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A critical exponent in a quasilinear Keller–Segel system with arbitrarily fast decaying diffusivities accounting for volume-filling effects

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Abstract. The quasilinear Keller–Segel system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$

endowed with homogeneous Neumann boundary conditions is considered in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with smooth boundary for sufficiently regular functions D and S satisfying $D > 0$ on $[0, \infty)$, $S > 0$ on $(0, \infty)$ and $S(0) = 0$. On the one hand, it is shown that if $\frac{S}{D}$ satisfies the subcritical growth condition

$$\frac{S(s)}{D(s)} \leq Cs^\alpha \qquad \text{for all } s \geq 1 \qquad \text{with some } \alpha < \frac{2}{n}$$

and $C > 0$, then for any sufficiently regular initial data there exists a global weak energy solution such that $\operatorname{ess\,sup}_{t>0} \|u(t)\|_{L^p(\Omega)} < \infty$ for some $p > \frac{2n}{n+2}$. On the other hand, if $\frac{S}{D}$ satisfies the supercritical growth condition

$$\frac{S(s)}{D(s)} \geq cs^\alpha \qquad \text{for all } s \geq 1 \qquad \text{with some } \alpha > \frac{2}{n}$$

and $c > 0$, then the nonexistence of a global weak energy solution having the boundedness property stated above is shown for some initial data in the radial setting. This establishes some criticality of the value $\alpha = \frac{2}{n}$ for $n \geq 3$, without any additional assumption on the behavior of $D(s)$ as $s \to \infty$, in particular without requiring any algebraic lower bound for D. When applied to the Keller–Segel system with volume-filling effect for probability distribution functions of the type $Q(s) = \exp(-s^\beta)$, $s \geq 0$, for global solvability the exponent $\beta = \frac{n-2}{n}$ is seen to be critical.

1. Introduction

In this work, we consider the quasilinear Keller–Segel system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ D(u)\frac{\partial u}{\partial \nu} - S(u)\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain with smooth boundary and outward unit normal ν on $\partial\Omega$,

$$\begin{cases} D \in C^2([0,\infty)) \text{ is such that } D > 0 \text{ on } [0,\infty) \quad \text{ and} \\ S \in C^2([0,\infty)) \text{ satisfies } S(0) = 0 \text{ and } S > 0 \text{ on } (0,\infty), \end{cases} \tag{1.2}$$

and where

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ is such that } u_0 > 0 \text{ in } \overline{\Omega} \text{ and} \\ v_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative.} \end{cases} \tag{1.3}$$

Systems of this type are quite often used to model the dynamics of chemotactically moving biological populations with density $u = u(x,t)$ that are attracted by a chemical signal $v = v(x,t)$ which they produce themselves. The analytical study of these systems was initiated by Keller and Segel in [19]. Particular variants of the quasilinear system (1.1) are the volume-filling models introduced in [22], which reflect that the ability of cells to move is decreased in regions with high cell densities as there is less space available due to the positive cell volume. For more details about the modeling of Keller–Segel systems, we refer to [30] and the surveys [1,15,16].

Concerning the question whether classical solutions to (1.1) are global and bounded or blow-up phenomena occur, it is known that the quotient $\frac{S}{D}$ of the chemotactic sensitivity function S and the diffusion rate D is important. More precisely, assuming that $n \geq 3$ and (1.2) and (1.3) are satisfied, the following results have been established. If

$$\frac{S(s)}{D(s)} \leq Cs^\alpha \qquad \text{for all } s \geq 1 \qquad \text{with some } \alpha < \frac{2}{n}, \tag{1.4}$$

and $C > 0$, then any solution to (1.1) is global and bounded, provided that the diffusivity $D(s)$ *has an algebraic lower bound* as $s \to \infty$ in the sense that

$$D(s) \geq cs^{-p} \qquad \text{for all } s \geq 1 \tag{1.5}$$

with some $p > 0$ and $c > 0$ (see [18,24] and also [2,7,20,21,23]).

However, if $\frac{S}{D}$ grows faster than $s^{2/n}$ as $s \to \infty$ in the sense that, e.g.,

$$\liminf_{s\to\infty} \frac{s\left(\frac{S}{D}\right)'(s)}{\left(\frac{S}{D}\right)(s)} > \frac{2}{n} \qquad \text{or} \qquad \lim_{s\to\infty} s^{-\alpha}\frac{S(s)}{D(s)} = c_0 \qquad \text{with some } \alpha > \frac{2}{n} \tag{1.6}$$

and $c_0 > 0$, then in the radial setting unbounded solutions to (1.1) exist *without any restriction on the behavior of D as s* $\to \infty$ (see [26]); in such cases, only some small-data solutions are known to exist globally ([12,13]). In addition, for the prototypical choice $D(s) = (s+1)^{-p}$ and $S(s) = s(s+1)^{q-1}$, $s \geq 0$, a regime for the parameters $p, q \in \mathbb{R}$ for which the blow-up takes place in finite time, and another regime leading to blow-up in infinite time, have been identified (see [8,9,28]). Various results in a similar flavor have been obtained for some parabolic–elliptic simplifications ([3,5,6,14,21]).

Concerning the question whether in case of (1.4) the solution to (1.1) is still global and bounded if (1.5) is violated, only some partial answers are available in the literature. If (1.4) and

$$K_1 e^{-\beta^- s} \leq D(s) \leq K_2 e^{-\beta^+ s} \qquad \text{for all } s \geq 0$$

for some $K_1, K_2 > 0$ and $\beta^- \geq \beta^+$ are fulfilled, then in case of $n = 2$ and $\beta^+ > 0$ all solutions to (1.1) are global and bounded, while in case of $n \geq 3$ and $\beta^+ > 0$ the global existence and boundedness has only be shown if the initial mass $\int_\Omega u_0$ is small enough (see [10,11]). In addition, if $\beta^- > 0$ and there is $\gamma \leq 0$ such that $\gamma \in [(\beta^+ - \beta^-)/2, \beta^+/2)$ and

$$\frac{S(s)}{D(s)} \leq Ce^{\gamma s} \qquad \text{for all } s \geq 0,$$

i.e. (1.4) is satisfied, only the global existence, but not the boundedness of the classical solution to (1.1) has been shown in [27].

In view of these results for exponentially decaying D, it seems to be questionable that in case of $n \geq 3$ and (1.4) the global existence and boundedness of any classical solution to (1.1) are valid without any restriction on the behavior of D for s large. Hence, it is the purpose of the present work to establish a type of global solvability for (1.1) which is valid for all initial data as soon as (1.4) holds and which is violated for some initial data if $\frac{S}{D}$ grows at least like s^α with some $\alpha > \frac{2}{n}$ as $s \to \infty$. We prove that in fact these results are true for the global weak energy solutions defined in Definition 2.2 below. To the best of our knowledge these are the first results showing that $\alpha = \frac{2}{n}$ is critical in (1.4) for global solvability *without any assumption on the behavior of D for s large* and in particular *without assuming an algebraic lower bound for D*. By applying our results to the volume-filling model introduced in [22], we further identify a critical exponent for global solvability in case of exponentially decreasing probability distribution functions.

Main results.

In order to state our main results, we define the functions

$$h(s) := \frac{S(s)}{D(s)}, \qquad s \geq 0, \tag{1.7}$$

and

$$G(s) := \int_1^s \int_1^\sigma \frac{d\tau d\sigma}{h(\tau)}, \qquad s > 0, \tag{1.8}$$

as well as

$$\ell(s) := \int_1^s \frac{\sigma d\sigma}{h(\sigma)}, \qquad s > 0. \tag{1.9}$$

If (1.10) and (1.12) hold, our first result establishes the existence of a global weak energy solution for all initial data without any assumption on the behavior of D for s large.

Theorem 1.1. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let D and S be such that besides (1.2) we have*

$$\frac{S(s)}{D(s)} \leq k_{SD} s^\alpha \qquad \text{for all } s \geq 1, \tag{1.10}$$

and

$$D(s) \leq K_D \qquad \text{for all } s \geq 0 \tag{1.11}$$

with some $k_{SD} > 0$, $K_D > 0$ and

$$\alpha < \frac{2}{n}. \tag{1.12}$$

Then for any choice of (u_0, v_0) fulfilling (1.3), the problem (1.1) possesses at least one global weak energy solution in the sense of Definition 2.2 below, which has the additional properties that

$$\begin{cases} u \in \bigcap_{p \geq 1} L^p_{loc}(\overline{\Omega} \times [0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \qquad \text{and} \\ v \in L^2_{loc}([0,\infty); W^{2,2}(\Omega)), \end{cases} \tag{1.13}$$

that

$$u > 0 \text{ and } v \geq 0 \text{ a.e. in } \Omega \times (0,\infty), \tag{1.14}$$

and that

$$\operatorname*{ess\,sup}_{t>0} \int_\Omega u^{2-\alpha}(\cdot,t) < \infty \tag{1.15}$$

as well as

$$\operatorname*{ess\,sup}_{t>0} \int_\Omega |\nabla v(\cdot,t)|^q < \infty \qquad \text{for all } q \in \left[1, \frac{n(2-\alpha)}{n+\alpha-2}\right), \tag{1.16}$$

Moreover, if $\Omega = B_R(0)$ with some $R > 0$ and

$$u_0 \text{ and } v_0 \text{ are radially symmetric with respect to } x = 0, \tag{1.17}$$

then (u, v) even is a radial global weak energy solution of (1.1) according to Definition 2.2.

Let us remark that the assumption (1.11) of the boundedness of D can be relaxed (by using some more involved arguments in some of the proofs). However, as we are focused on functions D decaying fast as $s \to \infty$, (1.11) does not seem to be a restriction of our result.

If (1.18) instead of (1.10) is satisfied, we establish the nonexistence of a global weak energy solution satisfying (1.21) for some initial data in the radial setting.

Theorem 1.2. *Let $n \geq 3$, $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, and suppose that D and S comply with* (1.2) *and are such that*

$$\frac{S(s)}{D(s)} \geq K_{SD} s^{\alpha} \qquad \text{for all } s \geq 1, \tag{1.18}$$

and that the functions ℓ and G defined in (1.9) *and* (1.8) *satisfy*

$$\ell(s) \leq \frac{n-2-\mu}{n} G(s) + K_{\ell G}\, s \qquad \text{for all } s \geq 1 \tag{1.19}$$

with some

$$\alpha > \frac{2}{n} \tag{1.20}$$

and some $\mu > 0$, $K_{SD} > 0$ and $K_{\ell G} > 0$. Then there exist initial data (u_0, v_0) such that (1.3) *and* (1.17) *hold, but that* (1.1) *does not possess any radial global weak energy solution (u, v) which is such that*

$$\operatorname*{ess\,sup}_{t>0} \int_\Omega u^p(\cdot, t) < \infty \tag{1.21}$$

for some $p > \frac{2n}{n+2}$.

In view of $n \geq 3$, we have $2-\alpha > \frac{2n}{n+2}$ for all $\alpha < \frac{2}{n}$. Hence, Theorems 1.1 and 1.2 show that $\alpha = \frac{2}{n}$ is indeed critical for the existence of global weak energy solutions to (1.1) satisfying (1.21) for all initial data.

Next we apply these results to concrete choices of D and S with D decaying faster than algebraically.

Corollary 1.3. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let*

$$D(s) := a(s+1)^{-p} e^{-bs^{\beta}}, \qquad S(s) := cs(s+1)^{q-1} e^{-ds^{\delta}}, \qquad s \geq 0, \tag{1.22}$$

with some constants $p, q \in \mathbb{R}$, $\beta > 0$, $\delta \geq 0$, and $a, b, c, d > 0$.

- *(i) If either $\beta = \delta$, $b = d$ and $p+q < \frac{2}{n}$ or $\beta = \delta$ and $b < d$ or $\beta < \delta$ is satisfied, then for any choice of (u_0, v_0) fulfilling* (1.3)*, the problem* (1.1) *possesses at least one global weak energy solution which satisfies in addition* (1.13)–(1.16) *as well as* (1.21) *for some $p > \frac{2n}{n+2}$.*
- *(ii) If either $\beta = \delta$, $b = d$ and $p+q > \frac{2}{n}$ or $\beta = \delta$ and $b > d$ or $\beta > \delta$ is satisfied, then for $\Omega = B_R(0)$ with some $R > 0$ there exist initial data (u_0, v_0) such that* (1.3) *and* (1.17) *hold, but that* (1.1) *does not possess any radial global weak energy solution (u, v) which satisfies* (1.21) *for some $p > \frac{2n}{n+2}$.*

If the two exponential terms in (1.22) coincide, the exponent $p+q = \frac{2}{n}$ is critical for the existence of global weak energy solutions. The same exponent is known to be critical in case of $\beta = \delta = 0$ for the existence of global bounded solutions (see, e.g., [18,24,26]).

We further consider the case of Keller–Segel systems with volume-filling effect, which are particularly relevant in applications and were introduced in [22]. This corresponds to the choice

$$D(s) := Q(s) - sQ'(s), \qquad S(s) := sQ(s), \qquad s \geq 0, \tag{1.23}$$

where $Q(s)$ represents the probability that a cell, which is located at a spatial position with cell density s, finds space in some neighboring site. In view of the original choice of Q in [22] such that $Q(s) = 0$ for all s large, it is favorable to choose Q decaying fast as $s \to \infty$ in order to reflect that cells can hardly move in presence of large cell densities. For exponentially decaying Q, we have the following result.

Corollary 1.4. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let D and S satisfy (1.23) with $Q(s) := ae^{-bs^\beta}$ with some positive constants a, b and β, i.e.,*

$$D(s) := ae^{-bs^\beta}\left(1 + b\beta s^\beta\right), \qquad S(s) := ase^{-bs^\beta}, \qquad s \geq 0. \tag{1.24}$$

(i) If $\beta > \frac{n-2}{n}$, then for any choice of (u_0, v_0) fulfilling (1.3), the problem (1.1) possesses at least one global weak energy solution which satisfies in addition (1.13)–(1.16) as well as (1.21) for some $p > \frac{2n}{n+2}$.

(ii) If $\beta \in (0, \frac{n-2}{n})$, then for $\Omega = B_R(0)$ with some $R > 0$ there exist initial data (u_0, v_0) such that (1.3) and (1.17) hold, but that (1.1) does not possess any radial global weak energy solution (u, v) which satisfies (1.21) for some $p > \frac{2n}{n+2}$.

This shows that for exponentially decreasing Q the decay $Q(s) := ae^{-bs^\beta}$ with $\beta = \frac{n-2}{n}$ is critical in the volume-filling model for the existence of global weak energy solutions in case of $n \geq 3$. The situation is quite different for $n = 2$, since there the algebraic decay $Q(s) = (1+s)^{-\gamma}$ has been shown to be critical for arbitrary $\gamma > 0$ (see [9]).

This paper is structured as follows: In Sect. 2, we give the definition of global weak energy solutions of (1.1). In particular, these solutions satisfy a slightly different energy inequality as compared to the energy identity commonly used in related problems. In Sect. 3, we prove the existence of a global weak energy solution of (1.1) as claimed in Theorem 1.1. To this end, we use approximate problems for (1.1) depending on a parameter $\varepsilon \in (0, 1)$ such that the diffusivity D_ε is uniformly positive for fixed ε. For the global classical solutions to the approximate problem, we prove a series of estimates which do not depend on ε. In order to establish some of these estimates on the whole time interval $(0, \infty)$, properties of the energy are proved and used. Finally, we establish the existence of a global weak energy solution to the original problem by a compactness argument relying on the Aubin–Lions lemma. In Sect. 4, we prove the nonexistence of a global weak energy solution satisfying (1.21) in the radial setting as claimed in Theorem 1.2. Assuming the existence of such a solution, we prove its convergence to a generalized stationary solution of (1.1) as $t \to \infty$ and use this to show that the initial energy $\mathcal{F}(u_0, v_0)$ is bounded from below by a constant depending on the initial mass $m = \int_\Omega u_0$. As the existence of initial data with initial energy smaller than this lower bound is already known, a global weak energy solution having the claimed properties cannot exist for these initial data.

2. Definition of global weak energy solutions

In order to define global weak energy solutions of (1.1), we start with the following elementary observation concerning some terms used in Definition 2.2 below.

Lemma 2.1. *Assume* (1.2), *and let*

$$\Sigma(s) := \begin{cases} K_\Sigma \sqrt{\frac{S(s)}{S(s)+1}}, & s \in [0,1], \\ \sqrt{\frac{S(s)}{S(s)+1}} \cdot \frac{h(s)}{h(s)+1}, & s > 1, \end{cases} \tag{2.1}$$

where h is defined in (1.7) *and* $K_\Sigma := \frac{h(1)}{h(1)+1} \in (0,1)$. *Then* $\Sigma \in C^0([0,\infty))$ *with* $\Sigma(s) > 0$ *for all* $s > 0$, *and we have*

$$\Sigma^2(s) \le S(s) \quad \textit{and} \quad \Sigma(s) \le 1 \qquad \textit{for all } s \ge 0 \tag{2.2}$$

as well as

$$\frac{\Sigma(s)}{h(s)} \le 1 \qquad \textit{for all } s > 1 \tag{2.3}$$

and

$$\sup_{s \in (0,1)} \frac{h(s)}{\Sigma(s)} < \infty. \tag{2.4}$$

Proof. In view of (1.2) and (1.7), we have $h > 0$ in $(0, \infty)$ and $h(0) = 0$. Hence, we immediately obtain (2.2), (2.3), $\Sigma > 0$ in $(0, \infty)$, and the continuity of Σ in view of the choice of K_Σ. Moreover, the positivity of D in $[0, 1]$ and

$$\frac{h(s)}{\Sigma(s)} = \frac{\sqrt{S(s)(S(s)+1)}}{K_\Sigma D(s)}, \qquad s \in (0,1),$$

imply (2.4). □

Next we define global weak energy solutions of (1.1) with a concept similar to the one in [29, Definition 4.2]. Let us remark that the dissipation rate $\mathcal{D}$ used in the energy inequality (2.7) slightly differs from that commonly used in related problems (see, e.g., [26]), as we use Σ instead of $\sqrt{S}$ in (2.9).

Definition 2.2. Assume (1.2) and let $u_0 \in L^1(\Omega)$ and $v_0 \in W^{1,2}(\Omega)$ be nonnegative, and suppose that

$$\begin{cases} u \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) & \text{and} \\ v \in L^1_{loc}([0,\infty); W^{1,2}(\Omega)) \end{cases}$$

are nonnegative functions such that

$$D(u) \in L^1_{loc}(\overline{\Omega} \times [0,\infty)),$$

that

$$D(u)\nabla u \text{ and } S(u)\nabla v \text{ lie in } L^1_{loc}(\overline{\Omega} \times [0,\infty); \mathbb{R}^n)$$

and

$$\frac{\Sigma(u)}{h(u)}\nabla u \in L^2_{loc}(\overline{\Omega} \times [0,\infty); \mathbb{R}^n),$$

and that

$$v_t \in L^2_{loc}(\overline{\Omega} \times [0,\infty)),$$

and that there exists a null set $N_\star \subset (0,\infty)$ such that

$$v(\cdot,t) \in W^{1,2}(\Omega), \quad u(\cdot,t)v(\cdot,t) \in L^1(\Omega)$$
$$\text{and} \quad G(u(\cdot,t)) \in L^1(\Omega) \qquad \text{for all } t \in (0,\infty) \setminus N_\star.$$

Then, (u, v) will be called a *global weak energy solution* of (1.1) if

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega D(u)\nabla u \cdot \nabla\varphi + \int_0^\infty \int_\Omega S(u)\nabla v \cdot \nabla\varphi \tag{2.5}$$

and

$$\int_0^\infty \int_\Omega v_t\varphi = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^\infty \int_\Omega v\varphi + \int_0^\infty \int_\Omega u\varphi \tag{2.6}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, if $\operatorname{ess\,lim}_{t\searrow 0} \|v(\cdot,t) - v_0\|_{L^2(\Omega)} = 0$, and if

$$\mathcal{F}(u(\cdot,t), v(\cdot,t)) + \int_0^t \mathcal{D}(s)ds \le \mathcal{F}(u_0, v_0) \qquad \text{for all } t \in (0,\infty) \setminus N_\star, \tag{2.7}$$

where

$$\mathcal{F}(\phi,\psi) := \frac{1}{2}\int_\Omega |\nabla\psi|^2 + \frac{1}{2}\int_\Omega \psi^2 - \int_\Omega \phi\psi + \int_\Omega G(\phi)$$
$$\text{for } \phi \in L^1(\Omega) \text{ and } \psi \in W^{1,2}(\Omega) \text{ such that } \phi > 0 \text{ a.e. in } \Omega, \tag{2.8}$$

and where

$$\mathcal{D}(t) := \int_\Omega v_t^2(\cdot,t) + \int_\Omega \left| \Sigma(u(\cdot,t)) \frac{\nabla u(\cdot,t)}{h(u(\cdot,t))} - \Sigma(u(\cdot,t))\nabla v(\cdot,t) \right|^2, \qquad t > 0. \tag{2.9}$$

If furthermore $u(\cdot,t)$ and $v(\cdot,t)$ are radially symmetric with respect to $x = 0$ for all $t \in (0,\infty) \setminus N_\star$, then we say that (u, v) is a radial global weak energy solution of (1.1).

3. Global existence. Proof of Theorem 1.1

In this section, we prove the existence of a global weak energy solution of (1.1), provided that the requirements stated in Theorem 1.1 are satisfied. Without loss of generality, we may assume that (1.10) is satisfied with some $\alpha \in [0, \frac{2}{n})$, since in case of $\alpha < 0$ it is fulfilled for $\alpha = 0$ as well. In order to have a uniformly parabolic PDE for u, we consider the approximate problem

$$\begin{cases} u_{\varepsilon t} = \nabla \cdot \Big((D(u_\varepsilon) + \varepsilon)\nabla u_\varepsilon \Big) - \nabla \cdot (S(u_\varepsilon)\nabla v_\varepsilon), & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \qquad v_\varepsilon(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

for $\varepsilon \in (0, 1)$.

The existence of a global classical solution to (3.1) satisfying the usual energy identity (3.3) below is a well-known result. We emphasize the difference between the definition of the dissipation rates $\mathcal{D}_\varepsilon$ in (3.5) and $\mathcal{D}$ in (2.9).

Lemma 3.1. *Suppose that (1.2), (1.10) and (1.11) hold with some $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ and $K_D > 0$, and assume (1.3). Then for each $\varepsilon \in (0, 1)$, the problem (3.1) admits a global classical solution $(u_\varepsilon, v_\varepsilon)$ with*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \quad \text{and} \\ v_\varepsilon \in \bigcap_{q \geq 1} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \end{cases}$$

which is such that $u_\varepsilon > 0$ and $v_\varepsilon \geq 0$ in $\overline{\Omega} \times [0,\infty)$, that

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \qquad \text{for all } t > 0, \tag{3.2}$$

and such that if $\Omega = B_R(0)$ with some $R > 0$, and if (1.17) holds, then $(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t))$ is radially symmetric with respect to $x = 0$ for all $t > 0$ and $\varepsilon \in (0, 1)$.
Moreover, this solution satisfies

$$\mathcal{F}_\varepsilon(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t)) + \int_0^t \mathcal{D}_\varepsilon(s)ds = \mathcal{F}_\varepsilon(u_0, v_0) \qquad \text{for all } t > 0, \tag{3.3}$$

with

$$\mathcal{F}_\varepsilon(\phi,\psi) := \frac{1}{2}\int_\Omega |\nabla\psi|^2 + \frac{1}{2}\int_\Omega \psi^2 - \int_\Omega \phi\psi + \int_\Omega G_\varepsilon(\phi),$$
$$\text{for } \phi \in L^1(\Omega) \text{ and } \psi \in W^{1,2}(\Omega) \text{ such that } \phi > 0 \text{ a.e. in } \Omega, \tag{3.4}$$

and

$$\mathcal{D}_\varepsilon(t) := \int_\Omega v_{\varepsilon t}^2(\cdot,t) + \int_\Omega \left| \sqrt{S(u_\varepsilon(\cdot,t)}\frac{\nabla u_\varepsilon(\cdot,t)}{h_\varepsilon(u_\varepsilon(\cdot,t))} - \sqrt{S(u_\varepsilon(\cdot,t))}\nabla v_\varepsilon(\cdot,t)\right|^2, \qquad t>0, \tag{3.5}$$

where

$$G_\varepsilon(s) := \int_1^s \int_1^\sigma \frac{d\tau d\sigma}{h_\varepsilon(\tau)}, \qquad s>0,\ \varepsilon\in(0,1), \tag{3.6}$$

with

$$h_\varepsilon(s) := \frac{S(s)}{D(s)+\varepsilon}, \qquad s>0,\ \varepsilon\in(0,1). \tag{3.7}$$

Proof. Since for each fixed $\varepsilon \in (0,1)$ the function $0 \le s \mapsto D(s)+\varepsilon$ is bounded from above and below by positive constants, in view of the observation that clearly $\frac{S(s)}{D(s)+\varepsilon} \le k_{SD}s^\alpha$ for all $s>1$ and $\varepsilon \in (0,1)$ the statements concerning existence follow from a well-established approach toward global existence in quasilinear Keller–Segel systems ([24], [18]). The identities (3.2) and (3.3) can thereafter be derived by straightforward computation (see, e.g., [26, Lemma 2.1]). □

3.1. Estimates for the approximate problems

We prove several estimates, which do not depend on $\varepsilon \in (0,1)$, for the solutions to (3.1) and start with a standard estimate for v_ε.

Lemma 3.2. *Suppose that (1.2), (1.10) and (1.11) hold with some $\alpha \in [0,\frac{2}{n})$, $k_{SD}>0$ and $K_D>0$, and assume (1.3). Then, for all $q \in [1,\frac{n}{n-2})$ there exists $C(q)>0$ such that*

$$\|v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le C(q) \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \tag{3.8}$$

Proof. This follows from (3.2) and well-known regularization features of the Neumann heat semigroup in a standard manner (cf. e.g. [17, Lemma 4.1]). □

In conjunction with an appropriate testing procedure and smoothing properties of the Neumann heat semigroup, the previous result implies bounds for u_ε.

Lemma 3.3. *Suppose that (1.2), (1.10) and (1.11) hold with some $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ and $K_D > 0$, and assume (1.3). Then there exists $p_0 > 2$ with the property that for all $p > p_0$ and each $T > 0$ one can find $C(p, T) > 0$ such that*

$$\int_0^T \int_\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 \le C(p, T) \qquad \textit{for all } \varepsilon \in (0, 1) \tag{3.9}$$

and

$$\int_0^T \int_\Omega u_\varepsilon^p \le C(p, T) \qquad \textit{for all } \varepsilon \in (0, 1). \tag{3.10}$$

Proof. We pick any $r > \max\{\frac{n}{2}, 2\}$ with $\frac{1}{r} > \alpha - \frac{1}{n}$ and may then use our assumption $\alpha < \frac{2}{n}$ to choose $p_0 > 2$ such that

$$\alpha < \frac{2}{n} - \frac{2(r-1)}{r(2-n+np)} \qquad \text{for all } p > p_0 \tag{3.11}$$

and

$$\frac{2(r-1)}{r(2-n+np)} < \frac{1}{n} \qquad \text{for all } p > p_0 \tag{3.12}$$

as well as

$$\frac{2r}{p} < \frac{2n}{n-2} \qquad \text{for all } p > p_0 \tag{3.13}$$

and

$$p > 3 - \frac{2}{r} \qquad \text{for all } p > p_0 \tag{3.14}$$

as well as

$$\frac{(n-2)(p+2\alpha-2)}{np} < 1 + \frac{2}{n} - \frac{2}{r} - \frac{4(r-1)}{r(2-n+np)} \qquad \text{for all } p > p_0. \tag{3.15}$$

Then fixing any $p > p_0$, from the restriction $p_0 > 2$ we particularly know that

$$\lambda := \frac{r(2-n+np)}{n(r-1)} \tag{3.16}$$

satisfies $\lambda > 1$, whereas (3.12) and (3.11) along with our requirement $r > 2$ warrant that

$$1 + \frac{2}{n} - \frac{2}{r} - \frac{4(r-1)}{r(2-n+np)} > 1 + \frac{2}{n} - \frac{2}{r} - \frac{2}{n} = 1 - \frac{2}{r} = \frac{r-2}{r} > 0$$

and that

$$2\alpha - \frac{2}{n} + 1 - \frac{2}{r} < \left\{ \frac{4}{n} - \frac{4(r-1)}{r(2-n+np)} \right\} - \frac{2}{n} + 1 - \frac{2}{r}$$

$$= 1 + \frac{2}{n} - \frac{2}{r} - \frac{4(r-1)}{r(2-n+np)}.$$

As, moreover, (3.14) together with $\alpha \geq 0$ implies that

$$2\alpha - \frac{2}{n} + 1 - \frac{2}{r} < p + 2\alpha - 2$$

as well as

$$\frac{r-2}{r} < p - 2 \leq p + 2\alpha - 2,$$

in view of (3.15) and $\frac{1}{r} > \alpha - \frac{1}{n}$ it is possible to select some $\theta > 1$ fulfilling $\theta < \frac{r}{r-2}$ as well as

$$\max\left\{2\alpha - \frac{2}{n} + 1 - \frac{2}{r}, \frac{(n-2)(p+2\alpha-2)}{np}\right\} < \frac{1}{\theta}$$
$$< \min\left\{1 + \frac{2}{n} - \frac{2}{r} - \frac{4(r-1)}{r(2-n+np)}, p + 2\alpha - 2\right\}. \tag{3.17}$$

We now rely on (1.10) and the fact that $\alpha < \frac{2}{n} \leq 1$ to find $c_1 > 0$ such that

$$\frac{S(s)}{D(s)} \leq c_1 s^\alpha \qquad \text{for all } s \geq 0,$$

and introducing

$$\Phi(s) := \int_0^s \int_0^\sigma \frac{\tau^{p-2}}{D(\tau)} d\tau d\sigma, \qquad s \geq 0,$$

we integrate by parts in (3.1) and use Young's inequality to see that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \Phi(u_\varepsilon) &= -\int_\Omega \Phi''(u_\varepsilon)(D(u_\varepsilon)+\varepsilon)|\nabla u_\varepsilon|^2 + \int_\Omega \Phi''(u_\varepsilon)S(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&= -\int_\Omega \frac{D(u_\varepsilon)+\varepsilon}{D(u_\varepsilon)} u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^{p-2}\frac{S(u_\varepsilon)}{D(u_\varepsilon)}\nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&\leq -\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^{p-2}\frac{S(u_\varepsilon)}{D(u_\varepsilon)}\nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&\leq -\frac{1}{2}\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + \frac{c_1^2}{2}\int_\Omega u_\varepsilon^{p+2\alpha-2}|\nabla v_\varepsilon|^2 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

Since Φ is nonnegative, fixing $T > 0$ we thus infer that with $c_2 := \frac{p^2}{2}\int_\Omega \Phi(u_0)$ and $c_3 := \frac{p^2 c_1^2}{4}$ we have

$$\int_0^T \int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \leq c_2 + c_3 \int_0^T \int_\Omega u_\varepsilon^{p+2\alpha-2}|\nabla v_\varepsilon|^2 \qquad \text{for all } \varepsilon \in (0,1), \tag{3.18}$$

where we use that $\theta > 1$ in applying the Hölder inequality to obtain that

$$\begin{aligned}
&\int_0^T \int_\Omega u_\varepsilon^{p+2\alpha-2} |\nabla v_\varepsilon|^2 \\
&\le \int_0^T \|u_\varepsilon(\cdot,t)\|_{L^{(p+2\alpha-2)\theta}(\Omega)}^{p+2\alpha-2} \|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{2\theta}{\theta-1}}(\Omega)}^2 dt \\
&\le \Big\{ \sup_{t\in(0,T)} \|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{2\theta}{\theta-1}}(\Omega)}^2 \Big\} \cdot \int_0^T \|u_\varepsilon(\cdot,t)\|_{L^{(p+2\alpha-2)\theta}(\Omega)}^{p+2\alpha-2} dt \qquad \text{for all } \varepsilon\in(0,1).
\end{aligned} \tag{3.19}$$

We next rely on the inequality $\theta < \frac{r}{r-2}$ in employing well-known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω as well as again the Hölder inequality to find $c_4 > 0$ such that

$$\begin{aligned}
&\|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{2\theta}{\theta-1}}(\Omega)}^2 \\
&= \Big\| \nabla e^{t(\Delta-1)} v_0 + \int_0^t \nabla e^{(t-s)(\Delta-1)} u_\varepsilon(\cdot,s)ds \Big\|_{L^{\frac{2\theta}{\theta-1}}(\Omega)}^2 \\
&\le c_4 + c_4 \cdot \Big\{ \int_0^t (t-s)^{-\frac12-\frac n2(\frac1r-\frac{\theta-1}{2\theta})} \|u_\varepsilon(\cdot,s)\|_{L^r(\Omega)} ds \Big\}^2 \\
&\le c_4 + c_4 \cdot \Big\{ \int_0^t (t-s)^{[-\frac12-\frac n2(\frac1r-\frac{\theta-1}{2\theta})]\cdot\frac{\lambda}{\lambda-1}} ds \Big\}^{\frac{2(\lambda-1)}{\lambda}} \cdot \Big\{ \int_0^t \|u_\varepsilon(\cdot,s)\|_{L^r(\Omega)}^\lambda ds \Big\}^{\frac2\lambda} \\
&\le c_4 + c_5(T) \cdot \Big\{ \int_0^T \|u_\varepsilon(\cdot,s)\|_{L^r(\Omega)}^\lambda ds \Big\}^{\frac2\lambda} \qquad \text{for all } \varepsilon\in(0,1),
\end{aligned} \tag{3.20}$$

with

$$c_5(T) := c_4 \cdot \Big\{ \int_0^T \sigma^{[-\frac12-\frac n2(\frac1r-\frac{\theta-1}{2\theta})]\cdot\frac{\lambda}{\lambda-1}} d\sigma \Big\}^{\frac{2(\lambda-1)}{\lambda}}$$

being finite, because by (3.16) and the second inequality in (3.17),

$$\begin{aligned}
&\Big\{\frac12 + \frac n2\Big(\frac1r - \frac{\theta-1}{2\theta}\Big)\Big\} - \Big\{1-\frac1\lambda\Big\} \\
&= -\frac12 + \frac{n}{2r} - \frac n4 + \frac{n}{4\theta} + \frac{n(r-1)}{r(2-n+np)} \\
&< -\frac12 + \frac{n}{2r} - \frac n4 + \frac n4 \cdot \Big\{1 + \frac2n - \frac2r - \frac{4(r-1)}{r(2-n+np)}\Big\} \\
&\quad + \frac{n(r-1)}{r(2-n+np)} \\
&= 0.
\end{aligned}$$

We now observe that by (3.13),

$$\frac2p < \frac{2r}{p} < \frac{2n}{n-2},$$

and that due to both inequalities in (3.17) and the fact that $p > 2$,

$$\frac{2}{p} < \frac{2(p+2\alpha-2)\theta}{p} < \frac{2(p+2\alpha-2)}{p} \cdot \frac{np}{(n-2)(p+2\alpha-2)} = \frac{2n}{n-2}.$$

Therefore, two applications of the Gagliardo–Nirenberg inequality together with (3.2) provide $c_6 > 0$, $c_7(T) > 0$ and $c_8 > 0$ such that

$$\begin{aligned}
&\left\{\int_0^T \|u_\varepsilon(\cdot,s)\|_{L^r(\Omega)}^{\lambda} ds\right\}^{\frac{2}{\lambda}} \\
&= \left\{\int_0^T \|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2r}{p}}(\Omega)}^{\frac{2\lambda}{p}} ds\right\}^{\frac{2}{\lambda}} \\
&\le c_6 \cdot \left\{\int_0^T \left\{\|\nabla u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 \|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2\lambda}{p}-2} + \|u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2\lambda}{p}}\right\} ds\right\}^{\frac{2}{\lambda}} \\
&\le c_7(T) \cdot \left\{\int_0^T \|\nabla u_\varepsilon^{\frac{p}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 ds + 1\right\}^{\frac{2}{\lambda}} \qquad \text{for all } \varepsilon \in (0,1)
\end{aligned} \tag{3.21}$$

and that, similarly,

$$\begin{aligned}
\int_0^T \|u_\varepsilon(\cdot,t)\|_{L^{(p+2\alpha-2)\theta}(\Omega)}^{p+2\alpha-2} dt &= \int_0^T \|u_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^{\frac{2(p+2\alpha-2)\theta}{p}}(\Omega)}^{\frac{2(p+2\alpha-2)}{p}} dt \\
&\le c_8 \int_0^T \left\{\|\nabla u_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^{\kappa} + 1\right\} dt \qquad \text{for all } \varepsilon \in (0,1)
\end{aligned} \tag{3.22}$$

with $\kappa := \frac{2n[(p+2\alpha-2)\theta-1]}{(2-n+np)\theta}$. Since (3.16) along with the first restriction contained in (3.17) ensures that

$$\begin{aligned}
\frac{4}{\lambda} + \kappa &= \frac{4n(r-1)}{r(2-n+np)} + \frac{2n(p+2\alpha-2)}{2-n+np} - \frac{2n}{2-n+np} \cdot \frac{1}{\theta} \\
&< \frac{4n(r-1)}{r(2-n+np)} + \frac{2n(p+2\alpha-2)}{2-n+np} - \frac{2n}{2-n+np} \cdot \left\{2\alpha - \frac{2}{n} + 1 - \frac{2}{r}\right\} \\
&= 2,
\end{aligned}$$

and that particularly also $\kappa < 2$, we may combine (3.19)-(3.22) with the Hölder inequality and Young's inequality to infer the existence of positive constants $c_9(T)$, $c_{10}(T)$ and $c_{11}(T)$ such that

$$\begin{aligned}
&c_3 \int_0^T \int_\Omega u_\varepsilon^{p+2\alpha-2} |\nabla v_\varepsilon|^2 \\
&\le c_9(T) \cdot \left\{\int_0^T \|\nabla u_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^2 dt + 1\right\}^{\frac{2}{\lambda}} \cdot \left\{\int_0^T \|\nabla u_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^{\kappa} dt + 1\right\}
\end{aligned}$$

$$\le c_{10}(T) \cdot \left\{ \int_0^T \|\nabla u_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^2 dt + 1 \right\}^{\frac{2}{\lambda}+\frac{\kappa}{2}}$$

$$\le \frac{1}{2} \int_0^T \|\nabla u_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^2 dt + c_{11}(T) \qquad \text{for all } \varepsilon \in (0,1).$$

Consequently, (3.18) implies that

$$\int_0^T \int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \le 2c_2 + 2c_{11}(T) \qquad \text{for all } \varepsilon \in (0,1)$$

and thus, in view of a Poincaré inequality and (3.2), establishes both (3.10) and (3.9). □

Using once more the properties of the heat semigroup, we obtain a bound for ∇v_ε in L^∞.

Lemma 3.4. *Suppose that* (1.2), (1.10) *and* (1.11) *hold with some* $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ *and* $K_D > 0$, *and assume* (1.3). *Then for all* $T > 0$ *there exists* $C(T) > 0$ *such that*

$$|\nabla v_\varepsilon(x,t)| \le C(T) \qquad \textit{for all } x \in \Omega,\, t \in (0,T) \textit{ and } \varepsilon \in (0,1). \tag{3.23}$$

Proof. We fix any $p > n+2$ and once more recall known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω to find $c_1 > 0$ such that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \left\| \nabla e^{t(\Delta-1)} v_0 + \int_0^t \nabla e^{(t-s)(\Delta-1)} u_\varepsilon(\cdot,s)ds \right\|_{L^\infty(\Omega)}$$
$$\le c_1 \|v_0\|_{W^{1,\infty}(\Omega)} + c_1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} ds.$$

Here by the Hölder inequality, for all $t > 0$ and $\varepsilon \in (0,1)$ we have

$$\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} ds$$
$$\le \left\{ \int_0^t (t-s)^{(-\frac{1}{2}-\frac{n}{2p})\cdot\frac{p}{p-1}} ds \right\}^{\frac{p-1}{p}} \cdot \left\{ \int_0^t \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)}^p ds \right\}^{\frac{1}{p}},$$

whence observing that $\int_0^T \sigma^{(-\frac{1}{2}-\frac{n}{2p})\cdot\frac{p}{p-1}} d\sigma$ is finite due to our restriction on p we obtain (3.23) as a consequence of Lemma 3.3. □

This further implies a bound for the cross-diffusion term.

Lemma 3.5. *Suppose that* (1.2), (1.10) *and* (1.11) *hold with some* $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ *and* $K_D > 0$, *and assume* (1.3). *Then for all* $p > 1$ *and* $T > 0$ *one can find* $C(p,T) > 0$ *such that*

$$\int_0^T \int_\Omega |S(u_\varepsilon)\nabla v_\varepsilon|^p \le C(p,T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.24}$$

Proof. Since (1.10) along with (1.11) asserts that with some $c_1 > 0$ we have

$$S^p(s) \leq c_1 \cdot (s^{p\alpha} + 1) \qquad \text{for all } s \geq 0,$$

this is an immediate consequence of Lemma 3.3 and Lemma 3.4. □

3.2. Further estimates involving the energy

In order to obtain estimates for u_ε and v_ε on the whole time interval $(0, \infty)$, we use the energy identity. The next two results provide a lower bound for the Lyapunov functional $\mathcal{F}_\varepsilon$.

Lemma 3.6. *Suppose that* (1.2), (1.10) *and* (1.11) *hold with some* $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ *and* $K_D > 0$, *and assume* (1.3). *Then there exists* $C > 0$ *such that with* $(G_\varepsilon)_{\varepsilon\in(0,1)}$ *as in (3.6),*

$$G_\varepsilon(s) \geq \frac{1}{C} \cdot s^{2-\alpha} - C \cdot (s + 1) \qquad \textit{for all } s > 0 \textit{ and } \varepsilon \in (0, 1). \tag{3.25}$$

Proof. According to (3.6) and (1.10), for each $s \geq 1$ and $\varepsilon \in (0, 1)$ we have

$$\begin{aligned} G_\varepsilon(s) &\geq \frac{1}{k_{SD}} \int_1^s \int_1^\sigma \tau^{-\alpha} d\tau d\sigma \\ &= \frac{1}{k_{SD}(1-\alpha)} \cdot \left\{ \frac{s^{2-\alpha} - 1}{2 - \alpha} - s + 1 \right\} \\ &\geq c_1 s^{2-\alpha} - c_2 \cdot (s + 1) \end{aligned}$$

with $c_1 := \frac{1}{k_{SD}(1-\alpha)(2-\alpha)}$ and $c_2 := \frac{1}{k_{SD}(1-\alpha)}$. As furthermore

$$G_\varepsilon(s) - c_1 s^{2-\alpha} \geq -c_1 s^{2-\alpha} \geq -c_1 \qquad \text{for all } s \in (0, 1) \text{ and any } \varepsilon \in (0, 1),$$

this entails (3.25) with $C := \max\{\frac{1}{c_1}, c_1, c_2\}$. □

Lemma 3.7. *Suppose that* (1.2) (1.10) *and* (1.11) *hold with some* $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ *and* $K_D > 0$, *and assume* (1.3). *Then there exists* $C > 0$ *such that with* $(\mathcal{F}_\varepsilon)_{\varepsilon\in(0,1)}$ *and* $(G_\varepsilon)_{\varepsilon\in(0,1)}$ *taken from (3.4) and (3.6), we have*

$$\mathcal{F}_\varepsilon(u_\varepsilon(\cdot, t), v_\varepsilon(\cdot, t)) \geq \frac{1}{2} \int_\Omega G_\varepsilon(u_\varepsilon(\cdot, t)) - C \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0, 1). \tag{3.26}$$

Proof. We first employ Lemma 3.6 to find $c_1 > 0$ and $c_2 > 0$ such that for all $\varepsilon \in (0, 1)$,

$$G_\varepsilon(s) \geq c_1 s^{2-\alpha} - c_2(s + 1) \qquad \text{for all } s > 0, \tag{3.27}$$

and use Young's inequality to fix $c_3 > 0$ such that

$$\xi\eta \leq \frac{c_1}{2} \xi^{2-\alpha} + c_3 \eta^{\frac{2-\alpha}{1-\alpha}} \qquad \text{for all } \xi \geq 0 \text{ and } \eta \geq 0. \tag{3.28}$$

We then note that our assumption $\alpha < \frac{2}{n} < 1$ warrants that $\frac{\alpha n}{2(1-\alpha)} < \frac{2-\alpha}{1-\alpha} < \frac{2n}{n-2}$ and $\frac{\alpha n}{2(1-\alpha)} < \frac{n}{n-2}$, and that hence we may fix $q > 1$ simultaneously fulfilling $q < \frac{n}{n-2}$ and $q < \frac{2-\alpha}{1-\alpha}$ as well as $q > \frac{\alpha n}{2(1-\alpha)}$, the latter implying that

$$\frac{(2-\alpha)n}{(1-\alpha)q} - n < 2 - n + \frac{2n}{q}.$$

This, namely, ensures that

$$a := \frac{\frac{n}{q} - \frac{n(1-\alpha)}{2-\alpha}}{1 - \frac{n}{2} + \frac{n}{q}}$$

belongs to $(0, 1)$ and moreover satisfies

$$\frac{2-\alpha}{1-\alpha} \cdot a = \frac{\frac{(2-\alpha)n}{(1-\alpha)q} - n}{1 - \frac{n}{2} + \frac{n}{q}} < 2,$$

and that hence combining the Gagliardo–Nirenberg inequality with Young's inequality provides $c_4 > 0$ and $c_5 > 0$ such that

$$\begin{aligned} c_3 \|\psi\|_{L^{\frac{2-\alpha}{1-\alpha}}(\Omega)}^{\frac{2-\alpha}{1-\alpha}} &\le c_4 \|\nabla\psi\|_{L^2(\Omega)}^{\frac{2-\alpha}{1-\alpha}\cdot a} \|\psi\|_{L^q(\Omega)}^{\frac{2-\alpha}{1-\alpha}\cdot(1-a)} + c_4 \|\psi\|_{L^q(\Omega)}^{\frac{2-\alpha}{1-\alpha}} \\ &\le \frac{1}{2}\|\nabla\psi\|_{L^2(\Omega)}^2 + c_5 \|\psi\|_{L^q(\Omega)}^{\frac{2(2-\alpha)(1-a)}{2(1-\alpha)-(2-\alpha)a}} + c_4 \|\psi\|_{L^q(\Omega)}^{\frac{2-\alpha}{1-\alpha}} \qquad \text{for all } \psi \in W^{1,2}(\Omega). \end{aligned} \tag{3.29}$$

Finally, due to the restriction $q < \frac{n}{n-2}$ we may invoke Lemma 3.2 to fix $c_6 > 0$ fulfilling

$$\|v_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \le c_6 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \tag{3.30}$$

Now successive application of (3.28), (3.29), (3.30) and (3.27) shows that for all $t > 0$ and $\varepsilon \in (0, 1)$ we have

$$\begin{aligned} \int_\Omega u_\varepsilon v_\varepsilon &\le \frac{c_1}{2}\int_\Omega u_\varepsilon^{2-\alpha} + c_3 \int_\Omega v_\varepsilon^{\frac{2-\alpha}{1-\alpha}} \\ &\le \frac{c_1}{2}\int_\Omega u_\varepsilon^{2-\alpha} + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + c_5 \|v_\varepsilon\|_{L^q(\Omega)}^{\frac{2(2-\alpha)(1-a)}{2(1-\alpha)-(2-\alpha)a}} + c_4 \|v_\varepsilon\|_{L^q(\Omega)}^{\frac{2-\alpha}{1-\alpha}} \\ &\le \frac{c_1}{2}\int_\Omega u_\varepsilon^{2-\alpha} + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + c_5 c_6^{\frac{2(2-\alpha)(1-a)}{2(1-\alpha)-(2-\alpha)a}} + c_4 c_6^{\frac{2-\alpha}{1-\alpha}} \\ &\le \frac{1}{2}\int_\Omega G_\varepsilon(u_\varepsilon) + \frac{c_2}{2}\int_\Omega (u_\varepsilon + 1) + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + c_5 c_6^{\frac{2(2-\alpha)(1-a)}{2(1-\alpha)-(2-\alpha)a}} + c_4 c_6^{\frac{2-\alpha}{1-\alpha}}, \end{aligned}$$

so that recalling (3.2) and our definition (3.4) of $\mathcal{F}_\varepsilon$ we obtain (3.26). □

In view of the energy identity (3.3), the previous estimate implies further useful bounds on the time interval $(0,\infty)$.

Lemma 3.8. *Suppose that (1.2), (1.10) and (1.11) hold with some* $\alpha \in [0,\frac{2}{n})$, $k_{SD}>0$ *and* $K_D>0$, *and assume (1.3). Then, there exists* $C>0$ *such that with* h_ε *as in (3.7) we have*

$$\int_\Omega u_\varepsilon^{2-\alpha}(\cdot,t) \le C \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1) \tag{3.31}$$

and

$$\int_0^t\int_\Omega v_{\varepsilon t}^2 \le C \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1) \tag{3.32}$$

as well as

$$\int_0^t\int_\Omega \left|\sqrt{S(u_\varepsilon)}\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \sqrt{S(u_\varepsilon)}\nabla v_\varepsilon\right|^2 \le C \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1) \tag{3.33}$$

and

$$\int_0^t\int_\Omega \left|\Sigma(u_\varepsilon)\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \Sigma(u_\varepsilon)\nabla v_\varepsilon\right|^2 \le C \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \tag{3.34}$$

Proof. According to (3.3), for all $t>0$ and $\varepsilon\in(0,1)$ we have

$$\mathcal{F}_\varepsilon(u_\varepsilon(\cdot,t),v_\varepsilon(\cdot,t)) + \int_0^t\int_\Omega v_{\varepsilon t}^2 + \int_0^t\int_\Omega \left|\sqrt{S(u_\varepsilon)}\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \sqrt{S(u_\varepsilon)}\nabla v_\varepsilon\right|^2 \\ = \mathcal{F}_\varepsilon(u_0,v_0), \tag{3.35}$$

where by (3.6), writing $c_1 := \min_{x\in\overline{\Omega}} u_0(x) > 0$ and $c_2 := \max_{x\in\overline{\Omega}} u_0(x)$ we can estimate

$$\mathcal{F}_\varepsilon(u_0,v_0) \le c_3 := \frac{1}{2}\|v_0\|^2_{W^{1,2}(\Omega)} \\ +|\Omega|\cdot \max\left\{\int_{c_1}^1\int_\sigma^1 \frac{D(\tau)+1}{S(\tau)}d\tau d\sigma\,,\ \int_1^{c_2}\int_1^\sigma \frac{D(\tau)+1}{S(\tau)}d\tau d\sigma\right\}$$

for all $\varepsilon\in(0,1)$, with c_3 being finite by positivity of S on $(0,\infty)$. Since furthermore, by Lemma 2.1,

$$\left|\Sigma(u_\varepsilon)\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \Sigma(u_\varepsilon)\nabla v_\varepsilon\right|^2 \le \left|\sqrt{S(u_\varepsilon)}\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \sqrt{S(u_\varepsilon)}\nabla v_\varepsilon\right|^2 \qquad \text{in } \Omega\times(0,\infty)$$

for each $\varepsilon\in(0,1)$, by using Lemma 3.7 and Lemma 3.6 we readily obtain (3.31)–(3.34) from (3.35). □

Once again the smoothing properties of the Neumann heat semigroup yield an improved estimate for ∇v_ε.

Lemma 3.9. *Suppose that (1.2), (1.10) and (1.11) hold with some* $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ *and* $K_D > 0$, *and assume (1.3). Then for each* $q \in [1, \frac{n(2-\alpha)}{n+\alpha-2})$ *one can find* $C(q) > 0$ *such that*

$$\int_\Omega |\nabla v_\varepsilon(\cdot, t)|^q \le C(q) \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1). \tag{3.36}$$

Proof. Since $\alpha < 2$ and hence $\frac{n(2-\alpha)}{n+\alpha-2} > 2-\alpha$, we may assume that $q > 2-\alpha$. Then once more relying on known regularization features of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω, with some $c_1 > 0$ we have

$$\begin{aligned} \|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} &= \left\| \nabla e^{t(\Delta-1)} v_0 + \int_0^t \nabla e^{(t-s)(\Delta-1)} u_\varepsilon(\cdot,s)ds \right\|_{L^q(\Omega)} \\ &\le c_1 \|v_0\|_{W^{1,\infty}(\Omega)} + c_1 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2-\alpha}-\frac{1}{q})}\right) e^{-(t-s)} \|u_\varepsilon(\cdot,s)\|_{L^{2-\alpha}(\Omega)} ds \end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. As $\int_0^\infty (1+\sigma^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2-\alpha}-\frac{1}{q})}) e^{-\sigma} d\sigma$ is finite due to the fact that

$$\frac{1}{2} + \frac{n}{2}\Big(\frac{1}{2-\alpha} - \frac{1}{q}\Big) < \frac{1}{2} + \frac{n}{2}\Big(\frac{1}{2-\alpha} - \frac{n+\alpha-2}{n(2-\alpha)}\Big) = 1,$$

the boundedness property in (3.31) thus directly implies (3.36). □

The previous results enable us to obtain another bound for ∇u_ε.

Lemma 3.10. *Suppose that (1.2), (1.10) and (1.11) hold with some* $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ *and* $K_D > 0$, *and assume (1.3). Then for all* $T > 0$ *there exists* $C(T) > 0$ *such that*

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.37}$$

Proof. According to (2.4), there exists $c_1 > 0$ such that $\frac{h(s)}{\Sigma(s)} \le c_1$ for all $s \in (0,1)$, and that thus, since $h_\varepsilon \le h$,

$$\frac{h_\varepsilon^2(s)}{\Sigma^2(s)} \le c_1^2 \qquad \text{for all } s \in (0,1) \text{ and } \varepsilon \in (0,1).$$

Therefore,

$$\begin{aligned} &\int\int_{(\Omega\times(0,T))\cap\{u_\varepsilon<1\}} |\nabla u_\varepsilon|^2 \\ &= \int\int_{(\Omega\times(0,T))\cap\{u_\varepsilon<1\}} \frac{h_\varepsilon^2(u_\varepsilon)}{\Sigma^2(u_\varepsilon)} \left| \Sigma(u_\varepsilon) \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} \right|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq c_1^2 \int_0^T \int_\Omega \left| \Sigma(u_\varepsilon) \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} \right|^2 \\
&\leq 2c_1^2 \int_0^T \int_\Omega \left| \Sigma(u_\varepsilon) \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \Sigma(u_\varepsilon)\nabla v_\varepsilon \right|^2 + 2c_1^2 \int_0^T \int_\Omega \left| \Sigma(u_\varepsilon)\nabla v_\varepsilon \right|^2 \\
&\leq 2c_1^2 \int_0^T \int_\Omega \left| \Sigma(u_\varepsilon) \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \Sigma(u_\varepsilon)\nabla v_\varepsilon \right|^2 + 2c_1^2 \int_0^T \int_\Omega |\nabla v_\varepsilon|^2
\end{aligned}
$$

for all $T > 0$ and $\varepsilon \in (0, 1)$, because $\Sigma^2 \leq 1$ due to Lemma 2.1. As a consequence of (3.34) and Lemma 3.9 when applied to $q := 2$, we thus infer the existence of $c_2 > 0$ such that

$$
\int\int_{(\Omega\times(0,T))\cap\{u_\varepsilon<1\}} |\nabla u_\varepsilon|^2 \leq c_2 \cdot (T+1) \qquad \text{for all } T > 0 \text{ and } \varepsilon \in (0, 1),
$$

so that (3.37) readily results upon employing (3.9) in Lemma 3.3 with an arbitrary fixed $p > 2$. □

The next estimate will imply the positivity of our solution to (1.1).

Lemma 3.11. *Suppose that (1.2), (1.10) and (1.11) hold with some $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ and $K_D > 0$, and assume (1.3). Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$
\int_\Omega \ln u_\varepsilon(\cdot, t) \geq -C(T) \qquad \textit{for all } t \in (0, T) \textit{ and any } \varepsilon \in (0, 1). \tag{3.38}
$$

Proof. By means of the first equation in (3.1) and Young's inequality, we see that

$$
\begin{aligned}
-\frac{d}{dt}\int_\Omega \ln u_\varepsilon &= -\int_\Omega \frac{D(u_\varepsilon)+\varepsilon}{u_\varepsilon^2}|\nabla u_\varepsilon|^2 + \int_\Omega \frac{S(u_\varepsilon)}{u_\varepsilon^2}\nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
&\leq \frac{1}{4}\int_\Omega \frac{S^2(u_\varepsilon)}{u_\varepsilon^2(D(u_\varepsilon)+\varepsilon)}|\nabla v_\varepsilon|^2 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),
\end{aligned} \tag{3.39}
$$

where we note that according to (1.10) and (1.11),

$$
\frac{S^2(s)}{s^2(D(s)+\varepsilon)} \leq \frac{S^2(s)}{s^{2\alpha}D(s)} \leq k_{SD}^2 D(s) \leq c_1 := k_{SD}^2 K_D \qquad \text{for all } s > 1 \text{ and } \varepsilon \in (0, 1),
$$

and where since $S \in C^1([0, \infty))$ with $S(0) = 0$,

$$
\frac{S^2(s)}{s^2(D(s)+\varepsilon)} \leq c_2 := \frac{\|S'\|^2_{L^\infty((0,1))}}{\min_{\sigma\in[0,1]} D(\sigma)} \qquad \text{for all } s \in [0, 1] \text{ and } \varepsilon \in (0, 1).
$$

Therefore, (3.39) implies that

$$
-\frac{d}{dt}\int_\Omega \ln u_\varepsilon \leq \frac{1}{4}\max\{c_1, c_2\} \cdot \int_\Omega |\nabla v_\varepsilon|^2 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),
$$

whence (3.38) follows upon a time integration using, e.g., Lemma 3.4. □

In order to ensure an appropriate compactness property of u_ε, we finally need an estimate for its time derivative.

Lemma 3.12. *Suppose that (1.2), (1.10) and (1.11) hold with some $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ and $K_D > 0$, and assume (1.3). Then there exists $C > 0$ such that*

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,\infty}(\Omega))^\star} dt \le C \qquad \text{for all } T > 0 \text{ and } \varepsilon \in (0,1). \tag{3.40}$$

Proof. For fixed $\psi \in W^{1,\infty}(\Omega)$, from (3.1) we obtain that thanks to the Cauchy–Schwarz inequality,

$$\begin{aligned}
\left| \int_\Omega u_{\varepsilon t}\psi \right| &= \left| -\int_\Omega \Big\{ (D(u_\varepsilon)+\varepsilon)\nabla u_\varepsilon - S(u_\varepsilon)\nabla v_\varepsilon \Big\} \cdot \nabla\psi \right| \\
&= \left| -\int_\Omega \sqrt{S(u_\varepsilon)} \left\{ \sqrt{S(u_\varepsilon)} \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \sqrt{S(u_\varepsilon)}\nabla v_\varepsilon \right\} \cdot \nabla\psi \right| \\
&\le \left\{ \int_\Omega S(u_\varepsilon) \right\}^{\frac12} \cdot \left\{ \int_\Omega \left| \sqrt{S(u_\varepsilon)} \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \sqrt{S(u_\varepsilon)}\nabla v_\varepsilon \right|^2 \right\}^{\frac12} \cdot \|\nabla\psi\|_{L^\infty(\Omega)}
\end{aligned} \tag{3.41}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Since according to (1.10), (1.11) and the fact that $\alpha \le 1$ we can find $c_1 > 0$ and $c_2 > 0$ such that

$$S(s) \le c_1 D(s)(s+1) \le c_2 \cdot (s+1) \qquad \text{for all } s \ge 0 \text{ and } \varepsilon \in (0,1),$$

and therefore

$$\int_\Omega S(u_\varepsilon) \le c_2 \int_\Omega u_0 + c_2|\Omega| \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1)$$

due to (3.2), from (3.41) we thus infer the existence of $c_3 > 0$ such that

$$\begin{aligned}
&\|u_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,\infty}(\Omega))^\star} \\
&\le c_3 \int_\Omega \left| \sqrt{S(u_\varepsilon)} \frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \sqrt{S(u_\varepsilon)}\nabla v_\varepsilon \right|^2 \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

The claim hence results from (3.33). □

3.3. Proof of Theorem 1.1

We are now in a position to prove the existence of a global weak energy solution to (1.1) with a compactness argument.

Lemma 3.13. *Suppose that (1.2), (1.10) and (1.11) hold with some $\alpha \in [0, \frac{2}{n})$, $k_{SD} > 0$ and $K_D > 0$, and assume (1.3). Then there exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$, and such that as $\varepsilon = \varepsilon_j \searrow 0$ we have*

$$u_\varepsilon \to u \qquad \text{a.e. in } \Omega \times (0,\infty) \text{ and in } L^p_{loc}(\overline{\Omega} \times [0,\infty)) \text{ for all } p \ge 1, \tag{3.42}$$

$$u_\varepsilon(\cdot,t) \to u(\cdot,t) \qquad \textit{a.e. in } \Omega \textit{ and in } L^p(\Omega) \textit{ for all } p \geq 1 \textit{ and a.e. } t > 0, \tag{3.43}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \qquad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{3.44}$$

$$v_\varepsilon \to v \qquad \textit{a.e. in } \Omega \times (0,\infty) \textit{ and in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{3.45}$$

$$\nabla v_\varepsilon \to \nabla v \qquad \textit{a.e. in } \Omega \times (0,\infty) \textit{ and in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), \tag{3.46}$$

$$v_\varepsilon(\cdot,t) \to v(\cdot,t) \qquad \textit{in } W^{1,2}(\Omega) \textit{ for a.e. } t > 0, \tag{3.47}$$

$$v_\varepsilon \rightharpoonup v \qquad \textit{in } L^2_{loc}([0,\infty); W^{2,2}(\Omega)), \tag{3.48}$$

$$S(u_\varepsilon)\nabla v_\varepsilon \rightharpoonup S(u)\nabla v \qquad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \textit{and} \tag{3.49}$$

$$v_{\varepsilon t} \rightharpoonup v_t \qquad \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) \tag{3.50}$$

with some functions u and v on $\Omega \times (0,\infty)$ which are such that (1.13), (1.14), (1.15) and (1.16) hold, that (u,v) is a global weak energy solution of (1.1) and that if $\Omega = B_R(0)$ with some $R > 0$ and (u_0, v_0) satisfies (1.17), then (u,v) even is a radial global weak energy solution of (1.1).

Proof. From Lemma 3.10 and Lemma 3.3, we know that

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,T); W^{1,2}(\Omega)) \text{ for all } T > 0,$$

while Lemma 3.12 asserts that

$$(u_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,T); (W^{1,\infty}(\Omega))^\star) \text{ for all } T > 0.$$

Apart from that, (3.32) along with Lemma 3.3, Lemma 3.9 and (3.1) shows that

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,T); W^{2,2}(\Omega)) \text{ for all } T > 0$$

and that

$$(v_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^2(\Omega \times (0,\infty)).$$

Two applications of an Aubin–Lions lemma ([25, Theorem III.2.1]) thus provide $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and that as $\varepsilon = \varepsilon_j \searrow 0$ we have (3.44), (3.45), (3.46), (3.47), (3.48), (3.50) as well as $u_\varepsilon \to u$ a.e. in $\Omega \times (0,\infty)$ with some nonnegative functions $u \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ and $v \in L^2_{loc}([0,\infty); W^{2,2}(\Omega))$. The completion of (3.42), (3.43) and (1.13) can thereafter be achieved by combining (3.10) with the Vitali convergence theorem, whereas (3.49) follows from Lemma 3.5 and the Egorov theorem due to the fact that $S(u_\varepsilon)\nabla v_\varepsilon \to S(u)\nabla v$ a.e. in $\Omega \times (0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$ by (3.42) and (3.46). Moreover, Lemma 3.11 along with (3.42) guarantees that

$$\operatorname*{ess\,inf}_{t\in(0,T)} \int_\Omega \ln u(\cdot,t) > -\infty \qquad \text{for all } T > 0,$$

implying that $\ln u > -\infty$ and hence $u > 0$ a.e. in $\Omega \times (0,\infty)$. The bounds in (1.15) and (1.16) are consequences of Lemma 3.8 and Lemma 3.9 when combined with

(3.42), (3.46) and Fatou's lemma, whereas the claim on radial symmetry is evident from Lemma 3.1, (3.42) and (3.45).

The verification of (2.5) is quite straightforward: Given $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, from (3.1) we obtain that

$$-\int_0^\infty \int_\Omega u_\varepsilon \varphi_t - \int_\Omega u_0 \varphi(\cdot,0) = -\int_0^\infty \int_\Omega (D(u_\varepsilon)+\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi + \int_0^\infty \int_\Omega S(u_\varepsilon)\nabla v_\varepsilon \cdot \nabla \varphi \qquad (3.51)$$

for all $\varepsilon \in (0,1)$, where by (3.42) and (3.49),

$$-\int_0^\infty \int_\Omega u_\varepsilon \varphi_t \to -\int_0^\infty \int_\Omega u\varphi_t \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0$$

and

$$\int_0^\infty \int_\Omega S(u_\varepsilon)\nabla v_\varepsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega S(u)\nabla v \cdot \nabla \varphi \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Since (1.11) along with (3.42) and the dominated convergence theorem ensures that $D(u_\varepsilon) + \varepsilon \to D(u)$ in $L^2_{loc}(\overline{\Omega} \times [0,\infty))$ as $\varepsilon = \varepsilon_j \searrow 0$, we furthermore have

$$-\int_0^\infty \int_\Omega (D(u_\varepsilon)+\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi \to -\int_0^\infty \int_\Omega D(u)\nabla u \cdot \nabla\varphi \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Therefore, (2.5) results from (3.51), while (2.6) can be derived similarly using (3.50), (3.45), (3.46) and (3.42). Moreover, $\operatorname{ess\,lim}_{t\searrow 0} \|v(\cdot,t) - v_0\|_{L^2(\Omega)} = 0$ is a consequence of (3.47).

Finally, from (3.3) and (2.2) we know that

$$\mathcal{F}_\varepsilon(u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t)) + \int_0^t \int_\Omega v_{\varepsilon t}^2 + \int_0^t \int_\Omega \left| \Sigma(u_\varepsilon)\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \Sigma(u_\varepsilon)\nabla v_\varepsilon \right|^2 \le \mathcal{F}_\varepsilon(u_0, v_0) \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \qquad (3.52)$$

where clearly

$$\mathcal{F}_\varepsilon(u_0, v_0) \to \mathcal{F}(u_0, v_0) \qquad \text{as } \varepsilon \searrow 0 \qquad (3.53)$$

according to the positivity of u_0 in $\overline{\Omega}$ and, e.g., the monotone convergence theorem, because $G_\varepsilon(s) \searrow G(s)$ for all $s > 0$ as $\varepsilon \searrow 0$. Moreover, (3.47) together with (3.43) warrants that for a.e. $t > 0$,

$$\frac{1}{2}\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 + \frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot,t) - \int_\Omega u_\varepsilon(\cdot,t)v_\varepsilon(\cdot,t) \to \frac{1}{2}\int_\Omega |\nabla v(\cdot,t)|^2 + \frac{1}{2}\int_\Omega v^2(\cdot,t) - \int_\Omega u(\cdot,t)v(\cdot,t) \qquad (3.54)$$

as $\varepsilon = \varepsilon_j \searrow 0$, and that thanks to Fatou's lemma,

$$\int_\Omega G(u(\cdot,t)) \le \liminf_{\varepsilon=\varepsilon_j\searrow 0} \int_\Omega G_\varepsilon(u_\varepsilon(\cdot,t)) \qquad \text{for a.e. } t>0, \tag{3.55}$$

because clearly $G_\varepsilon(u_\varepsilon(\cdot,t)) \to G(u(\cdot,t))$ a.e. in Ω for a.e. $t>0$ as $\varepsilon=\varepsilon_j \searrow 0$ due to (1.13) and (1.14). Since, apart from that,

$$\int_0^t \int_\Omega v_t^2 \le \liminf_{\varepsilon=\varepsilon_j\searrow 0} \int_0^t \int_\Omega v_{\varepsilon t}^2 \qquad \text{for all } t>0$$

by (3.50) and lower continuity of the norms in L^2 spaces with respect to weak convergence, the energy property (2.7) will thus result from (3.52)-(3.55) as soon as we have shown that for all $t>0$,

$$\int_0^t \int_\Omega \left| \Sigma(u)\frac{\nabla u}{h(u)} - \Sigma(u)\nabla v \right|^2 \le \liminf_{\varepsilon=\varepsilon_j\searrow 0} \int_0^t \int_\Omega \left| \Sigma(u_\varepsilon)\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)} - \Sigma(u_\varepsilon)\nabla v_\varepsilon \right|^2. \tag{3.56}$$

To accomplish this, we fix $(\chi_\delta)_{\delta\in(0,1)} \subset C^\infty([0,\infty))$ such that $0 \le \chi_\delta \le 1$ in $[0,\infty)$, $\chi_\delta \equiv 0$ in $[0,\frac{\delta}{2}] \cup [\frac{2}{\delta},\infty)$ and $\chi_\delta \equiv 1$ in $[\delta,\frac{1}{\delta}]$ for $\delta\in(0,1)$, and that $\chi_\delta \nearrow 1$ on $(0,\infty)$ as $\delta \searrow 0$, and note that then for each fixed $\delta\in(0,1)$,

$$\frac{\chi_\delta(u_\varepsilon)\Sigma(u_\varepsilon)}{h_\varepsilon(u_\varepsilon)}\nabla u_\varepsilon \rightharpoonup \frac{\chi_\delta(u)\Sigma(u)}{h(u)}\nabla u \quad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0. \tag{3.57}$$

Indeed, since

$$\begin{aligned} 0 \le \frac{\chi_\delta(s)\Sigma(s)}{h_\varepsilon(s)} &\le \frac{\chi_\delta(s)(D(s)+1)\Sigma(s)}{S(s)} \\ &\le c_1(\delta) := \max_{\sigma\in[\frac{\delta}{2},\frac{2}{\delta}]} \frac{(D(\sigma)+1)\Sigma(\sigma)}{S(\sigma)} \qquad \text{for all } s\ge 0 \text{ and } \varepsilon\in(0,1), \end{aligned}$$

with $c_1(\delta)$ being finite by continuity of D, S and Σ, and by positivity of S on $(0,\infty)$, it firstly follows from the dominated convergence theorem and (3.42) that

$$\frac{\chi_\delta(u_\varepsilon)\Sigma(u_\varepsilon)}{h_\varepsilon(u_\varepsilon)} \to \frac{\chi_\delta(u)\Sigma(u)}{h(u)} \quad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0,$$

which in conjunction with (3.44) ensures that

$$\frac{\chi_\delta(u_\varepsilon)\Sigma(u_\varepsilon)}{h_\varepsilon(u_\varepsilon)}\nabla u_\varepsilon \rightharpoonup \frac{\chi_\delta(u)\Sigma(u)}{h(u)}\nabla u \quad \text{in } L^1_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0.$$

The observation that due to Lemma 3.10,

$$\sup_{\varepsilon\in(0,1)}\int_0^T\int_\Omega\left|\frac{\chi_\delta(u_\varepsilon)\Sigma(u_\varepsilon)}{h_\varepsilon(u_\varepsilon)}\nabla u_\varepsilon\right|^2 \le c_1^2(\delta)\sup_{\varepsilon\in(0,1)}\int_0^T\int_\Omega|\nabla u_\varepsilon|^2<\infty \qquad \text{for all } T>0$$

therefore shows that indeed (3.57) holds. Since furthermore

$$\chi_\delta(u_\varepsilon)\Sigma(u_\varepsilon)\nabla v_\varepsilon \to \chi_\delta(u)\Sigma(u)\nabla v \quad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0$$

by (3.42), (3.46) and the fact that $0\le\chi_\delta(u_\varepsilon)\Sigma(u_\varepsilon)\le 1$ due to Lemma 2.1, again based on an argument based on lower semicontinuity of the norm in $L^2(\Omega\times(0,t))$, $t>0$, with respect to weak convergence, we infer from (3.57) that

$$\begin{aligned}
&\int_0^t\int_\Omega\chi_\delta^2(u)\left|\Sigma(u)\frac{\nabla u}{h(u)}-\Sigma(u)\nabla v\right|^2\\
&\le\liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^t\int_\Omega\chi_\delta^2(u_\varepsilon)\left|\Sigma(u_\varepsilon)\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)}-\Sigma(u_\varepsilon)\nabla v_\varepsilon\right|^2\\
&\le\liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^t\int_\Omega\left|\Sigma(u_\varepsilon)\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)}-\Sigma(u_\varepsilon)\nabla v_\varepsilon\right|^2 \qquad \text{for all } \delta\in(0,1) \text{ and } t>0.
\end{aligned}$$

Once more thanks to Beppo Levi's theorem, on taking $\delta\searrow 0$ we conclude that

$$\begin{aligned}
&\int\int_{(\Omega\times(0,t))\cap\{u>0\}}\left|\Sigma(u)\frac{\nabla u}{h(u)}-\Sigma(u)\nabla v\right|^2\\
&=\lim_{\delta\searrow 0}\int_0^t\int_\Omega\chi_\delta^2(u)\left|\Sigma(u)\frac{\nabla u}{h(u)}-\Sigma(u)\nabla v\right|^2\\
&\le\liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^t\int_\Omega\left|\Sigma(u_\varepsilon)\frac{\nabla u_\varepsilon}{h_\varepsilon(u_\varepsilon)}-\Sigma(u_\varepsilon)\nabla v_\varepsilon\right|^2 \qquad \text{for all } t>0,
\end{aligned}$$

and that thus indeed (3.56) holds according to (1.14). □

Proof of Theorem 1.1. The claim actually is a by-product of Lemma 3.13. □

4. Nonexistence. Proof of Theorem 1.2

In this section, we prove that for some initial data there does not exist a global weak energy solution of (1.1) satisfying (1.21) for some $p>\frac{2n}{n+2}$, provided that the requirements stated in Theorem 1.2 are satisfied.

We first show that each global weak energy solution has the property of mass conservation.

Lemma 4.1. *Assume that $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary, (1.2) is fulfilled, and let (u,v) be a global weak energy solution of (1.1) with some (u_0,v_0) fulfilling (1.3). Then,*

$$\int_\Omega u(\cdot,t)=\int_\Omega u_0 \qquad \textit{for a.e. } t>0. \tag{4.1}$$

Proof. Let $t_0 > 0$ be a Lebesgue point of $0 < t \mapsto \int_\Omega u(\cdot, t)$. Then since u and $J := D(u)\nabla u - S(u)\nabla v$ are locally integrable in $\overline{\Omega} \times [0, \infty)$, by means of a standard approximation argument it is possible to show that (2.5) continues to hold for $\varphi = \varphi_\delta$, $\delta > 0$, where $\varphi_\delta(x, t) := \zeta_\delta(t)$, $(x, t) \in \overline{\Omega} \times [0, \infty)$, with $\zeta_\delta(t) := 1$ for $t \in [0, t_0]$, $\zeta_\delta(t) := 1 - \frac{t - t_0}{\delta}$ for $t \in (t_0, t_0 + \delta)$ and $\zeta_\delta(t) := 0$ for $t \geq t_0 + \delta$, $\delta > 0$. From (2.5), we thereby obtain that

$$\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \int_\Omega u - \int_\Omega u_0 = - \int_{t_0}^{t_0+\delta} \int_\Omega J \cdot \nabla \varphi_\delta = 0, \tag{4.2}$$

where the Lebesgue point property of t_0 ensures that $\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \int_\Omega u \to \int_\Omega u(\cdot, t_0)$ as $\delta \searrow 0$. Therefore, (4.2) yields the claim due to the fact that the complement in $(0, \infty)$ of the set of all such Lebesgue points has measure zero. $\square$

The main step toward the claimed nonexistence result is an a priori lower bound for $\mathcal{F}(u_0, v_0)$ if a global weak energy solution evolves from (u_0, v_0). We use the strategy from [29, Sect. 5.2] which continues to hold in spite of a slightly different solution concept. For the reader's convenience, we give below the lemmata leading to the announced estimate, and refer to [29] for their proofs. Next we show that v satisfies an appropriate weak formulation of the respective PDE in (1.1) which allows an evaluation at almost any time.

Lemma 4.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, (1.2) is fulfilled, and let (u, v) be a global weak energy solution of (1.1) with some (u_0, v_0) fulfilling (1.3). Then, there exist $(\psi_l)_{l \in \mathbb{N}} \subset C^1(\overline{\Omega})$ and a null set $N \subset (0, \infty)$ such that $\{\psi_l \mid l \in \mathbb{N}\}$ is dense in $W^{1,2}(\Omega)$, and such that*

$$\begin{aligned} \int_\Omega v_t(\cdot, t)\psi_l = &- \int_\Omega \nabla v(\cdot, t) \cdot \nabla \psi_l - \int_\Omega v(\cdot, t)\psi_l \\ &+ \int_\Omega u(\cdot, t)\psi_l \qquad \text{for all } t \in (0, \infty) \setminus N \text{ and any } l \in \mathbb{N}. \end{aligned} \tag{4.3}$$

Proof. see [29, Lemma 5.2] $\square$

From now on, we only consider radial solutions in $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $n \geq 3$ and $R > 0$. By relying on the energy, we next show that $(u, v)(t)$ converges in an appropriate large time limit to some (u_∞, v_∞) which is a stationary weak solution to the second PDE in (1.1).

Lemma 4.3. *Assume that $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $n \geq 3$ and $R > 0$, that (1.2), (1.3) and (1.17) are satisfied, and suppose that (u, v) is a radial global weak energy solution of (1.1) such that*

$$\operatorname*{ess\,sup}_{t>0} \int_\Omega u^p(\cdot, t) < \infty \tag{4.4}$$

for some $p > \frac{2n}{n+2}$. *Then with* $N_\star \subset (0,\infty)$ *taken from Definition 2.2, we can find* $(t_k)_{k\in\mathbb{N}} \subset (0,\infty)\setminus N_\star$ *and nonnegative radially symmetric functions* $u_\infty \in L^p(\Omega)$ *and* $v_\infty \in W^{2,p}(\Omega)$ *such that* $t_k \to \infty$ *as* $k \to \infty$, *that*

$$u(\cdot,t_k) \rightharpoonup u_\infty \quad \text{in } L^p(\Omega) \qquad \text{as } k\to\infty \tag{4.5}$$

and

$$\sup_{k\in\mathbb{N}} \int_\Omega G(u(\cdot,t_k)) < \infty, \tag{4.6}$$

that

$$v(\cdot,t_k) \rightharpoonup v_\infty \quad \text{in } W^{2,p}(\Omega) \qquad \text{as } k\to\infty \tag{4.7}$$

and

$$v(\cdot,t_k) \to v_\infty \quad \text{in } W^{1,2}(\Omega) \qquad \text{as } k\to\infty, \tag{4.8}$$

and that

$$\Sigma(u(\cdot,t_k))\frac{\nabla u(\cdot,t_k)}{h(u(\cdot,t_k))} - \Sigma(u(\cdot,t_k))\nabla v(\cdot,t_k) \to 0 \quad \text{in } L^2(\Omega) \qquad \text{as } k\to\infty. \tag{4.9}$$

Moreover,

$$\int_\Omega u_\infty = \int_\Omega u_0 \tag{4.10}$$

and

$$\int_\Omega \nabla v_\infty\cdot\nabla\psi + \int_\Omega v_\infty\psi = \int_\Omega u_\infty\psi \qquad \text{for all } \psi \in W^{1,2}(\Omega). \tag{4.11}$$

Proof. see [29, Lemma 5.3] □

In order to show that (u_∞, v_∞) is also a stationary generalized solution to the first PDE in (1.1), we need two regularity properties of functions in Sobolev spaces.

Lemma 4.4. *Assume that* $\Omega = B_R(0) \subset \mathbb{R}^n$ *for some* $n \geq 3$ *and* $R > 0$, *let* $p > 1$ *and* $q \geq 1$ *be such that* $(n-2p)q < np$. *Then, there exist* $C(p) > 0$ *and* $C(p,q) > 0$ *such that if* $\psi \in W^{2,p}(\Omega)$ *is radially symmetric, then* $\psi \in C^1(\overline{\Omega}\setminus\{0\})$ *with*

$$|\nabla\psi(x)| \leq C(p)\cdot\|\psi\|_{W^{2,p}(\Omega)}\cdot|x|^{-\frac{n-p}{p}} \qquad \text{for all } x \in \overline{\Omega}\setminus\{0\} \tag{4.12}$$

and

$$\int_\Omega |x\cdot\nabla\psi|^q \leq C(p,q)\cdot\|\psi\|^q_{W^{2,p}(\Omega)}. \tag{4.13}$$

Proof. see [29, Lemma 5.4] □

Lemma 4.5. *Let $I \subset \mathbb{R}$ be an open interval, and suppose that $(\psi_k)_{k\in\mathbb{N}} \subset W^{1,1}(I)$ is such that as $k \to \infty$ we have $\psi_k(r) \to \psi(r)$ for all $r \in I$ and $\psi_k' \to \phi$ in $L^1(I)$ with some $\psi : I \to \mathbb{R}$ and some $\phi \in C^0(I)$. Then $\psi \in C^1(I)$ with $\psi' \equiv \phi$ in I.*

Proof. By [4, Theorem 8.2], we have (up to the choice of a continuous representative) $\psi_k \in C^0(I)$ and

$$\psi_k(x) - \psi_k(y) = \int_y^x \psi_k'(r)dr \qquad \text{for all } x, y \in I$$

and any $k \in \mathbb{N}$. In the limit $k \to \infty$ this implies

$$\psi(x) - \psi(y) = \int_y^x \phi(r)dr \qquad \text{for all } x, y \in I.$$

In view of $\phi \in C^0(I)$, the claim is proved. □

We now show additional properties of (u_∞, v_∞) as announced above.

Lemma 4.6. *Assume that $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $n \geq 3$ and $R > 0$, that (1.2), (1.3) and (1.17) are satisfied, and that (u, v) is a radial global weak energy solution of (1.1) such that (4.4) holds for some $p > \frac{2n}{n+2}$, and let $(t_k)_{k\in\mathbb{N}}$ as well as u_∞ and v_∞ be as in Lemma 4.3. Then, $u_\infty \in C^0(\overline{\Omega} \setminus \{0\}; [0, \infty])$ and $u_\infty \in C^1((\overline{\Omega} \setminus \{0\}) \cap \{u_\infty < \infty\})$ with*

$$\nabla u_\infty = h(u_\infty)\nabla v_\infty \qquad \text{in } (\overline{\Omega} \setminus \{0\}) \cap \{u_\infty < \infty\}. \tag{4.14}$$

Moreover, $G(u_\infty) \in L^1(\Omega)$ and

$$\mathcal{F}(u_\infty, v_\infty) \leq \mathcal{F}(u_0, v_0). \tag{4.15}$$

Proof. see [29, Lemma 5.6] □

Next we show that the initial energy $\mathcal{F}(u_0, v_0)$ is bounded from below by a constant depending on the initial mass $m = \int_\Omega u_0$. To this end, we need two preliminary results, the first being a useful identity, for which the condition $p > \frac{2n}{n+2}$ is important.

Lemma 4.7. *Assume that $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $n \geq 3$ and $R > 0$ and suppose that $p > \frac{2n}{n+2}$ and that $\psi \in W^{2,p}(\Omega)$ is radially symmetric with $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$. Then $\Delta\psi(x \cdot \nabla\psi) \in L^1(\Omega)$ with*

$$\int_\Omega \Delta\psi(x \cdot \nabla\psi) = \frac{n-2}{2}\int_\Omega |\nabla\psi|^2. \tag{4.16}$$

Proof. see [29, Lemma 5.8] □

For the next estimate, we particularly need condition (1.19).

Lemma 4.8. *Assume that $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $n \geq 3$ and $R > 0$, that (1.2), (1.3) and (1.17) are satisfied and suppose that (1.19) holds with some $\mu > 0$ and $K_{\ell G} > 0$. Moreover, assume that (u, v) is a radial global weak energy solution of (1.1) such that (4.4) holds for some $p > \frac{2n}{n+2}$, and let $(t_k)_{k\in\mathbb{N}}$ as well as u_∞ and v_∞ be as in Lemma 4.3. Then $u_\infty(x \cdot \nabla v_\infty)$ belongs to $L^1(\Omega)$ with*

$$-\int_\Omega u_\infty(x \cdot \nabla v_\infty) \leq n \int_{\{u_\infty \geq 1\}} \ell(u_\infty) + 2R \int_\Omega |\nabla v_\infty|. \tag{4.17}$$

Proof. see [29, Lemma 5.7] and notice that (1.19) implies [29, (1.20)] □

Finally, we are in a position to prove that the initial energy $\mathcal{F}(u_0, v_0)$ is bounded from below by a constant depending on the initial mass $m = \int_\Omega u_0$.

Lemma 4.9. *Assume that $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $n \geq 3$ and $R > 0$, that (1.2), (1.3) and (1.17) are satisfied, and suppose that (1.19) holds with some $\mu > 0$ and $K_{\ell G} > 0$. Then for all $m > 0$ there exists $C(m) > 0$ with the property that if (u_0, v_0) complies with (1.3) and (1.17) and is such that $\int_\Omega u_0 = m$, and if (1.1) admits a radial global weak energy solution of (1.1) fulfilling (4.4) with some $p > \frac{2n}{n+2}$, then necessarily*

$$\mathcal{F}(u_0, v_0) \geq -C(m). \tag{4.18}$$

Proof. With $\mu > 0$ and $K_{\ell G} > 0$ taken from (1.19), we fix $\eta \in (0, 1)$ small enough such that $n - 2 - \mu \leq (n - 2)(1 - \eta)$, and employ a Poincaré inequality in choosing $c_1 > 0$ suitably large such that

$$\left(\frac{2R^2}{(n-2)^2\eta} + \frac{1-\eta}{2}\right) \int_\Omega \psi^2 \leq \frac{\eta}{8} \int_\Omega |\nabla\psi|^2 + c_1 \cdot \left\{\int_\Omega |\psi|\right\}^2 \quad \text{for all } \psi \in W^{1,2}(\Omega). \tag{4.19}$$

We then suppose that $m > 0$ and that (1.3) and (1.17) hold with $\int_\Omega u_0 = m$, and that (u, v) is a radial global weak energy solution of (1.1) which satisfies (4.4) with some $p > \frac{2n}{n+2}$. Letting (u_∞, v_∞) be as provided by Lemma 4.3, we first observe upon taking $\psi \equiv 1$ in (4.11) that due to (4.10) we have

$$\int_\Omega v_\infty = \int_\Omega u_\infty = \int_\Omega u_0 = m, \tag{4.20}$$

and that since $v_\infty \in W^{2,p}(\Omega)$, a standard argument applied to (4.11) shows that $\Delta v_\infty = v_\infty - u_\infty$ a.e. in Ω. As a consequence of Lemma 4.7, multiplying this by $x \cdot \nabla v_\infty$ and integrating over Ω yields the identity

$$\frac{n-2}{2} \int_\Omega |\nabla v_\infty|^2 = \int_\Omega v_\infty(x \cdot \nabla v_\infty) - \int_\Omega u_\infty(x \cdot \nabla v_\infty), \tag{4.21}$$

in which thanks to Lemma 4.8 and Young's inequality,

$$\begin{aligned}
-\int_\Omega u_\infty(x\cdot\nabla v_\infty) &\le n\int_{\{u_\infty\ge 1\}}\ell(u_\infty)+2R\int_\Omega|\nabla v_\infty| \\
&\le n\int_{\{u_\infty\ge 1\}}\ell(u_\infty)+\frac{(n-2)\eta}{4}\int_\Omega|\nabla v_\infty|^2+\frac{4R^2|\Omega|}{(n-2)\eta}.
\end{aligned}$$

As furthermore, again by Young's inequality, and by (4.19) and (4.20),

$$\begin{aligned}
&\frac{1}{n-2}\int_\Omega v_\infty(x\cdot\nabla v_\infty)+\frac{1-\eta}{2}\int_\Omega v_\infty^2 \\
&\quad\le\frac{\eta}{8}\int_\Omega|\nabla v_\infty|^2+\Big(\frac{2R^2}{(n-2)^2\eta}+\frac{1-\eta}{2}\Big)\int_\Omega v_\infty^2 \\
&\quad\le\frac{\eta}{4}\int_\Omega|\nabla v_\infty|^2+c_1\cdot\Big\{\int_\Omega v_\infty\Big\}^2 \\
&\quad\le\frac{\eta}{4}\int_\Omega|\nabla v_\infty|^2+c_1m^2,
\end{aligned}$$

from (4.21) we thus obtain that

$$\begin{aligned}
\frac12\int_\Omega|\nabla v_\infty|^2+\frac{1-\eta}{2}\int_\Omega v_\infty^2 &= \frac{1}{n-2}\int_\Omega v_\infty(x\cdot\nabla v_\infty)+\frac{1-\eta}{2}\int_\Omega v_\infty^2 \\
&\quad-\frac{1}{n-2}\int_\Omega u_\infty(x\cdot\nabla v_\infty) \\
&\le\frac{\eta}{4}\int_\Omega|\nabla v_\infty|^2+c_1m^2 \\
&\quad+\frac{n}{n-2}\int_{\{u_\infty\ge 1\}}\ell(u_\infty)+\frac{\eta}{4}\int_\Omega|\nabla v_\infty|^2 \\
&\quad+\frac{4R^2|\Omega|}{(n-2)^2\eta},
\end{aligned}$$

which is equivalent to the inequality

$$\begin{aligned}
\frac12\int_\Omega|\nabla v_\infty|^2+\frac12\int_\Omega v_\infty^2 &\le\frac{n}{(n-2)(1-\eta)}\int_{\{u_\infty\ge 1\}}\ell(u_\infty)+\frac{c_1m^2}{1-\eta} \\
&\quad+\frac{4R^2|\Omega|}{(n-2)^2\eta(1-\eta)}.
\end{aligned}$$

Since (4.11) ensures that

$$\int_\Omega u_\infty v_\infty=\int_\Omega|\nabla v_\infty|^2+\int_\Omega v_\infty^2,$$

through (4.15), (1.19) and (4.20) this reveals that

$$
\begin{aligned}
\mathcal{F}(u_0, v_0) &\geq \mathcal{F}(u_\infty, v_\infty) \\
&= -\frac{1}{2}\int_\Omega |\nabla v_\infty|^2 - \frac{1}{2}\int_\Omega v_\infty^2 + \int_\Omega G(u_\infty) \\
&\geq -\frac{n}{(n-2)(1-\eta)}\int_{\{u_\infty \geq 1\}} \ell(u_\infty) - \frac{c_1 m^2}{1-\eta} \\
&\quad - \frac{4R^2|\Omega|}{(n-2)^2\eta(1-\eta)} + \int_\Omega G(u_\infty) \\
&\geq \left(1 - \frac{n-2-\mu}{(n-2)(1-\eta)}\right)\int_\Omega G(u_\infty) - \frac{nK_{\ell G} m}{(n-2)(1-\eta)} \\
&\quad - \frac{c_1 m^2}{1-\eta} - \frac{4R^2|\Omega|}{(n-2)^2\eta(1-\eta)},
\end{aligned}
$$

because G is nonnegative. As $1 - \frac{n-2-\mu}{(n-2)(1-\eta)} \geq 0$ according to our restriction on η, this establishes (4.18) with $C(m) := \frac{nK_{\ell G} m}{(n-2)(1-\eta)} + \frac{c_1 m^2}{1-\eta} + \frac{4R^2|\Omega|}{(n-2)^2\eta(1-\eta)}$. □

In contrast to the previous result, it is already known that there are initial data such that $\mathcal{F}(u_0, v_0)$ is arbitrarily small provided that (1.18) is fulfilled with some $\alpha > \frac{2}{n}$.

Lemma 4.10. *Assume that $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $n \geq 3$ and $R > 0$, and suppose that (1.18) holds with some $\alpha > \frac{2}{n}$ and $K_{SD} > 0$. Then for each $m > 0$ and $C > 0$ one can find functions u_0 and v_0 fulfilling (1.3) and (1.17) as well as $\int_\Omega u_0 = m$ and*

$$\mathcal{F}(u_0, v_0) < -C.$$

Proof. This has been verified by means of an explicit construction in [26, Lemma 4.1]. □

The previous two results finally show that Theorem 1.2 is valid.

Proof of Theorem 1.2. This statement is an immediate consequence of Lemma 4.9 when combined with Lemma 4.10. □

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