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On the separation property and the global attractor for the nonlocal Cahn-Hilliard equation in three dimensions

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Abstract. We consider the nonlocal Cahn-Hilliard equation with constant mobility and singular potential in three dimensional bounded and smooth domains. This model describes phase separation in binary fluid mixtures. Given any global solution (whose existence and uniqueness are already known), we prove the so-called *instantaneous* and *uniform* separation property: any global solution with initial finite energy is globally confined (in the L^∞ metric) in the interval $[-1+\delta, 1-\delta]$ on the time interval $[\tau, \infty)$ for any $\tau > 0$, where δ only depends on the norms of the initial datum, τ and the parameters of the system. We then exploit such result to improve the regularity of the global attractor for the dynamical system associated to the problem.

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1. Introduction and main results

We study the nonlocal Cahn-Hilliard equation (see [13,17,18])

$$\partial_t \phi = \Delta \left(F'(\phi) - J * \phi \right) \quad \text{in } \Omega \times (0, \infty), \tag{1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^3$. The state variable ϕ represents the difference of the concentrations of two fluids. This equation is commonly rewritten as

$$\partial_t \phi = \Delta \mu, \quad \mu = F'(\phi) - J * \phi \quad \text{in } \Omega \times (0, \infty), \tag{2}$$

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which is equipped with the following boundary and initial conditions

$$\partial_{\boldsymbol{n}}\mu = 0 \quad \text{on } \partial\Omega \times (0,T), \quad \phi(\cdot,0) = \phi_0 \quad \text{in } \Omega, \tag{3}$$

where $\boldsymbol{n}$ is the outward normal vector on $\partial\Omega$. The physically relevant form of the nonlinear function F is given by the convex part of the Flory-Huggins (also known as Boltzmann-Gibbs entropy) potential

$$F(s) = \frac{\theta}{2}\Big[(1+s)\ln(1+s) + (1-s)\ln(1-s)\Big], \quad s \in [-1,1]. \tag{4}$$

The function $J : \mathbb{R}^3 \to \mathbb{R}$ is a (sufficiently smooth) interaction kernel such that $J(x) = J(-x)$. The notation $(J * \phi)(x)$ stands for $\int_\Omega J(x-y)\phi(y)\,\mathrm{d}y$. The system (2–3) is a gradient flow with respect to the metric of $H^1_{(0)}(\Omega)'$, namely the dual of $H^1(\Omega)$ with zero mean value, associated to the free energy

$$\begin{aligned} E_{NL}(\phi) &= -\frac{1}{2}\int_{\Omega\times\Omega} J(x-y)\phi(y)\phi(x)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega F(\phi(x))\,\mathrm{d}x \\ &= \frac{1}{4}\int_{\Omega\times\Omega} J(x-y)|\phi(y)-\phi(x)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_\Omega F(\phi(x)) - \frac{a(x)}{2}\phi^2(x)\,\mathrm{d}x, \end{aligned} \tag{5}$$

where $a(x) = (J * 1)(x) = \int_\Omega J(x-y)\,\mathrm{d}y$ for $x \in \Omega$. The function μ appearing in (2) is the so-called chemical potential, which corresponds to $\frac{\delta E_{NL}(\phi)}{\delta \phi}$.

The analysis of the nonlocal Cahn-Hilliard equation with logarithmic potential (20) (actually a more general class of *singular* potentials) has been firstly studied in [13] (see also [11] for another proof of existence and [15] for the viscous case). In particular, the authors in [13] proved the existence and uniqueness of global weak solutions and their propagation of regularity for positive times (see proof of Theorem 1 below for more details). Such solutions satisfy

$$\phi \in L^\infty(\Omega \times (0,\infty)) \text{ with } |\phi(x,t)| < 1 \text{ for a.e. } x \in \Omega,\ \forall t > 0. \tag{6}$$

Such property has an important physical meaning since the solution ϕ takes value in the significant interval $[-1,1]$ (cf. definition of ϕ). Concerning the regularity of the global solutions, a main task consists in establishing L^p estimates of $F''(\phi)$ and $F'''(\phi)$, which are needed to prove the existence of *classical* solutions. This is a difficult question due to the growth conditions

$$F''(s) \leq C\mathrm{e}^{C|F'(s)|}, \quad |F'''(s)| \leq CF''(s)^2, \tag{7}$$

which prevent the possibility to control $F''(\phi)$ or $F'''(\phi)$ in L^p spaces in terms of some L^p norms of $F'(\phi)$ (as possible in the case of potential with polynomial growth). However, although L^p estimates of $F''(\phi)$ and $F'''(\phi)$ can be useful, this is not sufficient (in many cases) to prove higher order regularity, and it is necesssary to show the *instanteneous* (also called *strict*) *separation property*: for any $\tau > 0$, there exists $\delta = \delta(\tau) \in (0, 1)$ such that

$$|\phi(x,t)| \le 1 - \delta, \text{ for all } (x,t) \in \Omega \times (\tau, \infty). \tag{8}$$

We point out that the separation property is expected due to the gradient flow structure of the Cahn-Hilliard model, which drives the dynamics towards stationary states consisting of *separated* functional minima of the free energy $E_{NL}(\phi)$. A first proof of (8) has been established in [13, Theorem 5.2] in the two dimensional case. The argument hinges upon an iterative Alikakos-Moser argument for the powers of $|F'(\phi)|$ combined with Gagliardo-Nirenberg interpolation inequalities and the Trudinger-Moser inequality. A new proof of such result admitting a more general class of singular potentials has been proposed in [14, Sect. 4]. The latter relies on a De Giorgi's iterative argument. This method is usually employed to obtain an L^∞ estimate of the solution to a second order PDE, thereby the main achievement in [14] was to recast the method in order to get a *specific* bound (cf. (8) with (6)). More recently, the separation property has been proven in three dimensions in [28], which allowed to show the convergence to stationary states. The author in [28] improved the method in [14] in two ways: the truncated functions ϕ_n (see proof of Theorem 1 below) are shown to be bounded by 2δ (instead of 1 as in [14]), and a Poincaré type inequality for time-dependent functions is employed to avoid the integrals of $\overline{\phi_n}$ (see term Z_2 in [14, Sect. 4]). However, a main drawback of the argument, which is due to the latter ingredient, is that the value of δ in (8) depends on the particular solution. More precisely, δ cannot be estimated only in terms of *norm* of the initial data and the parameters of the system. The purpose of this work is to demonstrate that the De Giorgi iterative scheme in [14] and the observation $\|\phi_n\|_{L^\infty} \le 2\delta$ are sufficient to achieve (8) with a value δ which depends on τ, the initial energy $E_{NL}(\phi_0)$ and the parameters of the system (e.g. F, Ω, J). Beyond its intrinsic interest, this allows us to improve the regularity of the global attractor for the dynamical system associated to the system (2–3).

In order to present the main results of this work, let us formulate the assumptions for the admissible class of potentials:

(A1) $F \in C([-1,1]) \cap C^2(-1,1)$ such that $\lim_{|s|\to 1} F'(s) = \pm\infty$ and $F''(s) \ge \theta > 0$ for all $s \in (-1,1)$.

(A2) There exists $\varepsilon_0 > 0$ such that F'' is monotone non-decreasing on $[1-\varepsilon_0, 1)$ and non-increasing in $(-1, 1+\varepsilon_0]$.

(A3) There exist $\varepsilon_1 \in \left(0, \frac{1}{2}\right)$ and $C_F \ge 1$ such that

$$\frac{1}{F'(1-2\delta)} \le \frac{C_F}{|\ln(\delta)|}, \quad \frac{1}{F'(-1+2\delta)} \le \frac{C_F}{|\ln(\delta)|}, \quad \forall\, 0 < \delta \le \varepsilon_1 \tag{9}$$

and

$$\frac{1}{F''(1-2\delta)} \le C_F\delta, \quad \frac{1}{F''(-1+2\delta)} \le C_F\delta, \quad \forall\, 0 < \delta \le \varepsilon_1. \tag{10}$$

Remark 1. The assumptions (**A1**–**A3**) are satisfied by the convex part of the Flory-Huggins potential (4).

The main result reads as follows

Theorem 1. *Assume that (**A1**–**A3**) hold. Let J be $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)$ such that $J(x) = J(-x)$ for all $x \in \mathbb{R}^3$. Assume that $\phi_0 \in L^\infty(\Omega)$ such that $\|\phi_0\|_{L^\infty(\Omega)} \le 1$ and $|\overline{\phi_0}| = |\Omega|^{-1} \left|\int_\Omega \phi_0(x)\,\mathrm{d}x\right| < 1$. Then, for any $\tau > 0$, there exists $\delta \in (0,1)$ such that the unique global solution to (2–3) satisfies*

$$|\phi(x,t)| \le 1-\delta, \; \text{for a.e. } (x,t) \in \Omega \times [\tau,\infty). \tag{11}$$

In addition, there exists three positive constants C_1, C_2, C_3 and $\alpha \in (0,1)$ such that

$$\sup_{t\ge\tau} \|\mu(t)\|_{L^\infty(\Omega)} \le C_1, \quad \sup_{t\ge\tau} \|\partial_t \mu\|_{L^2(t,t+1;L^2(\Omega))} \le C_2, \tag{12}$$

and

$$|\phi(x_1,t_1) - \phi(x_2,t_2)| \le C_3 \left(|x_1-x_2|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}} \right), \tag{13}$$

for any $(x_1,t_1),(x_2,t_2) \in \Omega_t = \overline{\Omega} \times [t,t+1]$, for any $t \ge \tau$. The values of δ, C_1, C_2, C_3 and α only depend on τ, δ, the initial energy $E_{NL}(\phi_0)$, the initial mean $\overline{\phi_0}$ and the parameters of the system (i.e. F, J, Ω).

Remark 2. A combination of the separation property (11) and the Hölder regularity (13) gives the following stronger result

$$|\phi(x,t)| \le 1-\delta, \quad \forall\, (x,t) \in \overline{\Omega} \times [\tau,\infty). \tag{14}$$

As a direct consequence of Theorem 1, we infer additional features for the longtime behavior of the solutions to system (2–3). Let us introduce the dynamical system associated with problem (2–3). For any given $m \in (0,1)$, we define the phase space

$$\mathcal{H}_m = \left\{ \phi \in L^\infty(\Omega) : \|\phi\|_{L^\infty(\Omega)} \le 1 \text{ and } -1+m \le \overline{\phi} \le 1-m \right\} \tag{15}$$

endowed with the metric

$$\mathbf{d}(\phi_1,\phi_2) = \|\phi_1 - \phi_2\|_{L^2(\Omega)}. \tag{16}$$

The pair $(\mathcal{H}_m, \mathbf{d})$ is a complete metric space. Then, we define the map

$$S(t) : \mathcal{H}_m \to \mathcal{H}_m, \quad S(t)\phi_0 = \phi(t), \quad \forall\, t \ge 0,$$

where ϕ is the global (weak) solution (see [13, Theorem 3.4]) originating from the initial condition ϕ_0. It was shown in [13, Sect. 4] that $(\mathcal{H}_m, S(t))$ is a dissipative dynamical system and $S(t)$ is a closed semigroup on the phase space $\mathcal{H}_m$ (see [27] for the definition). Furthermore, the existence of the global attractor $\mathcal{A}_m$ was proven in [13, Theorem 4.4]. In particular, it is shown that $\mathcal{A}_m$ is a bounded set in $\mathcal{H}_m \cap H^1(\Omega)$. Our next result is concerned with the regularity of the global attractor $\mathcal{A}_m$.

Theorem 2. *Let (**A1**–**A3**) hold. Assume that $J \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)$ such that $J(x) = J(-x)$ for all $x \in \mathbb{R}^3$. Consider the connected global attractor $\mathcal{A}_m$ associated with the dynamical system $(\mathcal{H}_\kappa, S(t))$. Then, $\mathcal{A}_m \subset B_{L^\infty(\Omega)}(0, 1-\delta)$ and is bounded in $C^\alpha(\overline{\Omega})$.*

Before proceeding with the proofs of the main results, it worth presenting a wider picture concerning the validity of the separation property for other Cahn-Hilliard equations. First, we recall the nonlocal Cahn-Hilliard equation with non-constant degenerate mobility

$$\partial_t \phi = \operatorname{div}\Big((1-\phi^2)\nabla\mu\Big), \quad \mu = F'(\phi) - J * \phi \quad \text{in } \Omega \times (0, \infty), \tag{17}$$

which is completed with (3). In this case, the separation property has been previously proven by [23] in both two and three dimensions (see also [10]). Next, we consider the (local) Cahn-Hilliard equation [2–4] (see also [9,26]) with constant mobility

$$\partial_t \phi = \Delta\left(-\Delta\phi + \Psi'(\phi)\right) \quad \text{in } \Omega \times (0, T), \tag{18}$$

subject to the classical boundary and initial conditions

$$\partial_{\boldsymbol{n}}\phi = \partial_{\boldsymbol{n}}\Delta\phi = 0 \quad \text{on } \partial\Omega \times (0, T), \quad \phi(\cdot, 0) = \phi_0 \quad \text{in } \Omega, \tag{19}$$

where Ψ is the Flory-Huggins potential defined by

$$\begin{aligned}\Psi(s) &= F(s) - \frac{\theta_0}{2}s^2 \\ &= \frac{\theta}{2}\Big[(1+s)\ln(1+s) + (1-s)\ln(1-s)\Big] - \frac{\theta_0}{2}s^2, \quad s \in [-1, 1],\end{aligned} \tag{20}$$

with constant parameters θ and θ_0 fulfilling the conditions $0 < \theta < \theta_0$. The Cahn-Hilliard system (18) is the gradient flow with respect to the $H^1_{(0)}(\Omega)'$ metric of the total free energy

$$E_L(\phi) = \int_\Omega \frac{1}{2}|\nabla\phi|^2 + \Psi(\phi(x))\,\mathrm{d}x. \tag{21}$$

The separation property (8) for (18–19) was first established in [7] and [25] in one and two dimensions, respectively. The argument has been subsequently simplified in [19] and [20]. More recently, it was extended to a more general class of potential in [14]. In three dimensions, the separation property has been shown only in [1] on the time interval $[T_{SP}, \infty)$, where T_{SP} cannot be computed explicitly (see also [23] for a class of singular potentials different from (20)). However, it still remains a major challenge to demonstrate the separation property for (18–19) for all positive times in three dimensions. Finally, we mention the recent results regarding the nonlocal-to-local asymptotics obtained in [5,6,16]. That is, the weak solution to the nonlocal Cahn-Hillliard equation converges to the weak solution of the local Cahn-Hilliard equation, under suitable conditions on the data of the problem and a rescaling of the interaction kernel J.

2. Separation property and Hölder regularity

In this section we provide an improved proof of the separation property for the nonlocal Cahn-Hilliard equation in three dimensional domains. Then, we derive some consequences on the regularity of the solution.

Let us first recall the following well-known result.

Lemma 1. *Let* $\{y_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_+$ *satisfy the relation*

$$y_{n+1} \le Cb^n y_n^{1+\epsilon},$$

for some $C > 0$*,* $b > 1$ *and* $\epsilon > 0$*. Assume that* $y_0 \le C^{-\frac{1}{\epsilon}} b^{-\frac{1}{\epsilon^2}}$*. Then, we have*

$$y_n \le y_0 b^{-\frac{n}{\epsilon}}, \quad \forall\, n \ge 1.$$

In particular, $y_n \to 0$ *as* $n \to \infty$*.*

Proof of Theorem 1. Let us report the well-posedness results from [13, Theorems 3.4 and 4.1]: there exists a unique weak solution $\phi : \Omega \times [0,\infty) \to \mathbb{R}$ to the system (2–3) satisfying

$$\begin{aligned}
&\phi \in L^\infty(\Omega \times (0,\infty)) : |\phi(x,t)| < 1 \text{ a.e. in } \Omega,\ \forall\, t > 0,\\
&\phi \in L^2_{\mathrm{loc}}(0,\infty; H^1(\Omega)) \cap H^1_{\mathrm{loc}}(0,\infty; H^1(\Omega)'),\\
&\mu \in L^2_{\mathrm{loc}}(0,\infty; H^1(\Omega)), \quad F'(\phi) \in L^2_{\mathrm{loc}}(0,\infty; H^1(\Omega)),
\end{aligned} \tag{22}$$

such that

$$\langle \partial_t \phi, v\rangle + (\nabla\mu, \nabla v) = 0 \quad \forall\, v \in H^1(\Omega),\ \text{a.e. in } (0,\infty), \tag{23}$$

$$\mu = F'(\phi) - J * \phi \quad \text{a.e. in } \ \Omega \times (0,\infty), \tag{24}$$

and $\phi(\cdot,0) = \phi_0(\cdot)$ in Ω. Furthermore, for any $\tau \in (0,1)$,

$$\operatorname*{ess\,sup}_{t\ge\tau} \|\partial_t\phi(t)\|_{(H^1(\Omega))'} + \sup_{t\ge\tau} \|\partial_t\phi\|_{L^2(t,t+1;L^2(\Omega))} \le \frac{C_0}{\sqrt{\tau}}, \tag{25}$$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\ge\tau} \|\mu(t)\|_{H^1(\Omega)} + \sup_{t\ge\tau} \|\phi(t)\|_{H^1(\Omega)} \le \frac{C_0}{\sqrt{\tau}},\\
&\operatorname*{ess\,sup}_{t\ge\tau} \|F'(\phi)\|_{H^1(\Omega)} + \sup_{t\ge\tau} \|\mu\|_{L^2(t,t+1;H^2(\Omega))} \le \frac{C_0}{\sqrt{\tau}},
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\sup_{t\ge\tau} \|\nabla\mu\|_{L^q(t,t+1;L^p(\Omega))} + \|\nabla\phi\|_{L^q(t,t+1;L^p(\Omega))} \le C_1(\tau),\\
&\quad \text{with } \frac{3p-6}{2p} = \frac{2}{q},\ \forall\, p \in [2,6],
\end{aligned} \tag{27}$$

where the positive constant C_0 only depends on $E_{NL}(\phi_0)$, $\overline{\phi_0}$, Ω and the parameters of the system. The positive constant $C_1(\tau)$ also depends on the same quantities as C_0, in addition to τ. Furthermore, the constants C_0 and C_1 are uniformly bounded in $\overline{\phi_0}$ if $\overline{\phi_0}$ lies in a compact set of $(-1, 1)$.

In the first part of the proof, we show the separation property (11). To this end, we now introduce the iteration scheme à la De Giorgi devised in [14, Sect. 4]. Let $\tau > 0$ be fixed. We consider three positive parameters T, $\widetilde{\tau}$ and δ such that (cf. assumption (**A2**)-(**A3**))

$$T - 3\widetilde{\tau} \geq \frac{\tau}{2} \quad \text{and} \quad \delta \in \left(0, \min\{\frac{\varepsilon_0}{2}, \varepsilon_1\}\right).$$

The precise value of $\widetilde{\tau}$ and δ will be chosen afterwards. We define two sequences

$$\begin{cases} t_{-1} = T - 3\widetilde{\tau} \\ t_n = t_{n-1} + \frac{\widetilde{\tau}}{2^n} \end{cases} \quad \forall n \geq 0, \quad \text{and} \quad k_n = 1 - \delta - \frac{\delta}{2^n}, \quad \forall n \geq 0. \tag{28}$$

Notice that

$$t_{-1} < t_n < t_{n+1} < T - \widetilde{\tau}, \quad \forall n \geq 0, \quad t_n \to t_{-1} + 2\widetilde{\tau} = T - \widetilde{\tau} \quad \text{as } n \to \infty, \tag{29}$$

and

$$1 - 2\delta \leq k_n < k_{n+1} < 1 - \delta, \quad \forall n \geq 0, \quad k_n \to 1 - \delta \quad \text{as } n \to \infty. \tag{30}$$

For $n \geq 0$, we introduce $\eta_n \in C^1(\mathbb{R})$ such that

$$\eta_n(t) = \begin{cases} 1, & t \geq t_n \\ 0, & t \leq t_{n-1} \end{cases} \quad \text{and} \quad |\eta_n'(t)| \leq 2\frac{2^n}{\widetilde{\tau}}. \tag{31}$$

Next, for $n \geq 0$, we consider the function

$$\phi_n(x, t) = \max\{\phi(x, t) - k_n, 0\} = (\phi - k_n)_+.$$

Consequently, we introduce the sets

$$I_n = [t_{n-1}, T] \quad \text{and} \quad A_n(t) = \{x \in \Omega : \phi(x, t) - k_n \geq 0\}, \quad \forall t \in I_n.$$

If $t \in [0, t_{n-1})$, we set $A_n(t) = \emptyset$. We observe that we have

$$I_{n+1} \subseteq I_n, \quad \forall n \geq 0, \qquad I_n \to [T - \widetilde{\tau}, T] \quad \text{as } n \to \infty, \tag{32}$$

and

$$A_{n+1}(t) \subseteq A_n(t), \quad \forall n \geq 0, \ t \in I_{n+1}. \tag{33}$$

The last ingredient is

$$y_n = \int_{I_n} \int_{A_n(s)} 1 \, dx \, ds, \quad \forall n \geq 0.$$

For any $n \geq 0$, we choose as test function $v = \phi_n \eta_n^2$ in (23). Integrating over $[t_{n-1}, t]$, where $t_n \leq t \leq T$, we obtain the relation

$$\begin{aligned} &\int_{t_{n-1}}^{t} \left\langle \partial_t \phi, \phi_n\, \eta_n^2 \right\rangle \mathrm{d}s + \int_{t_{n-1}}^{t} \int_{A_n(s)} \nabla F'(\phi) \cdot \nabla \phi_n\, \eta_n^2 \,\mathrm{d}x\, \mathrm{d}s \\ &\quad = \int_{t_{n-1}}^{t} \int_{A_n(s)} (\nabla J * \phi) \cdot \nabla \phi_n\, \eta_n^2 \,\mathrm{d}x\, \mathrm{d}s. \end{aligned} \tag{34}$$

Since $F'(\phi) \in L^\infty(\tau, \infty; H^1(\Omega))$ and $|\{x \in \Omega : |\phi(x,t)| = 1\}| = 0$ for all $t \geq \tau$, we deduce from [24] that $h_k(F'(\phi)) \in L^\infty(\tau, \infty; H^1(\Omega) \cap L^\infty(\Omega))$, where

$$h_k : \mathbb{R} \to \mathbb{R}, \quad h_k(s) = \begin{cases} k, & s \geq k, \\ s, & s \in (-k, k), \\ k, & s \leq -k, \end{cases} \quad \forall\, k \in \mathbb{N}.$$

Then, it follows that $h_k(F'(\phi)) \to F'(\phi)$ almost everywhere in Ω and for all $t \geq \tau$, and $\nabla(h_k(F'(\phi))) = F''(\phi)\nabla\phi\, 1_{\{|F'(\phi)|<k\}}(\cdot) \to F''(\phi)\nabla\phi$ almost everywhere in Ω and for all $t \geq \tau$. Thus, by the monotone convergence theorem, $\int_\Omega |F''(\phi(t))\nabla\phi(t)|^2 \,\mathrm{d}x \leq \lim_{k\to\infty} \|h_k(F'(\phi(t)))\|_{H^1(\Omega)}^2 = \|F'(\phi(t))\|_{H^1(\Omega)}^2 < \infty$, for all $t \geq \tau$. As consequence, it is easily seen that $\nabla F'(\phi) = F''(\phi)\nabla\phi$ in distributional sense. Thanks to this, we rewrite (34) as

$$\begin{aligned} &\int_{t_{n-1}}^{t} \left\langle \partial_t \phi, \phi_n\, \eta_n^2 \right\rangle \mathrm{d}s + \int_{t_{n-1}}^{t} \int_{A_n(s)} F''(\phi)\nabla\phi \cdot \nabla \phi_n\, \eta_n^2 \,\mathrm{d}x\, \mathrm{d}s \\ &\quad = \int_{t_{n-1}}^{t} \int_{A_n(s)} (\nabla J * \phi) \cdot \nabla \phi_n\, \eta_n^2 \,\mathrm{d}x\, \mathrm{d}s. \end{aligned}$$

Notice that

$$\int_{t_{n-1}}^{t} \left\langle \partial_t \phi, \phi_n \eta_n^2 \right\rangle \mathrm{d}s = \frac{1}{2}\|\phi_n(t)\|_{L^2(\Omega)}^2 - \int_{t_{n-1}}^{t} \|\phi_n(s)\|_{L^2(\Omega)}^2\, \eta_n\, \partial_t \eta_n \,\mathrm{d}s.$$

Also, by the choice of δ, the assumption (**A2**) and the fact $A_n(t) \subseteq A_0(t)$ for $t \geq t_{n-1}$, we have

$$\int_{t_{n-1}}^{t} \int_{A_n(s)} F''(\phi)\nabla\phi \cdot \nabla\phi_n\, \eta_n^2 \,\mathrm{d}x\, \mathrm{d}s \geq F''(1-2\delta) \int_{t_{n-1}}^{t} \|\nabla\phi_n\|_{L^2(\Omega)}^2\, \eta_n^2 \,\mathrm{d}s.$$

Thus, we end up with

$$\begin{aligned} &\frac{1}{2}\|\phi_n(t)\|_{L^2(\Omega)}^2 + F''(1-2\delta) \int_{t_{n-1}}^{t} \|\nabla\phi_n\|_{L^2(\Omega)}^2\, \eta_n^2 \,\mathrm{d}s \\ &\qquad \leq \underbrace{\int_{t_{n-1}}^{t} \int_{A_n(s)} (\nabla J * \phi) \cdot \nabla\phi_n \eta_n^2 \,\mathrm{d}x\, \mathrm{d}s}_{I_1} \end{aligned}$$

$$+\underbrace{\int_{t_{n-1}}^{t}\|\phi_n(s)\|_{L^2(\Omega)}^2\eta_n\partial_t\eta_n\,\mathrm{d}s}_{I_2},\quad \forall\, t\in[t_n,T].$$

We now observe that

$$\begin{aligned}\sup_{x\in\Omega}|(\nabla J*\phi)(x)| &= \sup_{x\in\Omega}\left|\int_\Omega \nabla J(x-y)\phi(y)\,\mathrm{d}y\right| \le \sup_{x\in\Omega}\int_\Omega|\nabla J(x-y)|\,\mathrm{d}y\\ &= \sup_{x\in\Omega}\int_{x-\Omega}|\nabla J(z)|\,\mathrm{d}z.\end{aligned}$$

Since Ω is bounded, there exists $M>0$ such that $\Omega\subseteq B_M(\mathbf{0})$. Also, $\mathrm{diam}(\Omega)<\infty$. Then, there exists M_1 such that the set $x-\Omega\subset B_{M_1}(\mathbf{0})$ for any $x\in\Omega$. It follows that

$$\|\nabla J*\phi\|_{L^\infty(\Omega)}\le\int_{B_{M_1}(\mathbf{0})}|\nabla J(z)|\,\mathrm{d}z=\|\nabla J\|_{L^1(B_{M_1}(\mathbf{0}))}. \tag{35}$$

For simplicity of notation, we will use B_{M_1} to denote $B_{M_1}(\mathbf{0})$. A similar argument applies for $\|J*\phi\|_{L^\infty(\Omega)}$. Concerning the first term I_1, we obtain as in [14, Sect. 4] that

$$\begin{aligned}I_1 &= \int_{t_{n-1}}^{t}\int_{A_n(s)}(\nabla J*\phi)\,\eta_n\cdot\nabla\phi\,\eta_n\,\mathrm{d}x\,\mathrm{d}s\\
&\le\frac{1}{2}F''(1-2\delta)\int_{t_{n-1}}^{t}\|\nabla\phi_n\|_{L^2(\Omega)}^2\,\eta_n^2\,\mathrm{d}s+\frac{1}{2}\frac{1}{F''(1-2\delta)}\int_{t_{n-1}}^{t}\int_{A_n(s)}|\nabla J*\phi|^2\,\eta_n^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le\frac{1}{2}F''(1-2\delta)\int_{t_{n-1}}^{t}\|\nabla\phi_n\|_{L^2(\Omega)}^2\,\eta_n^2\,\mathrm{d}s+\frac{1}{2}\frac{1}{F''(1-2\delta)}\int_{t_{n-1}}^{t}\|\nabla J*\phi\|_{L^\infty(\Omega)}^2\int_{A_n(s)}1\,\mathrm{d}x\,\mathrm{d}s\\
&\le\frac{1}{2}F''(1-2\delta)\int_{t_{n-1}}^{t}\|\nabla\phi_n\|_{L^2(\Omega)}^2\,\eta_n^2\,\mathrm{d}s+\frac{1}{2}\frac{\|\nabla J\|_{L^1(B_{M_1})}^2}{F''(1-2\delta)}\int_{I_n}\int_{A_n(s)}1\,\mathrm{d}x\,\mathrm{d}s\\
&\le\frac{1}{2}F''\,(1-2\delta)\int_{t_{n-1}}^{t}\|\nabla\phi_n\|_{L^2(\Omega)}^2\,\eta_n^2\,\mathrm{d}s+\frac{1}{2}\frac{\|\nabla J\|_{L^1(B_{M_1})}^2}{F''\,(1-2\delta)}\,y_n.\end{aligned}$$

This is actually a correction of the argument in [13] and [28] where $\|\nabla J\|_{L^1(\Omega)}$ appears in the estimate analogous to the one above, instead of $\|\nabla J\|_{L^1(B_{M_1})}$. In order to handle the term I_2, we recall the main observation in [28]:

$$0\le\phi_n\le 2\delta\quad\text{a.e. in }\Omega,\ \forall\, t\in[T-2\widetilde{\tau},T]. \tag{36}$$

By exploiting (31) and (36), we simply have

$$I_2\le\frac{2^{n+1}}{\widetilde{\tau}}\int_{t_{n-1}}^{t}\int_{A_n(s)}\phi_n^2\,\mathrm{d}x\,\mathrm{d}s\le\frac{2^{n+1}}{\widetilde{\tau}}\int_{I_n}\int_{A_n(s)}(2\delta)^2\,\mathrm{d}x\,\mathrm{d}s=\frac{2^{n+3}}{\widetilde{\tau}}\delta^2 y_n.$$

Collecting the above estimates together, we infer that

$$\|\phi_n(t)\|_{L^2(\Omega)}^2+F''\,(1-2\delta)\int_{t_{n-1}}^{t}\|\nabla\phi_n\|_{L^2(\Omega)}^2\,\eta_n^2\,\mathrm{d}s$$

$$\leq \frac{\|\nabla J\|^2_{L^1(B_{M_1})}}{F''(1-2\delta)} y_n + 2^4 \frac{2^n}{\widetilde{\tau}} \delta^2 y_n, \quad \forall t \in [t_n, T]. \tag{37}$$

As a consequence,

$$\max_{t \in I_{n+1}} \|\phi_n(t)\|^2_{L^2(\Omega)} \leq X_n, \quad F''(1-2\delta) \int_{I_{n+1}} \|\nabla \phi_n\|^2_{L^2(\Omega)} \, \mathrm{d}s \leq X_n, \tag{38}$$

where

$$X_n := 2^n \max \left\{ \frac{\|\nabla J\|^2_{L^1(B_{M_1})}}{F''(1-2\delta)}, \frac{2^4 \delta^2}{\widetilde{\tau}} \right\} y_n. \tag{39}$$

Now, in light of (**A3**), we observe that $\frac{\|\nabla J\|^2_{L^1(B_{M_1})}}{F''(1-2\delta)} \leq C_F \delta \|\nabla J\|^2_{L^1(B_{M_1})}$, thereby

$$X_n = 2^n \frac{\|\nabla J\|^2_{L^1(B_{M_1})}}{F''(1-2\delta)} y_n, \quad \text{provided that} \quad \widetilde{\tau} \geq \frac{2^4 \delta}{C_F \|\nabla J\|^2_{L^1(B_{M_1})}}. \tag{40}$$

The latter constraint will be verified later on.

Next, for $t \in I_{n+1}$ and for almost every $x \in A_{n+1}(t)$, following [14, Sect. 4], we observe that

$$\begin{aligned} \phi_n(x,t) &= \phi(x,t) - \left[1 - \delta - \frac{\delta}{2^n}\right] \\ &= \underbrace{\phi(x,t) - \left[1 - \delta - \frac{\delta}{2^{n+1}}\right]}_{=\phi_{n+1}(x,t) \geq 0} + \delta \left[\frac{1}{2^n} - \frac{1}{2^{n+1}}\right] \geq \frac{\delta}{2^{n+1}}, \end{aligned}$$

which implies that

$$\begin{aligned} \int_{I_{n+1}} \int_{\Omega} |\phi_n|^{\frac{10}{3}} \, \mathrm{d}x \, \mathrm{d}s &\geq \int_{I_{n+1}} \int_{A_{n+1}(s)} |\phi_n|^{\frac{10}{3}} \, \mathrm{d}x \, \mathrm{d}s \\ &\geq \left(\frac{\delta}{2^{n+1}}\right)^{\frac{10}{3}} \int_{I_{n+1}} \int_{A_{n+1}(s)} 1 \, \mathrm{d}x \, \mathrm{d}s \\ &= \left(\frac{\delta}{2^{n+1}}\right)^{\frac{10}{3}} y_{n+1}. \end{aligned} \tag{41}$$

In order to proceed with the next step, we recall the following Gagliardo-Nirenberg inequality in three dimensions

$$\|u\|_{L^{\frac{10}{3}}(\Omega)} \leq C_\Omega \|u\|^{\frac{2}{5}}_{L^2(\Omega)} \|u\|^{\frac{3}{5}}_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega). \tag{42}$$

Exploiting the definition of y_n, (32) and (42), we have

$$\begin{aligned} y_{n+1}\left(\frac{\delta}{2^{n+1}}\right)^{\frac{10}{3}} &\le \int_{I_{n+1}}\int_{A_n(s)} |\phi_n|^{\frac{10}{3}}\,\mathrm{d}x\,\mathrm{d}s \\ &\le C_\Omega \int_{I_{n+1}} \|\phi_n\|^{\frac{4}{3}}_{L^2(\Omega)} \left(\|\nabla\phi_n\|^2_{L^2(\Omega)} + \|\phi_n\|^2_{L^2(\Omega)}\right)\mathrm{d}s \\ &\le \underbrace{C_\Omega \int_{I_{n+1}} \|\phi_n\|^{\frac{4}{3}}_{L^2(\Omega)} \|\nabla\phi_n\|^2_{L^2(\Omega)}\,\mathrm{d}s}_{A} \\ &\quad + \underbrace{C_\Omega \int_{I_{n+1}} \|\phi_n\|^{\frac{4}{3}}_{L^2(\Omega)} \|\phi_n\|^2_{L^2(\Omega)}\,\mathrm{d}s}_{B}. \end{aligned}$$

As in [14], we infer from (38) that

$$A \le \frac{1}{F''(1-2\delta)} \max_{t\in I_{n+1}} \|\phi_n(t)\|^{\frac{4}{3}}_{L^2(\Omega)} F''(1-2\delta) \int_{I_{n+1}} \|\nabla\phi_n\|^2_{L^2(\Omega)}\,\mathrm{d}s \le \frac{1}{F''(1-2\delta)} X_n^{\frac{5}{3}}.$$

On the other hand, by using (32) and (36), we notice that

$$B \le \max_{t\in I_{n+1}} \|\phi_n(t)\|^{\frac{4}{3}}_{L^2(\Omega)} \int_{I_n} \|\phi_n\|^2_{L^2(\Omega)}\,\mathrm{d}s \le (2\delta)^2 X_n^{\frac{2}{3}} \int_{I_n}\int_{A_n(s)} 1\,\mathrm{d}x\,\mathrm{d}s = (2\delta)^2 X_n^{\frac{2}{3}}\, y_n.$$

The above estimate of B is a major change compared to the argument in [28] which is based on a Poincareé-type inequality. Thus, thanks to (40), and making use of (**A3**), we find

$$\begin{aligned} y_{n+1}\left(\frac{\delta}{2^{n+1}}\right)^{\frac{10}{3}} &\le \left[\frac{C_\Omega \|\nabla J\|^{\frac{10}{3}}_{L^1(B_{M_1})}}{(F''(1-2\delta))^{\frac{8}{3}}}\, 2^{\frac{5}{3}n} + \frac{4C_\Omega \delta^2 \|\nabla J\|^{\frac{4}{3}}_{L^1(B_{M_1})}}{(F''(1-2\delta))^{\frac{2}{3}}}\, 2^{\frac{2}{3}n}\right] y_n^{\frac{5}{3}} \\ &\le 4C_\Omega C_F^{\frac{8}{3}} \underbrace{\max\left\{\|\nabla J\|^{\frac{10}{3}}_{L^1(B_{M_1})}, \|\nabla J\|^{\frac{4}{3}}_{L^1(B_{M_1})}\right\}}_{C_J} \delta^{\frac{8}{3}}\, 2^{\frac{5}{3}n} y_n^{\frac{5}{3}}, \end{aligned}$$

which is equivalent to

$$y_{n+1} \le \frac{2^{\frac{16}{3}} C_\Omega C_F^{\frac{8}{3}} C_J}{\delta^{\frac{2}{3}}}\, 2^{5n} y_n^{\frac{5}{3}}.$$

An application of Lemma 1 with

$$C = \frac{2^{\frac{16}{3}} C_\Omega C_F^{\frac{8}{3}} C_J}{\delta^{\frac{2}{3}}}, \quad b = 2^5, \quad \epsilon = \frac{2}{3}$$

entails that $y_n \to 0$ provided that

$$y_0 \le \frac{\delta}{2^8 C_\Omega^{\frac{3}{2}} C_F^4 C_J^{\frac{3}{2}}} \frac{1}{2^{\frac{45}{4}}} = \frac{\delta}{2^{\frac{77}{4}} C_\Omega^{\frac{3}{2}} C_F^4 C_J^{\frac{3}{2}}}. \tag{43}$$

We conclude from $y_n \to 0$ and $y_n \to \left|\left\{(x,t) \in \Omega \times [T-\widetilde{\tau}, T] : \phi(x,t) \geq 1-\delta\right\}\right|$, as $n \to \infty$, that

$$\|(\phi - (1-\delta))_+\|_{L^\infty(\Omega\times(T-\widetilde{\tau},T))} = 0. \tag{44}$$

We are left to show that (43) is satisfied. Recalling (**A3**), (26) and $y_0 = \int_{T-3\widetilde{\tau}}^T \int_{A_0(s)} 1\,\mathrm{d}x\,\mathrm{d}s$, we notice that (cf. [14,28])

$$\begin{aligned}\int_{T-3\widetilde{\tau}}^T \int_{A_0(s)} 1\,\mathrm{d}x\,\mathrm{d}s &\leq \frac{\int_{T-3\widetilde{\tau}}^T \|F'(\phi(s))\|_{L^1(\Omega)}\,\mathrm{d}s}{|F'(1-2\delta)|} \\ &\leq 3\widetilde{\tau}\|F'(\phi)\|_{L^\infty(\frac{\tau}{2},\infty;L^1(\Omega))}\frac{C_F}{|\ln(\delta)|} \\ &\leq \frac{3C_F\,C(E_{NL}(\phi_0),\tau)\widetilde{\tau}}{|\ln(\delta)|}.\end{aligned} \tag{45}$$

Thus, we impose that

$$\frac{3C_F\,C(E_{NL}(\phi_0),\tau)\widetilde{\tau}}{|\ln(\delta)|} \leq \frac{\delta}{2^{\frac{77}{4}}C_\Omega^{\frac{3}{2}}C_F^4 C_J^{\frac{3}{2}}}. \tag{46}$$

In light of (40) and (46), we choose δ sufficiently small such that $\widetilde{\tau}$ satisfies the relations

$$\frac{2^4\delta}{C_F\|\nabla J\|_{L^1(B_{M_1})}^2} \leq \widetilde{\tau} \leq \frac{\delta|\ln(\delta)|}{3\,2^{\frac{77}{4}}C_\Omega^{\frac{3}{2}}C_F^5 C_J^{\frac{3}{2}}C(E_{NL}(\phi_0),\tau)}. \tag{47}$$

Now, set $T = \tau + \frac{\widetilde{\tau}}{2}$. Up to eventually reducing δ to get $\widetilde{\tau}$ even smaller, we clearly have $\tau - \frac{5\widetilde{\tau}}{2} \geq \frac{\tau}{2}$. Therefore, by (44), we deduce that $\|(\phi-(1-\delta))_+\|_{L^\infty(\Omega\times(\tau-\frac{\widetilde{\tau}}{2},\tau+\frac{\widetilde{\tau}}{2}))} = 0$. We point out that the value of $\widetilde{\tau}$ is independent of the choice of T. Thus, repeating the same argument on intervals of size $\widetilde{\tau}$, we conclude that $\|(\phi-(1-\delta))_+\|_{L^\infty(\Omega\times(\tau-\frac{\widetilde{\tau}}{2},\infty)} = 0$. Finally, repeating the same argument for $(\phi+(-1+\delta))_-$, we arrive at the desired conclusion (11). It is important to highlight that the value of δ only depends on F, J, Ω, $E_{NL}(\phi_0)$ and τ.

The rest of the proof is devoted to the additional regularity results (12) and (13). Firstly, by definition of μ in (2), we observe that

$$\begin{aligned}\|\mu(t)\|_{L^\infty(\Omega\times[\tau,\infty))} &\leq \left(\|F'(\phi(t))\|_{L^\infty(\Omega\times[\tau,\infty))} + \sup_{t\geq\tau}\|J*\phi(t)\|_{L^\infty(\Omega)}\right) \\ &\leq |F'(1-\delta)| + \|J\|_{L^1(B_{M_1})} =: C_1.\end{aligned}$$

Let us observe that

$$\partial_t^h\mu(\cdot) = \partial_t^h\phi(\cdot)\left(\int_0^1 F''(s\phi(\cdot+h)+(1-s)\phi(\cdot))\,\mathrm{d}s\right)$$

$$-J * \partial_t^h \phi(\cdot), \quad 0 < t \leq T - h. \tag{48}$$

By (11), $\|s\phi(\cdot + h) + (1 - s)\phi(\cdot)\|_{L^\infty(\Omega\times(\tau,\infty))} \leq 1 - \delta$ for all $s \in (0, 1)$. Then, exploiting that $\|\partial_t^h \phi\|_{L^2(0,T-h;L^2(\Omega))} \leq \|\partial_t \phi\|_{L^2(0,T;L^2(\Omega))}$, we infer from (25) that $\sup_{t\geq\tau} \|\partial_t^h \mu\|_{L^2(t,t+1;L^2(\Omega))} \leq C_2$, where $C_2 > 0$ depends on C_0, τ, δ and J, but is independent of h,. This implies that $\partial_t \mu \in L^2(0, T; L^2(\Omega))$ for any $T > 0$, and $\sup_{t\geq\tau} \|\partial_t \mu\|_{L^2(t,t+1;L^2(\Omega)} \leq C_2$.

Secondly, we study the Hölder continuity in both time and space. We notice that (1) is a quasi-linear equation with principal part in divergence form. Following the notation in the book [21], we define $a_l(x, t, u, p) = \widetilde{F}''(u)p_l - (\partial_l J * \phi(\cdot, t))(x)$, where $\widetilde{F}$ is the restriction of F in $[-1 + \delta, 1 - \delta]$. In light of the convexity of F and $|\widetilde{F}''(s)| \leq |F''(1 - \delta)|$, for all $s \in [-1 + \delta, 1 - \delta]$, we deduce that

$$\begin{aligned} a_l(x, t, u, p)p_l &\geq \frac{\theta}{2}|p|^2 - \frac{1}{2\theta}\|\nabla J\|_{L^1(B_{M_1})}, \\ |a_l(x, t, u, p)| &\leq |F''(1 - \delta)||p| + \|\nabla J\|_{L^1(B_{M_1})}. \end{aligned}$$

We also note that the solution ϕ satisfying (22–27) is a *bounded generalized solution* in the sense of [21, Chapter V]. Thus, by [21, Theorem 1.1, Chapter V], we deduce that (13) holds in $\Omega' \times [t, t + 1]$, for any $t \geq \tau$ and $\Omega' \subset \Omega$ separated from $\partial\Omega$. In order to achieve (13) up to the boundary, we make use of [8, Corollary 4.2], which provides the desired conclusion under the same assumptions. It is worth noticing that the constant C_3 and the parameter α from both [21] and [8] only depends on δ, θ, $\|\nabla J\|_{L^1(B_{M_1})}$ and Ω. This completes the proof. □

3. On the regularity of the global attractor

This section is devoted to some regularity properties of the global attractor $\mathcal{A}_m$ for the dynamical system $(\mathcal{H}_m, S(t))$ stated in Theorem 2.

Proof of Theorem 2. Let us consider $\phi^\star \in \mathcal{A}_m$. It is clear that $\|\phi^\star\|_{L^\infty(\Omega)} \leq 1$ such $|\overline{\phi^\star}| \leq 1 - m$ and $\|\phi^\star\|_{H^1(\Omega)} \leq N_1$, where N_1 is a universal constant (namely, it does not depend on $\phi^\star$). We observe that $|E_{NL}(\phi^\star)| \leq N_2$, where N_2 is a universal constant depending only on $\|J\|_{L^1(B_{M_1})}$ (cf. (35)) and $\max_{s\in[-1,1]} |F(s)|$. Then, applying Theorem 1, we deduce that

$$\|S(t)\phi^\star\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall t \geq [1, \infty). \tag{49}$$

Here, δ depends on the constants in (47). In particular, since $|\overline{\phi^\star}| \leq 1 - m$, it is easily seen that $C(E_{NL}(\phi^\star), 1) \leq N_3$, where N_3 is a universal constant. This implies that δ is a universal constant. Since $\phi^\star$ is arbitrary in the above argument, we deduce that

$$\mathcal{A}_m = S(1)\mathcal{A}_m \subset B_{L^\infty(\Omega)}(0, 1 - \delta).$$

Next, by the second part of Theorem 1, we infer from (13) and (14) that

$$\begin{aligned}\|S(t)\phi^\star\|_{C^\alpha(\overline{\Omega})} &= \|S(t)\phi^\star\|_{C(\overline{\Omega})} + \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|(S(t)\phi^\star)(x) - (S(t)\phi^\star)(y)|}{|x-y|^\alpha} \\ &\leq 1 - \delta + C_3 =: N_4.\end{aligned}$$

Notice that N_4 is a universal constant which depends only on N_2, δ, m and the parameters of the system (namely, F, J, Ω). Thus, the constant N_4 is independent of $\phi^\star$, so we conclude that $\mathcal{A}_m = S(1)\mathcal{A}_m \subset B_{C^\alpha(\overline{\Omega})}(0, N_4)$. The proof is complete.

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