular function on S, and by $\widetilde{Q}_t^\sigma$ the fundamental solution to the extension problem

$$\widetilde{\Delta}\widetilde{v} - \frac{(1-2\sigma)}{t}\partial_t\widetilde{v} - \partial_{tt}^2\widetilde{v} = 0, \quad \widetilde{v}(\,\cdot\,,0) = \widetilde{v}_0, \quad t > 0. \tag{1.11}$$

Let $\widetilde{\varphi}_0 = \widetilde{\delta}^{1/2}\varphi_0$ be the modified ground spherical function and denote by $\widetilde{M} = \frac{\widetilde{v}_0 * \widetilde{\varphi}_0}{\widetilde{\varphi}_0}$ the mass function on S which generalizes the mass in the Euclidean case (see Sect. 5.2). Then, we show the following long-time asymptotic convergence results.

Theorem 1.4. *Let $\widetilde{v}_0$ belong to the class of continuous and compactly supported functions on S. Then, the solution to the extension problem (1.11) with initial data $\widetilde{v}_0$ satisfies*

$$\|\widetilde{v}(t,\,\cdot\,) - \widetilde{M}\,\widetilde{Q}_t^\sigma\|_{L^1(S)} \longrightarrow 0 \tag{1.12}$$

and

$$t^{\ell+|\Sigma_r^+|}\|\widetilde{v}(t,\,\cdot\,) - \widetilde{M}\,\widetilde{Q}_t^\sigma\|_{L^\infty(S)} \longrightarrow 0 \tag{1.13}$$

as $t \to +\infty$. Here ℓ denotes the rank of G/K and Σ_r^+ the set of positive reduced roots. Analogous L^p $(1 < p < \infty)$ norm estimates follow by interpolation.

Remark 1.5. Let us comment on (1.12) and (1.13). Firstly, notice that the L^1 convergence (1.12) holds without the restriction of bi-K-invariance, in contrast to Theorem 1.1, and that the sup norm estimate (1.13) is stronger than (1.9), as in the Euclidean setting for the Poisson semigroup. Secondly, the mass $\widetilde{M}$ is a bounded function and not necessarily a constant.

Thirdly, the power $\ell + |\Sigma_r^+|$ which occurs in time factor never coincides with the dimension at infinity $\nu = \ell + 2|\Sigma_r^+|$ and it is equal to the topological dimension $n = \ell + \sum_{\alpha\in\Sigma^+} m_\alpha$ if and only if the following equivalent conditions hold:

- the root system Σ is reduced and all roots have multiplicity $m_\alpha = 1$.
- G is a normal real form.

This paper is organized as follows. After the present introduction in Sect. 1 and preliminaries in Sect. 2, we discuss the extension problem associated with the Laplace–Beltrami operator on symmetric spaces in Sect. 3. In Sect. 4 we deal with the long-time asymptotic behavior of solutions to the extension problem associated with the Laplace–Beltrami operator on symmetric spaces. We first determine the critical region where the fractional Poisson kernel concentrates. Next, on the one hand, for continuous compactly supported initial data, we show that both the solution and the fractional Poisson kernel vanish asymptotically outside that critical region. On the other hand, inside the critical region we discuss the role of the additional assumption of the bi-K-invariance of the initial data. The rest of this section deals with problems for more general initial data in the L^p $(p \geq 1)$ setting. In Sect. 5, we investigate the asymptotic behavior of solutions to the extension problem associated with the distinguished Laplacian. After specifying the critical region in this context, we study the long-time convergence in L^1 and in L^∞ with compactly supported initial data and address some questions associated with other initial data at the end of the paper.

Throughout this paper, the notation $A \lesssim B$ between two positive expressions means that there is a constant $C > 0$ such that $A \leq CB$. The notation $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$. Also, $A(t) \sim B(t)$ means that $A(t)/B(t) \to 1$ as $t \to +\infty$.

2. Preliminaries

In this section, we review spherical Fourier analysis on Riemannian symmetric spaces of noncompact type. The notation is standard and follows [17,20,21]. Next, we recall bounds and asymptotics of the heat kernel, for which we refer to [5,6] for more details in this setting.

2.1. Noncompact Riemannian symmetric spaces

Let G be a semi-simple Lie group, connected, noncompact, with finite center, and K be a maximal compact subgroup of G. The homogeneous space $\mathbb{X} = G/K$ is a Riemannian symmetric space of noncompact type.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G. The Killing form of $\mathfrak{g}$ induces a K-invariant inner product $\langle .,. \rangle$ on $\mathfrak{p}$, hence a G-invariant Riemannian metric on G/K. We denote by $d(.,.)$ the Riemannian distance on $\mathbb{X}$.

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$. The rank of $\mathbb{X}$ is the dimension ℓ of $\mathfrak{a}$. We identify $\mathfrak{a}$ with its dual $\mathfrak{a}^*$ by means of the inner product inherited from $\mathfrak{p}$.

Let $\Sigma \subset \mathfrak{a}$ be the root system of $(\mathfrak{g}, \mathfrak{a})$ and denote by W the Weyl group associated with Σ.

Once a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ has been selected, Σ^+ (resp. Σ_r^+ or Σ_s^+) denotes the corresponding set of positive roots (resp. positive reduced, i.e., indivisible roots or simple roots).

Let n be the dimension and ν be the pseudo-dimension (or dimension at infinity) of $\mathbb{X}$:

$$n = \ell + \textstyle\sum_{\alpha \in \Sigma^+} m_\alpha \quad \text{and} \quad \nu = \ell + 2|\Sigma_r^+| \tag{2.1}$$

where m_α denotes the dimension of the positive root subspace

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \,|\, [H, X] = \langle \alpha, H \rangle X, \quad \forall H \in \mathfrak{a}\}.$$

Denote by $\rho \in \mathfrak{a}^+$ the half sum of all positive roots $\alpha \in \Sigma^+$ counted with their multiplicities m_α:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \, \alpha.$$

Sometimes we shall use coordinates on $\mathfrak{a}$. When we do, we always refer to the coordinates associated to the orthonormal basis $\delta_1, ..., \delta_{\ell-1}, \ \rho/|\rho|$, where $\delta_1, ..., \delta_{\ell-1}$, is any orthonormal basis of $\rho^\perp$.

Let $\mathfrak{n}$ be the nilpotent Lie subalgebra of $\mathfrak{g}$ associated with Σ^+, and let $N = \exp \mathfrak{n}$ be the corresponding Lie subgroup of G. We have the decompositions

$$\begin{cases} G = N\,(\exp \mathfrak{a})\,K & \text{(Iwasawa)}, \\ G = K\,(\exp \overline{\mathfrak{a}^+})\,K & \text{(Cartan)}. \end{cases}$$

Denote by $A(x) \in \mathfrak{a}$ and $x^+ \in \overline{\mathfrak{a}^+}$ the middle components of $x \in G$ in these two decompositions, respectively, and by $|x| = |x^+|$ the distance to the origin. In the Cartan decomposition, the Haar measure on G writes

$$\int_G \mathrm{d}x\, f(x) = |K/\mathbb{M}| \int_K \mathrm{d}k_1 \int_{\mathfrak{a}^+} \mathrm{d}x^+\, \delta(x^+) \int_K \mathrm{d}k_2\, f(k_1(\exp x^+)k_2)\,,$$

with density

$$\begin{aligned} \delta(x^+) &= \prod_{\alpha \in \Sigma^+} (\sinh\langle \alpha, x^+ \rangle)^{m_\alpha} \\ &\asymp \prod_{\alpha \in \Sigma^+} \Big(\frac{\langle \alpha, x^+ \rangle}{1 + \langle \alpha, x^+ \rangle} \Big)^{m_\alpha} e^{2\langle \rho, x^+ \rangle} \quad \forall\, x^+ \in \overline{\mathfrak{a}^+}. \end{aligned} \tag{2.2}$$

Here K is equipped with its normalized Haar measure, $\mathbb{M}$ denotes the centralizer of $\exp \mathfrak{a}$ in K, and the volume of $K/\mathbb{M}$ can be computed explicitly, see [5, Eq. (2.2.4)].

Finally, let us recall that

$$|x^+ - y^+| \leq d(xK, yK), \quad |(yx)^+ - y^+|, \ |(xy)^+ - y^+| \leq d(xK, eK), \tag{2.3}$$

see [5, Lemma 2.1.2] or [22, Lemma 2.1].

2.2. Spherical Fourier analysis

For this subsection, our main references are [17, Chap. 4] and [21, Chap. IV].

For every $\lambda \in \mathfrak{a}$, the spherical function φ_λ is a smooth bi-K-invariant eigenfunction of all G-invariant differential operators on $\mathbb{X}$, in particular of the Laplace–Beltrami operator:

$$-\Delta \varphi_\lambda(x) = (|\lambda|^2 + |\rho|^2)\, \varphi_\lambda(x).$$

It is symmetric in the sense that $\varphi_\lambda(x^{-1}) = \varphi_{-\lambda}(x)$, and is given by the integral representation

$$\varphi_\lambda(x) = \int_K \mathrm{d}k\, e^{\langle i\lambda+\rho,\, A(kx)\rangle}. \tag{2.4}$$

All the elementary spherical functions φ_λ with parameter $\lambda \in \mathfrak{a}$ are controlled by the ground spherical function φ_0, which satisfies the global estimate

$$\varphi_0(\exp x^+) \asymp \left\{ \prod_{\alpha\in\Sigma_r^+} 1 + \langle \alpha, x^+\rangle \right\} e^{-\langle \rho, x^+\rangle} \quad \forall x^+ \in \overline{\mathfrak{a}^+}. \tag{2.5}$$

Let $\mathcal{S}(K\backslash G/K)$ be the Schwartz space of bi-K-invariant functions on G. The spherical Fourier transform (Harish–Chandra transform) $\mathcal{H}$ is defined by

$$\mathcal{H}f(\lambda) = \int_G \mathrm{d}x\, \varphi_{-\lambda}(x)\, f(x) \quad \forall \lambda \in \mathfrak{a}, \quad \forall f \in \mathcal{S}(K\backslash G/K), \tag{2.6}$$

where $\varphi_\lambda \in \mathcal{C}^\infty(K\backslash G/K)$ is the spherical function of index $\lambda \in \mathfrak{a}$.

Denote by $\mathcal{S}(\mathfrak{a})^W$ the subspace of W-invariant functions in the Schwartz space $\mathcal{S}(\mathfrak{a})$. Then $\mathcal{H}$ is an isomorphism between $\mathcal{S}(K\backslash G/K)$ and $\mathcal{S}(\mathfrak{a})^W$. The inverse spherical Fourier transform is given by

$$f(x) = \frac{C_0}{|W|} \int_{\mathfrak{a}} \frac{\mathrm{d}\lambda}{|\mathbf{c}(\lambda)|^2}\, \varphi_\lambda(x)\, \mathcal{H}f(\lambda) \quad \forall x \in G, \quad \forall f \in \mathcal{S}(\mathfrak{a})^W, \tag{2.7}$$

where the constant $C_0 = 2^{n-\ell}/(2\pi)^\ell |K/\mathbb{M}|$ depends only on the geometry of $\mathbb{X}$, and $|\mathbf{c}(\lambda)|^{-2}$ is the so-called Plancherel density, given by an explicit formula by Gindikin–Karpelevič.

Finally, if f is a Schwartz function on $\mathbb{X}$, the Helgason–Fourier transform is defined by

$$\widehat{f}(\lambda, k\mathbb{M}) = \int_G \mathrm{d}g\, f(gK)\, e^{\langle -i\lambda+\rho,\, A(k^{-1}g)\rangle}, \tag{2.8}$$

which, in view of (2.4), boils down to the transform (2.6) when f is bi-K-invariant.

2.3. Heat kernel on symmetric spaces

The heat kernel on $\mathbb{X}$ is a positive bi-K-invariant right convolution kernel, i.e., $h_t(xK, yK) = h_t(y^{-1}x) > 0$, which is thus determined by its restriction to the positive Weyl chamber. In fact, it symmetric, i.e., $h_t(x) = h_t(x^{-1})$. According to the inversion formula (2.7) of the spherical Fourier transform, the heat kernel is given by

$$h_t(xK) = \frac{C_0}{|W|} \int_{\mathfrak{a}} \frac{\mathrm{d}\lambda}{|\mathbf{c}(\lambda)|^2}\, \varphi_\lambda(x)\, e^{-t(|\lambda|^2+|\rho|^2)} \tag{2.9}$$

and satisfies the global estimate

$$h_t(\exp H) \asymp t^{-\frac{n}{2}} \left\{ \prod_{\alpha\in\Sigma_r^+} (1+t+\langle\alpha, H\rangle)^{\frac{m_\alpha+m_{2\alpha}}{2}-1} \right\} \varphi_0(\exp H) e^{-|\rho|^2 t-\frac{|H|^2}{4t}} \tag{2.10}$$

for all $t > 0$ and $H \in \overline{\mathfrak{a}^+}$, see [5,6]. Recall that $\int_{\mathbb{X}} h_t = 1$.

Finally, in order to describe more accurately the asymptotic behavior of the ground spherical function and of the heat kernel on certain regions, let us introduce the following functions: consider

$$\boldsymbol{\pi}(i\lambda) = \prod_{\alpha\in\Sigma_r^+} \langle\alpha, \lambda\rangle \tag{2.11}$$

and

$$\mathbf{b}(\lambda) = \prod_{\alpha\in\Sigma_r^+} \mathbf{b}_\alpha\Big(\frac{\langle\alpha, \lambda\rangle}{\langle\alpha, \alpha\rangle}\Big)$$

where

$$\mathbf{b}_\alpha(z) = |\alpha|^2 \frac{\Gamma(\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle}+\frac{1}{2}m_\alpha)}{\Gamma(\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle})} \frac{\Gamma(\frac{1}{2}\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle}+\frac{1}{4}m_\alpha+\frac{1}{2}m_{2\alpha})}{\Gamma(\frac{1}{2}\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle}+\frac{1}{4}m_\alpha)} \frac{\Gamma(iz+1)}{\Gamma(iz+\frac{1}{2}m_\alpha)} \frac{\Gamma(\frac{i}{2}z+\frac{1}{4}m_\alpha)}{\Gamma(\frac{i}{2}z+\frac{1}{4}m_\alpha+\frac{1}{2}m_{2\alpha})}.$$

The function $\mathbf{b}(-\lambda)^{-1}$ is holomorphic for $\lambda \in \mathfrak{a} + i\overline{\mathfrak{a}^+}$ and positive for $\lambda \in i\overline{\mathfrak{a}^+}$. We recall that it has the following behavior

$$|\mathbf{b}(-\lambda)|^{-1} \asymp \prod_{\alpha\in\Sigma_r^+} (1+|\langle\alpha, \lambda\rangle|)^{\frac{m_\alpha+m_{2\alpha}}{2}-1} \tag{2.12}$$

and that its derivatives can be estimated by

$$p(\tfrac{\partial}{\partial\lambda})\mathbf{b}(-\lambda)^{-1} = \mathrm{O}\big(|\mathbf{b}(-\lambda)|^{-1}\big), \tag{2.13}$$

where $p(\frac{\partial}{\partial\lambda})$ is any differential polynomial, [5, pp. 1041–1042].

We are now ready to describe the asymptotic behavior of the ground spherical function away from the walls. More precisely, as $\mu(H) := \min_{\alpha\in\Sigma^+}\langle\alpha, H\rangle \to \infty$, we have

$$\varphi_0(\exp H) \sim C_1\, \boldsymbol{\pi}(H)\, e^{-\langle\rho, H\rangle}. \tag{2.14}$$

Here, $C_1 = \boldsymbol{\pi}(\widetilde{\rho})^{-1}\mathbf{b}(0)$ and $\widetilde{\rho} = \frac{1}{2}\sum_{\alpha\in\Sigma_r^+}\alpha$, see [5, Proposition 2.2.12(ii)].

As for the heat kernel, we have the following asymptotics [5, Theorem 5.1.1]:

$$h_t(\exp H) \sim C_2\, t^{-\frac{\nu}{2}}\, \mathbf{b}\big(-i\tfrac{H}{2t}\big)^{-1} \varphi_0(\exp H)\, e^{-|\rho|^2 t-\frac{|H|^2}{4t}} \tag{2.15}$$

as $t \to \infty$, provided $\mu(H) \to \infty$ or $|H| = \mathrm{O}(t)$. Here $C_2 = C_0 2^{-|\Sigma_r^+|}\pi^{\frac{\ell}{2}}\boldsymbol{\pi}(\widetilde{\rho})\mathbf{b}(0)^{-1}$.

3. The fractional Laplacian and the extension problem

This section deals with the notion of the fractional Laplacian and the extension problem which gives rise to a family of operators, containing the Poisson operator.

In recent years, there has been intensive research on various kinds of fractional order operators. Being nonlocal objects, local PDE techniques to treat nonlinear problems for the fractional operators do not apply. To overcome this difficulty, in the Euclidean case, Caffarelli and Silvestre [13] studied the extension problem associated with the Laplacian and realized the fractional power as the map taking Dirichlet data to Neumann data. In [27] Stinga and Torrea related the extension problem for the fractional Laplacian to the heat semigroup, via a flexible approach of functional calculus. On certain classes of noncompact manifolds, which include symmetric spaces of noncompact type, the extension problem has been studied by Banica, González and Sáez [9]. Interestingly, in the noncompact setting one needs to have a precise control of the behavior of the metric at infinity and geometry plays a crucial role.

From now on, we strictly work on symmetric spaces of noncompact type $\mathbb{X} = G/K$. To begin with, using the spectral theorem, one can define fractional powers of the Laplacian via the heat semigroup,

$$(-\Delta)^{\sigma} v_0(x) = \int_0^{\infty} \frac{\mathrm{d}t}{t^{1+\sigma}} \left(e^{t\Delta} v_0(x) - v_0(x)\right) \quad \text{in} \quad L^2(\mathbb{X}), \quad v_0 \in \mathrm{Dom}(-\Delta).$$

see [31, (5), p. 260], [9], or [27].

Then, by [27, Theorems 1.1 and 2.1] (see also [9, Theorem 1.1]) the relation between the fractional Laplacian and the extension problem (1.4) is the following.

Theorem 3.1. [27] *Let $\sigma \in (0,1)$. Then for $v_0 \in Dom((-\Delta)^{\sigma})$, a solution to the extension problem*

$$\Delta v + \frac{(1-2\sigma)}{t}\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial t^2} = 0, \quad v(0,x) = v_0(x), \quad t > 0, \quad x \in \mathbb{X},$$

is given by

$$v(t,x) = v(t,gK) = v_0 * Q_t^{\sigma}(gK) = \int_G \mathrm{d}y \, Q_t^{\sigma}(y^{-1}g)\, v_0(y), \quad g \in G,$$

where

$$Q_t^{\sigma}(g) = \frac{t^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)} \int_0^{+\infty} \frac{\mathrm{d}u}{u^{1+\sigma}} h_u(g)\, e^{-\frac{t^2}{4u}}, \tag{3.1}$$

Moreover, the fractional Laplacian on $\mathbb{X}$ can be recovered through

$$(-\Delta)^{\sigma} v_0(x) = -2^{2\sigma-1} \frac{\Gamma(\sigma)}{\Gamma(1-\sigma)} \lim_{t\to 0^+} t^{1-2\sigma} \frac{\partial v}{\partial t}(x,t).$$

It is worth mentioning that the existence of an integral (in fact, a right convolution) kernel on symmetric spaces for the extension problem follows from [27, pp. 2099–2101] (see also [9, Theorem 3.2]), since the heat kernel h_t for $t > 0$ and all $xK \in \mathbb{X}$ satisfies

$$\|h_t(xK,\,\cdot)\|_{L^2(\mathbb{X})} \lesssim t^{-\frac{n}{4}}(1+t^{\frac{n}{4}}), \quad \|\partial_t h_t(xK,\,\cdot)\|_{L^2(\mathbb{X})} \lesssim t^{-\frac{n}{4}-1}(1+t^{\frac{n}{4}+1}). \tag{3.2}$$

Indeed, recall that by [5, Proposition 4.1.1.], it holds

$$\|h_t(xK,\cdot K)\|_{L^2(G/K)} = \|h_t((\cdot)^{-1}x)\|_{L^2(G)} \asymp \begin{cases} t^{-n/4}, & 0 < t \le 1, \\ e^{-|\rho|^2 t} t^{-\nu/4}, & t > 1. \end{cases}$$

As for the time derivative, one can employ the pointwise estimate [4, Eq. (3.1)] and then estimate the $L^2(\mathbb{X})$ norm using the Cartan decomposition and (2.2).

Observe that due to the subordination (3.1) to the heat kernel, Q_t^σ is a positive, bi-K-invariant and symmetric (in the sense that $Q_t^\sigma(g) = Q_t^\sigma(g^{-1})$ for all $g \in G$) function on G.

We next recall some large-time upper and lower bounds for the kernel Q_t^σ proved in [10].

Theorem 3.2. [10, Theorem 3.2] *The fractional Poisson kernel Q_t^σ on $\mathbb{X}$, $0 < \sigma < 1$, satisfies the following upper and lower bounds*

$$Q_t^\sigma(\exp H) \asymp \frac{t^{2\sigma}}{4^\sigma\,\Gamma(\sigma)} \left(\sqrt{t^2+|H|^2}\right)^{-\sigma-\frac{1}{2}-\frac{\ell}{2}-|\Sigma_r^+|} \varphi_0(\exp H)\, e^{-|\rho|\sqrt{t^2+|H|^2}}, \tag{3.3}$$

if $t^2+|H|^2 \ge 1$.

For large-time sup norm estimates, we have the following result.

Proposition 3.3. *For $t > 1$, it holds*

$$\|\,Q_t^\sigma\,\|_{L^\infty(\mathbb{X})} \asymp t^{\sigma-\frac{1}{2}-\frac{\ell}{2}-|\Sigma_r^+|} e^{-|\rho| t}.$$

Proof. The lower bound follows immediately by the fact that $\|\,Q_t^\sigma\,\|_{L^\infty(\mathbb{X})} \ge Q_t^\sigma(eK)$, (3.3) and the fact that $\varphi_0(eK) = 1$. For the upper bound, we use that

$$\left(\sqrt{t^2+|H|^2}\right)^{-\sigma-\frac{1}{2}-\frac{\ell}{2}-|\Sigma_r^+|} \le t^{-\sigma-\frac{1}{2}-\frac{\ell}{2}-|\Sigma_r^+|}, \quad e^{-|\rho|\sqrt{t^2+|H|^2}} \le e^{-|\rho| t},$$

and the fact that $\varphi_0(\exp H) \lesssim 1$, for all $H \in \overline{\mathfrak{a}^+}$. □

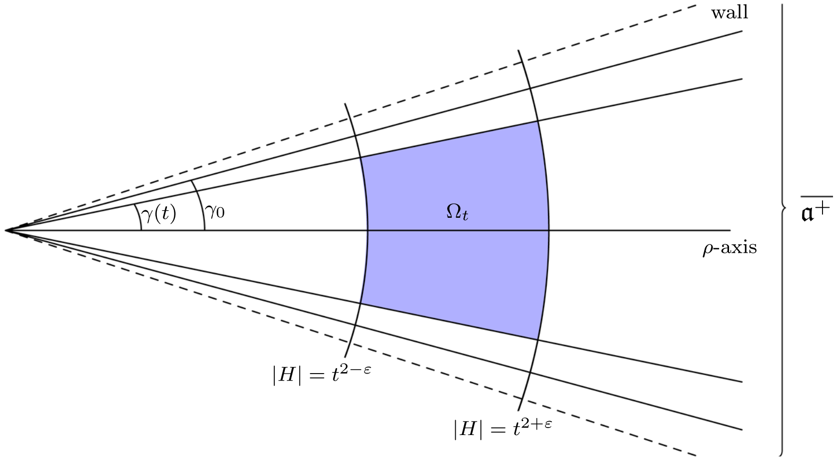

Figure 1. Flat part Ω_t of critical region

3.1. Large time behavior of the fractional Poisson kernel

Recall first that $\int_G \mathrm{d}x\, h_t(x) = 1$, $\forall t > 0$. This implies that

$$\int_G \mathrm{d}x\; Q_t^\sigma(x) = 1, \quad \forall t > 0, \quad \forall \sigma \in (0, 1),$$

by the subordination formula (3.1), the definition of the Gamma function and a Fubini argument. Motivated by this, let us introduce the notion of the *critical region* for the kernel Q_t^σ (Fig. 1).

Proposition 3.4. *Let* $0 < \varepsilon < 1$. *Consider in* $\mathfrak{a}$ *the annulus*

$$t^{2-\varepsilon} \le |H| \le t^{2+\varepsilon} \tag{3.4}$$

and the solid cone $\Gamma(t)$ *with angle*

$$\gamma(t) = t^{-\frac{\varepsilon}{2}}$$

around the ρ*-axis, and denote by* Ω_t *their intersection. Then, the critical region for the fractional Poisson kernel is* $K(\exp \Omega_t)K$*, in the sense that*

$$\int_{G \smallsetminus K(\exp \Omega_t)K} \mathrm{d}x\; Q_t^\sigma(x) \longrightarrow 0, \quad \textit{as} \quad t \to +\infty.$$

Proof. Let the rank ℓ be greater or equal to 2 (the rank one case is simpler, thus omitted). Let $0 \le a < b$. Using the bounds (3.3) and the fact that

$$\varphi_0(\exp H) \lesssim (1 + |H|)^{|\Sigma_r^+|} e^{-\langle \rho, H \rangle},$$

we have, by the Cartan decomposition and (2.2),

$$\int_{a<|x|<b} \mathrm{d}x\, Q_t^\sigma(x) \lesssim t^{2\sigma} \int_{\{a\leq|H|\leq b\}\cap\overline{\mathfrak{a}^+}} \mathrm{d}H\, \left(\sqrt{t^2+|H|^2}\right)^{-\frac{\ell}{2}-\frac{1}{2}-\sigma-|\Sigma_r^+|} (1+|H|)^{|\Sigma_r^+|}\, e^{\langle\rho,H\rangle}\, e^{-|\rho|\sqrt{t^2+|H|^2}}. \tag{3.5}$$

Take t large enough so that $\Gamma(t)$ is contained inside a small cone Γ_0 with fixed angle γ_0 around the ρ-axis and consider the regions

$$R_1 = \{x \in G : |x| < t^{2-\varepsilon}\},$$
$$R_2 = \{x \in G : |x| > t^{2-\varepsilon},\ x \notin K(\exp\Gamma_0)K\},$$
$$R_3 = \{x \in G : t^{2-\varepsilon} \leq |x| \leq t^{2+\varepsilon},\ x \in K(\exp\Gamma_0)K \smallsetminus K(\exp\Gamma(t))K\},$$
$$R_4 = \{x \in G : |x| > t^{2+\varepsilon},\ x \in K(\exp\Gamma_0)K\}.$$

First of all, we have

$$\int_{R_1} \mathrm{d}x\, Q_t^\sigma(x) \lesssim t^{2\sigma} e^{-\frac{|\rho|}{3}t^\varepsilon} \int_0^{t^{2-\varepsilon}} \mathrm{d}r\, (1+r)^{|\Sigma_r^+|}\, r^{\ell-1} \lesssim t^{-N\varepsilon} \quad \forall N>0,$$

using (3.5) and that if $|H| < t^{2-\varepsilon}$ then for t large enough,

$$e^{\langle\rho,H\rangle}\, e^{-|\rho|\sqrt{t^2+|H|^2}} \leq \exp\left\{-|\rho|\frac{t^2}{\sqrt{t^2+|H|^2}+|H|}\right\} \leq \exp\left\{-\frac{|\rho|}{3}t^\varepsilon\right\}.$$

Next, observe that if $H \notin \Gamma_0$, it holds

$$e^{\langle\rho,H\rangle}\, e^{-|\rho|\sqrt{t^2+|H|^2}} \leq e^{-|\rho||H|(1-\cos\gamma_0)},$$

which yields from (3.5) that

$$\begin{aligned}\int_{R_2} \mathrm{d}x\, Q_t^\sigma(x) &\lesssim t^{2\sigma} \int_{\{H\notin\Gamma_0:\, |H|>t^{2-\varepsilon}\}} \mathrm{d}H\, (1+|H|)^{|\Sigma_r^+|} e^{-|\rho||H|(1-\cos\gamma_0)} \\ &\lesssim t^{2\sigma} e^{-\frac{|\rho|}{2}t^{2-\varepsilon}(1-\cos\gamma_0)} \lesssim t^{-N\varepsilon} \quad \forall N>0.\end{aligned}$$

We next pass to the region R_3. Recall first the trivial inequality

$$\sin\theta \geq \frac{2}{\pi}\theta, \quad \theta\in[0,\pi/2].$$

Then, for some positive constant $c = c(|\rho|) > 0$, we have that

$$\begin{aligned} e^{\langle\rho,H\rangle}\, e^{-|\rho|\sqrt{t^2+|H|^2}} &\leq e^{-|\rho||H|(1-\cos\gamma(t))} \\ &= e^{-2|\rho||H|\sin^2(\gamma(t)/2)} \\ &\leq e^{-2|\rho||H|\gamma(t)^2/\pi^2} \\ &\leq e^{-c\,t^{2-2\varepsilon}}, \end{aligned}$$

since $|H| \geq t^{2-\varepsilon}$ and $\gamma(t) = t^{-\frac{\varepsilon}{2}}$. Since $0 < \varepsilon < 1$, passing to polar coordinates, we get by (3.5)

$$\int_{R_3} \mathrm{d}x\, Q_t^\sigma(x) \lesssim t^{2\sigma} \int_{t^{2-\varepsilon}}^{t^{2+\varepsilon}} \mathrm{d}r\, \left(\sqrt{t^2+r^2}\right)^{-\frac{\ell}{2}-\frac{1}{2}-\sigma-|\Sigma_r^+|} (1+r)^{|\Sigma_r^+|}\, r^{\ell-1} e^{-c\,t^{2-2\varepsilon}}$$
$$\lesssim t^{-N\varepsilon} \quad \forall N > 0.$$

To treat the integral in the remaining region R_4, in view of (3.5), let us write in polar coordinates:

$$\int_{R_4} \mathrm{d}x\, Q_t^\sigma(x) \lesssim t^{2\sigma} \int_{t^{2+\varepsilon}}^{+\infty} \mathrm{d}r\, \left(\sqrt{t^2+r^2}\right)^{-\frac{\ell}{2}-\frac{1}{2}-\sigma-|\Sigma_r^+|} (1+r)^{|\Sigma_r^+|}\, r^{\ell-1}$$
$$\times \int_0^{\gamma_0} \mathrm{d}\gamma\, e^{-|\rho|\frac{t^2}{\sqrt{t^2+r^2}+r\cos\gamma}}\, e^{-|\rho|\frac{r^2\sin^2\gamma}{\sqrt{t^2+r^2}+r\cos\gamma}} \sin^{\ell-2}\gamma. \tag{3.6}$$

Observe that

$$\int_0^{\gamma_0} \mathrm{d}\gamma\, e^{-|\rho|\frac{r^2\sin^2\gamma}{\sqrt{t^2+r^2}+r\cos\gamma}} \sin^{\ell-2}\gamma \leq \int_0^{\gamma_0} \mathrm{d}\gamma\, e^{-|\rho|\frac{r^2\gamma^2 4/\pi^2}{\sqrt{t^2+r^2}+r}}\, \gamma^{\ell-2}$$
$$\lesssim \left(\frac{r^2}{\sqrt{t^2+r^2}+r}\right)^{\frac{1-\ell}{2}}. \tag{3.7}$$

Thus, in R_4, where $r^2 + t^2 \asymp r^2$, (3.6) and (3.7) yield

$$\int_{R_4} \mathrm{d}x\, Q_t^\sigma(x) \lesssim_\sigma t^{2\sigma} \int_{t^{2+\varepsilon}}^{+\infty} \mathrm{d}r\, r^{-\frac{\ell}{2}-\frac{1}{2}-\sigma-|\Sigma_r^+|}(1+r)^{|\Sigma_r^+|}\, r^{\ell-1} r^{\frac{1-\ell}{2}}$$
$$\lesssim t^{2\sigma} \int_{t^{2+\varepsilon}}^{+\infty} \mathrm{d}r\, r^{-\sigma-1}$$
$$\lesssim t^{-\sigma\varepsilon}.$$

This completes the proof. □

Remark 3.5. The corresponding critical region for the Poisson kernel $\mathcal{Q}_t$ in the Euclidean case would be $B(0, t^{1+\varepsilon}) \smallsetminus B(0, t^{1-\varepsilon})$, as one can easily check using (1.5). On the other hand, the heat kernel h_t on a Riemannian symmetric space of the noncompact type is asymptotically concentrated along the (K-orbit) of the ρ-axis and an annulus centered at the origin, however, moving to infinity with finite speed $2|\rho|$, [8].

We now obtain precise long-time asymptotics of the kernel Q_t^σ which are crucial for our proof, by a slightly more general result.

Theorem 3.6. *Let $\sigma \in (0, 1)$. Then, as $t + |H| \to +\infty$, we have*

$$Q_t^\sigma(\exp H) \sim C(\sigma)\, t^{2\sigma}\, \boldsymbol{b}\left(-i|\rho|\frac{H}{\sqrt{t^2+|H|^2}}\right)^{-1} \left(\sqrt{t^2+|H|^2}\right)^{-\frac{\ell}{2}-\sigma-|\Sigma_r^+|-\frac{1}{2}}$$
$$\times \varphi_0(\exp H)\, e^{-|\rho|\sqrt{t^2+|H|^2}}, \tag{3.8}$$

where the constant is

$$C(\sigma) = \frac{1}{4^\sigma \Gamma(\sigma)}\, C_0\, 2^{\frac{\ell}{2}+\sigma+\frac{1}{2}}\, \pi^{\frac{\ell}{2}+\frac{1}{2}} \boldsymbol{\pi}(\widetilde{\rho})\, \boldsymbol{b}(0)^{-1} |\rho|^{\frac{\ell}{2}+\sigma+|\Sigma_r^+|-\frac{1}{2}},$$

with $\widetilde{\rho} = \frac{1}{2}\sum_{\alpha\in\Sigma_r^+}\alpha$ *and* $C_0 = 2^{n-\ell}/(2\pi)^\ell |K/\mathbb{M}|$.

Proof. The proof follows arguments for the asymptotics of the Poisson kernel ($\sigma = 1/2$) in [5, Section 5].

Consider a constant $\kappa > 4$. In view of the subordination formula (3.1), let us split

$$\begin{aligned} Q_t^\sigma(x) &= \frac{t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^{+\infty} \frac{\mathrm{d}u}{u^{1+\sigma}}\, h_u(x)\, e^{-\frac{t^2}{4u}}, \\ &= \frac{t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \{J_1 + J_2 + J_3\}, \end{aligned}$$

where the quantities J_1, J_2 and J_3 are defined by the integration over the intervals $\big[0, \kappa^{-1}b\big)$, $\big[\kappa^{-1}b, \kappa b\big)$ and $\big[\kappa b, \infty\big)$, respectively. Here, $b = \frac{\sqrt{t^2+|x|^2}}{2|\rho|}$.

We claim that the main contribution comes from the middle integral J_2. Indeed, for the first integral J_1, we get that for some $\delta > 0$ and some constants $d_1, d_2 > 0$, we have

$$J_1 \lesssim (1+|x|)^{d_2 - |\Sigma_r^+|} (t^2 + |x|^2)^{-\sigma - d_1}\, \varphi_0(x)\, e^{-(|\rho|+\delta)\sqrt{t^2+|x|^2}},$$

[10, p. 19]. For the third integral J_3, we get again by [10, p. 18] that

$$J_3 \lesssim \Big(\sqrt{t^2+|x|^2}\Big)^{-\frac{\ell}{2}-|\Sigma_r^+|-\frac{1}{2}}\, \varphi_0(x)\, e^{-(|\rho|+\eta)\sqrt{t^2+|x|^2}},$$

where $\eta = |\rho|\kappa/4 - |\rho| > 0$.

We now consider J_2. Define

$$h(t, H) = t^{\frac{\ell}{2}+|\Sigma_r^+|}\, \mathbf{b}\big(-i\tfrac{H}{2t}\big)\, \varphi_0(\exp H)^{-1}\, e^{|\rho|^2 t + \frac{|H|^2}{4t}}\, h_t(\exp H). \tag{3.9}$$

Then, by a change of variables and (3.9), we have

$$\begin{aligned} J_2 &= \int_{\kappa^{-1}\frac{\sqrt{t^2+|x|^2}}{2|\rho|}}^{\kappa\frac{\sqrt{t^2+|x|^2}}{2|\rho|}} \frac{\mathrm{d}u}{u^{1+\sigma}}\, h_u(x)\, e^{-\frac{t^2}{4u}} \\ &= \left(\frac{\sqrt{t^2+|H|^2}}{2|\rho|}\right)^{-\sigma} \int_{\kappa^{-1}}^{\kappa} \frac{\mathrm{d}u}{u^{1+\sigma}}\, h_{u\sqrt{t^2+|H|^2}/2|\rho|}(\exp H)\, e^{-t^2|\rho|/\left(2u\sqrt{t^2+|H|^2}\right)} \\ &= \varphi_0(\exp H)\, \left(\frac{\sqrt{t^2+|H|^2}}{2|\rho|}\right)^{-\frac{\ell}{2}-\sigma-|\Sigma_r^+|} \\ &\quad \times \int_{\kappa^{-1}}^{\kappa} \frac{\mathrm{d}u}{u^{\frac{\ell}{2}+|\Sigma_r^+|+1+\sigma}}\, \mathbf{b}\left(-i\,\frac{|\rho|H}{u\sqrt{t^2+|H|^2}}\right)^{-1} \\ &\quad e^{-\frac{|\rho|}{2}u\sqrt{t^2+|H|^2}-\frac{|\rho|}{2u}\sqrt{t^2+|H|^2}}\, h\left(u\frac{\sqrt{t^2+|H|^2}}{2|\rho|}, H\right) \\ &= \varphi_0(\exp H)\, \left(\frac{\sqrt{t^2+|H|^2}}{2|\rho|}\right)^{-\frac{\ell}{2}-\sigma-|\Sigma_r^+|} \mathbf{b}\left(-i\,\frac{|\rho|H}{\sqrt{t^2+|H|^2}}\right)^{-1} \\ &\quad \times \int_{\kappa^{-1}}^{\kappa} \frac{\mathrm{d}u}{u^{\frac{\ell}{2}+|\Sigma_r^+|+1+\sigma}}\, \frac{\mathbf{b}\left(-i\,\frac{|\rho|H}{u\sqrt{t^2+|H|^2}}\right)^{-1}}{\mathbf{b}\left(-i\,\frac{|\rho|H}{\sqrt{t^2+|H|^2}}\right)^{-1}} \\ &\quad e^{-|\rho|\sqrt{t^2+|H|^2}\left(\frac{u+u^{-1}}{2}\right)}\, h\left(u\frac{\sqrt{t^2+|H|^2}}{2|\rho|}, H\right). \end{aligned}$$

By the Laplace method we get that the last integral tends to

$$C_2\sqrt{\frac{2\pi}{|\rho|}}\,(t^2+|H|^2)^{-\frac{1}{2}}e^{-|\rho|\sqrt{t^2+|H|^2}},$$

due to the facts that

(i) $\mathbf{b}\left(-i\,\frac{|\rho|H}{u\sqrt{t^2+|H|^2}}\right)^{-1}$ is bounded above and below, uniformly in u and t, H;

(ii) $\mathbf{b}\left(-i\,\frac{|\rho|H}{u\sqrt{t^2+|H|^2}}\right)^{-1} \sim \mathbf{b}\left(-i\,\frac{|\rho|H}{\sqrt{t^2+|H|^2}}\right)^{-1}$ as $u \to 1$, uniformly in t and H;

(iii) $h\left(u\frac{\sqrt{t^2+|H|^2}}{2|\rho|}, H\right) \longrightarrow C_2$ as $t+|H| \to +\infty$, uniformly in u, by the asymptotics in (2.15) and contradiction (see [5, pp. 1085–1086]).

For the exact value of the constant C_2, we refer again to (2.15).

Since J_1, J_3 are very small compared to J_2 for t large, substituting the value of C_2 we finally get the claimed asymptotics. □

3.2. Asymptotics in the critical region

In this subsection we prove asymptotics for some quantities that will be later used in the proof. We introduce the notation $(\widehat{x^+, \rho})$ to denote the angle between $x^+ \in \overline{\mathfrak{a}^+}$ and ρ.

The first two lemmas describe the effect of a small translation on the critical region.

Lemma 3.7. *For all x in the critical region $K(\exp \Omega_t)K$, and for all $y \in G$ bounded, the following asymptotic behaviors hold as $t \to +\infty$:*

(i) $\frac{|(y^{-1}x)^+|}{|x^+|}$ *and* $\frac{|x^+|}{|(y^{-1}x)^+|}$ *are both equal to* $1 + \mathrm{O}\big(t^{-2+\varepsilon}\big)$.

(ii) $\frac{x^+}{|x^+|}$ *and* $\frac{(y^{-1}x)^+}{|(y^{-1}x)^+|}$ *are both equal to* $\frac{\rho}{|\rho|} + \mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big)$.

(iii) *For every* $\alpha \in \Sigma^+$, $\frac{\langle \alpha, (y^{-1}x)^+ \rangle}{\langle \alpha, x^+ \rangle} = 1 + \mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big)$.

(iv) $d(xK, eK) - d(xK, yK) = \langle \frac{\rho}{|\rho|}, A(k^{-1}y) \rangle + \mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big)$. *Here, k is the left component of x in the Cartan decomposition and* $\exp A(k^{-1}y)$ *is the middle component of $k^{-1}y$ in the Iwasawa decomposition.*

Proof. Assume that $t^{2-\varepsilon} \le d(xK, eK) \le t^{2+\varepsilon}$ and $d(yK, eK) \le \xi$, which implies by the triangle inequality that $\frac{1}{2}\, t^{2-\varepsilon} \le d(xK, yK) \le 2\, t^{2+\varepsilon}$, for t large enough.

We deduce first (i) by using

$$\frac{|(y^{-1}x)^+|}{|x^+|} = \frac{d(xK, yK)}{d(xK, eK)} = 1 + \mathrm{O}\big(t^{-2+\varepsilon}\big).$$

The second assertion follows similarly.

Next, for (ii), since the angle of x^+ with the ρ-axis is $\mathrm{O}(t^{-\frac{\varepsilon}{2}})$, we first have

$$\left| \frac{x^+}{|x^+|} - \frac{\rho}{|\rho|} \right|^2 = 2\left(1 - \frac{\langle \rho, x^+ \rangle}{|\rho||x^+|}\right) = \mathrm{O}(t^{-\varepsilon}).$$

For the second asymptotics in (ii), we work similarly, observing that since $(y^{-1}x)^+ = x^+ + \mathrm{O}(1)$, we have

$$\begin{aligned}
\left\langle \frac{\rho}{|\rho|}, \frac{(y^{-1}x)^+}{|(y^{-1}x)^+|} \right\rangle &= \frac{|x^+|}{|(y^{-1}x)^+|} \left\langle \frac{\rho}{|\rho|}, \frac{x^+}{|x^+|} \right\rangle + \mathrm{O}(|x|^{-1}) \\
&= \left\langle \frac{\rho}{|\rho|}, \frac{x^+}{|x^+|} \right\rangle + \mathrm{O}(t^{-2+\varepsilon}) \\
&= 1 + \mathrm{O}(t^{-\varepsilon}),
\end{aligned} \tag{3.10}$$

using (i) and that $\cos(\widehat{x^+, \rho}) = 1 + \mathrm{O}(t^{-\varepsilon})$.

Let us next deduce (iii) from (i) and (ii). For every positive root α,

$$\frac{\langle\alpha,(y^{-1}x)^{+}\rangle}{\langle\alpha,x^{+}\rangle} = \frac{\left\langle\alpha,\frac{(y^{-1}x)^{+}}{|(y^{-1}x)^{+}|}\right\rangle}{\left\langle\alpha,\frac{x^{+}}{|x^{+}|}\right\rangle}\,\frac{|(y^{-1}x)^{+}|}{|x^{+}|}$$
$$= \frac{\left\langle\alpha,\frac{\rho}{|\rho|}\right\rangle+\mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big)}{\left\langle\alpha,\frac{\rho}{|\rho|}\right\rangle+\mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big)}\left\{1+\mathrm{O}\big(t^{-2+\varepsilon}\big)\right\} = 1+\mathrm{O}(t^{-\frac{\varepsilon}{2}}).$$

It remains to prove (iv). For that, we follow [7, Lemma 3.8]. Let $x=k(\exp x^{+})k'$ in the Cartan decomposition and consider the Iwasawa decomposition $k^{-1}y=n(k^{-1}y)(\exp A(k^{-1}y))k''$ for some $k''\in K$. Then

$$d(xK,yK) = d\big(k(\exp x^{+})K,\,kn(k^{-1}y)(\exp A(k^{-1}y))K\big)$$
$$= d\big(\exp(-x^{+})[n(k^{-1}y)]^{-1}(\exp x^{+})K,\,\exp(A(k^{-1}y)-x^{+})K\big),$$

and we write

$$d(xK,eK)-d(xK,yK) = \overbrace{d(xK,eK)-d\big(\exp(A(k^{-1}y)-x^{+})K,eK\big)}^{I} + \underbrace{d\big(\exp(A(k^{-1}y)-x^{+})K,eK\big)-d(xK,yK)}_{II}.$$

On the one hand, $|II|$ tends exponentially fast to 0, see [7]. On the other hand, we have

$$I = |x^{+}|-|A(k^{-1}y)-x^{+}| = \frac{2\langle x^{+},A(k^{-1}y)\rangle-|A(k^{-1}y)|^{2}}{|x^{+}|+|A(k^{-1}y)-x^{+}|}$$
$$= \left\langle\frac{x^{+}}{|x^{+}|},A(k^{-1}y)\right\rangle+\mathrm{O}\left(\frac{1}{|x^{+}|}\right)$$
$$= \left\langle\frac{\rho}{|\rho|},A(k^{-1}y)\right\rangle+\mathrm{O}\left(t^{-\frac{\varepsilon}{2}}\right)$$

by using (ii), the fact that $\{A(k^{-1}y)\,|\,k\in K\}$ is a compact subset of $\mathfrak{a}$ and that $\mathrm{O}\big(\frac{1}{|x^{+}|}\big)=\mathrm{O}\big(t^{-2+\varepsilon}\big)$. This concludes the proof. □

Lemma 3.8. *Let $x\in K(\exp\Omega_t)K$ and let y be bounded. Then*

$$\langle\rho,x^{+}\rangle-\langle\rho,(y^{-1}x)^{+}\rangle = |\rho||x^{+}|-|\rho||(y^{-1}x)^{+}|+O(t^{-\frac{\varepsilon}{2}}).$$

Proof. Since the rank one case is trivial, let us consider $\ell\geq 2$. Observe first that the claim follows by

$$\cos\left(\widehat{(y^{-1}x)^{+},\rho}\right)=\cos\left(\widehat{x^{+},\rho}\right)+\mathrm{O}\left(t^{-\frac{\varepsilon}{2}}|x^{+}|^{-1}\right). \tag{3.11}$$

Indeed, by (3.11) and taking into account that $\cos(\widehat{x^+, \rho}) = 1 + \mathrm{O}(t^{-\varepsilon})$, we get

$$\begin{aligned}\langle \rho, x^+\rangle - \langle \rho, (y^{-1}x)^+\rangle &= |\rho||x^+|\cos(\widehat{x^+, \rho}) - |\rho||(y^{-1}x)^+|\cos(\widehat{(y^{-1}x)^+, \rho})\\ &= |\rho|\,|x^+|\cos(\widehat{x^+, \rho})\\ &\quad - |\rho|\,|(y^{-1}x)^+|\,(\cos(\widehat{x^+, \rho}) + \mathrm{O}(t^{-\frac{\varepsilon}{2}}|x^+|^{-1}))\\ &= |\rho||x^+| - |\rho||(y^{-1}x)^+| + \mathrm{O}(t^{-\frac{\varepsilon}{2}}).\end{aligned}$$

Therefore, it remains to prove (3.11). Observe that by (3.10), we have

$$\sin^2\left(\frac{(\widehat{(y^{-1}x)^+, \rho})}{2}\right) = \sin^2\left(\frac{(\widehat{x^+, \rho})}{2}\right) + \mathrm{O}(t^{-2+\varepsilon}),$$

thus $(\widehat{(y^{-1}x)^+, \rho}) = \mathrm{O}(t^{-\frac{\varepsilon}{2}})$. Next, recall the coordinates on $\mathfrak{a}$ with respect to the basis $\delta_1, \ldots, \delta_{\ell-1}, \rho/|\rho|$ introduced in Sect. 2, and write

$$x^+ = (\xi,\, \xi_\ell)\,, \quad (y^{-1}x)^+ = (\zeta,\, \zeta_\ell)\,.$$

Since $\langle x^+, \rho\rangle = \xi_\ell\,|\rho|$, we get

$$\xi_\ell = |x^+|\cos(\widehat{x^+, \rho}), \quad |\xi| = |x^+|\sin(\widehat{x^+, \rho}).$$

Similarly,

$$\zeta_\ell = |(y^{-1}x)^+|\cos(\widehat{(y^{-1}x)^+, \rho}), \quad |\zeta| = |(y^{-1}x)^+|\sin(\widehat{(y^{-1}x)^+, \rho}).$$

Therefore,

$$\frac{|\xi|}{\xi_\ell} = \tan(\widehat{x^+, \rho}) = \mathrm{O}(t^{-\frac{\varepsilon}{2}}), \quad |x^+| \asymp \xi_\ell \tag{3.12}$$

and

$$\frac{|\zeta|}{\zeta_\ell} = \tan(\widehat{(y^{-1}x)^+, \rho}) = \mathrm{O}(t^{-\frac{\varepsilon}{2}}), \quad |(y^{-1}x)^+| \asymp \zeta_\ell. \tag{3.13}$$

Thus, we have

$$\begin{aligned}\left|\cos(\widehat{(y^{-1}x)^+, \rho}) - \cos(\widehat{x^+, \rho})\right|\,|x^+| &= \frac{\left|\zeta_\ell\,|x^+| - \xi_\ell\,|(y^{-1}x)^+|\right|}{|(y^{-1}x)^+|}\\ &\asymp \frac{\left|\zeta_\ell\sqrt{|\xi|^2 + \xi_\ell^2} - \xi_\ell\sqrt{|\zeta|^2 + \zeta_\ell^2}\right|}{\zeta_\ell}\\ &\asymp \frac{\left|\zeta_\ell^2\,|\xi|^2 - \xi_\ell^2\,|\zeta|^2\right|}{\zeta_\ell^2\,\xi_\ell\left(\sqrt{\left(\frac{|\xi|}{\xi_\ell}\right)^2 + 1} + \sqrt{\left(\frac{|\zeta|}{\zeta_\ell}\right)^2 + 1}\right)}\\ &\asymp \frac{|\zeta_\ell\,|\xi| - \xi_\ell\,|\zeta||\,|\zeta_\ell\,|\xi| + \xi_\ell\,|\zeta||}{\zeta_\ell^2\,\xi_\ell},\end{aligned} \tag{3.14}$$

due to (3.12) and (3.13). The fact that $(y^{-1}x)^+ = x^+ + \mathrm{O}(1)$ implies $\zeta_\ell = \xi_\ell + \mathrm{O}(1)$ and $|\zeta| = |\xi| + \mathrm{O}(1)$; therefore,

$$\frac{\zeta_\ell\,|\xi| - \xi_\ell\,|\zeta|}{\xi_\ell} = \mathrm{O}(1), \quad \frac{\zeta_\ell\,|\xi| + \xi_\ell\,|\zeta|}{\zeta_\ell^2} = \mathrm{O}\left(\frac{|\zeta|}{\zeta_\ell}\right) = \mathrm{O}(t^{-\frac{\varepsilon}{2}}). \tag{3.15}$$

Altogether, we conclude by (3.14) and (3.15) that

$$\cos(\widehat{(y^{-1}x)^+, \rho}) - \cos(\widehat{x^+, \rho}) = \mathrm{O}(t^{-\frac{\varepsilon}{2}}|x^+|^{-1}).$$

□

The next lemma is the heart of the proof.

Lemma 3.9. *Assume that $x = k\exp(x^+)k'$ is in the critical region $K(\exp\Omega_t)K$ and that y is bounded. Then,*

$$\frac{Q_t^\sigma(xK, yK)}{Q_t^\sigma(xK, eK)} = e^{\langle 2\rho, A(k^{-1}y)\rangle} + O(t^{-\frac{\varepsilon}{2}}),$$

for $0 < \varepsilon < 2/(\nu + 2\sigma)$. Here, $\nu = \ell + 2|\Sigma_r^+|$ is the dimension at infinity.

Proof. By Theorem 3.6, we get

$$\frac{Q_t^\sigma(y^{-1}x)}{Q_t^\sigma(x)} \sim \frac{\mathbf{b}\left(-i|\rho|\frac{(y^{-1}x)^+}{\sqrt{t^2+|(y^{-1}x)^+|^2}}\right)^{-1}\sqrt{t^2+|(y^{-1}x)^+|^2}^{\,-\sigma-\frac{\ell}{2}-|\Sigma_r^+|-\frac{1}{2}}\, e^{-|\rho|\sqrt{t^2+|(y^{-1}x)^+|^2}}\,\varphi_0(y^{-1}x)}{\mathbf{b}\left(-i|\rho|\frac{x^+}{\sqrt{t^2+|x^+|^2}}\right)^{-1}\sqrt{t^2+|x^+|^2}^{\,-\sigma-\frac{\ell}{2}-|\Sigma_r^+|-\frac{1}{2}}\, e^{-|\rho|\sqrt{t^2+|x^+|^2}}\,\varphi_0(x)}.$$

Our aim is to show that for x inside the critical region and y bounded, the following asymptotics hold;

(i) $\dfrac{\mathbf{b}\left(-i|\rho|\frac{(y^{-1}x)^+}{\sqrt{t^2+|(y^{-1}x)^+|^2}}\right)^{-1}}{\mathbf{b}\left(-i|\rho|\frac{x^+}{\sqrt{t^2+|x^+|^2}}\right)^{-1}} = 1 + \mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big);$

(ii) $\left(\frac{t^2+|x^+|^2}{t^2+|(y^{-1}x)^+|^2}\right)^k = 1 + \mathrm{O}\left(t^{-2+\varepsilon(\nu+2\sigma)}\right)$, where $k = \frac{1}{2}\left(\sigma + \frac{\nu}{2} + \frac{1}{2}\right)$;

(iii) $\exp\left\{-|\rho|\left(\sqrt{t^2+|(y^{-1}x)^+|^2} - \sqrt{t^2+|x^+|^2}\right)\right\} = \langle\rho, A(k^{-1}y)\rangle + \mathrm{O}(t^{-2+2\varepsilon});$

(iv) $\frac{\varphi_0(y^{-1}x)}{\varphi_0(x)} = \langle\rho, A(k^{-1}y)\rangle + \mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big);$

thus, the claim for $Q_t^\sigma(y^{-1}x)/Q_t^\sigma(x)$ follows, choosing ε small enough.

We start the proof of (i)–(iv) by some preliminary observations. Write

$$r = |x^+| = d(xK, eK), \quad s = |(y^{-1}x)^+| = d(xK, yK),$$

and let $d(yK, eK) < \xi$, for some $\xi > 0$. Then, for x inside the critical region, and t large enough we have

$$t^{2-\varepsilon} \le r \le t^{2+\varepsilon}, \quad \frac{1}{2}t^{2-\varepsilon} \le s \le 2\,t^{2+\varepsilon}, \quad |r - s| \le \xi.$$

Also, we have $t^2+r^2 \asymp r^2$ and $t^2+s^2 \asymp s^2$. Finally, in the proof of Lemma 3.8 it was shown that the angle of $(y^{-1}x)^+$ with the ρ-axis is $\mathrm{O}(t^{-\frac{\varepsilon}{2}})$.

Proof of (i). Observe first that owing to (2.12), for all $g \in G$ such that $t^2+|g^+|^2 \asymp |g^+|^2$, we have

$$\mathbf{b}\left(-i|\rho|\frac{g^+}{\sqrt{t^2+|g|^2}}\right)^{-1} \asymp 1. \tag{3.16}$$

Therefore, for $x \in K(\exp \Omega_t)K$ and y bounded, by the mean value theorem we get

$$\begin{aligned}&\left|\mathbf{b}\left(-i\frac{|\rho|x^+}{\sqrt{t^2+|x^+|^2}}\right)^{-1} - \mathbf{b}\left(-i\frac{|\rho|(y^{-1}x)^+}{\sqrt{t^2+|(y^{-1}x)^+|^2}}\right)^{-1}\right| \\ &\quad\lesssim \left|\frac{x^+}{\sqrt{t^2+|x^+|^2}} - \frac{(y^{-1}x)^+}{\sqrt{t^2+|(y^{-1}x)^+|^2}}\right|,\end{aligned}$$

where we have used the derivative bound (2.13) and (3.16). Next, owing to Lemma 3.7(ii), we obtain

$$\begin{aligned}\frac{x^+}{\sqrt{t^2+|x^+|^2}} &= \frac{x^+}{|x^+|}\left(1+\frac{t^2}{|x^+|^2}\right)^{-1/2} = \left(\frac{\rho}{|\rho|}+\mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big)\right)\left(1+\mathrm{O}(t^{-2+2\varepsilon})\right) \\ &= \frac{\rho}{|\rho|}+\mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big),\end{aligned}$$

since $0<\varepsilon<2/(\nu+2\sigma)<2/3$. Working likewise for $\frac{(y^{-1}x)^+}{\sqrt{t^2+|(y^{-1}x)^+|^2}}$, the claim follows using (3.16).

Proof of (ii). We use a similar mean value argument applied to $(t^2+(.)^2)^k$, $k>1$, so that for some r_0 between r and s we have

$$\left|\frac{(t^2+r^2)^k}{(t^2+s^2)^k}-1\right| \lesssim \frac{r_0\,(t^2+r_0^2)^{k-1}}{(t^2+s^2)^k}. \tag{3.17}$$

Given that $r_0 \lesssim t^{2+\varepsilon}$, $s \gtrsim t^{2-\varepsilon}$ and $k=\frac{1}{2}\left(\sigma+\frac{\nu}{2}+\frac{1}{2}\right)$ we get the desired result.

Proof of (iii). We first claim that

$$\frac{r+s}{\sqrt{t^2+r^2}+\sqrt{t^2+s^2}} = 1+O\left(t^{-2+2\varepsilon}\right). \tag{3.18}$$

Indeed, consider the function $f(\tau)=\sqrt{\tau^2+r^2}+\sqrt{\tau^2+s^2}$, $\tau \geq 0$, and observe that the left-hand side of (3.18) is equal to $f(0)/f(t)$. Then, the mean value theorem for f in $[0,t]$ together with the fact that

$$f'(\tau) \lesssim \frac{\tau}{r}+\frac{\tau}{s} \lesssim t^{-1+\varepsilon}, \quad f(\tau) \gtrsim t^{2-\varepsilon}, \quad \forall \tau \in [0,t],$$

yields the claimed asymptotics (3.18). Finally, in Lemma 3.7(iv) it was shown that

$$r-s = d(xK,eK)-d(xK,yK) = \left\langle \frac{\rho}{|\rho|}, A(k^{-1}y)\right\rangle + \mathrm{O}\big(t^{-2+\varepsilon}\big). \tag{3.19}$$

Therefore, by (3.18) and (3.19), we get

$$\exp\left\{-|\rho|\left(\sqrt{t^2+s^2}-\sqrt{t^2+r^2}\right)\right\} = \exp\left\{|\rho|(r-s)\frac{r+s}{\sqrt{t^2+r^2}+\sqrt{t^2+s^2}}\right\}$$
$$= e^{\langle\rho, A(k^{-1}y)\rangle + \mathrm{O}(t^{-2+2\varepsilon})}$$
$$= e^{\langle\rho, A(k^{-1}y)\rangle} + \mathrm{O}(t^{-2+2\varepsilon}),$$

which proves (iii).

Proof of (iv). Since the angles of both x^+ and $(y^{-1}x)^+$ with the ρ-axis are $\mathrm{O}(t^{-\frac{\varepsilon}{2}})$, we may use the ground spherical asymptotics (2.14). On the one hand, by Lemma 3.7(iii), we have

$$\frac{\boldsymbol{\pi}((y^{-1}x)^+)}{\boldsymbol{\pi}(x^+)} = \prod_{\alpha\in\Sigma_r^+}\frac{\langle\alpha,(y^{-1}x)^+\rangle}{\langle\alpha,x^+\rangle} = 1+\mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big). \tag{3.20}$$

On the other hand, using Lemma 3.8 and Lemma 3.7(iv), we have

$$e^{\langle\rho,x^+\rangle-\langle\rho,(y^{-1}x)^+\rangle} = e^{\langle\rho,A(k^{-1}y)\rangle+\mathrm{O}(t^{-2+\varepsilon})} = e^{\langle\rho,A(k^{-1}y)\rangle} + \mathrm{O}(t^{-2+\varepsilon}),$$

with which the proof of (iv) is complete.

Altogether, we have

$$\frac{Q_t^\sigma(xK,yK)}{Q_t^\sigma(xK,eK)} = e^{\langle 2\rho, A(k^{-1}y)\rangle} + \mathrm{O}(t^{-\frac{\varepsilon}{2}}).$$

□

4. Asymptotic convergence associated with the extension problem for the Laplace–Beltrami operator

We first consider continuous compactly supported initial data v_0. We work separately outside and inside the critical region: we will show that

$$\|v_0 * Q_t^\sigma - M\,Q_t^\sigma\|_{L^1(G\smallsetminus K(\exp\Omega_t)K)} \to 0$$

but inside $K(\exp\Omega_t)K$, unless v_0 is bi-K-invariant, the convergence to the fundamental solution may break down.

4.1. Estimates outside the critical region

In this subsection, we show that the solution $v(t,x)$ to the extension problem vanishes asymptotically in $L^1(G\smallsetminus K(\exp\Omega_t)K)$ as $t\to+\infty$. Then the desired convergence follows by the triangle inequality.

Lemma 4.1. *Let* $x \in G \smallsetminus K(\exp \Omega_t)K$ *and* $y \in K(\exp B(0,\xi))K$. *Denote by* $\Gamma''(t)$ *the solid cone around the* ρ*-axis of angle* $\frac{1}{2}\,t^{-\frac{\varepsilon}{2}}$. *Consider in* $\mathfrak{a}$ *the set*

$$\Omega_t'' = \left(B(0, 2\,t^{2+\varepsilon}) \smallsetminus B\left(0, \frac{1}{2}\,t^{2-\varepsilon}\right)\right) \cap \Gamma''(t).$$

Then,

$$y^{-1}x \in G \smallsetminus K(\exp \Omega_t'')K.$$

Proof. Let $x \in G \smallsetminus K(\exp \Omega_t)K$ and $|y| < \xi$. Recall that by (2.3)

$$|(y^{-1}x)^+ - x^+| \le d(yK, eK) = |y| < \xi,$$

which implies that

$$\begin{cases} |(y^{-1}x)^+| \le |x^+| + \xi < t^{2-\varepsilon} + \xi < 2\,t^{2-\varepsilon}, \\ |(y^{-1}x)^+| \ge |x^+| - \xi > t^{2+\varepsilon} - \xi > \frac{1}{2}\,t^{2+\varepsilon}. \end{cases}$$

for t large enough. In other words,

$$x \in G \smallsetminus K(\exp\{B(0, t^{2+\varepsilon}) \smallsetminus B(0, t^{2-\varepsilon})\})K$$

implies

$$\Longrightarrow\ y^{-1}x \in G \smallsetminus K\left(\exp\left\{B(0, \frac{1}{2}t^{2+\varepsilon}) \smallsetminus B(0, 2\,t^{2-\varepsilon})\right\}\right)K.$$

We finally turn to the angles. Write $\phi = \widehat{(x^+, \rho)}$ and $\omega = \widehat{((y^{-1}x)^+, \rho)}$, and observe that by (3.10), we have

$$\sin^2\left(\frac{\phi}{2}\right) = \sin^2\left(\frac{\omega}{2}\right) + \mathrm{O}(t^{-2+\varepsilon}). \tag{4.1}$$

Using that

$$\sin^2\left(\frac{\phi}{2}\right) \ge \frac{1}{\pi^2}\,\phi^2 \ge \frac{1}{\pi^2}\,t^{-\varepsilon}, \quad \sin^2\left(\frac{\omega}{2}\right) \le \frac{1}{4}\,\omega^2,$$

we get that $\omega \ge \frac{1}{2}t^{-\frac{\varepsilon}{2}}$, for t large enough. This completes the proof. □

Proposition 4.2. *The solution to the extension problem satisfies*

$$\|v(t,\,\cdot)\|_{L^1(G \smallsetminus K(\exp \Omega_t)K)} \lesssim t^{-\sigma\varepsilon} \tag{4.2}$$

for $t > 0$ *large enough.*

Proof. Let $\xi > 0$ be a constant such that the compact support of v_0 belongs to $K(\exp B(0,\xi))K$. Then,

$$\begin{aligned}\int_{G \smallsetminus K(\exp \Omega_t)K} \mathrm{d}x\, |v(t,x)| &\lesssim \int_{K(\exp B(0,\xi))K} \mathrm{d}y\, |v_0(y)| \int_{G \smallsetminus K(\exp \Omega_t)K} \mathrm{d}x\, Q_t^{\sigma}(y^{-1}x) \\ &\lesssim \int_{K(\exp B(0,\xi))K} \mathrm{d}y\, |v_0(y)| \int_{G \smallsetminus K(\exp \Omega_t'')K} \mathrm{d}z\, Q_t^{\sigma}(z)\end{aligned}$$

where $\Omega_t'' \subseteq \mathfrak{a}$ is the region described in Lemma 4.1. Thus, working as in Proposition 3.4, one can show that the right-hand side of the inequality above is $\mathrm{O}(t^{-\sigma\varepsilon})$. In conclusion,

$$\int_{G \smallsetminus K(\exp \Omega_t)K} \mathrm{d}x\, |v(t,x)| \lesssim t^{-\sigma\varepsilon}.$$

□

4.2. Long-time behavior inside the critical region

Let now $x \in K(\exp \Omega_t)K$. By Lemma 3.9, the right-K-invariance of $A(k^{-1}.)$ and v_0, and the definition (2.8) of the Helgason–Fourier transform we have that

$$\begin{aligned} v_0 * Q_t^{\sigma}(x) - M\, Q_t^{\sigma}(x) &= \int_G \mathrm{d}y\, (Q_t^{\sigma}(y^{-1}x) - Q_t^{\sigma}(x))v_0(y) \\ &= Q_t^{\sigma}(x) \int_G \mathrm{d}y \left(\frac{Q_t^{\sigma}(y^{-1}x)}{Q_t^{\sigma}(x)} - 1 \right) v_0(y) \\ &= Q_t^{\sigma}(x) \left\{ \int_G \mathrm{d}y \left(e^{\langle 2\rho, A(k^{-1}y)\rangle} - 1 + \mathrm{O}(t^{-\frac{\varepsilon}{2}}) \right) v_0(y) \right\} \\ &= Q_t^{\sigma}(x) \left(\widehat{v_0}(i\rho, k\mathbb{M}) - \widehat{v_0}(-i\rho, k\mathbb{M}) + \mathrm{O}\big(t^{-\frac{\varepsilon}{2}}\big) \right). \end{aligned} \tag{4.3}$$

Notice that $\widehat{v_0}(\pm i\rho, k\mathbb{M}) = \mathcal{H}v_0(\pm i\rho) = M$ when v_0 is bi-K-invariant. Then we deduce the desired convergence by integrating (4.3) over the critical region:

$$\int_{K(\exp \Omega_t)K} \mathrm{d}x\, |v_0 * Q_t^{\sigma}(x) - M\, Q_t^{\sigma}(x)| = \mathrm{O}(t^{-\frac{\varepsilon}{2}}). \tag{4.4}$$

On the other hand, using again the Cartan decomposition we have

$$\begin{aligned} &\int_{K(\exp \Omega_t)K} \mathrm{d}x\, |v_0 * Q_t^{\sigma}(x) - M\, Q_t^{\sigma}(x)| \\ &\longrightarrow \int_K \mathrm{d}k \left| \int_G \mathrm{d}y\, v_0(y) \big(e^{\langle 2\rho, A(k^{-1}y)\rangle} - 1 \big) \right| \end{aligned}$$

as $t \to +\infty$. The last integral is not constantly zero when v_0 is not bi-K-invariant. For example, consider v_0 to be a Dirac measure supported on some point yK other than the origin, thus for $y \notin K$. In other words, the solution now coincides with $Q_t(., yK)$ and the mass is equal to 1. In this case, however, the last integral is equal to $\int_K \mathrm{d}k \left| e^{\langle 2\rho, A(k^{-1}y)\rangle} - 1 \right|$, and thus does not vanish identically.

4.3. Long-time convergence for general bi-K-invariant data

In this subsection, using the results of the previous two ones and a standard density argument, we prove Theorem 1.1 for the whole class of $L^1(\mathbb{X})$ functions that are bi-K-invariant. The argument is identical to that of [7, Section 3.3] but we include it for the reader's convenience.

Proof of Theorem 1.1. Let $\varepsilon > 0$, $v_0 \in L^1(K\backslash G/K)$ and $V_0 \in \mathcal{C}_c^\infty(K\backslash G/K)$ be such that $\|v_0 - V_0\|_{L^1(\mathbb{X})} < \frac{\varepsilon}{3}$.

Denote by $M = \int_G v_0$ and $M_V = \int_G V_0$ the masses of v_0 and V_0, respectively, then

$$|M - M_V| \leq \|v_0 - V_0\|_{L^1(\mathbb{X})} < \tfrac{\varepsilon}{3}.$$

Let $V(t,x) = V_0 * Q_t^\sigma(x)$ be the solution to the extension problem with initial data V_0. We deduce from (4.4), (4.2) and Proposition 3.3 that, there exists $T > 0$ such that

$$\begin{aligned}\|V(t,\cdot) - M_V\, Q_t^\sigma\|_{L^1(\mathbb{X})} \leq\ & \|V(t,\cdot) - M_V\, Q_t^\sigma\|_{L^1(K(\exp\Omega_t)K)} \\ &+ \|V(t,\cdot)\|_{L^1(G\smallsetminus K(\exp\Omega_t)K)} \\ &+ |M_V|\,\|Q_t^\sigma\|_{L^1(G\smallsetminus K(\exp\Omega_t)K)} \\ <\ & \tfrac{\varepsilon}{3}\end{aligned}$$

for all $t \geq T$. In conclusion,

$$\begin{aligned}\|v(t,\cdot) - M\, Q_t^\sigma\|_{L^1(\mathbb{X})} \leq\ & \overbrace{\|v(t,\cdot) - V(t,\cdot)\|_{L^1(\mathbb{X})}}^{\leq \|v_0 - V_0\|_{L^1}\,\|Q_t^\sigma\|_{L^1}} + \overbrace{\|M_V\, Q_t^\sigma - M\, Q_t^\sigma\|_{L^1(\mathbb{X})}}^{\leq |M-M_V|\,\|Q_t^\sigma\|_{L^1}} \\ &+ \|V(t,\cdot) - M_V\, Q_t^\sigma\|_{L^1(\mathbb{X})} \\ <\ & \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon\end{aligned}$$

for all $\varepsilon > 0$ and t large enough. □

Let us turn to the long-time convergence in $L^p(\mathbb{X})$ with $p > 1$. We first deal with the case $p = \infty$ and conclude for all $1 < p < \infty$ by convexity. Proposition 3.3 gives us the sup norm estimate:

$$\begin{aligned}\|v(t,\cdot) - M\, Q_t^\sigma\|_{L^\infty(\mathbb{X})} &\leq \|v_0\|_{L^1(\mathbb{X})}\,\|Q_t^\sigma\|_{L^\infty(\mathbb{X})} + |M|\,\|Q_t^\sigma\|_{L^\infty(\mathbb{X})} \\ &\lesssim t^{\sigma - \frac{1}{2} - \frac{\ell}{2} - |\Sigma_r^+|} e^{-|\rho| t}\end{aligned} \tag{4.5}$$

for t large and for all $f \in L^1(\mathbb{X})$. Notice that such an estimate holds without the bi-K-invariance assumption. By convexity, we obtain the following estimates in the $L^p(\mathbb{X})$ setting.

Corollary 4.3. *Under the assumptions of Theorem 1.1, we have*

$$\|v(t,\cdot) - M\, Q_t^\sigma\|_{L^p(\mathbb{X})} = \mathrm{o}\big(t^{-\frac{1}{p'}(-\sigma + \frac{1}{2} + \frac{\ell}{2} + |\Sigma_r^+|)} e^{-\frac{|\rho| t}{p'}}\big) \quad \text{as} \quad t \to +\infty \tag{4.6}$$

for all $1 < p < \infty$.

Remark 4.4. The sup norm estimate (4.5) is weaker compared to the results in the Euclidean setting. More precisely, on $\mathbb{R}^n$, the Poisson semigroup ($\sigma = 1/2$) satisfies the strong convergence (1.7) (recall that $\| \mathcal{Q}_t \|_{L^\infty(\mathbb{R}^n)} \asymp t^{-n}$). However, this is not true on noncompact symmetric spaces. Indeed, in the lines of [7, Remark 3.6], consider the Poisson kernel $Q_t^{1/2}$ as well as a "delayed" Poisson kernel $Q_{t+t'}^{1/2}$ for some $t' > 0$ to be determined later. Recall that $\nu = \ell + 2|\Sigma_r^+|$. Then

$$t^{\frac{\nu}{2}} e^{|\rho| t} \left\| Q_{t+t'}^{1/2} - Q_t^{1/2} \right\|_{L^\infty(\mathbb{X})} \geq t^{\frac{\nu}{2}} e^{|\rho| t} \left(Q_t^{1/2}(eK) - Q_{t+t'}^{1/2}(eK) \right)$$

since $Q_t^{1/2}(eK)$ is decreasing in t, as seen by the subordination formula.

According to (3.3), there exists a constant $C \geq 1$ such that

$$\begin{aligned} & t^{\frac{\nu}{2}} e^{|\rho| t} \left(Q_t^{1/2}(eK) - Q_{t+t'}^{1/2}(eK) \right) \\ & \geq t^{\frac{\nu}{2}} e^{|\rho| t} \left\{ \tfrac{1}{C} t^{-\frac{\nu}{2}} e^{-|\rho| t} - C(t+t')^{-\frac{\nu}{2}} e^{-|\rho|(t+t')} \right\} \\ & = C^{-1} - C \left(\tfrac{t}{t+t'} \right)^{\frac{\nu}{2}} e^{-|\rho| t'} \geq \tfrac{1}{2C}, \end{aligned}$$

provided that $t' > \frac{2 \ln C + \ln 2}{|\rho|}$. Hence

$$t^{\frac{\nu}{2}} e^{|\rho| t} \| Q_{t+t'}^{1/2} - Q_t^{1/2} \|_{L^\infty(\mathbb{X})} \not\to 0 \quad \text{as} \quad t \to +\infty.$$

4.3.1. Final remarks on the rate of convergence

For bi-K-invariant compactly supported initial data v_0, (4.2) and (4.4) imply that

$$\|v_0 * Q_t^\sigma - M\, Q_t^\sigma \|_{L^1(\mathbb{X})} = \mathrm{O}(t^{-\mu\varepsilon}), \quad \mu = \min\{\sigma, 1/2\}.$$

In the case of Euclidean space, and more generally, of manifolds $\mathcal{M}$ of nonnegative Ricci curvature and certain generalizations of those, following some ideas from the Euclidean setting in [28,29], it was shown in [24] that one can prescribe any rate of convergence to solutions of the extension problem by choosing appropriate initial data. More precisely, it was shown that given any decreasing and positive function $\phi(t)$ such that $\phi(t) \to 0$ as $t \to +\infty$, there is a solution u with mass $M = 1$ satisfying

$$\| \left| u(t,.) - Q_t^\sigma(., x_0) \right| V(., t) \|_{L^\infty(\mathcal{M})} \gtrsim k\phi(t_k), \tag{4.7}$$

for a sequence of times $t_k \to +\infty$ that can be chosen (here, x_0 is an arbitrary base point on the manifold and $V(x,t)$ denotes the volume of a geodesic ball centered at $x \in \mathcal{M}$ and of radius $t > 0$). This solution u corresponds to initial data being an infinite sum of weighted Dirac masses. However, this approach breaks down immediately for symmetric spaces of noncompact type: we proved in Sect. 4.2 that solutions corresponding to initial data being a Dirac mass supported on any point other that the origin fail to converge to the fundamental solution.

To demonstrate, however, the effect of different initial data on the rate of convergence, we give a specific example: for $\sigma = 1/2$, let us consider the solution $Q_{t+t'}^{1/2}$, where t' is fixed (for the standard heat equation on the three-dimensional real hyperbolic space, a "delayed" heat kernel $h_{t+t'}$ was considered in [30] as an attempt to discuss different rates of convergence on (real) hyperbolic space. However, on the one hand, no explicit calculations/rates were given in [30], and on the other hand, the result was limited to $\mathbb{H}^3(\mathbb{R})$.). Clearly, this solution is bi-K-invariant, nonnegative everywhere on $\mathbb{X}$, and with total mass 1. Now, since the annuli $\{H \in \mathfrak{a} : (t+t')^{2-\varepsilon} \leq |H| \leq (t+t')^{2+\varepsilon}\}$ and $\{H \in \mathfrak{a} : t^{2-\varepsilon} \leq |H| \leq t^{2+\varepsilon}\}$ are comparable for t large enough, it is not hard to show that, if $\Omega_t \subseteq \mathfrak{a}$ is the flat part of the critical region described in Proposition 3.4, then

$$\begin{aligned}\left\| Q_{t+t'}^{1/2} - Q_t^{1/2} \right\|_{L^1(G \setminus K(\exp \Omega_t)K)} &\leq \| Q_{t+t'}^{1/2} \|_{L^1(G \setminus K(\exp \Omega_t)K)} \\ &\quad + \| Q_t^{1/2} \|_{L^1(G \setminus K(\exp \Omega_t)K)} \\ &\lesssim t^{-\frac{\varepsilon}{2}}. \end{aligned} \tag{4.8}$$

Next, for $x = k(\exp H)k' \in K(\exp \Omega_t)K$, using Theorem 3.6, we have

$$\frac{Q_{t+t'}^{1/2}(\exp H)}{Q_t^{1/2}(\exp H)} \sim \frac{t+t'}{t} \frac{\boldsymbol{b}\left(-i|\rho|\frac{H}{\sqrt{(t+t')^2+|H|^2}}\right)^{-1}}{\boldsymbol{b}\left(-i|\rho|\frac{H}{\sqrt{t^2+|H|^2}}\right)^{-1}} \frac{\sqrt{t^2+|H|^2}^{\frac{\ell}{2}+|\Sigma_r^+|+1}}{\sqrt{(t+t')^2+|H|^2}^{\frac{\ell}{2}+|\Sigma_r^+|+1}}$$
$$\times \exp\left\{-|\rho|(\sqrt{(t+t')^2+|H|^2} - \sqrt{t^2+|H|^2})\right\}.$$

Clearly,

$$\frac{t+t'}{t} = 1 + \mathrm{O}(t^{-1}).$$

Next, since $t^2 + |H|^2 \asymp |H|^2$ for $H \in \Omega_t$, working similarly to Lemma 3.9, one can show that

$$\frac{\boldsymbol{b}\left(-i|\rho|\frac{H}{\sqrt{(t+t')^2+|H|^2}}\right)^{-1}}{\boldsymbol{b}\left(-i|\rho|\frac{H}{\sqrt{t^2+|H|^2}}\right)^{-1}} = 1 + \mathrm{O}(t^{-3+2\varepsilon}),$$

$$\frac{\sqrt{t^2+|H|^2}^{\frac{\ell}{2}+|\Sigma_r^+|+1}}{\sqrt{(t+t')^2+|H|^2}^{\frac{\ell}{2}+|\Sigma_r^+|+1}} = 1 + \mathrm{O}(t^{-3+2\varepsilon}),$$

where as usual, $\nu = \ell + 2|\Sigma_r^+|$ is the dimension at infinity. Finally,

$$\begin{aligned}&\exp\left\{-|\rho|(\sqrt{(t+t')^2+|H|^2} - \sqrt{t^2+|H|^2})\right\} \\ &= \exp\left\{-|\rho|\frac{2tt'+t'^2}{\sqrt{(t+t')^2+|H|^2}+\sqrt{t^2+|H|^2}}\right\} \\ &= \exp\left\{\mathrm{O}(t^{-1+\varepsilon})\right\} = 1 + \mathrm{O}(t^{-1+\varepsilon}).\end{aligned}$$

Therefore,

$$\frac{Q_{t+t'}^{1/2}(\exp H)}{Q_t^{1/2}(\exp H)} = 1 + \mathrm{O}(t^{-1+\varepsilon}), \quad H \in \Omega_t,$$

which in turn implies that

$$\left\| Q_{t+t'}^{1/2} - Q_t^{1/2} \right\|_{L^1(K(\exp \Omega_t)K)} \lesssim t^{-1+\varepsilon}. \tag{4.9}$$

Thus, for $\sigma = 1/2$, we conclude by (4.8) and (4.9) choosing $\varepsilon = 2/3$ that a "delayed" Poisson kernel converges to the fundamental solution in $L^1(\mathbb{X})$ in the following rate:

$$\left\| Q_{t+t'}^{1/2} - Q_t^{1/2} \right\|_{L^1(\mathbb{X})} \lesssim t^{-1/3}.$$

5. Asymptotic convergence associated with the extension problem for the distinguished Laplacian

Let $S = N(\exp \mathfrak{a}) = (\exp \mathfrak{a})N$ be the solvable group occurring in the Iwasawa decomposition $G = N(\exp \mathfrak{a})K$. Then S is identifiable, as a manifold, with the symmetric space $\mathbb{X} = G/K$. The distinguished Laplacian $\widetilde{\Delta}$ on S is given by the conjugation of the shifted Laplace–Beltrami operator $\Delta + |\rho|^2$ on $\mathbb{X}$:

$$\widetilde{\Delta} = \widetilde{\delta}^{\frac{1}{2}} \circ (\Delta + |\rho|^2) \circ \widetilde{\delta}^{-\frac{1}{2}} \tag{5.1}$$

where the modular function $\widetilde{\delta}$ of S is defined by

$$\widetilde{\delta}(g) = \widetilde{\delta}(n(\exp A)) = e^{-2\langle \rho, A \rangle} \quad \forall\, g \in S.$$

Here $n = n(g)$ and $A = A(g)$ denote, respectively, the N-component and the $\mathfrak{a}$-component of g in the Iwasawa decomposition.

The distinguished Laplacian $\widetilde{\Delta}$ is left-S-invariant and self-adjoint with respect to the right-invariant Haar measure on S:

$$\int_S \mathrm{d}_r g\, f(g) = \int_N \mathrm{d}n \int_{\mathfrak{a}} \mathrm{d}A\, f(n(\exp A)) = \int_{\mathfrak{a}} \mathrm{d}A\, e^{2\langle \rho, A \rangle} \int_N \mathrm{d}n f((\exp A)n).$$

The connection between the measures on S and the unimodular Haar measure on G is given as follows,

$$\int_S \mathrm{d}_r g\, f(g) = \int_G \mathrm{d}g\, e^{2\langle \rho, A(g) \rangle} f(g) \quad \text{and} \quad \int_S \mathrm{d}_\ell g\, f(g) = \int_G \mathrm{d}g\, f(g). \tag{5.2}$$

Recall the heat equation associated with the distinguished Laplacian:

$$\partial_t \widetilde{v}(t, g) = \widetilde{\Delta}_g \widetilde{v}(t, g), \quad \widetilde{v}(0, g) = f(g), \tag{5.3}$$

where the corresponding heat kernel is given by $\widetilde{h}_t = \widetilde{\delta}^{1/2}\, e^{|\rho|^2 t} h_t$ in the sense that

$$(e^{t\widetilde{\Delta}} f)(g) \;=\; (f * \widetilde{h}_t)(g) \;=\; \int_S \mathrm{d}_\ell y\; f(y)\, \widetilde{h}_t(y^{-1}g) \;=\; \int_S \mathrm{d}_r y\; f(gy^{-1})\, \widetilde{h}_t(y).$$

Here, we still denote by $*$ the convolution product on S or on G. We refer to [11,14] for more details about the distinguished Laplacian.

We now pass to fractional powers of the distinguished Laplacian as well to the associated extension problem, and for that we follow the approach by Stinga and Torrea [27], in terms of functional calculus. It is well-known that $-\widetilde{\Delta}$ is non-negative and (essentially) self-adjoint with respect to the right-invariant Haar measure, which will be our reference measure henceforth, [15]. Using the spectral resolution

$$-\widetilde{\Delta} = \int_0^\infty \xi\, dE(\xi),$$

where the $E(\xi)$ are self-adjoint projections, the spectral theorem implies that given a real measurable function m on $[0, \infty)$, the operator $m(-\widetilde{\Delta})$ can be given formally by

$$m(-\widetilde{\Delta}) = \int_0^\infty m(\xi)\, dE(\xi).$$

We can then define the fractional operators $(-\widetilde{\Delta})^\sigma$, $0 < \sigma < 1$ with domain $\mathrm{Dom}((-\widetilde{\Delta})^\sigma) \subseteq \mathrm{Dom}(-\widetilde{\Delta})$ by

$$(-\widetilde{\Delta})^\sigma = \int_0^\infty \xi^\sigma\, dE(\xi) = \int_0^\infty \frac{\mathrm{d}u}{u^{1+\sigma}}\, (e^{u\widetilde{\Delta}} - \mathrm{Id})\, e^{-\frac{t^2}{4u}}.$$

The general approach of [27, Theorem 1.1] thus applies and shows that if for $\widetilde{v}_0 \in \mathrm{Dom}(-\widetilde{\Delta})^\sigma$ one considers the boundary problem

$$\begin{cases} \widetilde{\Delta}\widetilde{v} - \frac{(1-2\sigma)}{t}\partial_t \widetilde{v} - \partial_{tt}^2 \widetilde{v} = 0, \quad t > 0, \\ \widetilde{u}(\,\cdot\,,0) = \widetilde{v}_0, \end{cases} \tag{5.4}$$

then up to a multiplicative constant, depending only on σ, one can recover the fractional powers of the Laplacian:

$$-\lim_{t\to 0^+} t^{1-2\sigma}\, \partial_t \widetilde{v}(t,x) = (-\widetilde{\Delta})^\sigma \widetilde{v}_0(x), \quad x \in S.$$

In order to pass to a fractional Poisson kernel related to the extension problem for the distinguished Laplacian, we again follow the approach of [27, Theorem 2.1], based on subordination to the heat semigroup and certain properties that the latter should possess, [27, p. 2100]. To this end, recall that the heat diffusion semigroup on S acts as follows:

$$(e^{t\widetilde{\Delta}} f)(g) = (f * \widetilde{h}_t)(g) = \int_S \mathrm{d}_\ell y\; f(y)\, \widetilde{h}_t(y^{-1}g) = \int_S \mathrm{d}_r y\; f(y)\, \widetilde{\delta}(y)\, \widetilde{h}_t(y^{-1}x),$$

where we used (5.2) to switch to the right-invariant Haar measure. Set

$$\widetilde{H}_t(x,y) := \widetilde{\delta}(y)\,\widetilde{h}_t(y^{-1}x) = e^{|\rho|^2 t}\,\widetilde{\delta}^{1/2}(x)\,\widetilde{\delta}^{1/2}(y)\,h_t(y^{-1}x), \quad x,y\in S.$$

Clearly, $\widetilde{H}_t(\cdot,\cdot)$ serves as an integral kernel for the action of $e^{t\widetilde{\Delta}}$ and it is symmetric, since $h_t(g) = h_t(g^{-1})$ for all $g \in G$).

It follows that

$$\begin{aligned}\|\widetilde{H}_t(x,\cdot)\|^2_{L^2(S,\mathrm{d}_r)} &= e^{2|\rho|^2 t}\int_S \mathrm{d}_r y\,\widetilde{\delta}(y)\,\widetilde{\delta}(x)\,h_t^2(y^{-1}x)\\ &= e^{2|\rho|^2 t}\,\widetilde{\delta}(x)\int_G \mathrm{d}y\,h_t^2(y^{-1}x)\\ &= e^{2|\rho|^2 t}\,\widetilde{\delta}(x)\,\|h_t(xK,\cdot Kx)\|^2_{L^2(G/K)}\end{aligned}$$

(notice that we used again (5.2)) which implies that by (3.2),

$$\|\widetilde{H}_t(x,\cdot)\|_{L^2(S,\mathrm{d}_r)} \lesssim \widetilde{\delta}^{1/2}(x)\,t^{-n/4}(1+t^{n/4}) \quad \forall x\in S, \quad \forall t>0.$$

Furthermore, by the $L^2(G/K)$ upper bound for the time derivative in (3.2) and the simple computation

$$\partial_t\widetilde{H}_t(x,y) = e^{|\rho|^2 t}\,\widetilde{\delta}^{1/2}(x)\,\widetilde{\delta}^{1/2}(y)\,(|\rho|^2 h_t(y^{-1}x) + \partial_t h_t(y^{-1}x)),$$

we obtain as before that

$$\|\partial_t\widetilde{H}_t(x,\cdot)\|_{L^2(S,d_r)} \lesssim \widetilde{\delta}^{1/2}(x)\,t^{-n/4-1}(1+t^{n/4+1}) \quad \forall x\in S, \quad \forall t>0.$$

Finally, observe that $\widetilde{H}_t$ satisfies the "distinguished" heat equation:

$$\begin{aligned}\partial_t\widetilde{H}_t(x,y) &= \widetilde{\delta}(y)\,\partial_t\widetilde{h}_t(y^{-1}x) = \widetilde{\delta}(y)\,\widetilde{\Delta}_x\widetilde{h}_t(y^{-1}x) = \widetilde{\Delta}_x(\widetilde{\delta}(y)\,\widetilde{h}_t(y^{-1}x))\\ &= \widetilde{\Delta}_x\widetilde{H}_t(x,y).\end{aligned}$$

Next, define the function $\widetilde{\mathcal{P}}_t^\sigma : S\times S \to (0,+\infty)$ by

$$\widetilde{\mathcal{P}}_t^\sigma(x,y) := \frac{t^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)}\int_0^\infty \frac{\mathrm{d}u}{u^{1+\sigma}}\,\widetilde{H}_u(x,y)\,e^{-\frac{t^2}{4u}}, \quad t>0, \quad 0<\sigma<1,$$

which is clearly well-defined, positive and symmetric, and define also the function $\widetilde{Q}_t^\sigma : S \to (0,+\infty)$ by

$$\widetilde{Q}_t^\sigma(g) = \frac{t^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)}\int_0^{+\infty}\frac{\mathrm{d}u}{u^{1+\sigma}}\,\widetilde{h}_u(g)\,e^{-\frac{t^2}{4u}} \tag{5.5}$$

$$= \widetilde{\delta}^{\frac{1}{2}}(g)\frac{t^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)}\int_0^{+\infty}\frac{\mathrm{d}u}{u^{1+\sigma}}\,e^{|\rho|^2 u}\,h_u(g)\,e^{-\frac{t^2}{4u}} =: \widetilde{\delta}^{\frac{1}{2}}(g)\,Q^{\sigma,0}(g). \tag{5.6}$$

Observe then that

$$\begin{aligned}\widetilde{T}_t^{\sigma} f(x) := f * \widetilde{Q}_t^{\sigma}(x) &= \int_S \mathrm{d}_\ell y\, \widetilde{Q}_t^{\sigma}(y^{-1}x) f(y)\\ &= \int_S \mathrm{d}_r y\, \widetilde{\delta}(y)\, \widetilde{Q}_t^{\sigma}(y^{-1}x) f(y)\\ &= \int_S \mathrm{d}_r y\, \widetilde{\delta}^{1/2}(x)\, \widetilde{\delta}^{1/2}(y)\, Q_t^{\sigma,0}(y^{-1}x) f(y)\\ &= \int_S \mathrm{d}_r y\, \widetilde{\mathcal{P}}_t^{\sigma}(x,y) f(y),\end{aligned}$$

where we used (5.2) for the first change of measures. Therefore, $\widetilde{\mathcal{P}}_t^{\sigma}(\cdot,\cdot)$ serves as an integral kernel for the action of $\widetilde{T}_t^{\sigma}$.

Having all the ingredients above, one can follow the approach of [27, Theorem 2.1] to show that $\widetilde{v}(\cdot,t) := \widetilde{T}_t^{\sigma}\widetilde{v}_0$ solves on S the boundary problem (5.4).

Remark 5.1. Notice that $\widetilde{Q}_t^{\sigma}(g)\mathrm{d}_r g$ is a probability measure on S. Indeed, this follows from the subordination formula (5.5) and the fact that $\int_S \mathrm{d}_r g\, \widetilde{h}_t(g) = 1$.

The first subsection is devoted to determine the critical region where the kernel $\widetilde{Q}_t^{\sigma}$ concentrates. In the next two subsections, we study, respectively, the L^1 and the L^∞ asymptotic convergences of solutions to (5.3) with compactly supported initial data (no bi-K-invariance required). We discuss the same questions for other initial data in the last subsection.

5.1. Asymptotic concentration of the fractional Poisson kernel associated to the distinguished Laplacian

We first give large-time asymptotics for the distinguished extension kernel. More precisely, we prove the following upper and lower bounds.

Proposition 5.2. *The fractional Poisson kernel $\widetilde{Q}_t^{\sigma}$ on S, $0<\sigma<1$, associated with the distinguished Laplacian, satisfies the upper and lower bounds*

$$\widetilde{Q}_t^{\sigma}(g) \asymp \widetilde{\delta}^{\frac{1}{2}}(g)\, \varphi_0(g)\, t^{2\sigma}\, (t+|g^+|)^{-\ell-2|\Sigma_r^+|-2\sigma}, \tag{5.7}$$

if $t^2+|g^+|^2 \geq 1$.

Proof. Recall that by the subordination formula (5.6), we may write

$$\begin{aligned}\frac{2^{2\sigma}\Gamma(\sigma)}{t^{2\sigma}}\widetilde{\delta}^{-\frac{1}{2}}(g)\, \widetilde{Q}_t^{\sigma}(g) &= \int_0^{t^2+|g|^2} \frac{\mathrm{d}u}{u^{1+\sigma}}\, h_u(g)\, e^{|\rho|^2 u}\, e^{-\frac{t^2}{4u}}\\ &\quad + \int_{t^2+|g|^2}^{+\infty} \frac{\mathrm{d}u}{u^{1+\sigma}}\, h_u(g)\, e^{|\rho|^2 u}\, e^{-\frac{t^2}{4u}},\end{aligned}$$

and observe that due to (2.10) we have

$$u^{-1-\sigma}\, h_u(g)\, e^{|\rho|^2 u}\, e^{-\frac{t^2}{4u}} \asymp \varphi_0(g)\, u^{-\frac{n}{2}-1-\sigma} \left\{ \prod_{\alpha\in\Sigma_r^+} (1+u+\langle\alpha, g^+\rangle)^{\frac{m_\alpha+m_{2\alpha}}{2}-1} \right\} e^{-\frac{t^2+|g^+|^2}{4u}}, \quad u>0.$$

Let $t^2+|g^+|^2 \geq 1$. As far as upper bounds are concerned, the claim follows in the first interval $(0, t^2+|g^+|^2]$ by estimating $1+u+\langle\alpha, g^+\rangle \lesssim t^2+|g^+|^2$. For the second interval $(t^2+|g^+|^2, +\infty)$ we estimate $1+u+\langle\alpha, g^+\rangle \lesssim u$, and take into account that $\sum_{\alpha\in\Sigma_r^+}(m_\alpha+m_{2\alpha}) = n-\ell$.

The lower bound follows writing

$$1+u+\langle\alpha, g^+\rangle \geq u$$

and integrating over $(t^2+|g|^2, +\infty)$. We omit the details. □

Recall that the extension kernel Q_t^σ associated with the Laplace–Beltrami operator concentrates in $K(\exp\Omega_t)K$, where Ω_t is described in Proposition 3.4.

The following proposition shows that the kernel $\widetilde{Q}_t^\sigma$ (associated with the extension problem for the distinguished Laplacian) concentrates in a different region.

Proposition 5.3. *Let* $0<\varepsilon<1$. *Consider in* $\mathfrak{a}$ *the annulus*

$$\widetilde{\Omega}_t = B(0, t^{1+\varepsilon}) \smallsetminus B(0, t^{1-\varepsilon}). \tag{5.8}$$

Then, the fractional Poisson kernel associated with the distinguished Laplacian on S *concentrates asymptotically in* $K(\exp\widetilde{\Omega}_t)K$. *In other words,*

$$\lim_{t\to+\infty} \int_{g\in S \text{ s.t. } g^+\in\overline{\mathfrak{a}^+}\smallsetminus\widetilde{\Omega}_t} d_r g\, \widetilde{Q}_t^\sigma(g) = 0$$

where g^+ *denotes the middle component of* g *in the Cartan decomposition.*

Proof. By using (5.2) and (5.6), write

$$I(t) = \int_{g\in S \text{ s.t. } g^+\in\overline{\mathfrak{a}^+}\smallsetminus\widetilde{\Omega}_t} \mathrm{d}_r g\, \widetilde{Q}_t^\sigma(g) = \int_{G\smallsetminus K(\exp\widetilde{\Omega}_t)K} \mathrm{d}g\, e^{\langle\rho, A(g)\rangle}\, Q_t^{\sigma,0}(g).$$

Since $Q_t^{\sigma,0}$ is bi-K-invariant on G, writing $\mathrm{d}k$ for the normalized Haar measure on the compact group K, and using Proposition 5.2 we have

$$\begin{aligned} I(t) &\asymp t^{2\sigma} \int_{G\smallsetminus K(\exp\widetilde{\Omega}_t)K} \mathrm{d}g\, \varphi_0(g)\, (t+|g|)^{-\ell-2|\Sigma_r^+|-2\sigma} \int_K \mathrm{d}k\, e^{\langle\rho, A(kg)\rangle} \\ &\asymp t^{2\sigma} \int_{G\smallsetminus K(\exp\widetilde{\Omega}_t)K} \mathrm{d}g\, \varphi_0(g)^2\, (t+|g|)^{-\ell-2|\Sigma_r^+|-2\sigma}. \end{aligned}$$

According to the Cartan decomposition, and to the estimates (2.2) and (2.5), we obtain

$$I(t) \asymp t^{2\sigma} \int_{\widetilde{\Omega}_t^c} \mathrm{d}g^+ \, \delta(g^+) \, (t+|g^+|)^{-\ell-2|\Sigma_r^+|-2\sigma} \, \varphi_0(\exp g^+)^2$$
$$\lesssim t^{2\sigma} \int_{\widetilde{\Omega}_t^c} \mathrm{d}g^+ \, (t+|g^+|)^{-\ell-2|\Sigma_r^+|-2\sigma} \, (1+|g^+|)^{2|\Sigma_r^+|}. \tag{5.9}$$

Next, let us study the right-hand side of (5.9) outside $\widetilde{\Omega}_t$. On the one hand, if $|g^+| < t^{1-\varepsilon}$ then $t+|g^+| \asymp t$, so

$$t^{2\sigma} \int_{|g^+|<t^{1-\varepsilon}} \mathrm{d}g^+ \, (t+|g^+|)^{-\ell-2|\Sigma_r^+|-2\sigma} (1+|g^+|)^{2|\Sigma_r^+|}$$
$$\lesssim t^{2\sigma} \int_0^{t^{1-\varepsilon}} \mathrm{d}r \, r^{\ell-1} t^{-\ell-2|\Sigma_r^+|-2\sigma} \, (1+r)^{2|\Sigma_r^+|}$$
$$\lesssim t^{-\varepsilon(\ell+2|\Sigma_r^+|)}.$$

On the other hand, if $|g^+| > t^{1+\varepsilon}$ then $t+|g^+| \asymp |g^+|$, so we have

$$t^{2\sigma} \int_{|g^+|>t^{1+\varepsilon}} \mathrm{d}g^+ \, (t+|g^+|)^{-\ell-2|\Sigma_r^+|-2\sigma} (1+|g^+|)^{2|\Sigma_r^+|}$$
$$\lesssim t^{2\sigma} \int_{t^{1+\varepsilon}}^{+\infty} \mathrm{d}r \, r^{\ell-1} r^{-\ell-2|\Sigma_r^+|-2\sigma} \, (1+r)^{2|\Sigma_r^+|}$$
$$\lesssim t^{-\varepsilon\sigma}.$$

In other words, we have proved that $I(t) = \mathrm{O}(t^{-\varepsilon\sigma})$; therefore, the Poisson kernel $\widetilde{Q}_t^\sigma$ associated with the distinguished Laplacian on S concentrates asymptotically in $K(\exp \widetilde{\Omega}_t)K$. □

Remark 5.4. The critical region for the fractional Poisson kernel associated with the distinguished Laplacian is similar to that of its Euclidean counterpart.

We now obtain precise long-time asymptotics of the kernel

$$Q_t^{\sigma,0}(g) = \frac{t^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)} \int_0^{+\infty} \frac{\mathrm{d}u}{u^{1+\sigma}} \, e^{|\rho|^2 u} \, h_u(g) \, e^{-\frac{t^2}{4u}},$$

which are crucial for our proof, by a slightly more general result.

Theorem 5.5. *Let $\sigma \in (0,1)$ and $g \in S$ such that $g^+ \in \widetilde{\Omega}_t$. Then, as $t+|g^+| \to +\infty$, we have*

$$Q_t^{\sigma,0}(g) \sim \widetilde{C}(\sigma) \, t^{2\sigma} \, \varphi_0(\exp g^+) \left(t^2+|g^+|^2\right)^{-\frac{\ell}{2}-|\Sigma_r^+|-\sigma}, \tag{5.10}$$

where the constant is

$$\widetilde{C}(\sigma) = \frac{1}{\Gamma(\sigma)} C_0 \, 2^{\ell+|\Sigma_r^+|} \pi^{\frac{\ell}{2}} \Gamma\left(\frac{\ell}{2}+|\Sigma_r^+|+\sigma\right) \boldsymbol{\pi}(\widetilde{\rho}) \boldsymbol{b}(0)^{-2},$$

with $\widetilde{\rho} = \frac{1}{2}\sum_{\alpha\in\Sigma_r^+} \alpha$ and $C_0 = 2^{n-\ell}/(2\pi)^\ell |K/\mathbb{M}|$.

Proof. The proof follows arguments for the asymptotics of the Poisson kernel ($\sigma = 1/2$) in [5, Theorem 5.3.1].

Let $0 < \eta < 1$ and $t^2 + |g^+|^2 > 1$. In view of the subordination formula (5.6), let us split

$$Q_t^{\sigma,0}(g) = \frac{t^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)} \int_0^{+\infty} \frac{\mathrm{d}u}{u^{1+\sigma}}\, e^{|\rho|^2 u}\, h_u(g)\, e^{-\frac{t^2}{4u}}$$
$$= \frac{t^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)} \{J_1 + J_2 + J_3\},$$

where the quantities J_1, J_2 and J_3 are defined by the integration over the intervals $\big[0, (t^2+|g|^2)^{1-\eta}\big)$, $\big[(t^2+|g|^2)^{1-\eta}, (t^2+|g|^2)^{1+\eta}\big)$ and $\big[(t^2+|g|^2)^{1+\eta}, \infty\big)$, respectively.

We claim that the main contribution comes from the middle integral J_2. Indeed, for the first integral J_1, working as in Proposition 5.2, we get $J_1 = \mathrm{O}((t^2+|g|^2)^{-\infty}\varphi_0(g))$ while, similarly, for the third integral we get $J_3 = \mathrm{O}((t^2+|g|^2)^{-(1+\eta)(\frac{\ell}{2}+|\Sigma_r^+|+\sigma)}\varphi_0(g))$.

We now consider J_2. Define

$$h'(t, g^+) = t^{\frac{\ell}{2}+|\Sigma_r^+|}\, \varphi_0(\exp g^+)^{-1}\, e^{|\rho|^2 t + \frac{|g^+|^2}{4t}}\, h_t(\exp g^+), \quad t > 0, \quad g^+ \in \overline{\mathfrak{a}^+}. \tag{5.11}$$

Then, by (5.11) we have

$$J_2 = \int_{(t^2+|g|^2)^{1-\eta}}^{(t^2+|g|^2)^{1+\eta}} \frac{\mathrm{d}u}{u^{1+\sigma}}\, e^{|\rho|^2 u} h_u(\exp g^+)\, e^{-\frac{t^2}{4u}}$$
$$= \varphi_0(\exp g^+) \int_{(t^2+|g|^2)^{1-\eta}}^{(t^2+|g|^2)^{1+\eta}} \mathrm{d}u\, u^{-\frac{\ell}{2}-|\Sigma_r^+|-\sigma-1} e^{-\frac{t^2+|g^+|^2}{4u}} h'(u, g^+)$$
$$= 2^{\ell+2|\Sigma_r^+|+2\sigma} (t^2+|g^+|^2)^{-\frac{\ell}{2}-|\Sigma_r^+|-\sigma}$$
$$\int_{\frac{1}{4}(t^2+|g|^2)^{-\eta}}^{\frac{1}{4}(t^2+|g|^2)^{\eta}} \mathrm{d}u\, u^{\frac{\ell}{2}+|\Sigma_r^+|+\sigma-1} e^{-u} h'\left(\frac{t^2+|g^+|^2}{4u}, g^+\right).$$

Since

$$h'\left(\frac{t^2+|g^+|^2}{4u}, g^+\right) \longrightarrow C_2\, \boldsymbol{b}(0)^{-1},$$

uniformly as $t^2 + |g^+|^2 \to +\infty$, by (2.15) and contradiction (see [5, p. 1086]), the Laplace method we obtain that the last integral tends to

$$C_2\, \boldsymbol{b}(0)^{-1} \Gamma\left(\frac{\ell}{2} + |\Sigma_r^+| + \sigma\right).$$

Since J_1, J_3 are very small compared to J_2 for t large, substituting the value of C_2 (see (2.15)) we finally get the claimed asymptotics. □

5.2. Heat asymptotics in L^1 for compactly supported initial data

In this subsection, we investigate the long-time asymptotic convergence in $L^1(S)$ of solutions to the Cauchy problem (5.3), where the initial data $\widetilde{v}_0$ is assumed continuous and compactly supported in $B(eK, \xi)$. Let $\widetilde{\varphi}_0 = \widetilde{\delta}^{\frac{1}{2}}\varphi_0$ be the modified ground spherical function. The mass function is defined by

$$\widetilde{M}(g) = \frac{(\widetilde{v}_0 * \widetilde{\varphi}_0)(g)}{\widetilde{\varphi}_0(g)} \quad \forall\, g \in S. \tag{5.12}$$

By using the fact that the modular function $\widetilde{\delta}$ is a character on S, we can also write the mass as

$$\widetilde{M}(g) = \frac{1}{\widetilde{\delta}(g)^{\frac{1}{2}}\,\varphi_0(g)} \int_S \mathrm{d}_\ell y\, v_0(gK)\, \underbrace{\widetilde{\delta}(y)^{\frac{1}{2}}\widetilde{\delta}(y^{-1}g)^{\frac{1}{2}}}_{\widetilde{\delta}(g)^{\frac{1}{2}}}\, \varphi_0(y^{-1}g) = \frac{(v_0 * \varphi_0)(g)}{\varphi_0(g)} \tag{5.13}$$

where $v_0(gK) = \widetilde{\delta}(g)^{-\frac{1}{2}}\widetilde{v}_0(g)$ is a right K-invariant function on G, with compact support $(\operatorname{supp}\widetilde{v}_0)K$.

The following properties of the mass function were already observed in [7, Remarks 4.5 and 4.6]:

Remark 5.6. 1. If $\widetilde{v}_0 \in \mathcal{C}_c(S)$, then the mass function $\widetilde{M}$ is bounded. This follows from the fact that

$$\frac{\varphi_0(y^{-1}g)}{\varphi_0(g)} \le C(\xi) \quad \text{if} \quad |y| < \xi. \tag{5.14}$$

2. The mass function $\widetilde{M}$ is a constant if v_0 is bi-K-invariant and $\widetilde{v}_0 = \widetilde{\delta}^{\frac{1}{2}}v_0$ belongs to $L^1(S)$:

$$\widetilde{M} = \int_G \mathrm{d}y\, v_0(y)\,\varphi_0(y) = \mathcal{H}v_0(0).$$

The following lemma plays a key role in the proof of Theorem 1.4.

Lemma 5.7. *For bounded $y \in G$ and for all g in the critical region $K(\exp\widetilde{\Omega}_t)K$, the following asymptotic behavior holds:*

$$\frac{Q_t^{\sigma,0}(y^{-1}g)}{Q_t^{\sigma,0}(g)} - \frac{\varphi_0(y^{-1}g)}{\varphi_0(g)} = \mathrm{O}\Big(t^{-1+\varepsilon(\nu+2\sigma-1)}\Big) \quad \text{as} \quad t \to +\infty,$$

for $0 < \varepsilon < 1/(\nu + 2\sigma - 1)$. *Here,* $\nu = \ell + 2|\Sigma_r^+|$.

Proof. Assume that $|y| \le \xi$ for some positive constant ξ. Recall that for every $H \in \widetilde{\Omega}_t$, we have $t^{1-\varepsilon} \le |H| \le t^{1+\varepsilon}$. Notice also that

$$|(y^{-1}g)^+ - g^+| \le |y| < \xi$$

according to (2.3). Then, for t large enough, we deduce the following estimates:

$$\begin{cases}|(y^{-1}g)^{+}| \leq |g^{+}|+\xi < t^{1+\varepsilon}+\xi < 2\,t^{1+\varepsilon},\\ |(y^{-1}g)^{+}| \geq |g^{+}|-\xi > t^{1-\varepsilon}-\xi > \frac{1}{2}\,t^{1-\varepsilon}.\end{cases}$$

In other words, we obtain

$$y^{-1}g \in K(\exp\widetilde{\Omega}'_t)K \qquad \forall\, g \in K(\exp\widetilde{\Omega}_t)K,\ \forall\,|y|<\xi,$$

where

$$\widetilde{\Omega}'_t = \left\{g \in S:\ \frac{1}{2}\,t^{1-\varepsilon} \leq |g^{+}| \leq 2\,t^{1+\varepsilon}\right\}.$$

Thus, the asymptotics of Theorem 5.5 yield

$$\frac{\widetilde{Q}_t^{\sigma}(y^{-1}g)}{\widetilde{Q}_t^{\sigma}(g)} - \frac{\varphi_0(y^{-1}g)}{\varphi_0(g)} \sim \frac{\varphi_0(y^{-1}g)}{\varphi_0(g)}\left(\frac{(t^2+|g^{+}|^2)^{\frac{\ell}{2}+|\Sigma_r^{+}|+\sigma}}{(t^2+|(y^{-1}g)^{+}|^2)^{\frac{\ell}{2}+|\Sigma_r^{+}|+\sigma}}-1\right).$$

On the one hand, the quotient of the ground spherical functions is bounded by the local Harnack inequality (5.14). On the other hand, using (3.17) for $r=|g^{+}|, s=|(y^{-1}g)^{+}|$ and $k=\frac{\ell}{2}+|\Sigma_r^{+}|+\sigma=\frac{\nu}{2}+\sigma>1$ (in the notation of (3.17), we now have $r_0 \lesssim t^{1+\varepsilon}$) and the trivial inequality $t^2+s^2 \geq t^2$, we get altogether

$$\frac{Q_t^{\sigma,0}(y^{-1}g)}{Q_t^{\sigma,0}(g)} - \frac{\varphi_0(y^{-1}g)}{\varphi_0(g)} = \mathrm{O}\Big(t^{-1+\varepsilon(\nu+2\sigma-1)}\Big) \quad \forall\, g \in K(\exp\widetilde{\Omega}_t)K, \quad \forall\,|y|<\xi.$$

□

Now, let us prove the first part of Theorem 1.4. The arguments follow those of [7] once Lemma 5.7 is at hand, but we include them for the reader's convenience.

Proof of (1.12) in Theorem 1.4. By using

$$(\widetilde{v}_0*\widetilde{\varphi}_0)(g) = \int_S \mathrm{d}_{\ell}y\, v_0(yK)\,\underbrace{\widetilde{\delta}(y)^{\frac{1}{2}}\,\widetilde{\delta}(y^{-1}g)^{\frac{1}{2}}}_{\widetilde{\delta}(g)^{\frac{1}{2}}}\,\varphi_0(y^{-1}g) = \widetilde{\delta}(g)^{\frac{1}{2}}(v_0*\varphi_0)(gK),$$

and the fact that $\widetilde{Q}_t^{\sigma}=\widetilde{\delta}^{\frac{1}{2}}\,Q_t^{\sigma,0}$, let us write the solution $\widetilde{v}$ to (5.4) as

$$\widetilde{v}(t,g) = (\widetilde{v}_0*\widetilde{Q}_t^{\sigma})(g) = \widetilde{\delta}(g)^{\frac{1}{2}}\,(v_0*Q_t^{\sigma,0})(g).$$

We aim to study the difference

$$\begin{aligned}\widetilde{v}(t,g)-\widetilde{M}(g)\widetilde{Q}_t^{\sigma}(g) &= \widetilde{Q}_t^{\sigma}(g)\,\frac{(v_0*Q_t^{\sigma,0})(g)}{Q_t^{\sigma,0}(g)} - \widetilde{Q}_t^{\sigma}(g)\,\frac{(v_0*\varphi_0)(g)}{\varphi_0(g)}\\ &= \widetilde{Q}_t^{\sigma}(g)\int_G \mathrm{d}y\, v_0(yK)\\ &\qquad\left\{\frac{Q_t^{\sigma,0}(y^{-1}g)}{Q_t^{\sigma,0}(g)} - \frac{\varphi_0(y^{-1}g)}{\varphi_0(g)}\right\}.\end{aligned} \tag{5.15}$$

According to the previous lemma, we have

$$\frac{Q_t^{\sigma,0}(y^{-1}g)}{Q_t^{\sigma,0}(g)} - \frac{\varphi_0(y^{-1}g)}{\varphi_0(g)} = \mathrm{O}\Big(t^{-1+\varepsilon(\nu+2\sigma-1)}\Big)$$
$$\forall\, g \in K(\exp\widetilde{\Omega}_t)K, \quad \forall\, y \in \operatorname{supp} v_0,$$

and therefore, the integral of $\widetilde{v}(t,\cdot) - \widetilde{M}\,\widetilde{Q}_t^\sigma$ over the critical region

$$\int_{S\cap K(\exp\widetilde{\Omega}_t)K} \mathrm{d}_r g\, |\widetilde{v}(t,g) - \widetilde{M}(g)\widetilde{Q}_t^\sigma(g)| \lesssim t^{-1+\varepsilon(\nu+2\sigma-1)} \underbrace{\int_S \mathrm{d}_r g\, \widetilde{Q}_t^\sigma(g)}_{1} \underbrace{\int_G \mathrm{d}y\, |v_0(yK)|}_{\text{const.}}$$

tends asymptotically to 0. Finally, we claim that the integral

$$\int_{S\smallsetminus K(\exp\widetilde{\Omega}_t)K} \mathrm{d}_r g\, |\widetilde{v}(t,g) - \widetilde{M}(g)\widetilde{Q}_t^\sigma(g)| \le \int_{S\smallsetminus K(\exp\widetilde{\Omega}_t)K} \mathrm{d}_r g\, |\widetilde{v}(t,g)| + \int_{S\smallsetminus K(\exp\widetilde{\Omega}_t)K} \mathrm{d}_r g\, |\widetilde{M}(g)|\widetilde{Q}_t^\sigma(g)$$

tends also to 0. On the one hand, we know that $\widetilde{M}$ is bounded and that the kernel $\widetilde{Q}_t^\sigma$ asymptotically concentrates in $K(\exp\widetilde{\Omega}_t)K$, hence

$$\int_{S\smallsetminus K(\exp\widetilde{\Omega}_t)K} \mathrm{d}_r g\, |\widetilde{M}(g)|\widetilde{Q}_t^\sigma(g) \longrightarrow 0$$

as $t \to +\infty$. On the other hand, notice that for all $y \in \operatorname{supp} v_0$ and for all $g \in G$ such that $g^+ \notin \widetilde{\Omega}_t$, using the triangle inequality one can show that

$$(y^{-1}g)^+ \notin \widetilde{\Omega}_t'' = \big\{H \in \overline{\mathfrak{a}^+} \,|\, 2\,t^{1-\varepsilon} \le |H| \le \tfrac{1}{2}\,t^{1+\varepsilon}\big\}. \tag{5.16}$$

Hence,

$$\int_{S\smallsetminus K(\exp\widetilde{\Omega}_t)K} \mathrm{d}_r g\, |\widetilde{v}(t,g)| \le \int_G \mathrm{d}y\, |v_0(yK)| \int_{G\smallsetminus K(\exp\widetilde{\Omega}_t)K} \mathrm{d}g\, \widetilde{\delta}(g)^{-\frac{1}{2}}\, Q_t^{\sigma,0}(y^{-1}g)$$
$$\lesssim \underbrace{\int_S \mathrm{d}_r y\, |\widetilde{v}_0(y)|}_{\|\widetilde{v}_0\|_{L^1(S)}} \underbrace{\int_{S\smallsetminus K(\exp\widetilde{\Omega}_t'')K} \mathrm{d}_r g\, \widetilde{Q}_t^\sigma(g)}_{\longrightarrow 0}.$$

This concludes the proof of the extension problem asymptotics in L^1 for the distinguished Laplacian $\widetilde{\Delta}$ on S and for initial data $\widetilde{v}_0 \in \mathcal{C}_c(S)$. □

5.3. Heat asymptotics in L^∞ for compactly supported initial data

We first recall the following lemma, which allows us to compare the "logarithms" of the middle components occurring in the Iwasawa decomposition and in the Cartan decomposition.

Lemma 5.8. [7, Lemma 4.8] *For all $g \in G$, we have*

$$\langle \rho, A(g) \rangle \leq \langle \rho, g^+ \rangle \tag{5.17}$$

where $A(g)$ denotes the $\mathfrak{a}$-component of g in the Iwasawa decomposition and g^+ denotes its $\overline{\mathfrak{a}^+}$-component in the Cartan decomposition.

In the following two propositions we collect some elementary properties of the extension problem kernel. The first one clarifies the lower and the upper bounds of $\widetilde{Q}_t^\sigma$, while the second one describes its critical region for the L^∞ norm.

Proposition 5.9. *The kernel $\widetilde{Q}_t^\sigma$ associated with the extension problem for the distinguished Laplacian satisfies*

$$\| \widetilde{Q}_t^\sigma \|_{L^\infty(S)} \asymp t^{-\ell - |\Sigma_r^+|} \tag{5.18}$$

for t large enough.

Proof. Using the global estimates (2.5) and (5.2), we have

$$\widetilde{Q}_t^\sigma(g) \asymp t^{2\sigma}\, e^{-\langle \rho, A(g) \rangle} e^{-\langle \rho, g^+ \rangle}\, (t + |g|)^{-\ell - 2|\Sigma_r^+| - 2\sigma} \left\{ \prod\nolimits_{\alpha \in \Sigma_r^+} 1 + \langle \alpha, g^+ \rangle \right\} \tag{5.19}$$

We obtain first the lower bound in (5.18) by evaluating the right-hand side of (5.19) at $g_0 = \exp(-t\rho)$ and by observing that

$$A(g_0) = -t\rho \quad \text{and} \quad g_0^+ = t\rho.$$

For the upper bound, notice that

$$e^{-\langle \rho, A(g) \rangle} e^{-\langle \rho, g^+ \rangle} \leq 1 \tag{5.20}$$

according to (5.17), and that

$$(t + |g|)^{-|\Sigma_r^+|} \left\{ \prod\nolimits_{\alpha \in \Sigma_r^+} 1 + \langle \alpha, g^+ \rangle \right\} \lesssim 1, \quad t^{2\sigma}\, (t + |g|)^{-\ell - |\Sigma_r^+| - 2\sigma} \lesssim t^{-\ell - |\Sigma_r^+|}$$

for t large enough, thus the claim follows from (5.19). □

Proposition 5.10. *The fractional Poisson kernel $\widetilde{Q}_t^\sigma$ concentrates asymptotically in the same critical region for the L^∞ norm as for the L^1 norm. In other words,*

$$t^{\ell + |\Sigma_r^+|}\, \| \widetilde{Q}_t^\sigma \|_{L^\infty(S \smallsetminus K(\exp \widetilde{\Omega}_t)K)} \longrightarrow 0 \quad \text{as} \quad t \to +\infty.$$

Proof. Let us study the sup norm of $\widetilde{Q}_t^{\sigma}$ outside the critical region. From (5.19) and (5.20) we deduce that

$$t^{\ell+|\Sigma_r^+|}\,\widetilde{Q}_t^{\sigma}(g) \lesssim t^{2\sigma+\ell+|\Sigma_r^+|}(t+|g^+|)^{-2\sigma-\ell-2|\Sigma_r^+|}(1+|g^+|^{|\Sigma_r^+|}). \tag{5.21}$$

Case 1: Assume that $|g^+| < t^{1-\varepsilon}$. Then $t+|g^+| \asymp t$ and $(1+|g^+|)^{|\Sigma_r^+|} \lesssim t^{(1-\varepsilon)|\Sigma_r^+|}$. Thus, we deduce from (5.21) that

$$t^{\ell+|\Sigma_r^+|}\,\widetilde{Q}_t^{\sigma}(g) \lesssim t^{-\varepsilon|\Sigma_r^+|}$$

which tends to 0.

Case 2: Assume that $|g^+| > t^{1+\varepsilon}$. Then $t+|g^+| \asymp |g^+|$; therefore,

$$t^{\ell+|\Sigma_r^+|}\,\widetilde{Q}_t^{\sigma}(g) \lesssim t^{2\sigma+\ell+|\Sigma_r^+|}|g^+|^{-2\sigma-\ell-|\Sigma_r^+|} \lesssim t^{-\varepsilon(\ell+|\Sigma_r^+|+2\sigma)}$$

which tends to 0. This completes the proof. □

Finally, let us prove the remaining part of Theorem 1.4.

Proof of (1.13) in Theorem 1.4. Fix $0<\varepsilon<\frac{1}{\nu+2\sigma-1}$. Consider the function

$$t \mapsto \epsilon(t), \quad \epsilon(t) = t^{-1+\varepsilon(\nu+2\sigma-1)} \longrightarrow 0, \quad \text{as} \quad t \to +\infty.$$

In the critical region $S \cap K(\exp\widetilde{\Omega}_t)K$, we have

$$|\widetilde{v}(t,g) - \widetilde{M}(g)\widetilde{Q}_t^{\sigma}(g)| \le \widetilde{Q}_t^{\sigma}(g)\int_{|y|<\xi} \mathrm{d}g\,|v_0(yK)|\left|\frac{Q_t^{\sigma,0}(y^{-1}g)}{Q_t^{\sigma,0}(g)} - \frac{\varphi_0(y^{-1}g)}{\varphi_0(g)}\right|$$

with

$$\left|\frac{Q_t^{\sigma,0}(y^{-1}g)}{Q_t^{\sigma,0}(g)} - \frac{\varphi_0(y^{-1}g)}{\varphi_0(g)}\right| \lesssim \epsilon(t)$$

according to (5.15) and to Lemma 5.7. Then we deduce from (5.18) that

$$t^{\ell+|\Sigma_r^+|}|\widetilde{v}(t,g) - \widetilde{M}(g)\widetilde{Q}_t^{\sigma}(g)| \lesssim \epsilon(t) \quad \forall\, g \in S \cap K(\exp\widetilde{\Omega}_t)K$$

where the right-hand side tends to 0 as $t \to +\infty$.

Outside the critical region, we estimate separately $\widetilde{v}(t,g)$ and $\widetilde{M}(g)\widetilde{Q}_t^{\sigma}(g)$. On the one hand, we know that $\widetilde{M}(g)$ is a bounded function and that $\widetilde{Q}_t^{\sigma}(g) = \mathrm{o}(t^{-\ell-|\Sigma_r^+|})$. Then $t^{\ell+|\Sigma_r^+|}\widetilde{M}(g)\widetilde{Q}_t^{\sigma}(g)$ tends to 0 as $t \to +\infty$.

On the other hand, since $g \notin K(\exp\widetilde{\Omega}_t)K$ and $|y| < \xi$ imply that $g^{-1}y \notin K(\exp\widetilde{\Omega}_t'')K$ (see (5.16)), we obtain

$$|\widetilde{v}(t,g)| \lesssim \int_G \mathrm{d}y\,|\widetilde{v}_0(yK)|\,|\widetilde{Q}_t^{\sigma}(g^{-1}y)|$$

which is $\mathrm{o}(t^{-\ell-|\Sigma_r^+|})$ outside the critical region. In conclusion,

$$t^{\ell+|\Sigma_r^+|}\|\widetilde{v}(t,\cdot) - \widetilde{M}\,\widetilde{Q}_t^{\sigma}\|_{L^\infty(S)} \longrightarrow 0$$

as $t \to +\infty$. □

The result for the L^p norm follows by convexity.

Corollary 5.11. *The solution* $\widetilde{v}$ *to the Cauchy problem* (5.3) *with initial data* $\widetilde{v}_0 \in \mathcal{C}_c(S)$ *satisfies*

$$t^{\frac{\ell+|\Sigma_r^+|}{p'}} \|\widetilde{v}(t,\,\cdot\,) - \widetilde{M}\,\widetilde{Q}_t^{\sigma}\|_{L^p(S)} \longrightarrow 0 \quad \text{as} \quad t \to +\infty, \tag{5.22}$$

for all $1 < p < \infty$.

5.4. Asymptotics for other initial data

We have obtained above the long-time asymptotic convergence in L^p $(1 \leq p \leq \infty)$ for the extension problem with compactly supported initial data. The following corollaries give some other functional spaces for which the convergence is true, but the question regarding the full $L^1(S)$ class remains open.

Corollary 5.12. *The asymptotic convergences* (1.12) *and* (1.13)*, hence* (5.22)*, still hold with initial data* $\widetilde{v}_0 = \widetilde{\delta}^{\frac{1}{2}} v_0 \in L^1(S)$ *when* v_0 *is bi-*K*-invariant.*

Corollary 5.13. *The asymptotic convergences* (1.12) *and* (1.13)*, hence* (5.22)*, still hold with no bi-*K*-invariance condition but under the assumption*

$$\int_G \mathrm{d}g\, |v_0(gK)| e^{\langle \rho, g^+ \rangle} < \infty. \tag{5.23}$$

The proofs of the above corollaries are similar to those of [7, Corollary 4.12] and [7, Corollary 4.13], respectively, thus omitted.

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