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A perturbative approach to Hölder continuity of solutions to a nonlocal p-parabolic equation

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Abstract. We study local boundedness and Hölder continuity of a parabolic equation involving the fractional p-Laplacian of order s, with $0 < s < 1, 2 \leq p < \infty$, with a general right-hand side. We focus on obtaining precise Hölder continuity estimates. The proof is based on a perturbative argument using the already known Hölder continuity estimate for solutions to the equation with zero right-hand side.

1. Introduction

In this paper, we study the local boundedness and Hölder regularity of solutions to the inhomogeneous equation

$$u_t + (-\Delta_p)^s u = f(x,t), \tag{1.1}$$

where $f \in L^r_{\mathrm{loc}}(I; L^q_{\mathrm{loc}}(\Omega))$ with $q \geq 1$, $r \geq 1$, $p \geq 2$ and $s \in (0,1)$. Here, $(-\Delta_p)^s$ is the fractional p-Laplacian, arising as the first variation of the Sobolev–Slobodeckiĭ seminorm

$$(-\Delta_p)^s u(x) := 2\,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,\mathrm{d}y.$$

Nonlocal equations involving operators of the above type, with a singular kernel, were first considered in [31] to the best of our knowledge.

In this study, continuing the work in [7], we perform a perturbative argument to obtain Hölder continuity estimates, with explicit exponents for the equations with a right-hand side. Our approach closely follows the arguments in [47] and [6]. In such perturbative arguments, it is often possible to establish Hölder regularity results for bounded solutions using only L^∞ estimates for the equations with zero right-hand side. Here, to estimate the Hölder seminorms of certain functions in the proof of Theorem 1.2 as well as to prove Theorem 3.6, we are led to prove Proposition 3.4. As a by-product, by combining Proposition 3.4 with the existing local boundedness results

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we obtain an L^∞ bound for equations with right-hand sides. This is Theorem 1.1. The proof is inspired by the work [5].

Below, we state the main results. For the definition of the tail and relevant function spaces, see Sect. 2. We use the following notation of parabolic cylinders

$$Q_{R,r}(x,T) := B_R(x_0) \times (T-r,T].$$

The exponent $p_s^\star = \frac{np}{n-sp}$ is the critical exponent for the Sobolev embedding theorem, see Proposition 2.5. We denote by p', the Hölder conjugate of p, that is $p' = \frac{p}{p-1}$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set, $I = (t_0, t_1]$, $p \geq 2$, $0 < s < 1$. Consider q and r such that*

$$\frac{1}{r} + \frac{n}{spq} < 1.$$

In addition, assume that $r \geq p'$,

$$q \geq 1 \quad \text{if } sp \neq n, \quad \text{and} \quad q > 1 \quad \text{if } sp = n.$$

Suppose u is a local weak solution of

$$u_t + (-\Delta_p)^s u = f \qquad \text{in } \Omega \times I,$$

such that

$$u \in L^p_{\mathrm{loc}}(I; L^{p-1}_{sp}(\mathbb{R}^n)) \quad \text{and} \quad f \in L^r_{\mathrm{loc}}(I; L^q_{\mathrm{loc}}(\Omega)).$$

then u is locally bounded in Ω. More specifically, if $Q_{2R,(2R)^{sp}}(x_0,T_0) \Subset \Omega \times I$, u is bounded in $Q_{R/2,(R/2)^{sp}}(x_0,T_0)$, and in the case $sp \neq n$, the estimate reads

$$\|u\|_{L^\infty\left(Q_{\frac{R}{2},(\frac{R}{2})^{sp}}(x_0,T_0)\right)} \leq C\Bigg[1 + \left(\fint_{Q_{R,R^{sp}}(x_0,T_0)} |u|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} + \left(\fint_{T_0-R^{sp}}^{T_0} \mathrm{Tail}_{p-1,sp}\left(u(\bullet,t);x_0,\frac{R}{2}\right)^p \mathrm{d}t\right)^{\frac{1}{p}} + \vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\left(1 + R^{sp\nu}\|f\|_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))}\right)\Bigg],$$

where $C = C(n,s,p)$, $\nu = 1 - \frac{1}{r} - \frac{n}{spq}$ and $\vartheta = 1 + \frac{sp\nu}{n}$.
In the case $sp = n$, given any l such that $\frac{p}{r'}(1 - \frac{1}{r} - \frac{1}{q})^{-1} < l < \infty$ we get

$$\|u\|_{L^\infty\left(Q_{\frac{R}{2},(\frac{R}{2})^{sp}}(x_0,T_0)\right)} \leq C\Bigg[1 + \left(\fint_{Q_{R,R^{sp}}(x_0,T_0)} |u|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} + \left(\fint_{T_0-R^{sp}}^{T_0} \mathrm{Tail}_{p-1,sp}\left(u(\bullet,t);x_0,\frac{R}{2}\right)^p \mathrm{d}t\right)^{\frac{1}{p}} + \vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\left(1 + R^{sp\nu}\|f\|_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))}\right)\Bigg],$$

where $C = C(n, s, p, q, l)$, $\vartheta = 2 - \frac{1}{r} - \frac{1}{q} - \frac{p}{lr'}$ *and* $\nu = 1 - \frac{1}{r} - \frac{1}{q}$.

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded and open set,* $I = (t_0, t_1]$, $p \geq 2$, $0 < s < 1$. *Consider* q *and* r *such that*

$$\frac{1}{r} + \frac{n}{spq} < 1.$$

In addition, assume that $r \geq p'$,

$$q \geq 1 \quad \text{if } sp \neq n, \quad \text{and} \quad q > 1 \quad \text{if } sp = n.$$

Define the exponent

$$\Theta(s, p) := \begin{cases} \dfrac{s\,p}{p-1}, & \text{if } s < \dfrac{p-1}{p}, \\[2ex] 1, & \text{if } s \geq \dfrac{p-1}{p}. \end{cases} \tag{1.2}$$

Suppose u *is a local weak solution of*

$$u_t + (-\Delta_p)^s u = f \qquad \text{in } \Omega \times I,$$

such that

$$u \in L^\infty_{\rm loc}(I; L^\infty_{\rm loc}(\Omega)) \cap L^\infty_{\rm loc}(I; L^{p-1}_{sp}(\mathbb{R}^n)), \quad \text{and} \quad f \in L^r_{\rm loc}(I; L^q_{\rm loc}(\Omega)).$$

Then

$$u \in C^\alpha_{x,\rm loc}(\Omega \times I) \cap C^{\frac{\alpha}{sp-(p-2)\alpha}}_{t,\rm loc}(\Omega \times I), \quad \text{for every } 0 < \alpha \leq \frac{r(spq - n) - spq}{q(r(p-1) - (p-2))} \text{ such that } \alpha < \Theta.$$

More precisely, given $\alpha < \Theta$ *satisfying*

$$\alpha \leq \frac{r(spq - n) - spq}{q(r(p-1) - (p-2))},$$

for every $R > 0$, $x_0 \in \Omega$ *and* T_0 *such that*

$$Q_{R,2R^{s\,p}}(x_0, T_0) \Subset \Omega \times (t_0, t_1],$$

there exists a constant $C = C(n, s, p, q, r, \alpha) > 0$ *such that*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C \left[\mathcal{M} \Big(\frac{|x_2 - x_1|}{R} \Big)^\alpha + \mathcal{M}^{p-1} \Big(\frac{|t_2 - t_1|}{R^{s\,p}} \Big)^{\frac{\alpha}{sp-(p-2)\alpha}} \right] \tag{1.3}$$

for any $(x_1, t_1),\ (x_2, t_2) \in Q_{R/2,(R/2)^{s\,p}}(x_0, T_0)$, *with*

$$\mathcal{M} = 1 + \|u\|_{L^\infty(Q_{R,2R^{sp}}(x_0,T_0))} + \sup_{T_0 - 2R^{sp} \leq t \leq T_0} \mathrm{Tail}_{p-1,sp}(u(\cdot, t); x_0, R) + \Big(R^{sp - \frac{n}{q} - \frac{sp}{r}} \|f\|_{L^{q,r}(Q_{R,2R^{sp}}(x_0,T_0))} \Big)^{\frac{1}{1 + \frac{p-2}{r'}}}.$$

1.1. Known results

Recently, there has been a growing interest in nonlocal problems of both elliptic and parabolic types. For studies of fractional p-Laplace operators with different (continuous) kernels, see [4]. Parabolic equations of the type (1.1) were first considered in [42] with a slightly different diffusion operator. See also [1,39,48] and [49] for studies of the existence, uniqueness and long time behavior of solutions.

A noteworthy area of investigation has been devoted to adapting the classical De Giorgi–Nash–Moser theory for nonlocal equations. Local boundedness, Hölder estimates and Harnack inequalities have been established in the elliptic case under general assumptions on the kernels; see, for instance, [19,23,24,32].

Here we seize the opportunity to mention [16–18] and [51] which contain regularity results for parabolic nonlocal equations.

Local boundedness for parabolic nonlocal equations has been studied, for instance, in [11,22,33,45]. In particular, the local boundedness of the solutions to equations modeled on (1.1) with zero right-hand side was obtained in [45]. The results concern operators of the form

$$L_K = P.V. \int_{R^n} K(x, y, t)|u(x) - u(y)|^{p-2}(u(x) - u(y))\, \mathrm{d}y, \tag{1.4}$$

where K is a measurable kernel, which is symmetric in the space variables and satisfies the ellipticity condition

$$\frac{\Lambda^{-1}}{|x-y|^{n+sp}} \leq K(x, y, t) \leq \frac{\Lambda}{|x-y|^{n+sp}}.$$

Later in [22], local boundedness for certain right-hand sides of the form $f(x, t, u)$ was established. See also [3] for a recent boundedness result in the setting of nonlocal kinetic Kolmogorov–Fokker–Planck equations. All the aforementioned local boundedness results have a particular unnatural assumption, $u \in L^\infty(I; L^{p-1}_{sp}(\mathbb{R}^n))$. It is more natural to assume $u \in L^{p-1}(I; L^{p-1}_{sp}(\mathbb{R}^n))$. This difficulty has been completely resolved in [34] when $p = 2$ and generalizes to the nonlinear setting in [10].

[46] contains a Harnack inequality for nonlinear parabolic equations with zero right-hand side, see also [34] for a full Harnack inequality with optimal tail assumption for $p = 2$. Hölder regularity has also been established in [13,27] for $p = 2$ and for locally bounded solutions in [2] and [37] for all $1 < p < \infty$ for equations with zero right-hand side.

The question of higher regularity of solutions to nonlocal equations has also been a subject of intensive study during the past few years. For instance, see [28,43] for a nonlocal Schauder-type theory. We also refer to [14,15] for nonlocal analogs of Krylov–Safanov and Evans–Krylov theorems. We refer to [6,7,11,12,26,40,41] for studies of higher regularity in the variational setting. In particular, in [7] they prove Hölder continuity of the solutions to (1.1) with explicit exponents (for $f = 0$ and $K = |x - y|^{-n-sp}$). Recently in [29], the same type of result has been established for nonlocal equations with double phase that is for diffusion operators involving two different degrees of homogeneity and differentiability.

Perturbative arguments have been very successful in obtaining sharp boundedness and Hölder regularity estimate at least in the elliptic setting, see, for instance, [25,35]. See also [36] for an overview of the local theory. In this study, continuing the work in [7], we perform a perturbative argument to obtain Hölder continuity estimates with explicit exponents for equations with a right-hand side. However, we have to say that the current work has some unnatural assumptions that have yet to be overcome.

1.1.1. Discussion of the results and comparison to some previous works

Our results contain an unnatural assumption $r \geq p'$, as well as the assumption $u \in L^p(I; L^{p-1}_{sp}(\mathbb{R}^n))$ in Theorem 1.1. We use these assumptions in two places. First and foremost these assumptions are needed to ensure the existence of a solution to (3.1), the so-called (s, p)-caloric replacement of our solution. This limitation comes from the regularity assumption on the boundary condition in Theorem 2.12 which is essentially the same as [7, Theorem A.3] see Remark 2.13. We also use the assumption $r \geq p'$ in obtaining the estimates in Lemma 3.2. We believe it is possible to overcome this issue by an interpolation argument, see Remark 3.3. It has to be mentioned that we also use the assumption $u \in L^p(I; L^{p-1}_{sp}(\mathbb{R}^n))$ to justify testing the equation with powers of the solution in Appendix B. Having said this, it is reasonable to expect Theorem 1.1 to hold for any weak solution under the assumption

$$\frac{1}{r} + \frac{n}{spq} < 1, \tag{1.5}$$

as this is the only assumption that appears in the estimates. The same extra assumptions on q and r are present in Theorem 3.6 due to the same reason as in Theorem 1.1. Furthermore, we assume our solutions to have bounded tail in time, that is, $u \in L^\infty(I; L^{p-1}_{sp}(\mathbb{R}^n))$. In light of the recent developments in [10,34], one can actually weaken the assumptions on the tail. In particular, by using [10, Theorem 1.2] instead of [7, Theorem 1.2] in the proof of Theorem 3.6, with some small modifications in the argument one should be able to obtain the Hölder continuity of the solutions under the assumption $u \in L^p(I; L^{p-1}_{sp}(\mathbb{R}^n))$, and the same assumptions on q, r as in Theorem 3.6. We also believe that it is possible to avoid using Proposition 3.4 in the proof of Theorem 3.6, by using [10, Theorem 1.1 and Theorem 1.2]. As improving upon this assumption does not improve our main result, Theorem 1.2, we do not complicate the article by going through the details of this issue. Furthermore, we actually expect the result to be true under the weaker assumption $u \in L^l(I; L^{p-1}_{sp}(\mathbb{R}^n))$ for some $l > p-1$ and without the assumption $r \geq p'$, but the current restrictions in the article especially with respect to the existence of the (s, p)-caloric replacement do not allow us to obtain such a result.

Let us also mention that the local boundedness and Hölder regularity results mentioned above hold for a more general class of equations with measurable coefficient

$u_t + L_k u = 0$, where L_k is as in 1.4. Although we write our results for the equation $u_t + (-\Delta_p)^s u = f$, the arguments in the proofs of Theorem 1.1 and Theorem 3.6 can be adapted to the equations $u_t + L_k u = f$ with measurable, asymmetric, uniformly elliptic coefficients easily. The only difference is that a dependence on the ellipticity coefficients will appear in the constants. But the question of what assumption is needed on the kernel to get higher Hölder regularity is subtle. We refer to [11,25,40] for a study of this issue.

The equation $u_t - \Delta_p u = f$ can be seen as a limit of the equation $u_t + (1 - s)c(n, p)(-\Delta_p)^s u = f$ as $s \nearrow 1$. A relevant question is whether the estimates provided here in the article are stable with respect to s as $s \nearrow 1$. We have to admit that we did not keep track of the dependence of the constants on s while writing this article, and we wrote the article for the operator $\partial_t + (-\Delta_p)^s$ instead of $\partial_t + (1 - s)(-\Delta_p)^s$. Still, we can say a few words on the dependence of our constants on s for those who might be interested in pursuing this question. The proofs of Theorem 1.1 and Theorem 3.6 are combinations of local boundedness estimates in [10, Theorem 1.1] and the Holder continuity result [7, Theorem 1.2] for the equations with zero right-hand side, together with the comparison estimates of Lemma 3.2 and Proposition 3.4. [7, Theorem 1.2] is stable as $s \nearrow 1$ see [7, Remark 1.7], as for [10, Theorem 1.1] they did not specify the dependence of their constants on s in their article. In Lemma 3.2 and Proposition 3.4, the dependence of the constants on s comes from the Sobolev and Morrey inequalities. The constants in these inequalities behave like $s(1 - s)$ with respect to s, but it has to be mentioned that we update the constants to be greater than one in several places. It might be the case that if one considers the operator $\partial_t + (1 - s)(-\Delta_p)^s$ instead, the estimates in Lemma 3.2 and Proposition 3.4 would become robust as $s \nearrow 1$. We cannot specify the dependence of the constant in Theorem 1.2 on s specifically. The main difficulty lies in the proof of Lemma 3.7, which is proved by contradiction.

Now we compare the main results of the article to some other works.

Local boundedness and continuity In the recent work [11], they address the issue of local boundedness when $p = 2$ for a more general class of operators by a direct proof. By avoiding the difficulty of the existence of the caloric replacement, their result does not contain the extra assumption $r \geq p'$, although they assume $u \in L^\infty(I; L^{p-1}_{sp}(\mathbb{R}^n))$.

We compare our boundedness result to [22]. Their result concerns more general right-hand sides depending on the solution as well. In the limiting case of $s \to 1$, they reproduce the local boundedness result contained in [21] for the evolution p-Laplacian equation. To compare the results, if we restrict their result to right-hand sides that are u-independent, their assumption on the integrability becomes $q, r > \frac{n+sp}{sp}(\frac{p(n+2s)}{2sp+(p-1)n})$. Their analysis is done with the same integrability assumption in time and space. Our local boundedness result, Theorem 1.1, contains this range of exponents.

In the limiting case when s goes to 1, 1.5 become $1 - \frac{1}{r} - \frac{n}{pq} > 0$. This is in accordance with the classical condition for boundedness of the evolution p-Laplace equation, see, for example, Remark 1 in [38], there they have a finer assumption formulated in terms of the Lorentz norm of the right-hand side, and moreover, they obtained estimates in terms of a parabolic version of Wolf potentials. It would be interesting to obtain finer estimates beyond L^p spaces, although we do not pursue this question in this article. If we assume the same integrability in time and space, the condition $1 - \frac{1}{r} - \frac{n}{spq} > 0$ reduces to $f \in L^{\hat{q}}$ with $\hat{q} > \frac{n+p}{p}$. This matches the condition in [50].

Now we turn our attention to the nonlocal elliptic (time-independent) case. For $r = \infty$, the condition for boundedness and basic Hölder continuity becomes

$$q > \frac{n}{sp}\,, \quad \text{if } sp < n, \quad \text{and} \quad q > 1\,, \quad \text{if } sp = n, \quad \text{and} \quad q \geq 1\,, \quad \text{if } sp = n.$$

In the case $sp < n$, this is the same condition for local boundedness and continuity contained in [8,35]. When $sp > n$ and $q \geq 1$, the boundedness and Hölder continuity for the time-independent equation is automatic using Morrey's inequality. The question of whether the solutions are locally bounded under the equality case of (1.5) is subtle. On the one hand, if $r = \infty$ even in the time-independent (elliptic) setting one requires the strict inequality $q > \frac{n}{sp}$ to obtain boundedness; on the other hand, local boundedness is obtained in the case $r = 1$ and $q = \infty$ in [34], see also [10].

There are actually local boundedness and Hölder continuity results available for the equations with zero right-hand side if $p < 2$. One could try to prove local boundedness and basic Hölder regularity of the solutions for the solutions of the equations with right-hand side in the singular case $p < 2$ as well. We have to warn the reader that some of the arguments in this article do not carry over to the singular case as they are written here. We use the condition $p \geq 2$ extensively, in particular in the Pointwise inequalities (2.1) and (2.3). We feel that it is better if we leave the study of the singular case to another work. We also have to mention that if one is only interested in local boundedness estimates, doing a nonperturbative argument is more suitable, as one can also deal with sub- and supersolutions.

Hölder continuity exponent: In the case $r = \infty$, the critical Hölder continuity exponent

$$\min\left\{\Theta, \frac{r(spq - n) - spq}{q(r(p-1) - (p-2))}\right\} = \min\left\{\Theta, sp\frac{1 - \frac{1}{r} - \frac{n}{spq}}{p - 1 - \frac{p-2}{r}}\right\}, \tag{1.6}$$

reduces to $\min\{\Theta, \frac{sp}{p-1}(1 - \frac{n}{spq})\}$ which matches the results in [6]. Although the results reported in [6] require a strict inequality $\alpha < \min\{\Theta, \frac{sp}{p-1}(1 - \frac{n}{spq})\}$, an inspection of the proofs reveals that the strict inequality is only needed when the minimum corresponds to Θ. The assumptions needed for their proof are actually $\alpha \leq \frac{sp}{p-1}(1 - \frac{n}{spq})$ and $\alpha < \Theta$. Through a finer estimate in [25], they have addressed this issue further and proved that given $\alpha \leq \Theta$, if the right-hand side f belongs to the Marcinkiewicz space $L^{\frac{n}{sp-\alpha(p-1)},\infty}(\Omega)$ then the solution is $C^{\alpha}_{\mathrm{loc}}(\Omega)$.

Let us also compare our results to the local p-parabolic equation studied in [47] where precise Hölder continuity exponents are obtained. If we send s to 1, (1.6) becomes

$$\min\left\{1, \frac{r(pq-n)-pq}{q(r(p-1)-(p-2))}\right\},$$

which is in accordance with the result in [47].

In [29], explicit Hölder continuity exponents for the more general case of double phase nonlocal diffusion operators were obtained. The ideas explored there are similar to the ones in [7], but their result allows for a bounded right-hand side instead of just zero. Their result implies the Hölder continuity exponent that we get in the case of $f \in L^\infty$, although with a slightly different estimate of the Hölder constants. Let us also mention that in the recent work [11] the conclusions of Theorem 1.2 have been obtained when $p = 2$, for a more general class of operators and kernels. Although [11, Theorem 1.2] does not contain the extra assumption $r \geq p'$, their argument is similar to our proof of Theorem 1.2 and the same difficulty regarding the existence of the (s, p)-caloric replacement is present in their proof. This difficulty has not been addressed properly in their article. In the assumptions [11, (A.1)] for their existence theorem, the regularity assumption $\xi_t \in L^2((0, T); W^{s,2}(\Omega'))^\star$ is present. We are not able to verify this assumption when ξ is a solution of

$$\xi_t + (-\Delta)^s \xi = f,$$

for $f \in L^{q,r}$ such that $\frac{n}{2sq} + \frac{1}{r} < 1$ as it is claimed by [11, Remark 6]. See Remark 2.13 for a possible strategy for resolving this issue.

Let us close this section with the question of how much regularity one should expect if the solution has a lower integrability of the tail in time. Namely given a weak solution u of the equation

$$u_t + (-\Delta_p)^s u = 0,$$

such that $u \in L^l(L^{p-1}_{sp}(\mathbb{R}^n))$ how much Hölder regularity does the solution have. Let us first mention that an example in [34, Example 5.2] shows that the assumption $u \in L^{p-1}(L^{p-1}_{sp}(\mathbb{R}^n))$ does not ensure the Hölder regularity of the solution. On the other hand, it is proved in [34] and [10] that if $l > p - 1$ then the solution is Hölder continuous, and the general strategy in these works is to treat the following nonlocal term

$$G(t) = \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|u(y,t)|^{p-1}}{|x_0 - y|^{n+sp}}\, \mathrm{d}y,$$

which appears in the Caccioppoli inequalities, as a right-hand side in $L^{\frac{l}{p-1}}(I; L^\infty(B))$. See [10, Section 1.2] for more details. Following this general philosophy, one can expect the solution to be C^α_x and $C_t^{\frac{\alpha}{sp-(p-2)\alpha}}$ with

$$\alpha = \min\left\{sp\frac{1-\frac{p-1}{l}}{p-1-\frac{(p-2)(p-1)}{l}}, 1\right\}.$$

However, at the moment we do not have a definite answer to how this can be shown rigorously.

1.2. Plan of the paper

In Sect. 2, we introduce some notations and preliminary lemmas. We also restate and adapt a result on the existence of solutions to our setting.

In Sect. 3, we establish basic local Hölder regularity and boundedness for local weak solutions.

Section 4 is devoted to proving Theorem 1.2. A so-called tangential analysis is performed to get specific Hölder continuity exponents in terms of q, r, s and p.

The article is also accompanied by two appendices. In the first one, Appendix A, we work out the details for a modified version of [7, Theorem 1.1]. The aim is to bound the Hölder seminorm of the solution in terms of the tail quantity.

In Appendix B, we justify using certain test functions in the weak formulation of (1.1).

2. Preliminaries

2.1. Notation

We define the monotone function $J_p : \mathbb{R} \to \mathbb{R}$ by

$$J_p(t) = |t|^{p-2}t.$$

We use the notation $B_R(x_0)$ for the open ball of radius R centered at x_0. If the center is the origin, we simply write B_R. We use the notation of ω_n for the surface area of the unit n-dimensional ball. For parabolic cylinders, we use the notation $Q_{r,T}(x_0, t_0) := B_r(x_0) \times (t_0 - T, t_0]$. If the center is the origin, we write $Q_{r,T}$.

We will work with the fractional Sobolev space extensively:

$$W^{\beta,q}(\mathbb{R}^n) := \{\psi \in L^q(\mathbb{R}^n) : [\psi]_{W^{\beta,q}(\mathbb{R}^n)} < \infty\}, \qquad 0 < \beta < 1, \quad 1 \le q < \infty,$$

where the seminorm $[\psi]_{W^{s,p}(\mathbb{R}^n)}$ is defined as below

$$[\psi]^q_{W^{\beta,q}(\mathbb{R}^n)} = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\psi(x) - \psi(y)|^q}{|x - y|^{n+\beta q}} \, \mathrm{d}x \, \mathrm{d}y.$$

We also need the space $W^{\beta,q}(\Omega)$ for a subset $\Omega \subset \mathbb{R}^n$, defined by

$$W^{\beta,q}(\Omega) := \{\psi \in L^q(\Omega) : [\psi]_{W^{\beta,q}(\Omega)} < \infty\}, \qquad 0 < \beta < 1, \quad 1 \le q < \infty,$$

where

$$[\psi]^q_{W^{\beta,q}(\Omega)} = \iint_{\Omega \times \Omega} \frac{|\psi(x) - \psi(y)|^q}{|x - y|^{n+\beta q}} \, \mathrm{d}x \, \mathrm{d}y.$$

In the following, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set in $\mathbb{R}^n$. We define the space of Sobolev functions taking boundary values $g \in L^{q-1}_{sp}(\mathbb{R}^n)$ by

$$X_g^{\beta,q}(\Omega, \Omega') = \{\psi \in W^{\alpha,q}(\Omega') \cap L^{q-1}_{sp}(\mathbb{R}^n) \,:\, \psi = g \text{ on } \mathbb{R}^n \setminus \Omega\},$$

where Ω' is an open set such that $\Omega \Subset \Omega'$.

We recall the definition of *tail space*

$$L^q_\alpha(\mathbb{R}^n) = \left\{ u \in L^q_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \int_{\mathbb{R}^n} \frac{|u|^q}{1+|x|^{n+\alpha}}\,\mathrm{d}x < +\infty \right\}, \qquad q \geq 1 \text{ and } \alpha > 0,$$

which is endowed with the norm

$$\|u\|_{L^q_\alpha(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \frac{|u|^q}{1+|x|^{n+\alpha}}\,\mathrm{d}x \right)^{\frac{1}{q}}.$$

For every $x_0 \in \mathbb{R}^n$, $R > 0$ and $u \in L^q_\alpha(\mathbb{R}^n)$, the following quantity

$$\mathrm{Tail}_{q,\alpha}(u; x_0, R) = \left[R^\alpha \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|u|^q}{|x-x_0|^{n+\alpha}}\,\mathrm{d}x \right]^{\frac{1}{q}}$$

plays an important role in regularity estimates for solutions to fractional problems.

Let $I \subset \mathbb{R}$ be an interval and let V be a separable, reflexive, Banach space endowed with a norm $\|\bullet\|_V$. We denote by $V^\star$ its topological dual space. Suppose that v is a mapping such that for almost every $t \in I$, we have $v(t) \in V$. If the function $t \to \|v(t)\|_V$ is measurable on I and $1 \leq p \leq \infty$, then v is an element of the Banach space $L^p(I; V)$ if and only if

$$\int_I \|v(t)\|_V \,\mathrm{d}t < \infty.$$

By [44, Theorem 1.5], the dual space of $L^p(I; V)$ can be characterized according to $(L^p(I; V))^\star = L^{p'}(I; V^\star)$. We write $v \in C(I; V)$ if the mapping $t \to v(t)$ is continuous with respect to the norm on V.

2.2. Pointwise inequalities

We will need the following pointwise inequality: Let $p \geq 2$, then for every $A, B \in \mathbb{R}$ we have

$$|A-B|^p \leq C\big(J_p(A) - J_p(B)\big)(A-B). \tag{2.1}$$

For a proof look at [7, Remark A.4], a close inspection of the proof reveals that the constant can be taken as $C = 3 \cdot 2^{p-1}$. Before stating the next inequality, we recall [8, Lemma A.2].

Lemma 2.1. *Let* $1 < p < \infty$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be an increasing function, and define*

$$G(t) = \int_0^t g'(\tau)^{\frac{1}{p}} \, \mathrm{d}\tau, \quad t \in \mathbb{R}.$$

Then

$$J_p(a-b)\big(g(a)-g(b)\big) \geq \big|G(a)-G(b)\big|^p.$$

Lemma 2.2. *For* $p \geq 2$ *and* $\beta \geq 1$,

$$\begin{aligned}&\big(J_p(a-b)-J_p(c-d)\big)\Big(((a-c)_M^+ + \delta)^\beta - ((b-d)_M^+ + \delta)^\beta\Big)\\ &\quad\geq \frac{1}{3\cdot 2^{p-1}}\frac{\beta p^p}{(\beta+p-1)^p}\Big|((a-c)_M^+ + \delta)^{\frac{\beta+p-1}{p}} - ((b-d)_M^+ + \delta)^{\frac{\beta+p-1}{p}}\Big|^p,\end{aligned} \tag{2.2}$$

where $(t)_M^+ := \min\{\max\{t,0\}, M\}$.

Proof. We consider three cases according to the sign of $a-b-c+d$. If $a-b-c+d=0$ both the left-hand side and the right-hand side of (2.2) are zero. Now we verify the inequality for $a-b-c+d>0$

First notice that using (2.1) with $A = a-b$ and $B = c-d$:

$$3\cdot 2^{p-1}(a-b-c+d)(J_p(a-b)-J_p(c-d)) \geq |a-b-c+d|^p,$$

using the fact that $a-b-c+d>0$, we arrive at

$$J_p(a-b)-J_p(c-d) \geq \frac{1}{3\cdot 2^{p-1}}\frac{|a-b-c+d|^p}{a-b-c+d} = \frac{1}{3\cdot 2^{p-1}}J_p((a-c)-(b-d)). \tag{2.3}$$

Now we use Lemma 2.1 with $g(t) = ((t)_M^+ + \delta)^\beta$. Then with $G = \int_0^t g'(\tau)^{\frac{1}{p}} \, \mathrm{d}\tau$,

$$G(t) = \frac{p\beta^{\frac{1}{p}}}{\beta+p-1}\Big((t_M^+ + \delta)^{\frac{\beta+p-1}{p}} - \delta^{\frac{\beta+p-1}{p}}\Big).$$

By Lemma 2.1,

$$J_p\big((a-c)-(b-d)\big)\big(g(a-c)-g(b-d)\big) \geq |G(a-c)-G(b-d)|^p.$$

Hence,

$$\begin{aligned}&J_p\big((a-c)-(b-d)\big)\Big(((a-c)_M^+ + \delta)^\beta - ((b-d)_M^+ + \delta)^\beta\Big)\\ &\quad\geq \frac{\beta p^p}{(\beta+p-1)}\Big|((a-c)_M^+ + \delta)^{\frac{\beta+p-1}{p}} - ((b-d)_M^+ + \delta)^{\frac{\beta+p-1}{p}}\Big|^p.\end{aligned}$$

Using (2.3) in the above inequality concludes the proof. It only remains to verify the case $a-b-c+d < 0$, now we are in the previous position and can use with (b, a, d, c) instead of (a, b, c, d) to obtain

$$\begin{aligned}&\big(J_p(b-a) - J_p(d-c)\big)\Big(((b-d)_M^+ + \delta)^\beta - ((a-c)_M^+ + \delta)^\beta\Big)\\&\geq \frac{1}{3\cdot 2^{p-1}}\frac{\beta p^p}{(\beta+p-1)^p}\left|((b-d)_M^+ + \delta)^{\frac{\beta+p-1}{p}} - ((a-c)_M^+ + \delta)^{\frac{\beta+p-1}{p}}\right|^p.\end{aligned}$$

As

$$J_p(b-a) - J_p(d-c) = -\big(J_p(a-b) - J_p(c-d)\big),$$

$$((b-d)_M^+ + \delta)^\beta - ((a-c)_M^+ + \delta)^\beta = -\Big(((a-c)_M^+ + \delta)^\beta - ((b-d)_M^+ + \delta)^\beta\Big),$$

and

$$\begin{aligned}&\left|((b-d)_M^+ + \delta)^{\frac{\beta+p-1}{p}} - ((a-c)_M^+ + \delta)^{\frac{\beta+p-1}{p}}\right|\\&= \left|((a-c)_M^+ + \delta)^{\frac{\beta+p-1}{p}} - ((b-d)_M^+ + \delta)^{\frac{\beta+p-1}{p}}\right|\end{aligned}$$

we obtain (2.2) □

The following pointwise inequality is a direct consequence of the convexity of the mapping $t \to |t|^\alpha$ for $\alpha \geq 1$.

$$|a^\alpha - b^\alpha| \geq \alpha \min\{a, b\}^{\alpha-1}|a-b|, \quad \text{for} \quad a, b \geq 0. \tag{2.4}$$

2.3. Functional inequalities

We need the following basic inequalities for the tail.

Lemma 2.3. *Let $\alpha > 0$, $1 \leq q < \infty$, and $u, v \in L^q_\alpha(\mathbb{R}^n)$ such that $u = v$ on $\mathbb{R}^n \setminus B_R(x_0)$. Then for any $\sigma < 1$,*

$$\mathrm{Tail}_{\alpha,q}(v; x_0, \sigma R) \leq 2\mathrm{Tail}_{\alpha,q}(u; x_0, \sigma R) + 2\sigma^{\frac{-n}{q}}\left(⨍_{B_R(x_0)} |u-v|^q \,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Proof.

$$
\begin{aligned}
&\mathrm{Tail}_{\alpha,q}(v; x_0, \sigma R)^q \\
&= (\sigma R)^{\alpha} \int_{\mathbb{R}^n \setminus B_{\sigma R}(x_0)} \frac{|v|^q}{|x-x_0|^{n+\alpha}}\,\mathrm{d}x \\
&= (\sigma R)^{\alpha} \Big(\int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|v|^q}{|x-x_0|^{n+\alpha}}\,\mathrm{d}x + \int_{B_R(x_0)\setminus B_{\sigma R}(x_0)} \frac{|v|^q}{|x-x_0|^{n+\alpha}}\,\mathrm{d}x \Big) \\
&= (\sigma R)^{\alpha} \Big(\int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|u|^q}{|x-x_0|^{n+\alpha}}\,\mathrm{d}x + \int_{B_R(x_0)\setminus B_{\sigma R}(x_0)} \frac{|v|^q}{|x-x_0|^{n+\alpha}}\,\mathrm{d}x \Big) \\
&\le (\sigma R)^{\alpha} \Big(\int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|u|^q}{|x-x_0|^{n+\alpha}}\,\mathrm{d}x + 2^{q-1} \int_{B_R(x_0)\setminus B_{\sigma R}(x_0)} \frac{|u|^q + |u-v|^q}{|x-x_0|^{n+\alpha}}\,\mathrm{d}x \Big) \\
&\le 2^{q-1} (\sigma R)^{\alpha} \Big(\int_{\mathbb{R}^n \setminus B_{\sigma R}(x_0)} \frac{|u|^q}{|x-x_0|^{n+\alpha}}\,\mathrm{d}x + \int_{B_R(x_0)\setminus B_{\sigma R}(x_0)} \frac{|u-v|^q}{|x-x_0|^{n+\alpha}}\,\mathrm{d}x \Big) \\
&\le 2^{q-1} \mathrm{Tail}_{\alpha,q}(u; x_0, \sigma R)^q + 2^{q-1} \sigma^{-n} \fint_{B_R(x_0)} |u-v|^q\,\mathrm{d}x.
\end{aligned}
$$

□

For a proof of the following result, see [6, Lemma 2.3].

Lemma 2.4. *Let $\alpha > 0$, $0 < q < \infty$. Suppose that $B_r(x_0) \subset B_R(x_1)$. Then for every $u \in L^q_\alpha(\mathbb{R}^n)$, we have*

$$
\mathrm{Tail}_{q,\alpha}(u; x_0, r)^q \le \Big(\frac{r}{R}\Big)^{\alpha} \Big(\frac{R}{R-|x-x_0|}\Big)^{n+\alpha} \mathrm{Tail}_{q,\alpha}(u; x_1, R)^q + r^{-n} \|u\|^q_{L^q(B_R(x_1))}.
$$

If in addition $u \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ for some $q < m \le \infty$, then

$$
\begin{aligned}
\mathrm{Tail}_{q,\alpha}(u; x_0, r)^q \; &\le \Big(\frac{r}{R}\Big)^{\alpha} \Big(\frac{R}{R-|x-x_0|}\Big)^{n+\alpha} \mathrm{Tail}_{q,\alpha}(u; x_1, R)^q \\
&\quad + \Big(\frac{(n\omega_n)m-q}{\alpha m + nq}\Big)^{\frac{m-q}{m}} r^{-\frac{qn}{m}} \|u\|_{L^m(B_R(x_1))},
\end{aligned}
$$

where ω_n is the measure of the n-dimensional open ball of radius 1.

We also recall the following Sobolev- and Morrey-type inequalities:

Proposition 2.5. *Suppose $1 < p < \infty$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Define $p^\star_s$ as*

$$
p^\star_s := \frac{np}{n-sp}. \tag{2.5}
$$

For every $u \in W^{s,p}(\mathbb{R}^n)$ vanishing almost everywhere in $\mathbb{R}^n \setminus \Omega$, we have

$$
\|u\|^p_{L^{p^\star_s}(\Omega)} \le C_1(n,s,p)\,[u]^p_{W^{s,p}(\mathbb{R}^n)}, \quad if \quad sp < n \tag{2.6}
$$

$$
\|u\|^p_{L^\infty(\Omega)} \le C_2(n,s,p)|\Omega|^{\frac{sp}{n}-1}[u]^p_{W^{s,p}(\mathbb{R}^n)}, \quad if \quad sp > n \tag{2.7}
$$

$$\|u\|^p_{L^l(\Omega)} \le C_3(n,s,p,l)|\Omega|^{\frac{p}{l}}[u]^p_{W^{s,p}(\mathbb{R}^n)}, \quad \textit{for every } 1 \le l < \infty, \quad \textit{if } sp = n. \tag{2.8}$$

In particular, the following Poincaré inequality holds true

$$\|u\|^p_{L^p(\Omega)} \le C\ |\Omega|^{\frac{sp}{n}}[u]_{W^{s,p}(\mathbb{R}^n)}, \tag{2.9}$$

for some $C = C(n,s,p)$. Furthermore, in the supercritical range of exponents functions in $W^{s,p}(\mathbb{R}^n)$ are Hölder continuous and the following inequality holds true:

$$[u]_{C^{s-\frac{n}{p}}(\Omega)} \le C_4(n,s,p)[u]_{W^{s,p}(\mathbb{R}^n)}, \quad \textit{if} \ \ sp > n. \tag{2.10}$$

Remark 2.6. The above Sobolev-type inequalities are also valid for functions $u \in X_0^{s,p}(\Omega,\Omega')$, where Ω is a bounded open set and Ω' is an open set such that $\Omega \Subset \Omega'$. This can be seen using the fact that there is an extension domain containing Ω and included in Ω'.

We will often use the following special application of Hölder's inequality

$$\begin{aligned} \|u(x,t)\|_{L^{q_1,r_1}(\Omega\times J)} &\le \| |\Omega|^{\frac{1}{q_1}-\frac{1}{q_2}} \|u(\bullet,t)\|_{L^{q_2}(\Omega)} \|_{L^{r_1}(J)} \\ &\le |\Omega|^{\frac{1}{q_1}-\frac{1}{q_2}} |J|^{\frac{1}{r_1}-\frac{1}{r_2}} \|u\|_{L^{q_2,r_2}(\Omega\times J)}, \end{aligned} \tag{2.11}$$

where $q_1 < q_2$, $r_1 \le r_2$. The following interpolation inequality (see, e.g., [5]) will be useful.

Lemma 2.7. *If w is contained in $L^{q_1,r_1}(\Omega\times J) \cap L^{q_2,r_2}(\Omega\times J)$, then w is contained in $L^{\tilde{q},\tilde{r}}(\Omega\times J)$, where*

$$\frac{1}{\tilde{r}} = \frac{\lambda}{r_1} + \frac{1-\lambda}{r_2}, \quad \frac{1}{\tilde{q}} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2}, \quad (0 \le \lambda \le 1).$$

Moreover,

$$\|w\|_{L^{\tilde{q},\tilde{r}}(\Omega\times J)} \le \|w\|^{\lambda}_{L^{q_1,r_1}(\Omega\times J)} \|w\|^{1-\lambda}_{L^{q_2,r_2}(\Omega\times J)}.$$

The following three lemmas will be needed in the proof of our local boundedness result (Proposition 3.4).

Lemma 2.8. *Let $sp \neq n$ and assume that w is in $L^p\big((T_0 - R^{sp}, T_0); W^{s,p}(\mathbb{R}^n)\big) \cap L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))$ and $w(x,t)$ is zero for all $x \in \mathbb{R}^n \backslash B_R(x_0)$, for almost every $t \in (T_0 - R^{sp}, T_0]$. Then w is in $L^{pq',pr'}(Q_{R,R^{sp}}(x_0,T_0))$ as long as q, r satisfy*

$$1 - \frac{1}{r} - \frac{n}{spq} \ge 0.$$

Moreover,

$$\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}}(x_0,T_0))}$$

$$\leq C R^{sp(1-\frac{1}{r}-\frac{n}{spq})}\Big(\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))}+\int_{T_0-R^{sp}}^{T_0}[w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\Big),$$

where C depends on n, s and p. In particular, in the case of $\frac{1}{r}+\frac{n}{spq}=1$ we have

$$\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}}(x_0,T_0))} \leq C(n,s,p)\left(\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))}+\int_{T_0-R^{sp}}^{T_0}[w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right).$$

Proof. Consider a pair of exponents $\tilde{r}=(\frac{1}{r'}-(1-\frac{1}{r}-\frac{n}{spq}))^{-1}=\frac{spq}{n}$, and $\tilde{q}=q'$ such that $\frac{1}{\tilde{r}'}+\frac{n}{sp\tilde{q}'}=1$. Using Hölder's inequality (2.11), we obtain

$$\begin{aligned}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}}(x_0,T_0))} &\leq (R^{sp})^{\frac{1}{r'}-\frac{1}{\tilde{r}}}\|w\|^p_{L^{p\tilde{q},p\tilde{r}}(Q_{R,R^{sp}}(x_0,T_0))}\\ &= R^{sp(1-\frac{1}{r}-\frac{n}{spq})}\|w\|^p_{L^{p\tilde{q},p\tilde{r}}(Q_{R,R^{sp}}(x_0,T_0))}.\end{aligned}$$

Now we split the proof into two cases depending on whether $sp<n$ or not.
Case $sp<n$: We use Lemma 2.7 with the choice

$$\frac{1}{p\tilde{r}}=\frac{\lambda}{p} \quad \text{and} \quad \frac{1}{p\tilde{q}}=\frac{\lambda}{p_s^\star}+\frac{1-\lambda}{p}, \quad (0\leq\lambda\leq 1).$$

This yields

$$\|w\|_{L^{p\tilde{q},p\tilde{r}}(Q_{R,R^{sp}}(x_0,T_0))}\leq \|w\|^{\lambda}_{L^{p_s^\star,p}(Q_{R,R^{sp}}(x_0,T_0))}\|w\|^{1-\lambda}_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))}.$$

The above relations hold for $\lambda=\frac{1}{\tilde{r}}=\frac{n}{sp\tilde{q}'}$, and using Sobolev's inequality 2.6, we arrive at

$$\begin{aligned}\|w\|^p_{L^{p\tilde{q},p\tilde{r}}(Q_{R,R^{sp}}(x_0,T_0))} &\leq \|w\|^{p\lambda}_{L^{p_s^\star,p}(Q_{R,R^{sp}}(x_0,T_0))}\|w\|^{p(1-\lambda)}_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))}\\ &\leq C\|w\|^{p(1-\lambda)}_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))}\left(\int_{T_0-R^{sp}}^{T_0}[w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right)^{\lambda}.\end{aligned}$$

By using Young's inequality, we get

$$\begin{aligned}&\|w\|^p_{L^{p\tilde{q},p\tilde{r}}(Q_{R,R^{sp}}(x_0,T_0))}\\ &\quad\leq C\|w\|^{p(1-\lambda)}_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))}\left(\int_{T_0-R^{sp}}^{T_0}[w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right)^{\lambda}\\ &\quad\leq C\left((1-\lambda)\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))}+\lambda\int_{T_0-R^{sp}}^{T_0}[w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right)\\ &\quad\leq C(n,s,p)\left(\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))}+\int_{T_0-R^{sp}}^{T_0}[w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right).\end{aligned}$$

Case $sp > n$: In this case, we use the following interpolation between Hölder and L^p spaces:

$$\|w\|_{L^\infty(B_R(x_0))} \leq C\|w\|^{\chi}_{L^p(B_R(x_0))}[w]^{1-\chi}_{C^\alpha(B_R(x_0))}, \quad \text{with} \quad \chi = \frac{\alpha}{\alpha + \frac{n}{p}}.$$

See [9, Lemma 2.2] for a proof. In light of the Morrey-type inequality (2.10), for almost every $t \in (T_0 - R^{sp}, T_0)$ we arrive at

$$\begin{aligned}\|w(\cdot,t)\|_{L^\infty(B_R(x_0))} &\leq C\|w(\cdot,t)\|^{1-\frac{n}{sp}}_{L^p(B_R(x_0))}[w(\cdot,t)]^{\frac{n}{sp}}_{C^{s-\frac{n}{p}}(B_R(x_0))} \\ &\leq C\|u(\cdot,t)\|^{1-\frac{n}{sp}}_{L^p(B_R(x_0))}[w(\cdot,t)]^{\frac{n}{sp}}_{W^{s,p}(\mathbb{R}^n)}. \end{aligned} \tag{2.12}$$

Now we interpolate once more between L^p and L^∞ to obtain

$$\begin{aligned}\|w(\cdot,t)\|_{L^{p\tilde{q}}(B_R(x_0))} &\leq \|w(\cdot,t)\|^{\frac{1}{\tilde{q}}}_{L^p(B_R(x_0))}\|w(\cdot,t)\|^{\frac{1}{\tilde{q}'}}_{L^\infty(B_R(x_0))} \\ \text{by (2.12)} \quad &\leq C\|w(\cdot,t)\|^{\frac{1}{\tilde{q}}}_{L^p(B_R(x_0))}\|w(\cdot,t)\|^{\frac{1}{\tilde{q}'}(1-\frac{n}{sp})}_{L^p(B_R(x_0))}[w(\cdot,t)]^{\frac{n}{sp\tilde{q}'}}_{W^{s,p}(\mathbb{R}^n)} \\ &\leq C\|w(\cdot,t)\|^{1-\frac{n}{sp\tilde{q}'}}_{L^p(B_R(x_0))}[w(\cdot,t)]^{\frac{n}{sp\tilde{q}'}}_{W^{s,p}(\mathbb{R}^n)}.\end{aligned}$$

We raise both sides to the power $p\tilde{r}$ and integrate with respect to t. Recalling that $\frac{1}{\tilde{r}} = \frac{n}{sp\tilde{q}'}$, we obtain

$$\begin{aligned}\|w\|^{p\tilde{r}}_{L^{p\tilde{q},p\tilde{r}}(Q_{R,R^{sp}}(x_0,T_0))} &\leq C^{p\tilde{r}}\int_{T_0-R^{sp}}^{T_0}\|w(\cdot,t)\|^{p(\tilde{r}-1)}_{L^p(B_R(x_0))}[w(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t \\ &\leq C^{p\tilde{r}}\sup_{T_0-R^{sp}\leq t\leq T_0}\|w(\cdot,t)\|^{p(\tilde{r}-1)}_{L^p(B_R(x_0))} \\ &\quad\times\int_{T_0-R^{sp}}^{T_0}[w(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t.\end{aligned}$$

Taking the $\tilde{r}$ root and applying Young's inequality, we obtain the desired estimate:

$$\begin{aligned}\|w\|^p_{L^{p\tilde{q},p\tilde{r}}(Q_{R,R^{sp}}(x_0,T_0))} &\leq C^p \|w\|^{p(\frac{\tilde{r}-1}{\tilde{r}})}_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))} \left(\int_{T_0-R^{sp}}^{T_0} [w]^p_{W^{s,p}(\mathbb{R}^n)}\, \mathrm{d}t\right)^{\frac{1}{\tilde{r}}} \\ &\leq C(n,s,p)\left(\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))} + \int_{T_0-R^{sp}}^{T_0} [w]^p_{W^{s,p}(\mathbb{R}^n)}\, \mathrm{d}t\right).\end{aligned}$$

□

Lemma 2.9. *Let $sp = n,\ q \geq 1,$ and $r \geq 1$ such that*

$$1 - \frac{1}{r} - \frac{1}{q} > 0.$$

Assume that $w \in L^{l,p}(Q_{R,R^{sp}}(x_0, T_0)) \cap L^{p,\infty}(Q_{R,R^{sp}}(x_0, T_0))$ for some l such that

$$l = \frac{p}{r'}\left(1 - \frac{1}{r} - \frac{1}{q}\right)^{-1}.$$

Then w belongs to $L^{pq',pr'}(Q_{R,R^{sp}}(x_0, T_0))$ and

$$\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}}(x_0,T_0))} \leq R^{\frac{np}{lr'}}\left(\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))} + R^{\frac{-np}{l}}\|w\|^p_{L^{l,p}(Q_{R,R^{sp}}(x_0,T_0))}\right).$$

Proof. We use Lemma 2.7 with the choice

$$\frac{1}{pr'} = \frac{\lambda}{p} \quad \text{and} \quad \frac{1}{pq'} = \frac{\lambda}{l} + \frac{1-\lambda}{p}, \quad (0 \leq \lambda \leq 1).$$

Due to the assumption $\frac{1}{l} = \frac{r'}{p}(1 - \frac{1}{r} - \frac{1}{q})$, the above equalities hold for $\lambda = \frac{1}{r'}$. Hence, we get

$$\|w\|_{L^{pq',pr'}(Q_{R,R^{sp}}(x_0,T_0))} \leq \|w\|^{\lambda}_{L^{l,p}(Q_{R,R^{sp}}(x_0,T_0))} \|w\|^{1-\lambda}_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))}.$$

Therefore, recalling that $\lambda = \frac{1}{r'}$

$$\begin{aligned}R^{\frac{-np}{lr'}}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}}(x_0,T_0))} &= R^{\frac{-\lambda np}{l}}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}}(x_0,T_0))} \\ &\leq \left(R^{\frac{-np}{l}}\|w\|^p_{L^{l,p}(Q_{R,R^{sp}}(x_0,T_0))}\right)^{\lambda}\left(\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))}\right)^{1-\lambda}.\end{aligned}$$

Using Young's inequality for the right-hand side, we can conclude

$$\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}}(x_0,T_0))} \leq R^{\frac{np}{lr'}}\left(\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}}(x_0,T_0))} + R^{\frac{-np}{l}}\|w\|^p_{L^{l,p}(Q_{R,R^{sp}}(x_0,T_0))}\right).$$

□

2.4. Weak solutions

Definition 2.10. For any $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$, we define $I = (t_0, t_1]$. Let

$$f \in \Big(L^p(I; X_0^{s,p}(\mathcal{K}, \Omega)) \cap L^\infty(I; L^2(\mathcal{K}))\Big)^\star,$$

for any open $\mathcal{K}$ such that $\mathcal{K} \Subset \Omega$. We say that u is a *local weak solution* to the equation

$$\partial_t u + (-\Delta_p)^s u = f, \qquad \text{in } \Omega \times I,$$

if for any closed interval $J = [T_0, T_1] \subset I$, the function u is such that

$$u \in L^p(J; W^{s,p}_{\mathrm{loc}}(\Omega)) \cap L^{p-1}(J; L^{p-1}_{s\,p}(\mathbb{R}^n)) \cap C(J; L^2_{\mathrm{loc}}(\Omega)),$$

and it satisfies

$$\begin{aligned} &-\int_J \int_\Omega u(x,t)\, \partial_t \varphi(x,t)\, \mathrm{d}x\, \mathrm{d}t + \int_J \iint_{\mathbb{R}^n \times \mathbb{R}^n} \\ &\quad \frac{J_p(u(x,t) - u(y,t))\, (\varphi(x,t) - \varphi(y,t))}{|x-y|^{n+s\,p}}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t \\ &= \int_\Omega u(x,T_0)\, \varphi(x,T_0)\, \mathrm{d}x - \int_\Omega u(x,T_1)\, \varphi(x,T_1)\, \mathrm{d}x + \langle f, \varphi \rangle, \end{aligned} \tag{2.13}$$

for any $\varphi \in L^p(J; W^{s,p}(\Omega)) \cap C^1(J; L^2(\Omega))$ which has spatial support compactly contained in Ω. In equation (2.13), the symbol $\langle \bullet, \bullet \rangle$ stands for the duality pairing between $W^{s,p}(\Omega)$ and its dual space $(W^{s,p}(\Omega))^*$.

Now, we define the notion of a weak solution to an initial boundary value problem.

Definition 2.11. Let $I = [t_0, t_1]$, $p \geq 2$, $0 < s < 1$, and $\Omega \Subset \Omega'$, where Ω' is a bounded open set in $\mathbb{R}^n$. Assume that the functions u_0, f and g satisfy

$$\begin{aligned} u_0 &\in L^2(\Omega), \\ f &\in \Big(L^p(I; X_0^{s,p}(\Omega, \Omega')) \cap L^\infty(I; L^2(\Omega))\Big)^\star, \\ g &\in L^p(I; W^{s,p}(\Omega')) \cap L^{p-1}(I; L^{p-1}_{s\,p}(\mathbb{R}^n)). \end{aligned}$$

We say that u is a *weak solution of the initial boundary value problem*

$$\begin{cases} \partial_t u + (-\Delta_p)^s u = f, & \text{in } \Omega \times I, \\ u = g, & \text{on } (\mathbb{R}^n \setminus \Omega) \times I, \\ u(\bullet, t_0) = u_0, & \text{on } \Omega, \end{cases} \tag{2.14}$$

if the following properties are verified:

- $u \in L^p(I; W^{s,p}(\Omega')) \cap L^{p-1}(I; L^{p-1}_{sp}(\mathbb{R}^n)) \cap C(I; L^2(\Omega))$;
- $u \in X_{\mathbf{g}(t)}(\Omega, \Omega')$ for almost every $t \in I$, where $(\mathbf{g}(t))(x) = g(x,t)$;

- $\lim_{t\to t_0} \|u(\bullet, t) - u_0\|_{L^2(\Omega)} = 0$;
- for every $J = [T_0, T_1] \subset I$ and every $\varphi \in L^p(J; X_0^{s,p}(\Omega, \Omega')) \cap C^1(J; L^2(\Omega))$

$$\begin{aligned} &-\int_J \int_\Omega u(x,t)\, \partial_t \varphi(x,t)\, \mathrm{d}x\, \mathrm{d}t \\ &+ \int_J \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{J_p(u(x,t)-u(y,t))\,(\varphi(x,t)-\varphi(y,t))}{|x-y|^{n+sp}}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t \\ = &\int_\Omega u(x,T_0)\,\varphi(x,T_0)\, \mathrm{d}x - \int_\Omega u(x,T_1)\,\varphi(x,T_1)\, \mathrm{d}x \\ &+ \langle f, \varphi\rangle. \end{aligned}$$

Let us mention that given a local weak solution in a cylinder $I \times \Omega'$, where $I = (t_0, t_1]$ and Ω' is a bounded, open subset of $\mathbb{R}^n$, by considering a smaller cylinder $J \times \Omega$ such that $\Omega \Subset \Omega'$ and J is a closed interval compactly contained in I we end up a weak solution in the smaller cylinder $J \times \Omega$.

Throughout the article, we work with right-hand sides $f \in L^{p'}(I; L^{(p_s^\star)'}(\Omega))$, where $p_s^\star$ is the Sobolev exponent and we consider it to be infinity if $sp > n$. An application of Hölder's inequality together with the Sobolev–Morrey inequalities ensures that $f \in f \in L^{p'}(I; (X_0^{s,p}(\Omega, \Omega'))^*) \subset \big(L^p(I; X_0^{s,p}(\Omega\,, \Omega')) \cap L^\infty(I; L^2(\Omega))\big)^\star$ with the duality pairing

$$\langle f, \varphi\rangle = \int_I \int_\Omega f(x,t)\varphi(x,t)\, \mathrm{d}x\, \mathrm{d}t.$$

Theorem 2.12. *Let $p \geq 2$, let $I = (T_0, T_1]$ and suppose that g satisfies*

$$\begin{aligned} &g \in L^p(I; W^{s,p}(\Omega')) \cap L^p(I; L^{p-1}_{s\,p}(\mathbb{R}^n)) \cap C(I; L^2(\Omega)), \\ &\partial_t g \in L^{p'}(I; (X_0^{s,p}(\Omega, \Omega'))^*), \\ &\lim_{t\to t_0} \|g(\bullet, t) - g_0\|_{L^2(\Omega)} = 0, \qquad \textit{for some } g_0 \in L^2(\Omega). \end{aligned}$$

Suppose also that

$$f \in L^{p'}(I; (X_0^{s,p}(\Omega, \Omega'))^*).$$

Then for any initial datum $g_0 \in L^2(\Omega)$, there exists a unique weak solution u to problem

$$\begin{cases} u_t + (-\Delta_p)^s u = f & in \;\; \Omega \times I \\ u = g & in \;\; (\mathbb{R}^n \setminus \Omega) \times I \\ u(x, T_0) = g(x, T_0) & in \;\; \Omega. \end{cases} \tag{2.15}$$

Proof. In [7, Theorem A.3], the same result is proved with a stronger condition $g_t \in L^{p'}(I; W^{s,p}(\Omega')^\star)$. The stronger condition is not needed in the proof. This condition can be replaced with $g_t \in L^{p'}(I, X_0^{s,p}(\Omega; \Omega')^\star)$ in all of the steps in the proof, except that the construction gives us a $C(I; L^2(\Omega))$ solution. There, the stronger assumption is used only to show that the boundary condition is in $C(I; L^2(\Omega))$, which we assume here. □

Remark 2.13. The condition $\partial_t g \in L^{p'}(I; (X_0^{s,p}(\Omega, \Omega'))^*)$ is too strong. This condition forces us to assume $r \geq p'$, $q \geq (p_s^\star)'$ and $u \in L^p(I; L_{sp}^{p-1}(\mathbb{R}^n))$ in Proposition 3.1 and hence in all our results. A more natural condition would be to assume $\partial_t g \in \big(L^p(I; X_0^{s,p}(\Omega, \Omega')) \cap L^\infty(I; L^2(\Omega))\big)^\star$. We believe it is possible to overcome this difficulty by pursuing an approximation procedure in the spirit of [35, Theorem 1.1 and Lemma 4.1].

3. Basic Hölder regularity and stability

Throughout the rest of the article, we assume $0 < s < 1$ and $2 \leq p < \infty$.

Here, we argue that the norm of the (s, p)-caloric replacement of u is close to u if f is small enough. By the (s, p)-caloric replacement of u in a cylinder $B_\rho(x_0) \times I$, we mean the solution to the following

$$\begin{cases} v_t + (-\Delta_p)^s v = 0 & in \quad B_\rho(x_0) \times I \\ v = u & in \quad (\mathbb{R}^n \setminus B_\rho(x_0)) \times I \\ v(x, \tau_0) = u(x, \tau_0) & in \quad B_\rho(x_0). \end{cases} \tag{3.1}$$

Here τ_0 is the initial point of the interval I. First we show the existence of a (s, p)-caloric replacement using Theorem 2.12

Proposition 3.1. *Let u be a local weak solution of $u_t + (-\Delta_p)^s u = f$ in the cylinder $B_\sigma \times J$, for some interval $J = (t_1, t_2]$ with $f \in L_{\mathrm{loc}}^{q,r}(B_\sigma \times J)$ such that $r \geq p'$,*

$$q \geq (p_s^\star)' \quad if \quad sp < n, \quad q \geq 1 \quad if \quad sp > n, \quad and \quad q > 1 \quad if \quad sp = n.$$

In addition, we assume that $u \in L^p(J; L_{sp}^{p-1}(R^n))$. Then for any $0 < \rho < \sigma$, and closed interval $I \Subset J$, the (s, p)-caloric replacement of u in $B_\rho(x_0) \times I$ (weak solution to (3.1)) exists.

Proof. We shall check the conditions in Theorem 2.12. If they are satisfied, there exists a unique weak solution $v \in L^p(I, W^{s,p}(B_\sigma)) \cap L^p(I; L_{sp}^{p-1}(\mathbb{R}^n)) \cap C(I; L^2(B_\rho))$ to the problem (3.1). The only condition on u that is not immediate from the fact that u is weak solution is $\partial_t u \in L^{p'}(I; X_0^{s,p}(B_\rho, B_\sigma)^\star)$. We have to show that for every function $\psi \in L^p(I; X_0^{s,p}(B_\rho, B_\sigma))$

$$\left| \int_I \langle u_t, \psi \rangle \, \mathrm{d}x \, \mathrm{d}t \right| \leq C \int_I \|\psi\|_{W^{s,p}(B_\sigma)}^p \, \mathrm{d}t. \tag{3.2}$$

Here we only write the proof for the case $sp < n$, the case of $sp \geq n$ is similar, except that one has to use the critical case of Sobolev inequality and the Morrey inequality instead of using the Sobolev inequality. We shall verify (3.2) for test functions belonging to the dense subspace, $\psi \in L^p(I; X_0^{s,p}(B_\rho, B_\sigma)) \cap C_0^1(I; L^2(B))$. We use the equation to do so. We have

$$\begin{aligned}\int_I \langle u_t, \psi\rangle \,\mathrm{d}x\,\mathrm{d}t &= \int_I \int_{B_\rho} u\psi_t \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_I \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{J_p(u(x,t)-u(y,t))(\psi(x,t)-\psi(y,t))}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\quad + \int_I \int_{B_r} f(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_I \iint_{B_\sigma\times B_\sigma} \frac{J_p(u(x,t)-u(y,t))(\psi(x,t)-\psi(y,t))}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\quad - 2\int_I \int_{\mathbb{R}^n\setminus B_\sigma} \int_{B_\rho} \frac{J_p(u(x,t)-u(y,t))\psi(x,t)}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\quad + \int_I \int_{B_\rho} f(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}&\int_I \iint_{B_\sigma\times B_\sigma} \frac{|J_p(u(x,t)-u(y,t))(\psi(x,t)-\psi(y,t))|}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\le \int_I \left\| \frac{J_p(u(x,t)-u(y,t))}{|x-y|^{\frac{n}{p'}+s(p-1)}} \right\|_{L^{p'}(B_\sigma\times B_\sigma)} \left\| \frac{\psi(x,t)-\psi(y,t)}{|x-y|^{\frac{n}{p}+s}} \right\|_{L^p(B_\sigma\times B_\sigma)} \mathrm{d}t \\ &\le [u]^{p-1}_{L^p(I;W^{s,p}(B_\sigma))} [\psi]_{L^p(I;W^{s,p}(B_\sigma))}. \end{aligned} \tag{3.3}$$

For the other nonlocal term, we note that for every $x \in B_\rho$ and $y \in \mathbb{R}^n \setminus B_\sigma$ we have $|y| \le \frac{\sigma}{\sigma-\rho}|x-y|$. Hence,

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B_\sigma} \frac{|J_p(u(x,t)-u(y,t))|}{|x-y|^{n+sp}} \,\mathrm{d}y \\ &\le \left(\frac{\sigma}{\sigma-\rho}\right)^{n+sp} C(p) \int_{\mathbb{R}^n\setminus B_\sigma} \frac{|u(x,t)|^{p-1}+|u(y,t)|^{p-1}}{|y|^{n+sp}} \,\mathrm{d}y \\ &\le C(\sigma,\rho,s,p,n)\Big(|u(x,t)|^{p-1} + \|u(\bullet,t)\|^{p-1}_{L^{p-1}_{sp}}\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_I \int_{B_\rho} \int_{\mathbb{R}^n\setminus B_\sigma} \frac{|J_p(u(x,t)-u(y,t))\psi(x,t)|}{|x-y|^{n+sp}} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\le C(\sigma,\rho,s,p,n)\Big(\int_I \int_{B_\rho} |\psi(x,t)||u(x,t)|^{p-1}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_I \|\psi(\bullet,t)\|_{L^1(B_\rho)} \|u(\bullet,t)\|^{p-1}_{L^{p-1}_{sp}(\mathbb{R}^n)}\,\mathrm{d}t\Big).\end{aligned}$$

By Hölder's inequality, we have

$$\int_I \int_{B_\rho} |\varphi(x,t)||u(x,t)|^{p-1} \, \mathrm{d}x \, \mathrm{d}t \le \int_I \|\psi(\cdot,t)\|_{L^p(B_\rho)} \|u(\cdot,t)\|_{L^p(B_\rho)}^{p-1}$$
$$\le \|\psi\|_{L^p(I;L^p(B_\rho))} \|u\|_{L^p(I;L^p(B_\rho))}^{p-1}. \quad (3.4)$$

For the other term,

$$\int_I \|\psi(\cdot,t)\|_{L^1(B_\rho)} \|u(\cdot,t)\|_{L^{p-1}_{sp}(\mathbb{R}^n)}^{p-1} \, \mathrm{d}t \le \|\psi\|_{L^p(I;L^1(B_\rho))} \|u(\cdot,t)\|_{L^p(I;L^{p-1}_{sp}(\mathbb{R}^n))}^{p-1}. \quad (3.5)$$

Since $f \in L^{p'}(I; L^{(p_s^\star)'}(B_\rho))$, by Hölder's inequality and Sobolev's inequality we obtain

$$\begin{aligned} \int_I \int_{B_\rho} |f\psi| \, \mathrm{d}x \, \mathrm{d}t &\le \int_I \|f\|_{L^{(p_s^\star)'}(B_\rho)} \|\psi\|_{L^{p_s^\star}(B_\rho)} \, \mathrm{d}t \\ &\le \int_I \|f\|_{L^{(p_s^\star)'}(B_\rho)} \|\psi\|_{W^{s,p}(B_\sigma)} \, \mathrm{d}t \\ &\le \|f\|_{L^{((p_s^\star)',p')}(B_\rho \times I)} \|\psi\|_{L^p(I;W^{s,p}(B_\sigma))}. \end{aligned} \quad (3.6)$$

Therefore, combining with (3.3), (3.4), and (3.5) we obtain

$$\left| \int_I \langle v_t, \psi \rangle \, \mathrm{d}t \right| \le C(\sigma, \rho, s, p, n, u, f) \|\psi\|_{L^p(I;W^{s,p}(B_\sigma))}.$$

□

Lemma 3.2. *Assume that $f \in L^{q,r}_{\mathrm{loc}}(Q_{\sigma,\sigma^{sp}}(x_0,T_0))$ with $r \ge p'$,*

$$q \ge (p_s^\star)' \quad \text{if} \quad sp < n, \quad q \ge 1 \quad \text{if} \quad sp > n, \quad \text{and} \quad q > 1 \quad \text{if} \quad sp = n.$$

Let u be a local weak solution of $\partial_t u + (-\Delta_p)^s u = f$ in $Q_{\sigma,\sigma^{sp}}(x_0,T_0)$, such that $u \in L^p_{\mathrm{loc}}((T_0 - \sigma^{sp}, T_0]; L^{p-1}_{sp}(\mathbb{R}^n))$. Let $\rho < \sigma$ and consider v to be the (s,p)-caloric replacement of u in $Q_{\rho,\rho^{sp}}(x_0,T_0)$. Then we have

$$\fint_{Q_{\rho,\rho^{s\,p}}(x_0,T_0)} |u-v|^p \, \mathrm{d}x \, \mathrm{d}t \le C\rho^{\xi} \, \|f\|_{L^{q,r}(Q_{\rho,\rho^{s\,p}}(x_0,T_0))}^{p'} \quad (3.7)$$

and

$$\|u-v\|_{L^{q',r'}(Q_{\rho,\rho^{sp}}(x_0,T_0))} \le C\rho^{\xi+n} \|f\|_{L^{q,r}(Q_{\rho,\rho^{sp}}(x_0,T_0))}^{\frac{1}{p-1}}, \quad (3.8)$$

with $\xi = spp'(1 - \frac{1}{r} - \frac{n}{spq})$ and $C = C(n,s,p)$, in the case $sp \ne n$. In the case $sp = n$, we can take $\xi = spp'(1 - \frac{1}{r} - \frac{1}{q})$, with $C = C(n,s,p,q)$ also depending on q.

Proof. Let $J := [T_0 - \rho^{sp}, T_0]$, throughout the proof, we drop the dependence of the balls on the center and write B_ρ instead of $B_\rho(x_0)$, and $Q_{\rho,\rho^{sp}}$ instead of $Q_{\rho,\rho^{sp}}(x_0, T_0)$.

By subtracting the weak formulation of the equations (2.13) for u and v with the same test function $\varphi(x,t) \in L^p(J; X_0^{s,p}(B_\rho, B_\sigma)) \cap C^1(J; L^2(B_\rho))$, we get

$$
\begin{aligned}
&-\int_J \int_{B_\rho} (u(x,t) - v(x,t)) \frac{\partial}{\partial t}\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_J \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\big[J_p\big(u(x,t) - u(y,t)\big) - J_p\big(v(x,t) - v(y,t)\big)\big](\varphi(x,t) - \varphi(y,t))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= \int_{B_\rho} ((u(x, T_0 - \rho^{sp}) - v(x, T_0 - \rho^{sp}))\varphi(x, T_0 - \rho^{sp})\,\mathrm{d}x \\
&\quad - \int_{B_\rho} ((u(x, T_0) - v(x, T_0))\varphi(x, T_0)\,\mathrm{d}x \\
&\quad + \int_J \int_{B_\rho} f(x,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Now we take $\varphi := u - v$, which belongs to $L^p(J; X_0^{s,p}(B_\rho; B_\sigma))$, but it may not be in $C^1(J; L^2(B_\rho))$. We justify taking this as a test function in Appendix B. By Proposition 6.1 with $F(t) = t$, we get

$$
\begin{aligned}
&\int_J \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\big[J_p\big(u(x,t) - v(x,t)\big) - J_p\big(u(y,t) - v(y,t)\big)\big]\big[\big(u(x,t) - u(y,t)\big) - \big(v(x,t) - v(y,t)\big)\big]}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad = \int_J \int_{B_\rho} f(x,t)(u(x,t) - v(x,t))\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \frac{1}{2}\int_{B_\rho} ((u(x,T_0) - v(x,T_0))^2 - ((u(x, T_0 - \rho^{sp}) - v(x, T_0 - \rho^{sp}))^2\,\mathrm{d}x \\
&\quad = \int_J \int_{B_\rho} f(x,t)(u(x,t) - v(x,t))\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_{B_\rho} ((u(x,T_0) - v(x,T_0))^2\,\mathrm{d}x \\
&\quad \le \int_J \int_{B_\rho} |f(x,t)(u(x,t) - v(x,t))|\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.9}
$$

where in the third line we have used $u(x, T_0 - \rho^{sp}) = v(x, T_0 - \rho^{sp})$. The left-hand side is essentially the $W^{s,p}$ seminorm. By the pointwise inequality (2.1),

$$
\begin{aligned}
&\int_J [u - v]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t \\
&\quad = \int_J \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x,t) - v(x,t) - (u(y,t) - v(y,t))|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad \le C(p) \int_J \iint_{\mathbb{R}^n \times \mathbb{R}^n} \\
&\qquad \times \frac{\big[J_p\big(u(x,t) - u(y,t)\big) - J_p\big(v(x,t) - v(y,t)\big)\big]\big[u(x,t) - u(y,t) - \big(v(x,t) - v(y,t)\big)\big]}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}
$$

Therefore, by (3.9) and Hölder's inequality

$$\begin{aligned}\int_J [u-v]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t &\le C(p)\int_J\int_{B_\rho}|f(x,t)(u(x,t)-v(x,t))|\,\mathrm{d}x\,\mathrm{d}t\\ &\le C(p)\int_J \|f(\cdot,t)\|_{L^q(B_\rho)}\|(u-v)(\cdot,t)\|_{L^{q'}(B_\rho)}\,\mathrm{d}t\\ &\le C(p)\|f\|_{L^{q,r}(Q_{\rho,\rho^{sp}})}\|u-v\|_{L^{q',r'}(Q_{\rho,\rho^{sp}})}. \end{aligned}\tag{3.10}$$

Now we consider three cases: $sp<n$, $sp>n$ and $sp=n$.

Case $sp<n$. By Hölder's inequality (2.11) and Sobolev's inequality (2.6), we have

$$\begin{aligned}\|u-v\|_{L^{q',r'}} &\le |B_\rho|^{\frac{1}{q'}-\frac{1}{p_s^\star}}\left(\int_J \|u-v\|^{r'}_{L^{p_s^\star}(B_\rho)}\,\mathrm{d}t\right)^{\frac{1}{r'}}\\ &\le C(n,s,p)|B_\rho|^{\frac{1}{q'}-\frac{1}{p_s^\star}}\left(\int_J [u-v]^{r'}_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right)^{\frac{1}{r'}}\\ &\le C(n,s,p)|B_\rho|^{\frac{1}{q'}-\frac{1}{p_s^\star}}|J|^{\frac{1}{r'}-\frac{1}{p}}\left(\int_J [u-v]^{p}_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right)^{\frac{1}{p}}. \end{aligned}\tag{3.11}$$

Combined with (3.10), this yields

$$\begin{aligned}\left(\int_J [u-v]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right)^{\frac{p-1}{p}} &\le C|B_\rho|^{\frac{1}{q'}-\frac{1}{p_s^\star}}|J|^{\frac{1}{r'}-\frac{1}{p}}\|f\|_{L^{q,r}(Q_{\rho,\rho^{sp}})}\\ &= C|B_\rho|^{\frac{1}{q'}-\frac{n-sp}{np}}|J|^{\frac{1}{r'}-\frac{1}{p}}\|f\|_{L^{q,r}(Q_{\rho,\rho^{sp}})}, \end{aligned}\tag{3.12}$$

where $C=C(n,s,p)$. By the Poincaré inequality,

$$⨍_J ⨍_{B_\rho}|u-v|^p\,\mathrm{d}x\,\mathrm{d}t \le C|B_\rho|^{\frac{p'}{q'}-p'\frac{(n-sp)}{np}+\frac{sp}{n}-1}|J|^{\frac{p'}{r'}-\frac{p'}{p}-1}\|f\|^{p'}_{L^{q,r}(Q_{\rho,\rho^{sp}})}.$$

Also from (3.12) and (3.11), we get

$$\|u-v\|_{L^{q',r'}(Q_{\rho,\rho^{sp}})} \le C(n,s,p)|B_\rho|^{\frac{p'}{q'}-p'\frac{n-sp}{np}}|J|^{\frac{p'}{r'}-\frac{p'}{p}}\|f\|^{\frac{1}{p-1}}_{L^{q,r}(Q_{\rho,\rho^{sp}})}.$$

Case $sp>n$. In this case, we use Morrey's inequality (2.7) and Hölder's inequality and obtain

$$\begin{aligned}\|u-v\|_{L^{q',r'}(Q_{\rho,\rho^{sp}})} &\le C|B_\rho|^{\frac{1}{q'}}\left(\int_J \|u-v\|^{r'}_{L^\infty(B_\rho)}\,\mathrm{d}t\right)^{\frac{1}{r'}}\\ &\le C|B_\rho|^{\frac{1}{q'}}|J|^{\frac{1}{r'}-\frac{1}{p}}\left(\int_J \|u-v\|^{p}_{L^\infty(B_\rho)}\,\mathrm{d}t\right)^{\frac{1}{p}}\\ &\le C|B_\rho|^{\frac{1}{q'}+\frac{sp-n}{np}}|J|^{\frac{1}{r'}-\frac{1}{p}}\left(\int_J [u-v]^{p}_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right)^{\frac{1}{p}}. \end{aligned}\tag{3.13}$$

Together with (3.10), this implies

$$\left(\int_J [u-v]^p_{W^{s,p}(\mathbb{R}^n)}\, dt\right)^{\frac{p-1}{p}} \le C|B_\rho|^{\frac{1}{q'}-\frac{n-sp}{np}}|J|^{\frac{1}{r'}-\frac{1}{p}}\|f\|_{L^{q,r}(Q_{\rho,\rho^{sp}})}. \tag{3.14}$$

By the Poincaré inequality,

$$\fint_J \fint_{B_\rho} |u-v|^p\, dx\, dt \le C|B_\rho|^{\frac{p'}{q'}-p'\frac{(n-sp)}{np}+\frac{sp}{n}-1}|J|^{\frac{p'}{r'}-\frac{p'}{p}-1}\|f\|^{p'}_{L^{q,r}(Q_{\rho,\rho^{sp}})}.$$

Combining (3.13) and (3.14), we get

$$\|u-v\|_{L^{q',r'}(Q_{\rho,\rho^{sp}})} \le C(n,s,p)|B_\rho|^{\frac{p'}{q'}-p'\frac{n-sp}{np}}|J|^{\frac{p'}{r'}-\frac{p'}{p}}\|f\|^{\frac{1}{p-1}}_{L^{q,r}(Q_{\rho,\rho^{sp}})}.$$

Case $sp = n$. In this case, we use the critical case of Sobolev's inequality (2.8) for $l = q'$ and obtain

$$\|u-v\|^p_{L^{q'}(B_\rho)} \le C(n,s,p,q)|B_\rho|^{\frac{p}{q'}}[u-v]^p_{W^{s,p}(\mathbb{R}^n)}.$$

Hence, using Hölder's inequality, we have for any $r \ge p'$

$$\begin{aligned}\|u-v\|_{L^{q',r'}(Q_{\rho,\rho^{sp}})} &= \left(\int_J \|u-v\|^{r'}_{L^{q'}(B_\rho)}\, dt\right)^{\frac{1}{r'}}\\ &\le C|B_\rho|^{\frac{1}{q'}}\left(\int_J [u-v]^{r'}_{W^{s,p}(\mathbb{R}^n)}\, dt\right)^{\frac{1}{r'}}\\ &\le C|B_\rho|^{\frac{1}{q'}}|J|^{\frac{1}{r'}-\frac{1}{p}}\left(\int_J [u-v]^p_{W^{s,p}(\mathbb{R}^n)}\, dt\right)^{\frac{1}{p}}.\end{aligned}$$

The above constant $C = C(n,s,p,q)$ does blow up as q goes to 1. In a similar way as in the prior cases, we get for $q > 1$ and $r \ge p'$

$$\fint_J \fint_{B_\rho} |u-v|^p \le C(n,s,p,q)|B_\rho|^{\frac{p'}{q'}}|J|^{\frac{p'}{r'}-\frac{p'}{p}-1}\|f\|^{p'}_{L^{q,r}(Q_{\rho,\rho^{sp}})}$$

and

$$\|u-v\|_{L^{q',r'}(Q_{\rho,\rho^{sp}})} \le C(n,s,p,q)|B_\rho|^{\frac{p'}{q'}}|J|^{\frac{p'}{r'}-\frac{p'}{p}}\|f\|^{\frac{1}{p-1}}_{L^{q,r}(Q_{\rho,\rho^{sp}})}.$$

Using that $|B_\rho| \sim \rho^n$ and $|I| \sim \rho^{sp}$, we can conclude that

$$\fint_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |u-v|^p\, dx\, dt \le C\rho^{\xi}\, \|f\|^{p'}_{L^{q,r}(Q_{\rho,\rho^{sp}})},$$

and

$$\|u-v\|_{L^{q',r'}(Q_{\rho,\rho^{sp}})} \le C\rho^{\xi+n}\|f\|^{\frac{1}{p-1}}_{L^{q,r}(Q_{\rho,\rho^{sp}})}.$$

Here in the case of $sp \neq n$,

$$\begin{aligned}\xi &= \frac{np'}{q'} - p'\frac{n-sp}{p} + sp - n + \frac{spp'}{r'} - \frac{spp'}{p} - sp \\ &= p'\left(\frac{n}{q'} - \frac{n}{p'} - \frac{n-sp}{p} + \frac{sp}{r'} - \frac{sp}{p}\right) \\ &= p'\left(\frac{n}{q'} - n + \frac{sp}{r'}\right) = p'\left(\frac{sp}{r'} - \frac{n}{q}\right) = spp'\left(1 - \frac{1}{r} - \frac{n}{spq}\right),\end{aligned}$$

and in the case $sp = n$,

$$\begin{aligned}\xi = p'\left(\frac{n}{q'} + \frac{sp}{r'} - \frac{sp}{p} - \frac{sp}{p'}\right) &= spp'\left(\frac{1}{q'} + \frac{1}{r'} - \frac{1}{p} - \frac{1}{p'}\right) \\ &= spp'\left(1 - \frac{1}{q} + 1 - \frac{1}{r} - 1\right) \\ &= spp'\left(1 - \frac{1}{r} - \frac{1}{q}\right).\end{aligned}$$

□

Remark 3.3. In Lemma 3.2, we assume the same conditions as in Proposition 3.1. These assumptions are used in the proof not only to ensure the existence of the (s, p)-caloric replacement but also to derive (3.11) and (3.13). As mentioned in Remark 2.13, one can expect the existence of the (s, p)-caloric replacement under a more general condition for the right-hand side. If such an existence theorem is available, one can expect the estimates in Lemma 3.2 to hold true for more general right-hand sides. In the proof of Lemma 3.2, we only used the diffusion term in (3.9), but the stronger estimate

$$\begin{aligned}&\sup_{t\in J} \|(u-v)(\bullet, t)\|^2_{L^2(B_\rho(x_0))} + \int_J [(u-v)(\bullet, t)]^p_{W^{s,p}(\mathbb{R}^n)}\, \mathrm{d}t \\ &\quad \leq C \int_J \int_{B_\rho(x_0)} |f(x,t)(u-v)(x,t)|\, \mathrm{d}x\, \mathrm{d}t,\end{aligned}$$

holds true. It might be possible to utilize an interpolation argument similar to Lemma 2.8 to replace the equations (3.11) and (3.13) and relax the assumptions on q and r. See also [12, Lemma 2.2]. However, the nonhomogeneity of the equation is for sure a challenge in pursuing this line of reasoning.

Next, we perform a Moser iteration to get an L^∞ bound for the difference between the solution and its (s, p)-caloric replacement.

Proposition 3.4. *Let u be a local weak solution of*

$$\partial_t u + (-\Delta_p)^s u = f, \quad in \quad Q_{\sigma,\sigma^{sp}}(x_0, T_0),$$

with $f \in L^{q,r}_{\mathrm{loc}}(Q_{\sigma,\sigma^{sp}}(x_0, T_0))$ *such that*

$$\frac{1}{r} + \frac{n}{spq} < 1.$$

In addition, assume that $u \in L^p_{\mathrm{loc}}\big((T_0 - \sigma^{sp}, T_0]; L^{p-1}_{sp}(\mathbb{R}^n)\big)$, $r \geq p'$,

$$q \geq 1 \quad \textit{if} \quad sp \neq n, \quad \textit{and} \quad q > 1 \quad \textit{if} \quad sp = n.$$

Let v *be the* (s, p)*-caloric replacement of* u *in* $Q_{R,R^{sp}}(x_0, T_0)$*, with* $R < \sigma$*. Then in the case of* $sp \neq n$*, we have*

$$\|(u - v)^+\|_{L^\infty(Q_{RR^{sp}}(x_0,T_0))} \leq C(n, s, p)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\Big(1 + R^{sp\nu}\|f\|_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))}\Big),$$

where $\nu = 1 - \frac{1}{r} - \frac{n}{spq}$ *and*

$$\vartheta = 1 + \frac{sp\nu}{n}.$$

In the case of $sp = n$*, given any* l *such that* $\frac{p}{r'}(1 - \frac{1}{r} - \frac{1}{q})^{-1} < l < \infty$ *we get*

$$\|(u-v)^+\|_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} \leq C(n, s, p, q, l)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\Big(1 + R^{sp\nu}\|f\|_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))}\Big),$$

where $\vartheta = 2 - \frac{1}{r} - \frac{1}{q} - \frac{p}{lr'}$ *and* $\nu = 1 - \frac{1}{r} - \frac{1}{q}$.

Proof. Throughout the proof, we write $Q_{R,R^{sp}}$ instead of $Q_{R,R^{s,p}}(x_0, T_0)$ and B_R instead of $B_R(x_0)$. We also define the interval J to be $J := (T_0 - R^{sp}, T_0]$. First, we verify that our assumptions ensure that the (s, p)-caloric replacement of u exists. If $sp \geq n$, we have explicitly assumed what is needed to use Proposition 3.1. If $sp < n$, we have to verify that $q \geq (p^\star_s)'$. This follows from the assumption $\frac{1}{r} + \frac{n}{spq} < 1$. Indeed

$$\frac{n}{sp} \leq q\left(1 - \frac{1}{r}\right) \leq q,$$

and it is straightforward to verify that $\frac{n}{sp} \geq (p^\star_s)'$. This shows that v, the (s, p)-caloric replacement of u exist. Let us also mention that the assumptions in Lemma 3.2 are the same as in Proposition 3.1, and we can use this lemma. Now, we test the equations with powers of $u - v$ and perform a Moser iteration. Using Proposition 6.1 with

$$F(t) = (\min\{t^+, M\} + \delta)^\beta - \delta^\beta,$$

and

$$\delta = \max\Big\{1, R^{sp\nu}\|f\|_{L^{q,r}(Q_{R,R^{sp}})}\Big\}, \tag{3.15}$$

we get

$$\sup_{t\in J}\int_{B_R}\mathcal{F}(u - v)\,\mathrm{d}x + \int_J\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{J_p(u(x,t) - u(y,t)) - J_p(v(x,t) - v(y,t))}{|x - y|^{n+sp}}$$

$$
\begin{aligned}
&\times(F(u(x,t)-v(x,t))-F(u(y,t)-v(y,t)))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\le \int_J\int_{B_R}|f(x,t)|F(u(x,t)-v(x,t))\\
&\le \|f\|_{L^{q,r}(B_R\times J)}\|((u-v)_M^++\delta)^\beta\|_{L^{q',r'}(B_R\times J)}.
\end{aligned}
\tag{3.16}
$$

In the last line, we have used Hölder's inequality. Here $\mathcal{F}(t)=\int_0^t F(t)\,\mathrm{d}t$ is

$$
\mathcal{F}(t)=\begin{cases}0 & \text{if}\quad t\le 0,\\ \frac{1}{\beta+1}(t+\delta)^{\beta+1}-\frac{\delta^{\beta+1}}{\beta+1}-t\delta^\beta & \text{if}\quad 0\le t\le M,\\ \frac{1}{\beta+1}(M+\delta)^{\beta+1}-\frac{\delta^{\beta+1}}{\beta+1}-t\delta^\beta+(t-M)(M+\delta)^\beta & \text{if}\quad t\ge M.\end{cases}
$$

Notice that by Young's inequality, for $t\ge 0$

$$
\frac{(t+\delta)^{\beta+1}}{2(\beta+1)}+\frac{\beta}{\beta+1}2\delta^{\beta+1}\ge\frac{t+\delta}{2^{\frac{1}{\beta+1}}}2^{\frac{\beta}{\beta+1}}\delta^\beta\ge t\delta^\beta .
$$

In particular, for $0\le t\le M$

$$
\mathcal{F}(t)\ge\frac{(t+\delta)^{\beta+1}}{2(\beta+1)}-\frac{2\beta+1}{\beta+1}\delta^{\beta+1}\ge\frac{(t+\delta)^{\beta+1}}{2(\beta+1)}-2\delta^{\beta+1},
$$

and for $t\ge M$

$$
\begin{aligned}
&\frac{1}{\beta+1}(M+\delta)^{\beta+1}-\frac{\delta^{\beta+1}}{\beta+1}-t\delta^\beta+(t-M)(M+\delta)^\beta\\
&=\frac{1}{\beta+1}(M+\delta)^{\beta+1}-\frac{\delta^{\beta+1}}{\beta+1}-M\delta^\beta+(t-M)\big((M+\delta)^\beta-\delta^\beta\big)\ge\mathcal{F}(M)\\
&\ge\frac{(M+\delta)^{\beta+1}}{2(\beta+1)}-2\delta^{\beta+1}.
\end{aligned}
$$

Hence,

$$
\mathcal{F}(t)\ge\frac{(t_M^++\delta)^{\beta+1}}{2(\beta+1)}-2\delta^{\beta+1}.
\tag{3.17}
$$

Using Lemma 2.2 for the second term in the left-hand side of (3.16) and (3.17) in the first term, we obtain

$$
\begin{aligned}
&\frac{1}{2(\beta+1)}\sup_{t\in J}\int_{B_R}((u-v)_M^++\delta)^{\beta+1}\,\mathrm{d}x\\
&\quad+\frac{1}{3\cdot 2^{p-1}}\frac{\beta p^p}{(\beta+p-1)^p}\int_J\left[((u-v)_M^++\delta)^{\frac{\beta+p-1}{p}}\right]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\\
&\le\sup_{t\in J}\int_{B_R}\mathcal{F}(u-v)\,\mathrm{d}x+2\delta^{\beta+1}|B_R|\\
&\quad+\int_J\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{J_p(u(x,t)-u(y,t))-J_p(v(x,t)-v(y,t))}{|x-y|^{n+sp}}
\end{aligned}
$$

$$\times\,(F(u(x,t)-v(x,t))-F(u(y,t)-v(y,t)))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$\le \|f\|_{L^{q,r}(B_R\times J)}\|((u-v)_M^+ +\delta)^\beta\|_{L^{q',r'}(B_R\times J)}+2\delta^{\beta+1}|B_R|. \tag{3.18}$$

Let $w(x,t)=((u-v)_M^+ +\delta)^{\frac{\beta}{p}}$. Since $\delta\le (u-v)_M^+ +\delta$, we see that

$$\delta^\beta \le \frac{\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}}{|B_R|^{1-\frac{1}{q}}|J|^{1-\frac{1}{r}}}. \tag{3.19}$$

Using (3.19) in (3.18), we get

$$\frac{\delta}{2(\beta+1)}\|w\|^p_{L^{p,\infty}(B_R\times J)}+\frac{1}{3\cdot 2^{p-1}}\frac{\beta p^p}{(\beta+p-1)^p}\int_J\left[w^{\frac{\beta+p-1}{\beta}}\right]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t$$
$$\le \|f\|_{L^{q,r}(B_R\times J)}\|w\|^p_{L^{pq',pr'}(B_R\times J)}+2\delta|B_R|\frac{\|w\|^p_{L^{pq',pr'}(B_R\times J)}}{|B_R|^{1-\frac{1}{q}}|J|^{1-\frac{1}{r}}}. \tag{3.20}$$

By (2.4), we have

$$\begin{aligned}
{}^{p}_{W^{s,p}(\mathbb{R}^n)} &= \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|w(x)^{\frac{\beta+p-1}{\beta}}-w(y)^{\frac{\beta+p-1}{\beta}}|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\ge \left(\frac{\beta+p-1}{\beta}\right)^p\min\{w(x),w(y)\}^{p(\frac{\beta+p-1}{\beta}-1)}\\
&\quad\times\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\ge \left(\frac{\beta+p-1}{\beta}\right)^p\delta^{p-1}[w]^p_{W^{s,p}(\mathbb{R}^n)}.
\end{aligned}$$

Using this in (3.20) and since J has length R^{sp}, we arrive at

$$\frac{\delta}{2(\beta+1)}\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}})}+\frac{1}{3\cdot 2^{p-1}}\frac{\delta^{p-1}p^p}{\beta^{p-1}}\int_J[w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t$$
$$\le \|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\left(\|f\|_{L^{q,r}(Q_{R,R^{sp}})}+\frac{2(n\omega_n)^{\frac{1}{q}}\delta R^n}{R^{n(1-\frac{1}{q})+sp(1-\frac{1}{r})}}\right).$$

Upon multiplying both sides by $\frac{3\cdot 2^{p-1}\cdot\beta^{p-1}}{\delta}$, this implies

$$3\cdot 2^{p-2}\frac{\beta^{p-1}}{\beta+1}\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}})}+\delta^{p-2}p^p\int_J[w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t$$
$$\le 3\cdot 2^{p-1}\times\frac{\beta^{p-1}}{\delta}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\big(\|f\|_{L^{q,r}(Q_{R,R^{sp}})}+2(n\omega_n)^{\frac{1}{q}}\delta R^{-sp\nu}\big). \tag{3.21}$$

Since $\delta \geq 1$ and $p \geq 2$, for $\beta \geq 1$ we have

$$\begin{aligned}
&\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}})} + \int_J [w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t \\
&\quad \leq 3\cdot 2^{p-2}\frac{\beta^{p-1}}{\beta+1}\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}})} + \delta^{p-2}p^p\int_J [w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t.
\end{aligned}$$

Using this in (3.21), we get

$$\begin{aligned}
&\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}})} + \int_J [w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t \\
&\leq 3\cdot 2^{p-1}\beta^{p-1}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\Big(\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta} + 2(n\omega_n)^{\frac{1}{q}}R^{-sp\nu}\Big) \\
&\leq C\beta^{p-1}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\Big(\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta} + R^{-sp\nu}\Big), \qquad (3.22)
\end{aligned}$$

where $C = C(n, p)$. Now we consider two cases depending on whether $sp \neq n$ or $sp = n$.

Case $sp \neq n$: Notice that since $\nu > 0$, if we take $\vartheta = 1 + \frac{sp\nu}{n}$, the exponents $(\vartheta r')'$, $(\vartheta q')'$ satisfy the condition of Lemma 2.8. Indeed,

$$\begin{aligned}
1 - \frac{1}{(\vartheta r')'} - \frac{n}{sp(\vartheta q')'} &= \frac{1}{\vartheta r'} + \frac{n}{sp\vartheta q'} - \frac{n}{sp} \\
&= \frac{1}{\vartheta}\left(\frac{1}{r'} + \frac{n}{spq'} - \frac{\vartheta n}{sp}\right) = \frac{1}{\vartheta}\left(\nu + \frac{n}{sp} - \frac{\vartheta n}{sp}\right) = 0.
\end{aligned}$$

As $w - \delta^{\frac{\beta}{p}}$ does vanish in $B_R(x_0)^c$, using Lemma 2.8 for the exponents $(\vartheta q')'$ and $(\vartheta r')'$ we get

$$\begin{aligned}
&\|w - \delta^{\frac{\beta}{p}}\|^p_{L^{\vartheta pq',\vartheta pr'}(Q_{R,R^{sp}})} \\
&\leq C(n,s,p)\left(\|w - \delta^{\frac{\beta}{p}}\|^p_{L^{p,\infty}(Q_{R,R^{sp}})} + \int_{T_0-R^{sp}}^{T_0} [w - \delta^{\frac{\beta}{p}}]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right) \\
&\leq C(n,s,p)\left(\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}})} + \int_{T_0-R^{sp}}^{T_0} [w]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t\right) \\
\text{by (3.22)}\quad &\leq C(n,s,p)\beta^{p-1}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\Big(\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta} + R^{-sp\nu}\Big). \qquad (3.23)
\end{aligned}$$

Here we have used that $w - \delta^{\frac{\beta}{p}}$ is nonnegative as well as the fact that $[w]_{W^{s,p}(\mathbb{R}^n)}$ does not change by subtracting a constant from w. Hence, by (3.19) and (3.23) we obtain

$$\begin{aligned}
&\|w^{\vartheta}\|^{\frac{p}{\vartheta}}_{L^{pq',pr'}(Q_{R,R^{sp}})} \\
&= \|w\|^p_{L^{\vartheta pq',\vartheta pr'}(Q_{R,R^{sp}})} \leq \left(\|w - \delta^{\frac{\beta}{p}}\|_{L^{\vartheta pq',\vartheta pr'}(Q_{R,R^{sp}})} + \delta^{\frac{\beta}{p}}R^{\frac{n}{\vartheta pq'}+\frac{sp}{\vartheta pr'}}\right)^p
\end{aligned}$$

$$
\begin{aligned}
&\le 2^{p-1}\left(\|w-\delta^{\frac{\beta}{p}}\|^p_{L^{\vartheta pq',\vartheta pr'}(Q_{R,R^{sp}})}+\delta^{\beta}R^{\frac{n}{\vartheta q'}+\frac{sp}{\vartheta r'}}\right)\\
&\le C\beta^{p-1}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\left(\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta}+R^{-sp\nu}\right)\\
&\quad +2^{p-1}\delta^{\beta}R^{\frac{n}{\vartheta q'}+\frac{sp}{\vartheta r'}}\\
&\le C\beta^{p-1}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\left(\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta}+R^{-sp\nu}\right)\\
&\quad +2^{p-1}(n\omega_n)^{\frac{1}{q}-1}\frac{\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}}{R^{\frac{n}{q'}+\frac{sp}{r'}-\frac{n}{\vartheta q'}-\frac{sp}{\vartheta r'}}}\\
&\le C\beta^{p-1}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\left(\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta}+R^{-sp\nu}+R^{(\frac{1}{\vartheta}-1)(sp\nu+n)}\right).
\end{aligned}
\tag{3.24}
$$

Observe that $(\frac{1}{\vartheta}-1)(sp\nu+n)=-\frac{\vartheta-1}{\vartheta}(sp\nu+n)=-sp\nu$. Furthermore, recalling the definition of δ (3.15) whenever $\delta>1$ we have

$$
\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta}=R^{-sp\nu}.
$$

When $\delta=1$, it is straightforward to verify that

$$
\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta}\le R^{-sp\nu}.
$$

Inserting these into (3.24), we arrive at

$$
\|w^{\vartheta}\|^{\frac{p}{\vartheta}}_{L^{pq',pr'}(Q_{R,R^{sp}})}\le C\beta^{p-1}R^{-sp\nu}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}. \tag{3.25}
$$

Now we iterate this inequality with the following choice of exponents

$$
\beta_0=1,\qquad \beta_{m+1}=\vartheta\beta_m=\vartheta^{m+1}.
$$

With the notation

$$
\varphi_m:=\|((u-v)^+_M+\delta)^{\frac{\beta_m}{p}}\|^{\frac{p}{\beta_m}}_{L^{pq',pr'}(Q_{R,R^{sp}})}=\|(u-v)^+_M+\delta\|_{L^{\beta_m q',\beta_m r'}(Q_{R,R^{sp}})},
$$

(3.25) reads

$$
\varphi_{m+1}\le\left(C\,R^{-sp\nu}\right)^{\frac{1}{\vartheta^m}}\vartheta^{\frac{(p-1)m}{\vartheta^m}}\varphi_m.
$$

Iterating this yields

$$
\varphi_{m+1}\le\left(C\,R^{-sp\nu}\right)^{\sum_{j=0}^{m}\vartheta^{-j}}\vartheta^{(p-1)\sum_{j=0}^{m}j\vartheta^{-j}}\varphi_0. \tag{3.26}
$$

Since $\vartheta>1$, we have the following convergent series

$$
\sum_{j=0}^{\infty}\vartheta^{-j}=\frac{\vartheta}{\vartheta-1}=\frac{n+sp\nu}{sp\nu},
$$

and

$$\sum_{j=0}^{\infty} j\vartheta^{-j} = \frac{\vartheta}{(\vartheta-1)^2} = \frac{n^2+nspv}{s^2p^2v^2}.$$

By (3.8) in Lemma 3.2,

$$\begin{aligned}\varphi_0 &= \|(u-v)_M^+ + \delta\|_{L^{q',r'}(Q_{R,R^{sp}})} \le C(n,s,p)R^{spp'v+n}\|f\|_{L^{q,r}(Q_{R,R^{sp}})}^{\frac{1}{p-1}} + \delta(n\omega_n)^{\frac{1}{q'}} R^{\frac{n}{q'}+\frac{sp}{r'}} \\ &= C(n,s,p)R^{n+spv}(R^{\frac{spv}{p-1}}\|f\|_{L^{q,r}(Q_{R,R^{sp}})}^{\frac{1}{p-1}} + \delta) \le C(n,s,p)R^{n+spv}\left(\delta^{\frac{1}{p-1}} + \delta\right) \\ &\le C(n,s,p)R^{n+spv}\delta. \end{aligned} \tag{3.27}$$

In the last line, we have used that $p-1 \ge 1$ and $\delta \ge 1$. Inserting (3.27) to (3.26) and sending m to infinity, we obtain

$$\begin{aligned}\|(u-v)_M^+ + \delta\|_{L^\infty(Q_{R,R^{sp}})} &\le C\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} R^{-n-spv} R^{spv+n}\delta \\ &= C\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\delta \\ &\le C\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} \max\{1, R^{spv}\|f\|_{L^{q,r}(Q_{R,R^{sp}})}\} \\ &\le C(n,s,p)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\left(1 + R^{spv}\|f\|_{L^{q,r}(Q_{R,R^{sp}})}\right).\end{aligned}$$

Since the above estimate is independent of M, we get

$$\|(u-v)^+\|_{L^\infty(Q_{R,R^{sp}})} \le C(n,s,p)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\left(1 + R^{spv}\|f\|_{L^{q,r}(Q_{R,R^{sp}})}\right),$$

which is the desired result.

Case sp=n. Here we use the critical case of Sobolev–Morrey inequality, (2.8) with

$$\max\left\{\frac{p}{r'}\left(1-\frac{1}{r}-\frac{1}{q}\right)^{-1}, q'\right\} < l < \infty. \tag{3.28}$$

This applied for the second term in the left-hand side of (3.22) implies

$$\begin{aligned}&\|w\|_{L^{p,\infty}(Q_{R,R^{sp}})}^p + \left(C(n,s,p,l)R^{\frac{np}{l}}\right)^{-1}\left[\|w\|_{L^{l,p}(Q_{R,R^{sp}}))}^p - \delta^\beta|B_R|^{\frac{p}{l}}|J|\right] \\ &\le \|w\|_{L^{p,\infty}(Q_{R,R^{sp}}))}^p + \left(C(n,s,p,l)R^{\frac{np}{l}}\right)^{-1}\int_J \|w-\delta^{\frac{\beta}{p}}\|_{L^l(B_R)}^p \\ &\le \|w\|_{L^{p,\infty}(Q_{R,R^{sp}})}^p + \int_J [w-\delta^{\frac{\beta}{p}}]_{W^{s,p}(\mathbb{R}^n)}^p\,\mathrm{d}t \\ &= \|w\|_{L^{p,\infty}(Q_{R,R^{sp}})}^p + \int_J [w]_{W^{s,p}(\mathbb{R}^n)}^p\,\mathrm{d}t \\ &\le C(n,p)\beta^{p-1}\|w\|_{L^{pq',pr'}(Q_{R,R^{sp}})}^p\left(\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta} + R^{-spv}\right).\end{aligned}$$

We replace the constant $C(n,s,p,l)$ with $\max\{1, C(n,s,p,l)\}$, and multiply both sides with it to arrive at

$$\begin{aligned}
&\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}})} + R^{\frac{-np}{l}}\|w\|^p_{L^{l,p}(Q_{R,R^{sp}})} \\
&\leq C(n,s,p,l)\beta^{p-1}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\Big(\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta}+R^{-sp\nu}\Big)+C(n,p,l)\delta^\beta|J| \\
\text{(using } \beta\geq 1)\ &\leq C(n,s,p,l)\beta^{p-1}\left(\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\Big(\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta}+R^{-sp\nu}\Big)+\delta^\beta R^{sp}\right) \\
\text{using (3.19)}\ &\leq C(n,s,p,l)\beta^{p-1}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}\left(\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta}+R^{-sp\nu}+\frac{R^{sp}}{R^{\frac{n}{q'}+\frac{sp}{r'}}}\right) \\
&\leq C(n,s,p,l)\beta^{p-1}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}R^{-sp\nu}.
\end{aligned} \tag{3.29}$$

In the last line, we have used that since $sp = n$ we have

$$sp - \frac{n}{q'} - \frac{sp}{r'} = sp - n\left(1-\frac{1}{q}\right) - sp\left(1-\frac{1}{r}\right) = sp\left(\frac{1}{r}+\frac{1}{q}-1\right) = -sp\nu.$$

We have also used the following inequality which we have discussed in the case $sp \neq n$:

$$\frac{\|f\|_{L^{q,r}(Q_{R,R^{sp}})}}{\delta} \leq R^{-sp\nu}.$$

Now we choose $\vartheta = 2 - \frac{1}{r} - \frac{1}{q} - \frac{p}{lr'}$. Notice that due to the choice of l, (3.28), we have $\vartheta > 1$. Then the exponents $(\vartheta r')'$ and $(\vartheta q')'$ satisfy

$$1 - \frac{1}{(\vartheta r')'} - \frac{1}{(\vartheta q')'} = \frac{p}{l\vartheta r'}.$$

Therefore, we can apply Lemma 2.9 with the exponents $(\vartheta r')'$ and $(\vartheta q')'$ to (3.29) to arrive at

$$\begin{aligned}
\|w^\vartheta\|^{\frac{p}{\vartheta}}_{L^{pq',pr'}(Q_{R,R^{sp}})} &= \|w\|^p_{L^{p\vartheta q',p\vartheta r'}(Q_{R,R^{sp}})} \leq R^{sp(1-\frac{1}{(\vartheta r')'}-\frac{1}{(\vartheta q')'})} \\
&\quad\times\left(\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}})} + R^{\frac{-np}{l}}\|w\|^p_{L^{l,p}(Q_{R,R^{sp}})}\right) \\
&= R^{\frac{np}{l\vartheta r'}}\left(\|w\|^p_{L^{p,\infty}(Q_{R,R^{sp}})} + R^{\frac{-np}{l}}\|w\|^p_{L^{\infty,p}(Q_{R,R^{sp}})}\right) \\
&\leq C(n,s,p,l)\beta^{p-1}R^{\frac{np}{l\vartheta r'}-sp\nu}\|w\|^p_{L^{pq',pr'}(Q_{R,R^{sp}})}.
\end{aligned} \tag{3.30}$$

We apply (3.30) with the exponents

$$\beta_0 = 1, \qquad \beta_{m+1} = \vartheta\beta_m = \vartheta^{m+1}.$$

Let

$$\varphi_m := \|((u-v)^+_M+\delta)^{\frac{\beta_m}{p}}\|^{\frac{p}{\beta_m}}_{L^{pq',pr'}(Q_{R,R^{sp}})} = \|(u-v)^+_M+\delta\|_{L^{\beta_m q',\beta_m r'}(Q_{R,R^{sp}})}.$$

Then (3.30) reads

$$\varphi_{m+1} \leq \left(C\, R^{\frac{np}{l\vartheta r'} - spv}\right)^{\frac{1}{\vartheta^m}} \theta^{\frac{(p-1)m}{\vartheta^m}} \varphi_m.$$

By iterating the above inequality, we get

$$\varphi_{m+1} \leq \left(C\, R^{\frac{np}{l\vartheta r'} - spv}\right)^{\sum_{j=0}^{m} \vartheta^{-j}} \vartheta^{(p-1)\sum_{j=0}^{m} j\vartheta^{-j}} \varphi_0. \tag{3.31}$$

Since $\vartheta > 1$, we have the following convergent series

$$\sum_{j=0}^{\infty} \vartheta^{-j} = \frac{\vartheta}{\vartheta - 1}$$

and

$$\sum_{j=0}^{\infty} j\vartheta^{-j} = \frac{\vartheta}{(\vartheta - 1)^2}.$$

By (3.8) in Lemma 3.2, we obtain

$$\begin{aligned}\varphi_0 = \|(u-v)_M^+\|_{L^{q',r'}(Q_{R,R^{sp}})} &\leq C(n,s,p,q) R^{spp'v+n} \|f\|_{L^{q,r}(Q_{R,R^{sp}})}^{\frac{1}{p-1}} + \delta R^{\frac{n}{q'} + \frac{sp}{r'}} \\ &\leq C(n,s,p,q) R^{n+spv}\delta.\end{aligned}$$

Inserting this into (3.30), and sending m to infinity, we get

$$\begin{aligned}\|(u-v)_M^+ + \delta\|_{L^\infty(Q_{R,R^{sp}})} &\leq C(n,s,p,q,l)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} R^{\frac{\vartheta}{\vartheta-1}\left(\frac{np}{l\vartheta r'} - spv\right)} R^{n+spv}\delta \\ &= C(n,s,p,q,l)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} R^{\frac{np}{(\vartheta-1)lr'} - \frac{spv}{\vartheta-1} + n}\delta \\ &= C(n,s,p,q,l)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} R^{\frac{n}{(\vartheta-1)}(\vartheta - 1 + \frac{p}{lr'}) - \frac{spv}{\vartheta-1}}\delta \\ &= C(n,s,p,q,l)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} R^{\frac{nv}{(\vartheta-1)} - \frac{spv}{\vartheta-1}}\delta \\ &\leq C(n,s,p,q,l)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} \left(1 + R^{spv}\|f\|_{L^{q,r}(Q_{R,R^{sp}})}\right).\end{aligned}$$

Hence, we arrive at the desired estimate

$$\|(u-v)^+\|_{L^\infty(Q_{R,R^{sp}})} \leq C(n,s,p,q,l)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} \left(1 + R^{spv}\|f\|_{L^{q,r}(Q_{R,R^{sp}})}\right).$$

□

Notice that $-u$ is a solution to the same type of problem, and we can apply the above proposition to $-u$. Since $-v$ is the (s,p)-caloric replacement of $-u$, we get the same bound on $\|(-u+v)^+\|_{L^\infty(Q_{R,R^{sp}})}$; as a result, we get a bound on the $\|u-v\|_{L^\infty(Q_{R,R^{sp}})}$.

Corollary 3.5. *Let u be a solution of $\partial_t u + (-\Delta_p)^s u = f$ in $Q_{\sigma,\sigma^{sp}}(x_0, T_0)$ with $f \in L^{q,r}_{\mathrm{loc}}(Q_{\sigma,\sigma^{sp}}(x_0, T_0))$ such that*

$$\frac{1}{r} + \frac{n}{spq} < 1.$$

In addition, assume that $r \geq p'$,

$$q \geq 1 \quad \text{if } sp \neq n, \quad \text{and} \quad q > 1 \quad \text{if } sp = n.$$

Let v be the (s, p)-caloric replacement of u in $Q_{R,R^{sp}}(x_0, T_0)$, with $R < \sigma$.
If $sp \neq n$, then

$$\|u - v\|_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} \leq C(n, s, p)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} \left(1 + R^{sp\nu} \|f\|_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))}\right),$$

where $\nu = 1 - \frac{1}{r} - \frac{n}{spq}$ and $\vartheta = 1 + \frac{sp\nu}{n}$.
If $sp = n$, then for any l such that $\frac{p}{r'}(1 - \frac{1}{r} - \frac{1}{q})^{-1} < l < \infty$, we have

$$\|u - v\|_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} \leq C(n, s, p, q, l)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} \left(1 + R^{sp\nu} \|f\|_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))}\right),$$

where $\vartheta = 2 - \frac{1}{r} - \frac{1}{q} - \frac{p}{lr'}$ and $\nu = 1 - \frac{1}{r} - \frac{1}{q}$.

Now we combine the local boundedness results for the equations with zero right-hand side (see [45] and also [22]) with Proposition 3.4 to prove local boundedness for the equation with nonzero right-hand side.

By [10, Theorem 1.1] with $q = p$ and $\sigma = \frac{1}{2}$, we have

$$\|v\|_{L^\infty(Q_{\frac{R}{2},(\frac{R}{2})^{sp}}(x_0,T_0))} \leq C\left[1 + \left(\fint_{Q_{R,R^{sp}}(x_0,T_0)} |v|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} + \left(2^{sp}\fint_{T_0-R^{sp}}^{T_0} \mathrm{Tail}_{p-1,sp}(v(\cdot,t); x_0, \frac{R}{2})^p\right)^{\frac{\beta}{(\beta-1)p^2}}\right],$$

where $\beta = \frac{2s+3n-\frac{2n}{p}}{n+s}$ and C depends on n, s and p. By Hölder's inequality, we have

$$\left(\fint_{Q_{R,R^{sp}}(x_0,T_0)} |v|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} \leq \left(\fint_{Q_{R,R^{sp}}(x_0,T_0)} |v|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p^2}}$$

As $p \geq 2$, we have $\frac{2}{p^2} \leq \frac{1}{p}$ and $\frac{\beta}{(\beta-1)p^2} \leq \frac{1}{p}$. Hence, we arrive at

$$\|v\|_{L^\infty(Q_{\frac{R}{2},(\frac{R}{2})^{sp}}(x_0,T_0))} \leq C\left[1 + \left(\fint_{Q_{R,R^{sp}}(x_0,T_0)} |v|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} + \left(\fint_{T_0-R^{sp}}^{T_0} \mathrm{Tail}_{p-1,sp}\left(v(\cdot,t); x_0, \frac{R}{2}\right)^p\right)^{\frac{1}{p}}\right]. \tag{3.32}$$

Proof of Theorem 1.1. For u, a local weak solution of

$$\partial_t u + (-\Delta_p)^s u = f(x,t), \qquad in\ Q_{2R,(2R)^{sp}}(x_0, T_0),$$

we consider v to be the (s, p)-caloric replacement in $Q_{R,R^{sp}}(x_0, T_0)$,

$$\begin{cases} v_t + (-\Delta_p)^s v = 0 & in\ \ Q_{R,R^{sp}}(x_0, T_0), \\ v = u & in\ \ (\mathbb{R}^n \setminus B_R(x_0)) \times [T_0 - R^{sp}, T_0], \\ v(x, T_0 - R^{sp}) = u(x, T_0 - R^{sp}) & in\ \ B_R(x_0). \end{cases}$$

As mentioned in the proof of Proposition 3.5, our assumptions ensure that we can use Proposition 3.1 and v exists. Using (3.32), we arrive at

$$\begin{aligned}
&\|u\|_{L^\infty(Q_{\frac{R}{2},(\frac{R}{2})^{sp}}(x_0,T_0))} \le \|u-v\|_{L^\infty(Q_{\frac{R}{2},(\frac{R}{2})^{sp}}(x_0,T_0))} + \|v\|_{L^\infty(Q_{\frac{R}{2},(\frac{R}{2})^{sp}}(x_0,T_0))} \\
&\le C\left[1 + \left(⨍_{Q_{R,R^{sp}}(x_0,T_0)} |v|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} + \left(⨍_{T_0-R^{sp}}^{T_0} \mathrm{Tail}_{p-1,sp}\left(v(\bullet,t); x_0, \frac{R}{2}\right)^p \,\mathrm{d}t\right)^{\frac{1}{p}}\right] \\
&\quad + \|u-v\|_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} \\
&\le C\Bigg[1 + \left(2^{p-1} ⨍_{Q_{R,R^{sp}}(x_0,T_0)} |u|^p \,\mathrm{d}x\,\mathrm{d}t + 2^{p-1} ⨍_{Q_{R,R^{sp}}(x_0,T_0)} |u-v|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\quad + \left(⨍_{T_0-R^{sp}}^{T_0} \mathrm{Tail}_{p-1,sp}\left(v(\bullet,t); x_0, \frac{R}{2}\right)^p \,\mathrm{d}t\right)^{\frac{1}{p}}\Bigg] \\
&\quad + \|u-v\|_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} \\
&\le C\Bigg[1 + \left(2^{p-1} ⨍_{Q_{R,R^{sp}}(x_0,T_0)} |u|^p \,\mathrm{d}x\,\mathrm{d}t + 2^{p-1} \|u-v\|^p_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))}\right)^{\frac{1}{p}} \\
&\quad + \left(⨍_{T_0-R^{sp}}^{T_0} \mathrm{Tail}_{p-1,sp}\left(v(\bullet,t); x_0, \frac{R}{2}\right)^p \,\mathrm{d}t\right)^{\frac{1}{p}}\Bigg] \\
&\quad + \|u-v\|_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} \\
&\le C\Bigg[1 + \left(⨍_{Q_{R,R^{sp}}(x_0,T_0)} |u|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} + \left(⨍_{T_0-R^{sp}}^{T_0} \mathrm{Tail}_{p-1,sp}\left(v(\bullet,t); x_0, \frac{R}{2}\right)^p \,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\quad + \|u-v\|_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))}\Bigg].
\end{aligned} \tag{3.33}$$

Using Lemma 2.3 in (3.33), we arrive at

$$\begin{aligned}
&\|u\|_{L^\infty(Q_{\frac{R}{2},(\frac{R}{2})^{sp}})} \\
&\le C\Bigg[1 + \left(⨍_{Q_{R,R^{sp}}(x_0,T_0)} |u|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} + \|(u-v)\|_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} \\
&\quad + \Bigg(⨍_{T_0-R^{sp}}^{T_0} \Bigg(2\mathrm{Tail}_{p-1,sp}\left(u(\bullet,t); x_0, \frac{R}{2}\right)
\end{aligned}$$

$$+2^{1+\frac{n}{p-1}}\left(\fint_{B_R(x_0)}|u-v|^{p-1}\,\mathrm{d}x\right)^{\frac{1}{p-1}}\Bigg)^p\,\mathrm{d}t\Bigg)^{\frac{1}{p}}\Bigg]$$

$$\leq C\Bigg[1+\left(\fint_{Q_{R,R^{sp}}(x_0,T_0)}|u|^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}+\|(u-v)\|_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))}$$

$$+\left(\fint_{T_0-R^{sp}}^{T_0}2^p\mathrm{Tail}_{p-1,sp}\left(u(\bullet,t);x_0,\frac{R}{2}\right)^p\,\mathrm{d}t+2^{p+\frac{n}{p-1}}\|u-v\|^p_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))}\right)^{\frac{1}{p}}\Bigg]$$

$$\leq C\Bigg[1+\left(\fint_{Q_{R,R^{sp}}(x_0,T_0)}|u|^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}+\left(\fint_{T_0-R^{sp}}^{T_0}\mathrm{Tail}_{p-1,sp}\left(u(\bullet,t);x_0,\frac{R}{2}\right)^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$

$$+\|u-v\|_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))}\Bigg], \tag{3.34}$$

where $C=C(n,s,p)$. Finally, using Proposition 3.4 to estimate the term $\|u-v\|_{L^\infty(Q_{R,R^{sp}})}$, in (3.34) we get the desired result. Here the estimate is written in the case $sp\neq n$

$$\|u\|_{L^\infty(Q_{\frac{R}{2},(\frac{R}{2})^{sp}}(x_0,T_0))}\leq C\Bigg[1+\left(\fint_{Q_{R,R^{sp}}(x_0,T_0)}|u|^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}$$

$$+\left(\fint_{T_0-R^{sp}}^{T_0}\mathrm{Tail}_{p-1,sp}\left(u(\bullet,t);x_0,\frac{R}{2}\right)^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$

$$+\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\left(1+R^{sp\nu}\|f\|_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))}\right)\Bigg].$$

□

Theorem 3.6. *Let $f\in L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))$ with*

$$\frac{1}{r}+\frac{n}{spq}<1.$$

In addition, assume that $r\geq p'$,

$$q\geq 1\quad\text{if}\quad sp\neq n,\quad\text{and}\quad q>1\quad\text{if}\quad sp=n.$$

If u is a weak solution of the equation

$$\partial_t u+(-\Delta_p)^s u=f\qquad\text{in } Q_{R_1,R_1^{sp}}(z,T_1),$$

such that

$$u\in L^p(I;W^{s,p}(B_{R_1}(z)))\cap C(I;L^2(B_{R_1}(z)))\cap L^\infty(I;L^{p-1}_{sp}(\mathbb{R}^n))$$

$$\cap L^\infty(Q_{R_1,R_1^{sp}}(z,T_1)),$$

then u is locally Hölder continuous in time and space. In particular, there exists a $\zeta > 0$, such that for $\sigma < 1$, $(x_1,t_1),\ (x_2,t_2) \in Q_{\sigma R_1,(\sigma R_1)^{sp}}(z,T_1)$, there holds

$$|u(x_1,t_1) - u(x_2,t_2)| \le C\mathcal{M}(|x_1-x_2|^\zeta + |t_1-t_2|^{\frac{\zeta}{sp}}),$$

with C depending on n, s, p and σ, and

$$\mathcal{M} := \Bigg[1 + \|u\|_{L^\infty(Q_{R_1,R_1^{sp}}(z,T_1))} + \vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\big(1 + d^{sp\nu}\|f\|_{L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))}\big) + \sup_{T_1-R_1^{sp}<t\le T_1} \mathrm{Tail}_{p-1,sp}(u(\bullet,t);z,R_1)\Bigg]^{p-1} + \min\{1,d\}^{-1}\|u\|_{L^\infty(Q_{R_1,R_1^{sp}}(z,T_1))} + \|f\|^{\frac{1}{p-1}}_{L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))}.$$

Proof. Take a cylinder $Q_{\sigma R_1,(\sigma R_1)^{sp}}(z,T_1) \subset Q_{R_1,R_1^{sp}}(z,T_1)$ and let $d := \min\{R_1(1-\sigma), R_1(1-\sigma^{sp})^{\frac{1}{sp}}\} > 0$. For any point, $(x_0,T_0) \in Q_{\sigma R_1,(\sigma R_1)^{sp}}(z,T_1)$ consider the (s,p)-caloric replacement of u in the cylinder $Q_{R,R^{sp}}(x_0,T_0)$ with $R \le \min\{1,d\}$. The choice of d implies that $Q_{R,R^{sp}}(x_0,T_0) \subset Q_{R_1,R_1^{sp}}(z,t)$. First, we observe that:

$$\begin{aligned}
⨍_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |u - \bar{u}_{(x_0,T_0),\rho}|^p \,\mathrm{d}x\,\mathrm{d}t &\le C(p) ⨍_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |u-v|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C(p) ⨍_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |\bar{u}_{(x_0,T_0),\rho} - \bar{v}_{(x_0,T_0),\rho}|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C(p) ⨍_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |v - \bar{v}_{(x_0,T_0),\rho}|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\le 2C(p) ⨍_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |u-v|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C(p) ⨍_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |v - \bar{v}_{(x_0,T_0),\rho}|^p \,\mathrm{d}x\,\mathrm{d}t. \qquad (3.35)
\end{aligned}$$

For $\rho \le \frac{R}{2}$, v is Hölder continuous in $Q_{\rho,\rho^{sp}}(x_0,T_0)$ by Theorem 5.1, and by the mean value theorem, there is a point $(\tilde{x}_0,\tilde{t}_0) \in Q_{\rho,\rho^{sp}}$ such that $\bar{v}_{x_0,t_0} = v(\tilde{x}_0,\tilde{t}_0)$. With the notation

$$\mathcal{M} := 1 + \|v\|_{L^\infty(Q_{R,(R)^{sp}})} + \sup_{T_0-R^{sp}<t\le T_0} \mathrm{Tail}_{p-1,sp}(v(\bullet,t);x_0,R),$$

Theorem 5.1 implies:

$$\begin{aligned}
|v(x,t) - \bar{v}_{(x_0,t_0),\rho}| &\le C\Big(\mathcal{M}\Big(\frac{x-\tilde{x}_0}{R}\Big)^{\frac{\Theta}{2}} + \mathcal{M}^{p-1}\Big(\frac{t-\tilde{t}_0}{R^{sp}}\Big)^{\frac{\Gamma}{2}}\Big) \\
&\le C\mathcal{M}^{p-1}\Big(\Big(\frac{2\rho}{R}\Big)^{\frac{\Theta}{2}} + \Big(\Big(\frac{\rho}{R}\Big)^{sp}\Big)^{\frac{\Gamma}{2}}\Big), \quad \text{for } (x,t) \in Q_{\rho,\rho^{sp}}(x_0,T_0)
\end{aligned}$$

with $C = C(n, s, p)$. Therefore,

$$
\begin{aligned}
&\fint_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |v - \bar{v}_{(x_0,t_0),\rho}|^p \, dx \, dt \\
&\qquad \le C\mathcal{M}^{p(p-1)} \fint_{Q_{\rho,\rho^{sp}}(x_0,T_0)} \Big(\frac{2\rho}{R}\Big)^{\frac{\Theta p}{2}} + \Big(\Big(\frac{\rho}{R}\Big)^{sp}\Big)^{\frac{\Gamma p}{2}} \, dx \, dt \\
&\qquad \le C\mathcal{M}^{p(p-1)} \Big(\Big(\frac{\rho}{R}\Big)^{\frac{\Theta p}{2}} + \Big(\frac{\rho}{R}\Big)^{\frac{\Gamma p}{2}}\Big) \\
&\qquad \le C\Big(\frac{\rho}{R}\Big)^{\delta p} \Big[1 + \|v\|^p_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} \\
&\qquad\quad + \sup_{T_0 - R^{sp} < t \le T_0} \mathrm{Tail}_{p-1,sp}(v(\bullet, t); x_0, R)^p\Big]^{p-1},
\end{aligned} \tag{3.36}
$$

where the constants C depends on n, s and p, and we have defined $\delta := \min\left\{\frac{\Theta}{2}, \frac{\Gamma}{2}\right\}$.

Moreover, by Lemma 3.2

$$
\begin{aligned}
\fint_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |u - v|^p \, dx \, dt &\le \Big(\frac{R}{\rho}\Big)^n \frac{R^{sp}}{\rho^{sp}} \fint_{Q_{R,R^{sp}}} |u - v|^p \, dx \, dt \\
&\le C(n, s, p)\Big(\frac{R}{\rho}\Big)^{n+sp} R^\xi \, \|f\|^{p'}_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))},
\end{aligned} \tag{3.37}
$$

where ξ is defined in Lemma 3.2. Notice that $\xi > 0$ by our assumptions on q and r. Inserting (3.37) and (3.36) in (3.35), we arrive at

$$
\begin{aligned}
&\fint_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |u - \bar{u}_{(x_0,T_0),\rho}|^p \, dx \, dt \le C(n, s, p)\Big(\frac{R}{\rho}\Big)^{n+sp} R^\xi \, \|f\|^{p'}_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))} \\
&\quad + C(n, s, p)\Big(\frac{\rho}{R}\Big)^{\delta p}\Big(1 + \|v\|^p_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} + \sup_{T_0 - R^{sp} \le t \le T_0} \mathrm{Tail}_{p-1,sp}(u(\bullet, t); x_0, R)^p\Big)^{p-1} \\
&\le C(n, s, p)\Big(\frac{R}{\rho}\Big)^{n+sp} R^\xi \, \|f\|^{p'}_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))} + C(n, s, p)\Big(\frac{\rho}{R}\Big)^{\delta p}\Big(\|u\|^p_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} \\
&\quad + \|u - v\|^p_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} + \sup_{T_0 - R^{sp} \le t \le T_0} \mathrm{Tail}_{p-1,sp}(u(\bullet, t); x_0, R)^p\Big)^{p-1}.
\end{aligned}
$$

Using Corollary 3.5, we get:

$$
\begin{aligned}
&\fint_{Q_{\rho,\rho^{sp}}(x_0,T_0)} |u - \bar{u}_{(x_0,T_0),r}|^p \, dx \, dt \le C(n, s, p) \left(\frac{R}{\rho}\right)^{n+sp} R^\xi \|f\|^{p'}_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))} \\
&\quad + C(n, s, p)\left(\frac{\rho}{R}\right)^{\delta p} \Big[\|u\|^p_{L^\infty(Q_{R,R^{sp}}(x_0,T_0))} + \sup_{T_0 - R^{sp} \le t \le T_0} \mathrm{Tail}_{p-1,sp}(u(\bullet, t); x_0, R)^p \\
&\quad + C(n, s, p)\Big(\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} \big(1 + R^{sp\nu} \|f\|_{L^{q,r}(Q_{R,R^{sp}}(x_0,T_0))}\big)\Big)^p\Big]^{p-1},
\end{aligned}
$$

with ϑ and ν defined in Corollary 3.5; here, the estimate is only written in the case $sp \neq n$ for simplicity. Since $Q_{R,R^{sp}}(x_0, T_0) \subset Q_{R_1,R_1^{sp}}(z, T_1)$, the above expression is less than

$$
\begin{aligned}
&\leq C(n,s,p)\left(\frac{R}{\rho}\right)^{n+sp} R^{\xi}\|f\|^{p'}_{L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))}\\
&\quad + C(n,s,p)\left(\frac{\rho}{R}\right)^{\delta p}\Big[1+\|u\|^{p}_{L^{\infty}(Q_{R_1,R_1^{sp}}(z,T_1))}\\
&\quad + \sup_{T_0-R^{sp}<t\leq T_0} \mathrm{Tail}_{p-1,sp}(u(\bullet,t);x_0,R)^{p}\\
&\quad + \left(\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\left(1+d^{sp\nu}\|f\|_{L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))}\right)\right)^{p}\Big]^{p-1}.
\end{aligned}
$$

Concerning the tail term, since $B_R(x_0)\subset B_{R_1}(z)$, using Lemma 2.4 we have

$$
\begin{aligned}
&\mathrm{Tail}_{p-1,sp}(u(\bullet,t);x_0,R)^{p-1}\\
&\leq\left(\frac{R}{R_1}\right)^{sp}\left(\frac{R_1}{R_1-|x_0-z|}\right)^{n+s\,p}\mathrm{Tail}_{p-1,sp}(u(\bullet,t);z,R_1)^{p-1}+\|u(\bullet,t)\|^{p-1}_{L^{\infty}(B_{R_1}(z))},
\end{aligned}
\tag{3.38}
$$

and by the choice of the radii, we have

$$
\frac{R}{R_1}<\frac{d}{R_1}<1-\sigma \quad \textit{and} \quad \frac{R_1}{R_1-|x_0-z|}\leq\frac{R_1}{R_1-\sigma R_1}\leq\frac{1}{1-\sigma}.
$$

Hence, taking the supremum in time and using Minkowski's inequality in (3.38), we arrive at

$$
\begin{aligned}
&\sup_{T_0-R^{sp}<t\leq T_0} \mathrm{Tail}_{p-1,sp}(u;x_0,R)^{p}\\
&\leq C\frac{1}{(1-\sigma)^n}\left(\sup_{T_0-R^{sp}<t\leq T_0}\mathrm{Tail}_{p-1,sp}(u(\bullet,t);z,R_1)^{p}+\|u\|^{p}_{L^{\infty}([T_0-R^{sp},T_0]\times B_{R_1}(z))}\right)\\
&\leq C\frac{1}{(1-\sigma)^n}\left(\|u\|^{p}_{L^{\infty}(Q_{R_1,R_1^{sp}}(z,T_1))}+\sup_{T_1-R_1^{sp}<t\leq T_1}\mathrm{Tail}_{p-1,sp}(u(\bullet,t);z,R_1)^{p}\right),
\end{aligned}
$$

where the above constant C depends on n, $s$and p. In conclusion,

$$
\begin{aligned}
&⨍_{Q_{\rho,\rho^{s\,p}}(x_0,T_0)}|u-\bar{u}_{(x_0,T_0),\rho}|^{p}\,\mathrm{d}x\,\mathrm{d}t\leq C(n,s,p)\left(\frac{R}{\rho}\right)^{n+s\,p}R^{\xi}\|f\|^{p'}_{L^{q,r}(Q_{R_1,R_1^{s\,p}}(z,T_1))}\\
&\quad + C(n,s,p,\sigma)\left(\frac{\rho}{R}\right)^{\delta p}\Big[1+\|u\|^{p}_{L^{\infty}(Q_{R_1,R_1^{sp}}(z,T_1))}\\
&\quad + \sup_{T_1-R_1^{sp}<t\leq T_1}\mathrm{Tail}_{p-1,sp}(u(\bullet,t);z,R_1)^{p}\\
&\quad + \left(\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\left(1+d^{sp\nu}\|f\|_{L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))}\right)\right)^{p}\Big]^{p-1}.
\end{aligned}
$$

Now we make the choice $\rho=\frac{R^{\theta}}{2}$ with

$$
\theta:=1+\frac{\xi}{\delta p+n+sp}.
$$

This yields

$$\rho^{-\zeta p} \fint_{Q_{\rho,\rho^s p}(x_0,T_0)} |u-\bar{u}_{(x_0,T_0),\rho}|^p \,\mathrm{d}x\,\mathrm{d}t \le C(n,s,p,\sigma)\Big[\Big(1+\|u\|^p_{L^\infty(Q_{R_1,R_1^{sp}}(z,T_1))} + \Big(\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\big(1+d^{sp\nu}\|f\|_{L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))}\big)\Big)^p + \sup_{T_1-R_1^{sp}<t\le T_1} \mathrm{Tail}_{p-1,sp}(u(\bullet,t);z,R_1)^p\Big)^{p-1} + \|f\|^{p'}_{L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))}\Big],$$

for any $0<\rho<\frac{\min\{1,d\}^\theta}{2}$, where

$$\zeta = \frac{\xi\delta}{n+sp+\delta p+\xi}.$$

For values of $\rho \ge \frac{\min\{1,d\}^\theta}{2}$,

$$\rho^{-\zeta p} \fint_{Q_{\rho,\rho^{sp}}(x_0,T_0)\cap Q_{R_1,R_1^{sp}}(z,T_1)} |u-\bar{u}_{(x_0,T_0),\rho}|^p \,\mathrm{d}x\,\mathrm{d}t \le 2^{(1+\zeta)p}\min\{1,d\}^{-\theta\zeta p}\|u\|^p_{L^\infty(Q_{R_1,R_1^{sp}}(z,T_1))}.$$

We can then conclude that for any cylinder of arbitrary size we have

$$\rho^{-\zeta p} \fint_{Q_{\rho,\rho^{sp}}(x_0,T_0)\cap Q_{R_1,R_1^{sp}}(z,T_1)} |u-\bar{u}_{(x_0,T_0),\rho}|^p \,\mathrm{d}x\,\mathrm{d}t \le C\ ,$$

with C depending on

$$n,\ s,\ p,\ R_1,\ \sigma,\ \sup_{T_1-R_1^{sp}<t\le T_1} \mathrm{Tail}_{p-1,sp}(u(\bullet,t);z,R_1),\ \|f\|_{L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))},$$
$$\text{and } \|u\|_{L^\infty(Q_{R_1,R_1^{sp}}(z,T_1))}.$$

In particular, one can obtain

$$\rho^{-\zeta}\left(\fint_{Q_{\rho,\rho^{sp}}(x_0,T_0)\cap Q_{R_1,R_1^{sp}}(z,T_1)} |u-\bar{u}_{(x_0,T_0),\rho}|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} \le C(n,s,p,\sigma)\mathcal{M},$$

$$\text{where } \mathcal{M} := \Big[1+\|u\|_{L^\infty(Q_{R_1,R_1^{sp}}(z,T_1))} + \vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}}\big(1+d^{sp\nu}\|f\|_{L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))}\big) + \sup_{T_1-R_1^{sp}<t\le T_1}\mathrm{Tail}_{p-1,sp}(u(\bullet,t);z,R_1)\Big]^{p-1} + \min\{1,d\}^{-1}\|u\|_{L^\infty(Q_{R_1,R_1^{sp}}(z,T_1))} + \|f\|^{\frac{1}{p-1}}_{L^{q,r}(Q_{R_1,R_1^{sp}}(z,T_1))}. \tag{3.39}$$

Now we use the characterization of the Campanato spaces in $\mathbb{R}^{n+1}$ with a general metric in [30], see also [20]. Our setting does not fit directly in the context considered there, since we only work with cylinders that are one-sided in the time direction that is $(t-r^{sp}, t] \times B_r(x)$ instead of $(t-r^{sp}, t+r^{sp}) \times B_r(x)$. Still, if you follow the proof in [30] with small modifications, you can also conclude the result in this setting.

In the case of $sp \geq 1$, using [30, Theorem 3.2] we get the Hölder continuity of u with exponent ζ in $Q_{\sigma R,(\sigma R)^{sp}}$ with respect to the metric

$$d((x,\tau_1),(y,\tau_2)) = \max\{|x-y|, |\tau_2-\tau_1|^{\frac{1}{sp}}\},$$

for which the balls of radius r are of the form $(t-r^{sp}, t+r^{sp}) \times B_r(x)$, which means

$$\begin{aligned}|u(x_1,t_1)-u(x_2,t_2)| &\leq C\mathcal{M}\big(|x_1-x_2|+|t_1-t_2|^{\frac{1}{sp}}\big)^{\zeta}\\ &\leq C\mathcal{M}\Big(|x_1-x_2|^{\zeta}+|t_1-t_2|^{\frac{\zeta}{sp}}\Big).\end{aligned}$$

Here C depends on n, s, p and σ. In the case of $sp<1$, we use the metric

$$d((x,\tau_1),(y,\tau_2)) = \max\{|x-y|^{sp}, |\tau_2-\tau_1|\}.$$

The balls of radius r are of the form $(t-r, t+r) \times B_{r^{\frac{1}{sp}}}(x)$. Hence, we have a decay of order $r^{\frac{\xi}{sp}p}$ of the average of u on the half balls. [30, Theorem 3.2] implies the following Hölder continuity on $Q_{\sigma R_1,(\sigma R_1)^{sp}}$

$$|u(x_1,t_1)-u(x_2,t_2)| \leq C\big(|x_1-x_2|^{sp}+|t_1-t_2|\big)^{\frac{\zeta}{sp}} \leq C\Big((|x_1-x_2|^{\zeta}+|t_1-t_2|^{\frac{\zeta}{sp}}\Big).$$

□

Lemma 3.7. *(Stability in L^∞) Let $f \in L^{q,r}_{\mathrm{loc}}(Q_{2R,(2R)^{sp}})$ with*

$$\frac{1}{r}+\frac{n}{spq}<1.$$

In addition, assume that $r \geq p'$,

$$q \geq 1 \quad \text{if} \quad sp \neq n, \quad \text{and} \quad q>1 \quad \text{if} \quad sp=n.$$

Let u be a local weak solution to the equation

$$u_t+(-\Delta_p)^s u = f \quad \text{in } Q_{2R,(2R)^{sp}},$$

with

$$\|u\|_{L^\infty(Q_{R,R^{sp}})}+\sup_{-R^{sp}<t\leq 0} \mathrm{Tail}_{p-1,sp}(u(\cdot,t);0,R) \leq M,$$

and

$$\|f\|_{L^{q,r}(Q_{R,R^{sp}})} \leq \omega.$$

Consider the (s, p)-caloric replacement

$$\begin{cases} \varphi_t + (-\Delta_p)^s \varphi = 0 & in \;\; Q_{R,R^{sp}} \\ \varphi = u & in \;\; (\mathbb{R}^n \setminus B_R) \times [-R^{sp}, 0] \\ \varphi(x, -R^{sp}) = u(x, -R^{sp}) & in \;\; B_R. \end{cases}$$

Then for $\sigma < 1$, there is a $\delta_{M,R,\sigma}(\omega)$ such that

$$\|u - \varphi\|_{L^\infty(Q_{\sigma R,(\sigma R)^{sp}})} < \delta_{M,R,\sigma}(\omega),$$

and $\delta_{M,R,\sigma}(\omega)$ converges to 0 as ω goes to 0.

Proof. The existence of such a bound follows immediately from Corollary 3.5.

To show the convergence of $\sigma_{M,R,\sigma}$ to zero, we argue by contradiction, suppose that there is a sequence $f_n \in L^{q,r}(Q_{R,R^{sp}})$ and u_n such that

$$\begin{aligned} &\|u_n\|_{L^\infty(Q_{R,R^{sp}})} + \sup_{T_0 - R^{sp} \le t \le T_0} \mathrm{Tail}_{p-1,sp}(u_n(\cdot, t); 0, R) \le M \\ &\text{and} \quad \|f_n\|_{L^{q,r}(Q_{R,R^{sp}})} \to 0, \end{aligned}$$

but

$$\|u_n - \varphi_n\|_{L^\infty(Q_{\sigma R,(\sigma R)^{sp}})} > \varepsilon > 0. \tag{3.40}$$

Using (3.12) from Lemma 3.2, we have

$$\lim_{n\to\infty} \int_{-R^{sp}}^0 [u_n - \varphi_n]^p_{W^{s,p}(\mathbb{R}^n)} \,\mathrm{d}t \le C(n, s, p, q, r, R) \lim_{n\to\infty} \|f_n\|^{p'}_{L^{q,r}(Q_{R,R^{sp}})} = 0. \tag{3.41}$$

By assumption, u_n is uniformly bounded in $L^\infty(Q_{R,R^{sp}})$. Now we show that φ_n is also uniformly bounded in $L^\infty(Q_{R,R^{sp}})$.

$$\begin{aligned} \|\varphi_n\|_{L^\infty(Q_{R,R^{sp}})} &\le \|u_n\|_{L^\infty(Q_{R,(\sigma R)^{sp}})} + \|u_n - \varphi_n\|_{L^\infty(Q_{R,R^{sp}})} \\ &\le M + \|u_n - \varphi_n\|_{L^\infty(Q_{R,R^{sp}})}. \end{aligned} \tag{3.42}$$

By Corollary 3.5,

$$\|u_n - \varphi_n\|_{L^\infty(Q_{R,R^{sp}})} \le C(n, s, p)\vartheta^{\frac{(p-1)\vartheta}{(\vartheta-1)^2}} \left(1 + R^{sp\nu} \|f_n\|_{L^{q,r}(Q_{R,R^{sp}})}\right). \tag{3.43}$$

Since $\|f_n\|^{p'}_{L^{q,r}(Q_{R,R^{sp}})}$ is uniformly bounded, (3.43) and (3.42) give us a uniform bound on $\|\varphi_n\|_{L^\infty(Q_{R,R^{sp}})}$.

Now we are in a position to use Theorem 3.6 for both of the sequences u_n and φ_n, which gives us a uniform bound on the Hölder seminorms of u_n and φ_n in $Q_{\sigma R,(\sigma R)^{sp}}$. Therefore, by Arzela–Ascoli's theorem $u_n - \varphi_n$ has a uniformly convergent subsequence in $Q_{\sigma R,(\sigma R)^{sp}}$. By (3.41), the limit is 0, contradicting (3.40). □

4. Improved Hölder regularity for nonhomogeneous equation

Proposition 4.1. *Let* $f \in L^{q,r}(Q_{1,2})$ *with* q, r *satisfying*

$$\frac{1}{r} + \frac{n}{spq} < 1.$$

In addition, assume that $r \geq p'$,

$$q \geq 1 \quad \textit{if} \quad sp \neq n, \quad \textit{and} \quad q > 1 \quad \textit{if} \quad sp = n.$$

Let u *be a weak solution of* $u_t + (-\Delta_p)^s u = f$ *in* $Q_{1,2}$ *that satisfies*

$$\|u\|_{L^\infty(Q_{1,2})} \leq 1\ , \qquad \sup_{-2\leq t\leq 0} \mathrm{Tail}_{p-1,sp}(u; 0, 1) \leq 1.$$

Then there exists ω *such that if*

$$\|f\|_{L^{q,r}(Q_{1,2})} \leq \omega(n, s, p, q, r, \alpha),$$

u is locally Hölder continuous in $Q_{\frac{1}{2},\frac{1}{2^{sp}}}$ *with exponents* α *in space and* $\frac{\alpha}{sp-(p-2)\alpha}$ *in time, as long as*

$$\alpha \leq \frac{r(spq-n) - spq}{q(r(p-1) - (p-2))} \quad \textit{and} \quad \alpha < \Theta. \tag{4.1}$$

Recall that $\Theta = \min\left\{\frac{sp}{p-1}, 1\right\}$.

More precisely, for $(x_1, t_1),\ (x_2, t_2) \in Q_{\frac{1}{2},\frac{1}{2^{sp}}}$ *we have*

$$|u(x_2, t_2) - u(x_1, t_1)| \leq C(n, s, p, q, r, \alpha)\left(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\frac{\alpha}{sp-(p-2)\alpha}}\right).$$

Proof. **Step 1**: Decay at the origin.

For this part, we prove a decay at the origin for u under the assumptions

$$\|u\|_{L^\infty(Q_{1,1})} \leq 1\ , \qquad \sup_{-1\leq t\leq 0} \mathrm{Tail}_{p-1,sp}(u; 0, 1) \leq 1, \quad \text{and} \quad \|f\|_{L^{q,r}(Q_{1,1})} \leq \omega. \tag{4.2}$$

Here $\omega > 0$ is a small number to be determined later which depends on n, s, p and α. We introduce the parabolic cylinder

$$G_r := B_r(0) \times (-r^\beta, 0],$$

with $\beta = sp - (p-2)\alpha$. We show that for any exponent α satisfying (4.1), the following holds for $r < 1$

$$\|u(x, t) - u(0, 0)\|_{L^\infty(G_r)} \leq Cr^\alpha.$$

It is enough to prove the inequality for a sequence of radii $(r_k)_{k=0}^{\infty}$, $r_k = \lambda^k$, for some $\lambda < 1$. Without loss of generality, we assume $u(0,0) = 0$. Consider the rescaled functions

$$v_k(x,t) := \frac{u(\lambda^k x, \lambda^{k\beta} t)}{\lambda^{\alpha k}},$$

with λ small enough to be determined later. We will prove the following by induction,

$$\|v_k(x,t)\|_{L^\infty(G_1)} \le 1 \qquad \text{and} \qquad \sup_{-1\le t\le 0} \int_{\mathbb{R}^n\setminus B_1} \frac{|v_k(x,t)|}{|x|^{n+sp}}\,\mathrm{d}x \le 1. \tag{4.3}$$

For $k=0$, (4.3) follows from our assumptions (4.2).

Observe that

$$\begin{cases} \frac{\partial v_k(x,t)}{\partial t} = \lambda^{\beta k - \alpha k} u_t(\lambda^k x, \lambda^{\beta k} t) \\ (-\Delta_p)^s v_k(x,t) = \lambda^{k[sp-(p-1)\alpha]} (-\Delta_p)^s u(\lambda^k x, \lambda^{\beta k} t). \end{cases}$$

With $\beta = sp - (p-2)\alpha$, $v_k(x,t)$ solves

$$\frac{\partial v_k}{\partial t} + (-\Delta_p)^s v_k = \lambda^{k[sp-(p-1)\alpha]} f(\lambda^k x, \lambda^{\beta k} t) =: f_k(x,t) \qquad \text{in } Q_{\frac{1}{\lambda^k}, \frac{1}{\lambda^{\beta k}}}.$$

Moreover,

$$\begin{aligned} \|f_k\|_{L^{q,r}(G_1)}^r &= \int_{-1}^0 \left(\int_{B_1} |f_k(x,r)|^q \,\mathrm{d}x \right)^{\frac{r}{q}} \mathrm{d}t \\ &= \int_{-1}^0 \left(\int_{B_{\lambda^k}} \lambda^{kq[sp-(p-1)\alpha]-kn} |f(x, \lambda^{\beta k} t)|^q \,\mathrm{d}x \right)^{\frac{r}{q}} \mathrm{d}t \\ &= \int_{-1}^0 \lambda^{rk[sp-(p-1)\alpha]-\frac{krn}{q}} \left(\int_{B_{\lambda^k}} |f(x, \lambda^{\beta k} t)|^q \,\mathrm{d}x \right)^{\frac{r}{q}} \mathrm{d}t \\ &= \lambda^{rk[sp-(p-1)\alpha]-\frac{krn}{q}-\beta k} \|f\|_{L^{q,r}(G_{\lambda^k})}. \end{aligned}$$

Since $\lambda < 1$, and the exponent of λ is nonnegative by (4.1), we get $\|f_k\|_{L^{q,r}(G_1)} \le \omega$.

Assume that (4.3) holds for k. Now we prove that it holds for $k+1$. Consider the (s,p)-caloric replacement of $v_k(x,t)$ in $Q_{1,1}$, say $\varphi_k(x,t)$. Then

$$|v_k(x,t)| \le |v_k(x,t) - \varphi_k(x,t)| + |\varphi_k(x,t) - \varphi_k(0,t)| + |\varphi_k(0,t) - v_k(0,t)|.$$

By Theorem 5.1, φ_k is locally Hölder continuous in $Q_{1,1}$, and for $(x,t) \in Q_{\frac{1}{2}, \frac{1}{2^{sp}}}$,

$$|\varphi_k(x,t) - \varphi_k(0,0)| \le C_1 |x|^{\Theta-\varepsilon} + C_2 |t|^{\Gamma - \frac{\varepsilon}{\beta}}.$$

Here we take $\varepsilon = \frac{\Theta-\alpha}{2}$. Since $\|f_k\|_{L^{q,r}(Q_{1,1})} \le \omega$, Lemma 3.7 implies

$$|v_k(x,t)| \le 2\delta(\omega) + C_1 |x|^{\Theta-\varepsilon} + C_2 |t|^{\Gamma-\frac{\varepsilon}{\beta}}, \qquad \text{in} \quad Q_{\frac{1}{4}, \frac{1}{4^{sp}}}. \tag{4.4}$$

In Theorem 5.1, the Hölder constants are bounded by

$$
\begin{aligned}
(C_2)^{\frac{1}{p-1}} \le C_1 &\le C\big(1 + \|\varphi_k\|_{L^\infty(Q_{1,1})} + \sup_{-\frac{1}{2^{sp}} \le t \le 0} \mathrm{Tail}_{p-1,sp}(\varphi_k; 0, 1)\big) \\
&\le C\big(1 + \|\varphi_k\|_{L^\infty(Q_{1,1})} + \sup_{-1 \le t \le 0} \mathrm{Tail}_{p-1,sp}(v_k; 0, 1)\big) \\
&\le C\big(1 + \|v_k - \varphi_k\|_{L^\infty(Q_{1,1})} + \|v_k\|_{L^\infty(Q_{1,1})} \\
&\quad + \sup_{-1 \le t \le 0} \mathrm{Tail}_{p-1,sp}(v_k; 0, 1)\big).
\end{aligned}
$$

Therefore, by (4.3) we have

$$
(C_2)^{\frac{1}{p-1}} \le C_1 \le C(n, s, p, \alpha)(3 + \|v_k - \varphi_k\|_{L^\infty(Q_{1,1})}).
$$

By Corollary 3.5,

$$
C_1 \le C\left(3 + C(n, s, p, q, r)(1 + \|f_k\|_{L^{q,r}(Q_{1,1})})\right) \le C(3 + C(n, s, p, q, r)(1 + \omega)).
$$

This is a bound independent of k. We can take ω to be less than 1 and take $C_1 = C(n, s, p)(3+2C(n, s, p, q, r)$, with the $C(n, s, p, q, r)$ coming from Corollary 3.5, so that the constants $C_1,\ C_2$ are independent of ω as well.

Now we proceed and prove (4.3) for $k + 1$. First, we state our choice of λ

$$
\begin{aligned}
\lambda \le \min\Bigg\{&\frac{1}{4}, \frac{1}{4^{\frac{sp}{\beta}}}, \\
&\times \frac{1}{(2C_1 + 2C_2)^{\frac{2}{\Theta-\alpha}}}, \left(1 + \frac{\omega_n(4^{sp} - 1)}{sp} + \frac{(1 + C_1 + C_2)^{p-1}}{(p-1)(\Theta - \alpha)/2}\right)^{\frac{2}{(p-1)(\Theta-\alpha)}}\Bigg\}.
\end{aligned}
\tag{4.5}
$$

Since $\lambda < \frac{1}{4}$, and $\lambda^\beta < \frac{1}{4^{sp}}$, $Q_{\lambda,\lambda^\beta} \subset Q_{\frac{1}{4},\frac{1}{4^{sp}}}$. Therefore, from (4.4) we obtain

$$
\|v_k(x, t)\|_{L^\infty(G_\lambda)} \le \delta(\omega) + C_1\lambda^{\Theta-\varepsilon} + C_2\lambda^{\beta(\Gamma-\frac{\varepsilon}{\beta})}.
$$

Notice that $\beta\Gamma \ge \Theta$, by the above choice of β. Thus,

$$
\|v_k(x, t)\|_{L^\infty(G_\lambda)} \le \delta(\omega) + (C_1 + C_2)\lambda^{\Theta-\varepsilon}. \tag{4.6}
$$

Recall that $\varepsilon = \frac{\Theta-\alpha}{2}$ and by the assumption (4.5)

$$
(C_1 + C_2)\lambda^{\Theta-\varepsilon} < \frac{1}{2}\lambda^\alpha.
$$

Now we choose ω so that

$$
2\delta(\omega) \le \frac{1}{2}\lambda^\Theta \le \frac{1}{2}\lambda^\alpha.
$$

This is possible since $\delta(\omega)$ converges to zero as $\omega \to 0$. Then, (4.6) implies

$$\|v_k(x,t)\|_{L^\infty(G_\lambda)} \le \lambda^\alpha,$$

which translates to

$$\|v_{k+1}(x,t)\|_{L^\infty(G_1)} = \left\| \frac{v_k(\lambda x, \lambda^\beta t)}{\lambda^\alpha} \right\|_{L^\infty(G_1)} \le 1, \tag{4.7}$$

which is the first part of (4.3). For the second part, we want to show

$$\sup_{-1<t<0} \int_{\mathbb{R}^n \setminus B_1} \frac{|v_{k+1}(x,t)|^{p-1}}{|x|^{n+sp}} \, \mathrm{d}x \le 1.$$

We split the integral into three parts. Using the induction hypothesis,

$$\begin{aligned} \sup_{-1<t<0} \int_{\mathbb{R}^n \setminus B_{\frac{1}{\lambda}}} \frac{|v_{k+1}(x,t)|^{p-1}}{|x|^{n+sp}} \, \mathrm{d}x &\le \sup_{-\lambda^{-\beta} \le t \le 0} \int_{\mathbb{R}^n \setminus B_{\frac{1}{\lambda}}} \frac{|v_{k+1}(x,t)|^{p-1}}{|x|^{n+sp}} \, \mathrm{d}x \\ &= \lambda^{sp-\alpha(p-1)} \sup_{-1<t<0} \int_{\mathbb{R}^n \setminus B_1} \frac{|v_k(x,t)|^{p-1}}{|x|^{n+sp}} \, \mathrm{d}x \\ \left(\text{using } \Theta \le \frac{sp}{p-1} \right) \quad &\le \lambda^{(p-1)(\Theta-\alpha)}. \end{aligned}$$

Moreover, $\|v_k\|_{L^\infty(G_1)} \le 1$, and hence,

$$\begin{aligned} \sup_{-1<t<0} \int_{B_{\frac{1}{\lambda}} \setminus B_{\frac{1}{4\lambda}}} \frac{|v_{k+1}(x,t)|^{p-1}}{|x|^{n+sp}} \, \mathrm{d}x &\le \lambda^{sp-\alpha(p-1)} \sup_{-\lambda^\beta<t<0} \int_{B_1 \setminus B_{\frac{1}{4}}} \frac{|v_k(x,t)|^{p-1}}{|x|^{n+sp}} \, \mathrm{d}x \\ &\le \lambda^{sp-\alpha(p-1)} \int_{B_1 \setminus B_{\frac{1}{4}}} \frac{1}{|x|^{n+sp}} \, \mathrm{d}x \\ &\le \lambda^{(p-1)(\Theta-\alpha)} \frac{\omega_n(4^{sp}-1)}{sp} := C_3 \lambda^{2(p-1)\varepsilon}. \end{aligned}$$

For remaining part, we transfer the estimate (4.4) to v_{k+1} and obtain

$$|v_{k+1}(x,t)| \le \delta(\omega)\lambda^{-\alpha} + C_1 \lambda^{\Theta-\varepsilon-\alpha}|x|^{\Theta-\varepsilon} + C_2 \lambda^{\beta\Gamma-\varepsilon-\alpha}|t|^{\Gamma-\frac{\varepsilon}{\beta}} \quad \text{in } Q_{\frac{1}{4\lambda},\frac{1}{4^{sp}\lambda^\beta}}.$$

In particular, since $\lambda^\beta \le \frac{1}{4^{sp}}$, $Q_{\frac{1}{4\lambda},1} \subset Q_{\frac{1}{4\lambda},\frac{1}{4^{sp}\lambda^\beta}}$, and $\delta(\omega) \le \lambda^\Theta \le \lambda^{\Theta-\varepsilon}$ we get

$$\sup_{-1\le t\le 0} |v(x,t)| \le \lambda^{\Theta-\varepsilon-\alpha}(1 + C_2\lambda^{\beta\Gamma-\Theta} + C_1|x|^{\Theta-\varepsilon}).$$

Therefore,

$$
\begin{aligned}
&\sup_{-1<t<0}\int_{B_{\frac{1}{4\lambda}}\setminus B_1}\frac{|v_{k+1}(x,t)|^{p-1}}{|x|^{n+sp}}\,\mathrm{d}x\\
&\le \lambda^{(p-1)(\Theta-\varepsilon-\alpha)}\int_{B_{\frac{1}{4\lambda}}\setminus B_1}\frac{|1+C_2\lambda^{\beta\Gamma-\Theta}+C_1|x|^{\Theta-\varepsilon}|^{p-1}}{|x|^{n+sp}}\,\mathrm{d}x\\
&(\text{using } |x|\ge 1)\le (1+C_2\lambda^{\beta\Gamma-\Theta}+C_1)^{p-1}\lambda^{(p-1)(\Theta-\varepsilon-\alpha)}\\
&\quad\times\int_{B_{\frac{1}{4\lambda}}\setminus B_1}\frac{1}{|x|^{n+sp-(p-1)(\Theta-\varepsilon)}}\,\mathrm{d}x\\
&(\text{using } sp\ge (p-1)\Theta)\le (1+C_2+C_1)^{p-1}\lambda^{(p-1)(\Theta-\varepsilon-\alpha)}\int_{\mathbb{R}^n\setminus B_1}\frac{1}{|x|^{n+\varepsilon(p-1)}}\,\mathrm{d}x\\
&\le \frac{(1+C_1+C_2)^{p-1}}{\varepsilon(p-1)}\lambda^{(p-1)(\Theta-\varepsilon-\alpha)}:=C_4\lambda^{(p-1)\varepsilon}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sup_{-1<t<0}\int_{\mathbb{R}^n\setminus B_1}\frac{|v_{k+1}(x,t)|^{p-1}}{|x|^{n+sp}}\,\mathrm{d}x &\le \lambda^{2(p-1)\varepsilon}+C_3\lambda^{2(p-1)\varepsilon}+C_4\lambda^{(p-1)\varepsilon}\\
&\le \lambda^{(p-1)\varepsilon}\left(1+C_3+\frac{(1+C_1+C_2)^{p-1}}{\varepsilon(p-1)}\right).
\end{aligned}
$$

Using the assumption (4.5) on λ, we obtain

$$
\sup_{-1<t<0}\int_{\mathbb{R}^n\setminus B_1}\frac{|v_{k+1}(x,t)|^{p-1}}{|x|^{n+sp}}\,\mathrm{d}x\le 1.
$$

Step 2: Regularity in a cylinder. We choose α as in (4.1) and let ω be as in Step 1. For a point $(x_0,t_0)\in Q_{\frac{1}{2},\frac{1}{2^{sp}}}$, define

$$
\tilde{u}(x,t)=\frac{1}{L}u\Big(\frac{x}{2}+x_0,\,L^{2-p}\frac{1}{2^{sp}}t+t_0\Big),
$$

where $L=2^{\frac{n}{p-1}}(1+|B_1|)^{\frac{1}{p-1}}$. Then $\tilde{u}$ is a solution of

$$
\partial_t\tilde{u}+(-\Delta_p)^s\tilde{u}=\frac{L^{-(p-1)}}{2^{sp}}f\Big(\frac{x}{2}+x_0,\,L^{2-p}\frac{1}{2^{sp}}t+t_0\Big):=\tilde{f}\quad\text{in } Q_{1,2^{sp-1}L^{p-2}}.
$$

By the choice of L, $\tilde{u}$ satisfies the conditions (4.2) in Step 1. Since $L\ge 1$, we immediately have

$$
\|\tilde{u}\|_{L^\infty(Q_{1,1}(0,0))}\le\frac{1}{L}\|u\|_{L^\infty(Q_{\frac{1}{2},\frac{L^{2-p}}{2^{sp}}}(x_0,t_0))}\le\|u\|_{L^\infty(Q_{1,2})}\le 1,
$$

since $Q_{\frac{1}{2},\frac{L^{2-p}}{2^{sp}}}(x_0,t_0) \subset Q_{1,2}$. As for the $L^{q,r}$ norm of $\tilde{f}$, we have

$$\begin{aligned}\|\tilde{f}\|_{L^{q,r}(Q_{1,1})} &= \frac{L^{-(p-1)}}{2^{sp}}\big(2^{\frac{n}{q}+\frac{sp}{r}}L^{\frac{p-2}{r}}\|f\|_{L^{q,r}(Q_{\frac{1}{2},\frac{L^{2-p}}{2^{sp}}}(x_0,t_0))}\big)\\ &\le L^{-(p-1)}(1/2)^{sp(1-\frac{1}{r}-\frac{n}{spq})}\|f\|_{Q_{1,2}}\\ &\le L^{-(p-1)}(1/2)^{sp(1-\frac{1}{r}-\frac{n}{spq})}\omega \le \omega.\end{aligned}$$

Here we have used $1-\frac{1}{r}-\frac{n}{spq} > 0$. Notice that in the case of $sp \ge n$, we are assuming $1-\frac{1}{r}-\frac{1}{q} > 0$ which is a stronger assumption. Now we verify the assumption on the tail.

$$\begin{aligned}\sup_{-1\le t\le 0}\int_{\mathbb{R}^n\setminus B_1}\frac{|\tilde{u}|^{p-1}}{|x|^{n+sp}}\,\mathrm{d}x &= \frac{2^{-sp}}{L^{p-1}}\sup_{t_0-\frac{L^{2-p}}{2^{sp}}\le t\le t_0}\int_{\mathbb{R}^n\setminus B_{\frac{1}{2}}(x_0)}\frac{|u(y)|^{p-1}}{|y-x_0|^{n+sp}}\,\mathrm{d}y\\ &\le \frac{1}{L^{p-1}}\sup_{-2\le t\le 0}\mathrm{Tail}_{p-1,sp}(u(\bullet,t);x_0,\frac{1}{2})^{p-1}\\ &\le \frac{1}{L^{p-1}}(\frac{1}{2})^{sp}\left(\frac{1}{1-|x-x_0|}\right)^{n+sp}\\ &\quad\times\sup_{-2\le t\le 0}\mathrm{Tail}_{p-1,sp}(u(\bullet,t);0,1)^{p-1}\\ &\quad+\frac{2^n}{L^{p-1}}\sup_{-2\le t\le 0}\|u(\bullet,t)\|^{p-1}_{L^{p-1}(B_1(0))}\\ &\le \frac{2^n}{L^{p-1}}\big(1+|B_1|\|u\|_{L^\infty(Q_{1,2})}\big) \le \frac{2^n(1+|B_1|)}{L^{p-1}} \le 1.\end{aligned}$$

Now we can apply Step 1 to $\tilde{u}$ and we get the decay

$$\|\tilde{u}-\tilde{u}(0,0)\|_{L^\infty(G_r)} \le Cr^\alpha, \quad \text{for } 0<r<1$$

or in other words

$$|\tilde{u}(x,t)-\tilde{u}(0,0)| \le C(|x|^\alpha+|t|^{\frac{\alpha}{\beta}}), \quad \text{for } (x,t)\in Q_{1,1}.$$

In terms of u, this means

$$\begin{aligned}&|u(x,t)-u(x_0,t_0)| \le CL(2^\alpha|x-x_0|^\alpha+(2^{sp}L^{p-2})^{\frac{\alpha}{\beta}}|t-t_0|^{\frac{\alpha}{\beta}}), \quad \text{for}\\ &(x,t)\in Q_{\frac{1}{2},\frac{1}{2^{sp}L^{p-2}}}(x_0,t_0).\end{aligned} \tag{4.8}$$

Now take two points $(x_1,t_1),(x_2,t_2) \in Q_{\frac{1}{2},\frac{1}{2^{sp}}}$ and split the line joining them into $1+[L^{p-2}]$ pieces, say $(y_i,\tau_i)_{i=0}^{1+[L^{p-2}]}$ with $(x_1,t_1)=(y_0,\tau_0)$, $(x_2,t_2)=(y_{1+[L^{p-2}]},\tau_{1+[L^{p-2}]})$, $|y_{i+1}-y_i| = \frac{|x_2-x_1|}{1+[L^{p-2}]} < \frac{1}{2}$ and $|\tau_{i+1}-\tau_i| = \frac{|t_2-t_1|}{1+[L^{p-2}]} < \frac{1}{2^{sp}L^{p-2}}$ so that $(y_{i+1},\tau_{i+1}) \in Q_{\frac{1}{2},\frac{1}{2^{sp}L^{p-2}}}(y_i,\tau_i)$. By (4.8) applied in each of

$Q_{\frac{1}{2},\frac{1}{2^{sp}L^{p-2}}}(y_i,\tau_i)$ obtain

$$
\begin{aligned}
|u(x_2,t_2)-u(x_1,t_1)| &\leq \sum_{i=0}^{[L^{p-2}]} |u(y_{i+1},\tau_{i+1})-u(y_i,\tau_i)| \\
&\leq CL\sum_{i=0}^{[L^{p-2}]} 2^{\alpha}|y_{i+1}-y_i|^{\alpha} + (2^{sp}L^{p-2})^{\frac{\alpha}{\beta}}|\tau_{i+1}-\tau_i|^{\frac{\alpha}{\beta}} \\
&\leq C(1+L)^{p-1}\left(\left(2\frac{|x_2-x_1|}{1+[L^{p-2}]}\right)^{\alpha}+\left(2^{sp}L^{p-2}\frac{|t_2-t_1|}{1+\lfloor L^{p-2}\rfloor}\right)^{\frac{\alpha}{\beta}}\right) \\
&\leq C(n,s,p,q,r,\alpha)(|x_2-x_1|^{\alpha}+|t_2-t_1|^{\frac{\alpha}{\beta}}).
\end{aligned}
$$

□

Now we prove the Hölder regularity at any scale.

Proof of Theorem 1.2. We will consider the rescaled functions

$$
\tilde{u}_\iota(x,t)=\frac{1}{\mu}u(Rx+x_0,\mu^{2-p}R^{sp}t+\iota+T_0)
$$

with

$$
\begin{aligned}
\mu =&1+\|u\|_{L^\infty(Q_{R,2R^{sp}}(x_0,T_0))}+\sup_{T_0-2R^{sp}\leq t\leq T_0}\mathrm{Tail}_{p-1,sp}(u(\bullet,t);x_0,R) \\
&+\left(\frac{R^{sp-\frac{n}{q}-\frac{sp}{r}}\|f\|_{L^{q,r}(Q_{R,2R^{sp}}(x_0,T_0))}}{\omega}\right)^{\frac{1}{p-1+\frac{p-2}{r}}},
\end{aligned}
$$

where $\omega=\omega(n,s,p,q,r,\alpha)$ is the same as in the proof of Proposition 4.1 and $\iota\in[-(R/2)^{sp}(1-\mu^{2-p}),0]$. The interval $[-(R/2)^{sp}(1-\mu^{2-p}),0]$ is chosen so that the cylinders $Q_{\frac{R}{2},\frac{\mu^{2-p}R^{sp}}{2^{sp}}}(x_0,T_0+\iota)$ cover all of $Q_{\frac{R}{2},(\frac{R}{2})^{sp}}(x_0,T_0)$ by varying ι over. Note that for these choices of ι we have $Q_{R,2\mu^{2-p}R^{sp}}(x_0,T_0+\iota)\subset Q_{R,2R^{sp}}(x_0,T_0)$. Then $\tilde{u}$ is a solution of

$$
\partial_t\tilde{u}_\iota+(-\Delta_p^s)\tilde{u}_\iota=R^{sp}\frac{f(Rx+x_0,\mu^{2-p}R^{sp}t+\iota+T_0)}{\mu^{p-1}},\quad \text{in}\quad Q_{1,2}.
$$

We now verify that $\tilde{u}_\iota$ satisfies the conditions of Proposition 4.1. The $L^{q,r}$ norm of the right-hand side is

$$
\begin{aligned}
\left\|R^{sp}\frac{f(Rx,\mu^{2-p}R^{sp}t+\iota)}{\mu^{p-1}}\right\|_{L^{q,r}(Q_{1,2})} &= \frac{\mu^{\frac{p-2}{r}}}{2^{\frac{1}{r}}\mu^{(p-1)}}R^{sp-\frac{n}{q}-\frac{sp}{r}}\|f\|_{L^{q,r}(Q_{R,2\mu^{2-p}R^{sp}}(x_0,T_0+\iota))} \\
&\leq \frac{R^{sp-\frac{n}{q}-\frac{sp}{r}}\|f\|_{L^{q,r}(Q_{R,2R^{sp}}(x_0,T_0))}}{2^{\frac{1}{r}}\mu^{p-1-\frac{p-2}{r}}} \\
&\leq \frac{\omega}{2^{\frac{1}{r}}}<\omega.
\end{aligned}
$$

The L^∞ norm of $\tilde{u}_\iota$ satisfies

$$\|\tilde{u}_\iota\|_{L^\infty(Q_{1,2}(0,0))} = \frac{1}{\mu}\|u\|_{L^\infty(Q_{R,2\mu^{2-p}R^{sp}}(x_0,T_0+\iota))} \le \frac{1}{\mu}\|u\|_{L^\infty(Q_{R,2R^{sp}})} \le 1.$$

Similarly

$$\begin{aligned}\sup_{-2\le t\le 0}\mathrm{Tail}_{p-1,sp}(\tilde{u}(\cdot,t);0,1) &\le \frac{1}{\mu}\sup_{T_0+\iota-2\mu^{2-p}R^{sp}\le t\le T_0+\iota}\mathrm{Tail}_{p-1,sp}(u(\cdot,t);x_0,R)\\ &\le \frac{1}{\mu}\sup_{T_0-2R^{sp}\le t\le T_0}\mathrm{Tail}_{p-1,sp}(u(\cdot,t);x_0,R)\le 1.\end{aligned}$$

Hence, using Proposition 4.1 for $\tilde{u}_\iota$, we get

$$\begin{aligned}&|\tilde{u}_\iota(\tilde{x}_2,\tilde{t}_2)-\tilde{u}_\iota(\tilde{x}_1,\tilde{t}_1)|\le C(|\tilde{x}_2-\tilde{x}_1|^\alpha+|\tilde{t}_2-\tilde{t}_1|^{\frac{\alpha}{sp-(p-2)\alpha}})\\ &\text{for } (\tilde{x}_1,\tilde{t}_1),\ (\tilde{x}_2,\tilde{t}_2)\in Q_{\frac{1}{2},\frac{1}{2^{sp}}}(0,0),\end{aligned}$$

with $C=C(n,s,p,q,r,\alpha)$. This translates to

$$|u(x_2,\tau_2)-u(x_1,\tau_1)|\le \mu C\Big[\Big(\frac{|x_2-x_1|}{R}\Big)^\alpha+\Big(\frac{|\tau_2-\tau_1|}{R^{sp}\mu^{2-p}}\Big)^{\frac{\alpha}{sp-(p-2)\alpha}}\Big], \quad (4.9)$$

for $(x_1,\tau_1),\ (x_2,\tau_2)\in Q_{\frac{R}{2},\frac{R^{sp}\mu^{2-p}}{2^{sp}}}(x_0,T_0+\iota)$. Now we vary ι to obtain an estimate in the whole $Q_{\frac{R}{2},(\frac{R}{2})^{sp}}$. Specifically we split the interval $[t_1,t_2]$ into $1+\lfloor\mu^{p-2}\rfloor$ pieces, say $[\tau_{i+1},\tau_i]$, with $\tau_i-\tau_{i+1}=\frac{|t_2-t_1|}{1+\lfloor\mu^{p-2}\rfloor}$, $\tau_0=t_2$, and $\tau_{\lfloor 1+\mu^{p-2}\rfloor}=t_1$. Using (4.9), we obtain

$$\begin{aligned}|u(x_2,t_2)-u(x_1,t_1)| &\le |u(x_2,t_1)-u(x_1,t_1)|+|u(x_2,t_2)-u(x_2,t_1)|\\ &\le \mu C\Big(\frac{|x_2-x_1|}{R}\Big)^\alpha+\sum_{i=0}^{\lfloor\mu^{p-2}\rfloor}|u(x_2,\tau_i)-u(x_2,\tau_{i+1})|\\ &\le \mu C\Big[\Big(\frac{|x_2-x_1|}{R}\Big)^\alpha+\sum_{i=0}^{\lfloor\mu^{p-2}\rfloor}\Big(\frac{|\tau_i-\tau_{i+1}|}{R^{sp}\mu^{2-p}}\Big)^{\frac{\alpha}{sp-(p-2)\alpha}}\Big]\\ &= \mu C\Big[\Big(\frac{|x_2-x_1|}{R}\Big)^\alpha+\sum_{i=0}^{\lfloor\mu^{p-2}\rfloor}\Big(\frac{|t_2-t_1|}{R^{sp}\mu^{2-p}(1+\lfloor\mu^{p-2}\rfloor)}\Big)^{\frac{\alpha}{sp-(p-2)\alpha}}\Big]\\ &\le \mu C\Big[\Big(\frac{|x_2-x_1|}{R}\Big)^\alpha+\sum_{i=0}^{\lfloor\mu^{p-2}\rfloor}\Big(\frac{|t_2-t_1|}{R^{sp}}\Big)^{\frac{\alpha}{sp-(p-2)\alpha}}\Big]\\ &\le \mu C\Big[\Big(\frac{|x_2-x_1|}{R}\Big)^\alpha+2\mu^{p-2}\Big(\frac{|t_2-t_1|}{R^{sp}}\Big)^{\frac{\alpha}{sp-(p-2)\alpha}}\Big],\end{aligned}$$

which concludes the desired result. □

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Declarations

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5. Appendix A

In this section, we spell out the necessary modifications to prove the following theorem 5.1 which is a modified version of [7, Theorem 1.2]. As it is explained in [7, Remark 1.4] one can obtain the conclusions of [7, Theorem 1.2] under the weaker assumptions $u \in L^\infty_{\mathrm{loc}}(I; L^\infty_{\mathrm{loc}}(\Omega)) \cap L^\infty_{\mathrm{loc}}(I; L^{p-1}_{sp}(\mathbb{R}^n))$, instead of $u \in L^\infty_{\mathrm{loc}}(I; L^\infty(\mathbb{R}^n))$.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set, $I = (t_0, t_1]$, $p \geq 2$ and $0 < s < 1$. Suppose u is a local weak solution of*

$$u_t + (-\Delta_p)^s u = 0 \qquad \text{in } \Omega \times I,$$

such that

$$u \in L^\infty_{\mathrm{loc}}(I; L^\infty_{\mathrm{loc}}(\Omega)) \cap L^\infty_{\mathrm{loc}}(I; L^{p-1}_{sp}(\mathbb{R}^n)). \tag{5.1}$$

Define the exponents

$$\Theta(s,p) := \begin{cases} \dfrac{s\,p}{p-1}, & \text{if } s < \dfrac{p-1}{p}, \\[2ex] 1, & \text{if } s \geq \dfrac{p-1}{p}, \end{cases}$$

$$\text{and} \quad \Gamma(s,p) := \begin{cases} 1, & \text{if } s < \dfrac{p-1}{p}, \\[2ex] \dfrac{1}{s\,p-(p-2)}, & \text{if } s \geq \dfrac{p-1}{p}. \end{cases} \tag{5.2}$$

Then

$u \in C^{\delta}_{x,\mathrm{loc}}(\Omega \times I) \cap C^{\gamma}_{t,\mathrm{loc}}(\Omega \times I)$, *for every* $0 < \delta < \Theta(s,p)$ *and* $0 < \gamma < \Gamma(s,p)$.

More precisely, for every $0 < \delta < \Theta(s,p)$, $0 < \gamma < \Gamma(s,p)$, $R > 0$, $x_0 \in \Omega$ *and* T_0 *such that*

$$Q_{R,R^{s\,p}}(x_0,T_0) \Subset \Omega \times (t_0,t_1],$$

there exists a constant $C = C(n,s,p,\delta,\gamma,\sigma) > 0$ *such that*

$$\begin{aligned} |u(x_1,\tau_1)-u(x_2,\tau_2)| \leq C \Big(&\|u\|_{L^\infty(Q_{R,R^{sp}(x_0,T_0)})} \\ &+ \sup_{t\in[T_0-R^{sp},T_0]} \mathrm{Tail}_{p-1,sp}(u;x_0,R)+1\Big) \left(\frac{|x_1-x_2|}{R}\right)^{\delta} \\ &+ C\Big(\|u\|_{L^\infty(Q_{R,R^{sp}(x_0,T_0)})} \\ &+ \sup_{t\in[T_0-R^{sp},T_0]} \mathrm{Tail}_{p-1,sp}(u;x_0,R)+1\Big)^{p-1} \left(\frac{|\tau_1-\tau_2|}{R^{s\,p}}\right)^{\gamma}, \end{aligned} \tag{5.3}$$

for any $(x_1,\tau_1),\ (x_2,\tau_2) \in Q_{\sigma R,(\sigma R)^{s\,p}}(x_0,T_0)$.

First we reproduce a modified version of [7, Proposition 4.1], where instead of a global L^∞ bound we assume $\|u\|_{L^\infty(B_1\times[-1,0])} + \sup_{t\in[-1,0]} \mathrm{Tail}_{p-1,sp}(u;0,1)) \leq 1$. Before stating the proposition, let us recall the following notations from [7]:

$$u_h(x,t) := u(x+h,t), \qquad \delta_h u(x,t) := u_h(x,t) - u(x,t),$$

and

$$\delta^h u(x,t) := \delta_h(\delta_h u(x,t)) = u_{2h}(x,t) + u(x,t) - 2u_h(x,t).$$

Proposition 5.2. *Assume $p \geq 2$ and $0 < s < 1$. Let u be a local weak solution of $u_t + (-\Delta_p)^s u = 0$ in $B_2 \times (-2, 0]$. We assume that*

$$\|u\|_{L^\infty(B_1 \times [-1,0])} + \sup_{t \in [-1,0]} \mathrm{Tail}_{p-1,sp}(u(\bullet, t); 0, 1) \leq 1,$$

and that, for some $q \geq p$ and $0 < h_0 < 1/10$, we have

$$\int_{T_0}^{T_1} \sup_{0<|h|<h_0} \left\| \frac{\delta_h^2 u}{|h|^s} \right\|_{L^q(B_{R+4h_0})}^q \mathrm{d}t < +\infty,$$

for a radius $4 h_0 < R \leq 1 - 5 h_0$ and two time instants $-1 < T_0 < T_1 \leq 0$. Then we have

$$\int_{T_0+\mu}^{T_1} \sup_{0<|h|<h_0} \left\| \frac{\delta_h^2 u}{|h|^s} \right\|_{L^{q+1}(B_{R-4h_0})}^{q+1} \mathrm{d}t + \frac{1}{q+3-p} \sup_{0<|h|<h_0} \left\| \frac{\delta_h u(\bullet, T_1)}{|h|^{\frac{(q+2-p)s}{q+3-p}}} \right\|_{L^{q+3-p}(B_{R-4h_0})}^{q+3-p}$$
$$\leq C \int_{T_0}^{T_1} \left(\sup_{0<|h|<h_0} \left\| \frac{\delta_h^2 u}{|h|^s} \right\|_{L^q(B_{R+4h_0})}^q + 1 \right) \mathrm{d}t, \tag{5.4}$$

for every $0 < \mu < T_1 - T_0$. Here $C = C(n, s, p, q, h_0, \mu) > 0$ and $C \nearrow +\infty$ as $h_0 \searrow 0$ or $\mu \searrow 0$.

Proof. In the proof of [7, Proposition 4.1], the $L^\infty(\mathbb{R}^n \times [0, 1])$ boundedness is only used in Step 3, in the estimation of the nonlocal terms $\mathcal{I}_2$ and $\mathcal{I}_3$, which are defined by

$$\mathcal{I}_2(t) := \int_{B_{\frac{R+r}{2}} \times (\mathbb{R}^n \setminus B_R)} \frac{\Big(J_p(u_h(x) - u_h(y)) - J_p(u(x) - u(y)) \Big)}{|h|^{1+\vartheta\beta}} \times J_{\beta+1}(u_h(x) - u(x))\, \eta(x)^p \,\mathrm{d}\mu,$$

and

$$\mathcal{I}_3(t) := - \iint_{(\mathbb{R}^n \setminus B_R) \times B_{\frac{R+r}{2}}} \frac{\Big(J_p(u_h(x) - u_h(y)) - J_p(u(x) - u(y)) \Big)}{|h|^{1+\vartheta\beta}} \times J_{\beta+1}(u_h(y) - u(y))\, \eta(y)^p \,\mathrm{d}\mu.$$

We also recall the definition of $\tilde{\mathcal{I}}_2$ and $\tilde{\mathcal{I}}_3$

$$\tilde{\mathcal{I}}_i := \int_{T_0}^{T_1} \mathcal{I}_i(t)\, \tau(t) \,\mathrm{d}t, \qquad i = 2, 3,$$

where τ is smooth function $0 \leq \tau \leq 1$ such that

$$\tau \equiv 1 \quad \text{on } [T_0 + \mu, +\infty), \qquad \tau \equiv 0 \quad \text{on } (-\infty, T_0], \qquad \tau' \leq \frac{C}{\mu}.$$

The general argument is the same, but instead of using the L^∞ norm of $u(y)$ we can keep the inequality as it is and write

$$\begin{aligned}&\left|(J_p(u_h(x)-u_h(y))-J_p(u(x)-u(y)))J_{\beta+1}(\delta_h u(x))\right|\\&\quad\le C(1+|u_h(y)|^{p-1}+|u(y)|)|\delta_h u(x)|^{\beta},\end{aligned}$$

where $x\in B_{R-2h_0}$ and $4h_0<R<1-5h_0$. Therefore, $|x-y|\ge(1-\frac{R-2h_0}{R})|y|\ge C(h_0)|y|$ and we get

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B_R}\frac{1+|u(y)|^{p-1}+|u_h(y)|^{p-1}}{|x-y|^{n+sp}}\,\mathrm{d}y\\&\le\ C(n,s,p,h_0)+(C(h_0))^{n+sp}\int_{\mathbb{R}^n\setminus B_R}\frac{|u(y)|^{p-1}}{|y|^{n+sp}}\,\mathrm{d}y+(C(h_0))^{n+sp}\\&\quad\times\int_{\mathbb{R}^n\setminus B_R}\frac{|u_h(y)|^{p-1}}{|y|^{n+sp}}\,\mathrm{d}y.\end{aligned}$$

Now

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_R}\frac{|u(y)|^{p-1}}{|y|^{n+sp}}\,\mathrm{d}y&\le\int_{\mathbb{R}^n\setminus B_1}\frac{|u(y)|^{p-1}}{|y|^{n+sp}}\,\mathrm{d}y+R^{-n-sp}\int_{B_1}|u|^{p-1}\,\mathrm{d}y\\&\le 1+n\omega_n R^{-n-sp}\le 1+n\omega_n(4h_0)^{-n-sp},\end{aligned}$$

and for u_h

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B_R}\frac{|u(y+h)|^{p-1}}{|y|^{n+sp}}\,\mathrm{d}y\\&\quad\le\int_{\mathbb{R}^n\setminus B_R(h)}\frac{|u(y)|^{p-1}}{|y-h|^{n+sp}}\,\mathrm{d}y\le\left(\frac{3}{2}\right)^{n+sp}\int_{\mathbb{R}^n\setminus B_R(h)}\frac{|u(y)|^{p-1}}{|y|^{n+sp}}\,\mathrm{d}y\\&\quad\le\left(\frac{3}{2}\right)^{n+sp}\Big[\int_{\mathbb{R}^n\setminus B_1}\frac{|u(y)|^{p-1}}{|y|^{n+sp}}\,\mathrm{d}y+R^{-n-sp}\int_{B_1}|u(y)|^{p-1}\,\mathrm{d}y\Big]\\&\quad\le\Big(\frac{3}{2}\Big)^{n+sp}(1+n\omega_n R^{-n-sp})\le\Big(\frac{3}{2}\Big)^{n+sp}(1+n\omega_n(4h_0)^{-n-sp}).\end{aligned}$$

Here we have used $B_R(h)\subset B_1$, and $\frac{|y-h|}{|y|}=|\frac{y}{|y|}-\frac{h}{|y|}|\ge|\frac{y}{|y|}|-|\frac{h}{|y|}|\ge 1-|\frac{h_0}{R-h_0}|\ge\frac{2}{3}$. Using this, we get

$$\int_{\mathbb{R}^n\setminus B_R}\frac{1+|u(y)|^{p-1}+|u_h(y)|^{p-1}}{|x-y|^{n+sp}}\,\mathrm{d}y\le C(n,s,p,h_0),$$

and we can conclude

$$\begin{aligned}|\tilde{\mathcal{I}}_2|+|\tilde{\mathcal{I}}_3|&\le C(n,s,p,h_0)\int_{T_0}^{T_1}\int_{B_{\frac{R+r}{2}}}\frac{|\delta_h u|^{\beta}}{|h|^{1+\vartheta\beta}}\tau\,\mathrm{d}x\,\mathrm{d}t\le C(h_0,n,s,p,q,\beta)\\&\int_{T_0}^{T_1}\Big(1+\int_{B_R}\Big|\frac{\delta_h u}{|h|^{\frac{1+\vartheta\beta}{\beta}}}\Big|^{\frac{\beta q}{q-p+2}}\Big)\tau\,\mathrm{d}t,\end{aligned}$$

which is the same as equation (4.6) in [7]. □

We can estimate the $W^{s,p}$ seminorm of a solution as follows. The proof follows the argument in [7, Lemma 7.1].

Lemma 5.3. *Let $p \geq 2$ and $0 < s < 1$. Let u be a local weak solution of*

$$\partial_t u + (-\Delta_p)^s u = 0, \quad \text{in } B_{2R} \times (-2\,R^{s\,p}, 0],$$

such that $u \in L^\infty(B_{2R} \times [-R^{s\,p}, 0])$. Then

$$\left(R^{-n} \int_{-\frac{7}{8}\,R^{s\,p}}^{0} [u]^p_{W^{s,p}(B_R(x_0))}\ \mathrm{d}t\right)^{\frac{1}{p}} \leq C\left(\|u\|_{L^\infty(B_{2R}\times[-R^{s\,p},0])} + \sup_{t\in[-R^{sp},0]} \mathrm{Tail}_{p-1,sp}(u;0,2R) + 1\right),$$

for some $C = C(n, s, p) > 0$.

Proof. Without loss of generality, we may suppose that $x_0 = 0$. Let

$$k = \|u\|_{L^\infty(B_{2R}\times[-R^{s\,p},0])} + \sup_{t\in[-R^{sp},0]} \mathrm{Tail}_{p-1,sp}(u(\bullet,t);0,2R) + 1 \text{ and } \widetilde{u} = u + k.$$

Then $\widetilde{u}$ is a local weak solution in $B_2 \times (-2\,R^{s\,p}, 0]$ and $\widetilde{u} \geq 1$ in $B_{2R} \times [-R^{s\,p}, 0]$. We choose φ and ψ exactly as in [7, Lemma 7.1], that is,

$$\eta \in C_0^\infty(2R),\ \eta \equiv 1 \text{ in } B_R,\ |\nabla \eta| \leq \frac{C}{R} \text{ and } \eta \equiv 0 \text{ in } \mathbb{R}^n \setminus B_{\frac{3}{2}R};$$

and

$$\psi \in C^\infty(\mathbb{R}),\ \psi(t) = 0 \text{ for } t \leq -R^{s\,p},\ \psi \equiv 1 \text{ in } \left[-\frac{7}{8}\,R^{s\,p}, 0\right] \text{ and } |\psi'| \leq \frac{C}{R^{s\,p}}.$$

Then for $\varphi(x,t) = \eta(x)\varphi(t)$, we get

$$\begin{aligned}
&\int_{-\frac{7}{8}R^{s\,p}}^{0} [\tilde{u}(\bullet,t)]^p_{W^{s,p}(B_R)}\ \mathrm{d}t \leq \int_{-R^{s\,p}}^{0} \left[\widetilde{u}(\bullet,t)\,\varphi(\bullet,t)\right]^p_{W^{s,p}(B_{2R})}\ \mathrm{d}t \\
&\leq C \int_{-R^{s\,p}}^{0} \iint_{B_{2R}\times B_{2R}} \max\left\{\widetilde{u}(x,t),\ \widetilde{u}(y,t)\right\}^p |\varphi(x,t) - \varphi(y,t)|^p\ \mathrm{d}\mu\ \mathrm{d}t \\
&\quad + C\left(\sup_{x\in\operatorname{supp}\eta} \int_{\mathbb{R}^n\setminus B_{2R}} \frac{\mathrm{d}y}{|x-y|^{n+s\,p}}\right)\left(\int_{-R^{s\,p}}^{0} \int_{B_{2R}} \widetilde{u}(x,t)^p\,\varphi(x,t)^p\ \mathrm{d}x\ \mathrm{d}t\right) \\
&\quad + C\Big(\sup_{t\in[-R^{s\,p},0]}\ \sup_{x\in\operatorname{supp}\eta} \int_{\mathbb{R}^n\setminus B_{2R}} \frac{(u(y,t)^+)^{p-1}}{|x-y|^{n+s\,p}}\ \mathrm{d}y\Big) \\
&\quad \times \int_{-R^{s\,p}}^{0} \int_{B_{2R}} \widetilde{u}(x,t)\,\varphi(x,t)^p\ \mathrm{d}x\ \mathrm{d}t \\
&\quad + \frac{1}{2}\int_{-R^{s\,p}}^{0} \int_{B_{2R}} \widetilde{u}(x,t)^2 \left(\frac{\partial \varphi^p}{\partial t}\right)^+\ \mathrm{d}x\ \mathrm{d}t + \int_{B_{2R}} \widetilde{u}(x,0)\ \mathrm{d}x \\
&\leq C\,R^n\,(k^p + k^2 + k) \leq C\,R^n\,k^p.
\end{aligned}$$

The only difference in the proof is in estimating the term

$$\sup_{x\in \operatorname{supp}\eta} \int_{\mathbb{R}^n\setminus B_{2R}} \frac{(u(y,t)^+)^{p-1}}{|x-y|^{n+s\,p}}\,\mathrm{d}y.$$

Noticing that for $x \in \operatorname{supp}\eta \subset B_{\frac{3}{2}R}$ we have $\frac{|x-y|}{|y|} \geq 1 - \frac{|x|}{|y|} \geq 1 - \frac{3/2R}{2R} = \frac{1}{4}$, we get

$$\int_{\mathbb{R}^n\setminus B_{2R}} \frac{(u(y,t)^+)^{p-1}}{|x-y|^{n+s\,p}}\,\mathrm{d}y \leq 4^{n+s\,p} R^{-s\,p}\ \mathrm{Tail}_{p-1,sp}^{p-1}(u;0,2R) \leq C\,R^{-s\,p} k^{p-1}.$$

□

We can now prove the following modified version of [7, Theorem 4.2].

Theorem 5.4. *(Spatial almost C^s regularity) Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set, $I = (t_0, t_1]$, $p \geq 2$ and $0 < s < 1$. Suppose u is a local weak solution of*

$$u_t + (-\Delta_p)^s u = 0 \qquad \text{in } \Omega \times I,$$

such that $u \in L^\infty_{\mathrm{loc}}(I; L^\infty(\Omega)) \cap L^\infty_{\mathrm{loc}}(I; L^{p-1}_{sp}(\mathbb{R}^n))$. Then $u \in C^\delta_{x,\mathrm{loc}}(\Omega \times I)$ for every $0 < \delta < s$.

More precisely, for every $0 < \delta < s$, $R > 0$ and every (x_0, T_0) such that

$$Q_{2R,2R^{s\,p}}(x_0, T_0) \Subset \Omega \times (t_0, t_1],$$

there exists a constant $C = C(n, s, p, \delta) > 0$ such that

$$\begin{aligned}&\sup_{t\in\left[T_0-\frac{R^{s\,p}}{2},T_0\right]} [u(\bullet,t)]_{C^\delta(B_{R/2}(x_0))}\\ &\quad\leq \frac{C}{R^\delta}\left(1 + \|u\|_{L^\infty(B_{2R}(x_0)\times[T_0-R^{s\,p},T_0])} + \sup_{t\in[T_0-R^{sp},T_0]} \mathrm{Tail}_{p-1,sp}(u;x_0,2R)\right)\end{aligned} \tag{5.5}$$

Proof. The proof is essentially the same as the proof of [7, Theorem 4.2]. We assume for simplicity that $x_0 = 0$ and $T_0 = 0$, and set

$$\begin{aligned}\mathcal{M}_R = \|u\|_{L^\infty(B_{2R}\times[-R^{s\,p},0])} + \sup_{t\in[-R^{sp},0]} \mathrm{Tail}_{p-1,sp}(u;0,R)\\ + \left(R^{-n}\int_{-\frac{7}{8}R^{s\,p}}^{0} [u]^p_{W^{s,p}(B_R)}\,\mathrm{d}t\right)^{\frac{1}{p}} + 1.\end{aligned}$$

Notice that by Lemma 5.3 we have

$$\mathcal{M}_R \leq C\Big(\|u\|_{L^\infty(B_{2R}\times[-R^{s\,p},0])} + \sup_{t\in[-R^{sp},0]} \mathrm{Tail}_{p-1,sp}(u;0,2R)\Big) + 1.$$

Let $\alpha \in [-R^{s\,p}(1-\mathcal{M}_R^{2-p}), 0]$ and define

$$u_{R,\alpha}(x,t) := \frac{1}{\mathcal{M}_R}\, u\left(R\,x, \frac{1}{\mathcal{M}_R^{p-2}}\, R^{s\,p}\, t+\alpha\right), \qquad \text{for } x \in B_2,\ t \in (-2,0].$$

Then $u_{R,\alpha}(x,t)$ is a local weak solution of

$$u_t + (-\Delta_p)^s u = 0, \qquad \text{in } B_2 \times (-2,0],$$

that satisfies

$$\|u_{R,\alpha}\|_{L^\infty(B_2\times[-1,0])} + \sup_{t\in[-1,0]} \mathrm{Tail}_{p-1,sp}(u(\bullet,t);0,1) \le 1,$$
$$\int_{-\frac{7}{8}}^{0} [u_{R,\alpha}]^p_{W^{s,p}(B_1)}\ \mathrm{d}t \le 1.$$

This function satisfies the assumption of Proposition 5.2, and we can do the same argument as in [7] to obtain

$$\sup_{t\in[-1/2,0]} [u_{R,\alpha}(\bullet,t)]_{C^\delta(B_{1/2})} \le C(n,s,p,\delta),$$

for a C independent of α and by scaling back we get

$$\sup_{\alpha-\frac{1}{2}\mathcal{M}_R^{2-p}\, R^{s\,p} \le t \le 0} [u(\bullet,t)]_{C^\delta(B_{R/2})} \le \frac{C}{R^\delta}\mathcal{M}_R.$$

By varying $\alpha \in [-R^{s\,p}(1-\mathcal{M}_R^{2-p}), 0]$, we get the desired result. □

We now address the improved regularity and start with the following modified version of [7, Proposition 5.1].

Proposition 5.5. *Assume* $p \ge 2$ *and* $0 < s < 1$. *Let* u *be a local weak solution of* $u_t + (-\Delta_p)^s u = 0$ *in* $B_2 \times (-2,0]$, *such that*

$$\|u\|_{L^\infty(B_2\times[-1,0])} + \sup_{t\in[-1,0]} \mathrm{Tail}_{p-1,sp}(u;0,2) \le 1.$$

Assume further that for some $0 < h_0 < 1/10$ *and* $\vartheta < 1$, $\beta \ge 2$ *such that* $(1+\vartheta\,\beta)/\beta < 1$, *we have*

$$\int_{T_0}^{T_1} \sup_{0<|h|\le h_0} \left\| \frac{\delta_h^2 u}{|h|^{\frac{1+\vartheta\,\beta}{\beta}}} \right\|^{\beta}_{L^\beta(B_{R+4\,h_0})} \mathrm{d}t < +\infty,$$

for a radius $4\,h_0 < R \le 1-5\,h_0$ *and two time instants* $-1 < T_0 < T_1 \le 0$. *Then*

$$\int_{T_0+\mu}^{T_1} \sup_{0<|h|<h_0} \left\| \frac{\delta_h^2 u}{|h|^{\frac{1+s\,p+\vartheta\,\beta}{\beta-1+p}}} \right\|^{\beta-1+p}_{L^{\beta-1+p}(B_{R-4\,h_0})} \mathrm{d}t + \frac{1}{\beta+1} \sup_{0<|h|<h_0} \left\| \frac{\delta_h u(\bullet,T_1)}{|h|^{\frac{1+\vartheta\,\beta}{\beta+1}}} \right\|^{\beta+1}_{L^{\beta+1}(B_{R-4\,h_0})}$$

$$\leq C \int_{T_0}^{T_1} \sup_{0<|h|<h_0} \left(\left\| \frac{\delta_h^2 u}{|h|^{\frac{1+\vartheta\,\beta}{\beta}}} \right\|_{L^\beta(B_{R+4\,h_0})}^{\beta} + 1 \right) \, \mathrm{d}t, \tag{5.6}$$

for every $0 < \mu < T_1 - T_0$. *Here* C *depends on the* n, h_0, s, p, μ *and* β.

Proof. The only major difference from the proof of Proposition 5.2 is in the estimation of term $\mathcal{I}_{11}$ and it can be treated in the exact same way as in the proof of [7, Proposition 5.1]. □

Using the previous proposition with the same type of modifications as in the proof of Theorem 5.4, we can state the following version of [7, Theorem 5.2].

Theorem 5.6. *Let* Ω *be a bounded and open set, let* $I = (t_0, t_1]$, $p \geq 2$ *and* $0 < s < 1$. *Suppose* u *is a local weak solution of*

$$u_t + (-\Delta_p)^s u = 0 \qquad \text{in } \Omega \times I,$$

such that $u \in L^\infty_{\mathrm{loc}}(I; L^\infty_{\mathrm{loc}}(\Omega)) \cap L^\infty_{\mathrm{loc}}(I; L^{p-1}_{sp}(\mathbb{R}^n))$. *Then* $u \in C^\delta_{x,\mathrm{loc}}(\Omega \times I)$ *for every* $0 < \delta < \Theta(s, p)$, *where* $\Theta(s, p)$ *is defined in (5.2).*
More precisely, for every $0 < \delta < \Theta(s, p)$, $R > 0$, $x_0 \in \Omega$ *and* T_0 *such that*

$$B_{2R}(x_0) \times [T_0 - 2\,R^{s\,p}, T_0] \Subset \Omega \times (t_0, t_1],$$

there exists a constant $C = C(n, s, p, \delta) > 0$ *such that*

$$\sup_{t \in \left[T_0 - \frac{R^{s\,p}}{2}, T_0\right]} [u(\cdot, t)]_{C^\delta(B_{R/2}(x_0))} \leq \frac{C}{R^\delta} \left(\|u\|_{L^\infty(B_{2R} \times [T_0 - R^{s\,p}, T_0])} + \sup_{t \in [T_0 - R^{sp}, T_0]} \mathrm{Tail}_{p-1,sp}(u; x_0, 2R) + 1 \right). \tag{5.7}$$

Now we modify the argument regarding the regularity in time (see [7, Proposition 6.2]).

Proposition 5.7. *Suppose that* u *is a local weak solution of*

$$\partial_t u + (-\Delta_p)^s u = 0, \qquad \text{in } B_2 \times (-2, 0],$$

such that

$$\|u\|_{L^\infty(B_2 \times [-1,0])} + \sup_{t \in [-1,0]} \mathrm{Tail}_{p-1,sp}(u; 0, 2) \leq 1,$$

and

$$\sup_{t \in [-1/2,0]} [u(\cdot, t)]_{C^\delta(B_{1/2})} \leq K_\delta, \qquad \text{for any } s < \delta < \Theta(s, p), \tag{5.8}$$

where $\Theta(s,p)$ is the exponent defined in (5.2). Then there is a constant $C = C(n, s, p, K_\delta, \delta) > 0$ such that

$$|u(x,t) - u(x,\tau)| \le C\,|t-\tau|^\gamma, \qquad \text{for every } (x,t), (x,\tau) \in Q_{\frac14,\frac14},$$

where

$$\gamma = \frac{1}{\frac{s\,p}{\delta} - (p-2)}.$$

In particular, $u \in C_t^\gamma(Q_{\frac14,\frac14})$ for any $\gamma < \Gamma(s,p)$, where $\Gamma(s,p)$ is the exponent defined in (5.2).

Proof. The only part that needs to be modified is the estimation of the nonlocal term J_2

$$J_2 := \int_{T_0}^{T_1} \iint_{(\mathbb{R}^n \setminus B_r(x_0)) \times B_{r/2}(x_0)} J_p(u(x,\tau) - u(y,\tau))\, \eta(x) \; \mathrm{d}\mu(x,y)\, \mathrm{d}\tau,$$

here $T_0, T_1 \in (t_0 - \theta, t_0)$ with $T_0 < T_1$. We recall that $0 < \theta < \frac18$, $x_0 \in B_{\frac14}$, and $r < \frac18$. Thus, $x \in B_{\frac{r}{2}}(x_0)$ implies $x \in B_{\frac{5}{16}}$.

For $y \in B_{\frac12}(0)$, assumption (5.8) implies

$$|u(x,\tau) - u(y,\tau)| \le K_\delta |x-y|^\delta.$$

For $y \in B_2(0) \setminus B_{\frac12}(0)$, the L^∞ bound on u implies

$$|u(x,\tau) - u(y,\tau)| \le 2 \le C(\delta)|x-y|^\delta.$$

Also notice that for $x \in B_{r/2}(x_0)$ and $y \in \mathbb{R}^n \setminus B_r(x_0)$, we have $|x-y| \ge \frac12 |y - x_0|$. Using these, we obtain

$$\begin{aligned}
J_2 &\le 2\,(T_1 - T_0)\,\|\eta\|_{L^\infty(B_{r/2}(x_0))} \sup_{t \in [-\frac12, 0]} \iint_{(\mathbb{R}^n \setminus B_r(x_0)) \times B_{r/2}(x_0)} \frac{|u(x,t) - u(y,t)|^{p-1}}{|x-y|^{n+s\,p}} \; \mathrm{d}y\, \mathrm{d}x \\
&\le 2\,(T_1 - T_0)\,\|\eta\|_{L^\infty(B_{r/2}(x_0))} \Big(\sup_{t \in [-\frac12, 0]} \iint_{(\mathbb{R}^n \setminus B_2) \times B_{r/2}(x_0)} \frac{|u(x,t) - u(y,t)|^{p-1}}{|x-y|^{n+sp}} \; \mathrm{d}y\, \mathrm{d}x \\
&\quad + C(\delta, K_\delta) \iint_{(B_2 \setminus B_r(x_0)) \times B_{r/2}(x_0))} |x-y|^{\delta(p-1)-n-sp}\, \mathrm{d}y\, \mathrm{d}x \Big) \\
&\le C\theta \Big(\sup_{t \in [-\frac12, 0]} \iint_{(\mathbb{R}^n \setminus B_2) \times B_{r/2}(x_0)} \frac{1 + |u(y,t)|^{p-1}}{|x_0 - y|^{n+sp}}\, \mathrm{d}y\, \mathrm{d}x \\
&\quad + \iint_{(B_2(0) \setminus B_r(x_0) \times B_{r/2}(x_0))} |x_0 - y|^{\delta(p-1)-n-sp}\, \mathrm{d}y\, \mathrm{d}x \Big) \\
&\le C\theta \int_{B_{r/2}(x_0)} \Big(\sup_{t \in [-\frac12, 0]} \int_{\mathbb{R}^n \setminus B_2} \frac{1 + |u(y,t)|^{p-1}}{|y|^{n+sp}}\, \mathrm{d}y
\end{aligned}$$

$$+\int_{B_2\setminus B_r(x_0)}|x_0-y|^{\delta(p-1)-n-sp}\,\mathrm{d}y\Big)\,\mathrm{d}x$$
$$\le C\,\theta\,r^n\Big(2^{-sp}+1+r^{\delta(p-1)-sp}\Big)\le C\,\theta\,r^{n-sp+\delta(p-1)}.$$
(since $\delta(p-1)-sp$ is not positive)

□

Finally, we are ready to prove a modified version of [7, Theorem 1.1], which is Theorem 5.1.

Proof of Theorem 5.1. Consider a cylinder $Q_{2\rho,2\rho^{sp}}(\tilde{x},\tau)\Subset\Omega\times I$, first, we prove the following type of bound on the Hölder seminorm in $Q_{\rho/4,\rho^{sp}/4}(\tilde{x},\tau)$, and later with the aid of a covering argument, we conclude the claim of the theorem.

Claim: For any $(x_1,\tau_1),\ (x_2,\tau_2)\in Q_{\rho/4,\rho^{sp}/4}(\tilde{x},\tau)$ we have

$$\begin{aligned}|u(x_1,\tau_1)-u(x_2,\tau_2)|&\le C\,\Big(\|u\|_{L^\infty(B_{2\rho}\times[T_0-\rho^{sp},T_0])}\\&+\sup_{t\in[T_0-\rho^{sp},T_0]}\mathrm{Tail}_{p-1,sp}(u;x_0,2\rho)+1\Big)\left(\frac{|x_1-x_2|}{\rho}\right)^{\delta}\\&+C\,\Big(\|u\|_{L^\infty(B_{2\rho}\times[T_0-R^{sp},T_0])}\\&+\sup_{t\in[T_0-\rho^{sp},T_0]}\mathrm{Tail}_{p-1,sp}(u;x_0,2\rho)+1\Big)^{p-1}\left(\frac{|\tau_1-\tau_2|}{\rho^{sp}}\right)^{\gamma}.\end{aligned}\tag{5.9}$$

The regularity in space variable has been proved in Theorem 5.6. To prove the part on time regularity, we set

$$\mathcal{M}_\rho(\tilde{x},\tau):=1+\|u\|_{L^\infty(Q_{2\rho,\rho^{sp}}(\tilde{x},\tau))}+\sup_{\tau-\rho^s\le t\le\tau}\mathrm{Tail}_{p-1,sp}(u;\tilde{x},2\rho)$$

and consider the rescaled functions

$$\tilde{u}_{\rho,\iota}(x,t):=\frac{1}{\mathcal{M}_\rho(\tilde{x},\tau)}u(\rho x+\tilde{x},\mathcal{M}_\rho(\tilde{x},\tau)^{2-p}\rho^{sp}t+\tau+\iota),$$

for $\iota\in(-\frac{\rho^{sp}}{4}(1-\mathcal{M}_\rho^{2-p}),0)$. Then $\tilde{u}_{\rho,\iota}(x,t)$ is a solution of

$$\partial_t\tilde{u}_{\rho,\iota}+(-\Delta_p)\tilde{u}_{\rho,\iota}=0,\qquad\text{in}\qquad Q_{2,2}.$$

Moreover, $\tilde{u}_{\rho,\iota}(x,t)$ satisfies the conditions of Proposition 5.7. Indeed by construction

$$\|\tilde{u}_{\rho,\iota}\|_{L^\infty(B_2\times[-1,0])}+\sup_{t\in[-1,0]}\mathrm{Tail}_{p-1,sp}(\tilde{u}_{\rho,\iota};0,2)\le 1$$

and the estimate (5.8) follows from (5.7) in Theorem 5.6. From Proposition 5.7, we obtain

$$\sup_{x\in B_{\frac{1}{4}}}[\tilde{u}_{\rho,\iota}(x,\bullet)]_{C^\gamma[-\frac{1}{4},0]}\le C,$$

with $C = C(n, s, p, \gamma)$ for every $0 < \gamma < \Gamma(s, p)$. By scaling back, this translates to

$$|u(x,t_1) - u(x,t_2)| \le \mathcal{M}_\rho(\tilde{x},\tau) C\Big(\frac{|t_1 - t_2|}{\rho^{sp}\mathcal{M}_\rho^{2-p}}\Big)^\gamma \quad \text{for}$$
$$(x,t_1),(x,t_2) \in Q_{\frac{\rho}{4},\frac{\rho^{sp}}{4}\mathcal{M}_\rho^{2-p}}(\tilde{x},\tau+\iota). \tag{5.10}$$

By varying ι with an argument similar to the proof of Theorem 1.2, we arrive at the claim (5.9). We have to point out that the Hölder constant does change, unlike what is suggested in the proof of [7, Theorem 1.1]. Here is a detailed computation

We split the time interval $[t_1, t_2]$ into $1 + \lfloor \mathcal{M}_\rho(\tilde{x},\tau)^{p-2} \rfloor$ pieces, say $[\tau_{i+1}, \tau_i]$, with $\tau_i - \tau_{i+1} = \frac{|t_2 - t_1|}{1 + \lfloor \mathcal{M}_\rho(\tilde{x},\tau)^{p-2} \rfloor}$, $\tau_0 = t_2$, and $\tau_{\lfloor 1+\mu^{p-2} \rfloor} = t_1$. Then using (5.10) and the triangle inequality, we get

$$\begin{aligned}
|u(x,t_2) - u(x,t_1)| &\le |u(x,t_2) - u(x,t_1)| \\
&\le \sum_{i=0}^{\lfloor \mathcal{M}_\rho^{p-2} \rfloor} |u(x_2,\tau_i) - u(x_2,\tau_{i+1})| \\
&\le C\mathcal{M}_\rho \sum_{i=0}^{\lfloor \mathcal{M}_\rho^{p-2} \rfloor} \Big(\frac{|\tau_i - \tau_{i+1}|}{R^{s\,p}\mathcal{M}_\rho^{2-p}}\Big)^\gamma \\
&= C\mathcal{M}_\rho(\tilde{x},\tau) \sum_{i=0}^{\lfloor \mathcal{M}_\rho^{p-2} \rfloor} \Big(\frac{|t_2 - t_1|}{R^{s\,p}\mathcal{M}_\rho^{2-p}(1 + \lfloor \mathcal{M}_\rho^{p-2} \rfloor)}\Big)^\gamma \\
&\le C\mathcal{M}_\rho \sum_{i=0}^{\lfloor \mathcal{M}_\rho^{p-2} \rfloor} \Big(\frac{|t_2 - t_1|}{R^{s\,p}}\Big)^\gamma \\
&\le C\mathcal{M}_\rho \Big[\mathcal{M}_\rho^{p-2}\Big(\frac{|t_2 - t_1|}{R^{s\,p}}\Big)^\gamma\Big] \le C\mathcal{M}_\rho^{p-1}\Big(\frac{|t_2 - t_1|}{R^{s\,p}}\Big)^\gamma.
\end{aligned}$$

Now use (5.9) in cylinders of the form

$$Q_{\frac{r}{4},\frac{r^{sp}}{4}}(y,t), \quad \text{for } (y,t) \in Q_{\sigma R,(\sigma R)^{sp}},$$

where the radius $r = \frac{R}{C(n,s,p,\sigma)}$ is so small, such that

$$Q_{2r,2r^{s\,p}}(y,t) \subset Q_{R,R^{sp}}.$$

Consider a sequence of points $(\tilde{x}_i, \tilde{\tau}_i)$ on the segment joining (x_1, τ_1) and (x_2, τ_2) such that

$$(\tilde{x}_i, \tilde{\tau}_i) \in Q_{\frac{r}{4},\frac{r^{sp}}{4}}(x_{i-1}, \tau_{i-1}).$$

Using (5.9) together with the triangle inequality, we obtain

$$\begin{aligned}|u(x_1,\tau_1)-u(x_2,\tau_2)| \le C\,\Big(&\|u\|_{L^\infty(Q_{R,R^{sp}(x_0,T_0)})}\\ &+\sup_{t\in[T_0-R^{sp},T_0]}\mathrm{Tail}_{p-1,sp}(u;x_0,R)+1\Big)\left(\frac{|x_1-x_2|}{R}\right)^{\delta}\\ &+C\,\Big(\|u\|_{L^\infty(Q_{R,R^{sp}(x_0,T_0)}}\\ &+\sup_{t\in[T_0-R^{sp},T_0]}\mathrm{Tail}_{p-1,sp}(u;x_0,R)+1\Big)^{p-1}\left(\frac{|\tau_1-\tau_2|}{R^{s\,p}}\right)^{\gamma},\end{aligned}$$

with $C = C(n, s, p, \delta, \gamma, \sigma)$, which is the desired result. □

6. Appendix B

Here we will justify the insertion of $u - v$ and $|u - v|^{p-2}(u - v)$ as test functions.

Proposition 6.1. *Let $B = B_R(x_0)$ be a ball of radius r, $B_2 = B_{\sigma R}(x_0)$ with $\sigma > 1$, and $I = (\tau_0, \tau_1]$ be an interval. Let $f \in L^{(p_s^\star)',p'}(B \times I)$ and assume that $u \in L^p(I, W^{s,p}(B_2)) \cap L^{p-1}(I; L^{p-1}_{sp}(\mathbb{R}^n)) \cap C(I; L^2(B))$ is a local weak solution of*

$$u_t + (-\Delta_p)^s u = f, \quad in\ B_2 \times I$$

with

$$\mathrm{Tail}_{p-1,sp}(u(\bullet, t); x_0, R) \in L^p(I).$$

(in particular, this will be the case under the stronger assumption $\sup_{t\in I} \mathrm{Tail}_{p,sp}(u(\bullet, t); x_0, R) < \infty$ that we use in this article.) Let $[T_0, T_1] \Subset I$ and let $v \in L^p([T_0, T_1], W^{s,p}(B_2)) \cap L^{p-1}([T_0, T_1]; L^{p-1}_{sp}(\mathbb{R}^n)) \cap C([T_0, T_1]; L^2(B))$ be a weak solution to

$$\begin{cases} v_t + (-\Delta_p)^s v = 0 & in\ \ B \times [T_0, T_1]\\ v = u & in\ \ (\mathbb{R}^n \setminus B) \times [T_0, T_1]\\ v(x, T_0) = u(x, T_0) & in\ \ B.\end{cases}$$

In addition, assume that F is a globally Lipschitz function with $F(0) = 0$, which is either bounded or $F(a) = a$. Then we have:

$$\begin{aligned}\int_{T_0}^{T_1}\iint_{\mathbb{R}^n\times\mathbb{R}^n}&\Big(J_p(u(x,t)-u(y,t)) - J_p(v(x,t)-v(y,t))\Big)\\ &\times\Big(F(u(x,t)-v(x,t)) - F(u(y,t)-v(y,t))\Big)\,\mathrm{d}\mu\,\mathrm{d}t\\ &+\int_B \mathcal{F}(u(x,t)-v(x,t))\,\mathrm{d}x\Big|_{T_0}^{T_1}\\ &=\int_{T_0}^{T_1}\int_B F(u-v)f\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\mathcal{F}(a) := \int_0^a f(t)\,\mathrm{d}t$ is the primitive function of F.

Proof. The proof is essentially the same as [7, Lemma 3], except that here we do not use a cutoff function and do not have the global boundedness of u in the ball. For simplicity, we assume $x_0 = 0$, $R = 1$ and $\sigma = 2$.

For a function $\varphi \in C((T_0, T_1); L^2(B)) \cap L^p((T_0, T_1); X_0^{s,p}(B, B_2))$, we use the following regularization of functions

$$\varphi^\varepsilon(x,t) := \frac{1}{\varepsilon}\int_{t-\frac{\varepsilon}{2}}^{t+\frac{\varepsilon}{2}} \zeta\left(\frac{t-\ell}{\varepsilon}\right)\varphi(x,\ell)\,\mathrm{d}\ell = \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta(-\sigma)\varphi(x, t+\varepsilon\sigma)\,\mathrm{d}\sigma,$$

where $\zeta(\sigma)$ is a smooth function with compact support in $(-\frac{1}{2}, \frac{1}{2})$ satisfying

$$|\zeta| \leq 1, \qquad \text{and} \qquad |\zeta'| \leq 8.$$

This regularization process gives us a test function $\varphi^\varepsilon \in C^1((T_0+\varepsilon, T_1-\varepsilon); L^2(B)) \cap L^p((T_0+\varepsilon, T_1-\varepsilon); X_0^{s,p}(B, B_2))$. Let $t_0 = T_0 + \varepsilon_0$ and $t_1 = T_1 - \varepsilon_0$ and we test the equation with φ^ε as above, for $\varepsilon < \frac{\varepsilon_0}{2}$. First, we will show the claim for the smaller interval $[t_0, t_1] \subset [T_0, T_1]$, and then through a limiting argument, prove the result for the whole interval. As in equation (3.5) in [7], we get

$$\begin{aligned}
&\int_{t_0}^{t_1}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\Big(J_p(u(x,t)-u(y,t))(\varphi^\varepsilon(x,t)-\varphi^\varepsilon(y,t)\Big)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad+\int_B\int_{t_0+\frac{\varepsilon}{2}}^{t_1-\frac{\varepsilon}{2}}\partial_t u^\varepsilon(x,t)\varphi(x,t)\,\mathrm{d}t\,\mathrm{d}x+\Sigma_u(\varepsilon)\\
&\quad=\int_B\big[u(x,t_0)\varphi(x,t_0)-u^\varepsilon\left(x,t_0+\frac{\varepsilon}{2}\right)\varphi(x,t_0+\frac{\varepsilon}{2})\big]\,\mathrm{d}x\\
&\qquad-\int_B\big[u(x,t_1)\varphi(x,t_1)-u^\varepsilon(x,t_1-\frac{\varepsilon}{2})\varphi(x,t_1-\frac{\varepsilon}{2})\big]\,\mathrm{d}x+\int_{t_0}^{t_1}\int_B\varphi^\varepsilon f\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

and we obtain a similar identity for v without $\int_{t_0}^{t_1}\int_B \varphi^\varepsilon f\,\mathrm{d}x\,\mathrm{d}t$ in the right-hand side. Here Σ_u is defined by

$$\begin{aligned}
\Sigma_u(\varepsilon) = &-\int_B\int_{t_0-\frac{\varepsilon}{2}}^{t_0+\frac{\varepsilon}{2}}\left(\frac{1}{\varepsilon}\int_{t_0}^{\ell+\frac{\varepsilon}{2}}u(x,t)\,\zeta\left(\frac{\ell-t}{\varepsilon}\right)\,\mathrm{d}t\right)\partial_\ell\varphi(x,\ell)\,\mathrm{d}\ell\,\mathrm{d}x\\
&-\int_B\int_{t_1-\frac{\varepsilon}{2}}^{t_1+\frac{\varepsilon}{2}}\left(\frac{1}{\varepsilon}\int_{\ell-\frac{\varepsilon}{2}}^{t_1}u(x,t)\,\zeta\left(\frac{\ell-t}{\varepsilon}\right)\,\mathrm{d}t\right)\partial_\ell\varphi(x,\ell)\,\mathrm{d}\ell\,\mathrm{d}x.
\end{aligned}$$

Observe that by using an integration by parts, the term $\Sigma_u(\varepsilon)$ can be rewritten as

$$\begin{aligned}
\Sigma_u(\varepsilon) = &-\int_B\left(\frac{1}{\varepsilon}\int_{T_0}^{T_0+\varepsilon}u(x,t)\,\zeta\left(\frac{T_0-t}{\varepsilon}+\frac{1}{2}\right)\,\mathrm{d}t\right)\varphi\left(x,T_0+\frac{\varepsilon}{2}\right)\,\mathrm{d}x\\
&+\int_B\int_{T_0-\frac{\varepsilon}{2}}^{T_0+\frac{\varepsilon}{2}}\left(\frac{1}{\varepsilon^2}\int_{T_0}^{\ell+\frac{\varepsilon}{2}}u(x,t)\,\zeta'\left(\frac{\ell-t}{\varepsilon}\right)\,\mathrm{d}t\right)\varphi(x,\ell)\,\mathrm{d}\ell\,\mathrm{d}x
\end{aligned}$$

$$+\int_B \left(\frac{1}{\varepsilon}\int_{T_1-\varepsilon}^{T_1} u(x,t)\,\zeta\left(\frac{T_1-t}{\varepsilon}-\frac{1}{2}\right)\,\mathrm{d}t\right)\varphi\left(x,T_1-\frac{\varepsilon}{2}\right)\,\mathrm{d}x$$
$$-\int_B\int_{T_1-\frac{\varepsilon}{2}}^{T_1+\frac{\varepsilon}{2}}\left(\frac{1}{\varepsilon^2}\int_{\ell-\frac{\varepsilon}{2}}^{T_1} u(x,t)\,\zeta'\left(\frac{\ell-t}{\varepsilon}\right)\,\mathrm{d}t\right)\varphi(x,\ell)\,\mathrm{d}\ell\,\mathrm{d}x, \tag{6.1}$$

where we also used that ζ has compact support in $(-1/2, 1/2)$. By subtracting the identities for u and v, we obtain

$$\int_{t_0}^{t_1}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\Big(J_p(u(x,t)-u(y,t))-J_p(v(x,t)-v(y,t))\Big)(\varphi^\varepsilon(x,t)-\varphi^\varepsilon(y,t))\,\mathrm{d}\mu\,\mathrm{d}t$$
$$+\int_B\int_{t_0+\frac{\varepsilon}{2}}^{t_1-\frac{\varepsilon}{2}}\partial_t(u-v)^\varepsilon(x,t)\varphi(x,t)\,\mathrm{d}t\,\mathrm{d}x+\Sigma_u(\varepsilon)-\Sigma_v(\varepsilon)$$
$$=\int_B\big[(u-v)(x,t_0)\varphi(x,t_0)-(u-v)^\varepsilon(x,t_0+\frac{\varepsilon}{2})\varphi(x,t_0+\frac{\varepsilon}{2})\big]\,\mathrm{d}x$$
$$-\int_B\big[(u-v)(x,t_1)\varphi(x,t_1)-(u-v)^\varepsilon(x,t_1-\varepsilon)\varphi(x,t_1-\frac{\varepsilon}{2})\big]\,\mathrm{d}x$$
$$+\int_{t_0}^{t_1}\int_B\varphi^\varepsilon(x,t)f(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Now we take φ to be $F(u^\varepsilon-v^\varepsilon)$. Observe that

$$\partial_t(u-v)^\varepsilon F(u^\varepsilon-v^\varepsilon)=\partial_t\mathcal{F}(u^\varepsilon-v^\varepsilon).$$

After an integration by parts, we get

$$\int_{t_0}^{t_1}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\Big(J_p(u(x,t)-u(y,t))-J_p(v(x,t)-v(y,t))\Big)$$
$$\times\left([F(u^\varepsilon-v^\varepsilon)(x,t)]^\varepsilon-[F(u^\varepsilon-v^\varepsilon)(y,t)]^\varepsilon\right)\,\mathrm{d}\mu\,\mathrm{d}t$$
$$+\int_B\mathcal{F}(u^\varepsilon-v^\varepsilon)\,\mathrm{d}x\Big]_{t_0+\frac{\varepsilon}{2}}^{t_1-\frac{\varepsilon}{2}}+\Sigma_u(\varepsilon)-\Sigma_v(\varepsilon)$$
$$=\int_B\big[(u-v)(x,t_0)F(u^\varepsilon-v^\varepsilon)(x,t_0)-(u-v)^\varepsilon\left(x,t_0+\frac{\varepsilon}{2}\right)$$
$$\times F(u^\varepsilon-v^\varepsilon)\left(x,t_0+\frac{\varepsilon}{2}\right)\big]\,\mathrm{d}x$$
$$-\int_B\big[(u-v)(x,t_1)F(u^\varepsilon-v^\varepsilon)(x,t_1)$$
$$-(u-v)^\varepsilon\left(x,t_1-\frac{\varepsilon}{2}\right)F(u^\varepsilon-v^\varepsilon)\left(x,t_1-\frac{\varepsilon}{2}\right)\big]\,\mathrm{d}x$$
$$+\int_{t_0}^{t_1}\int_B(F(u^\varepsilon-v^\varepsilon))^\varepsilon(x,t)f(x,t)\,\mathrm{d}x\,\mathrm{d}t:=\mathcal{I}_1-\mathcal{I}_2+\mathcal{I}_3. \tag{6.2}$$

We now wish to pass to the limit in $\mathcal{I}_1,\mathcal{I}_2$ and $\mathcal{I}_3$. Let $w=u-v$, we now treat $\mathcal{I}_1$. The fact that F is globally Lipschitz together with $F(0)=0$ implies $|F(t)|\leq C|t|$.

Therefore,

$$
\begin{aligned}
&|w(x,t_0)F(w^\varepsilon)(x,t_0) - w^\varepsilon(x,t_0+\frac{\varepsilon}{2})F(w^\varepsilon(x,t_0+\frac{\varepsilon}{2}))| \\
&\quad \le |\left(w(x,t_0) - w^\varepsilon\left(x,t_0+\frac{\varepsilon}{2}\right)\right) F(w^\varepsilon(x,t_0))| \\
&\qquad + |w^\varepsilon\left(x,t_0+\frac{\varepsilon}{2}\right)\left(F(w^\varepsilon(x,t_0)) - F\left(w^\varepsilon\left(x,t_0+\frac{\varepsilon}{2}\right)\right)\right)| \\
&\quad \le C\big[\,|(w(x,t_0) - w^\varepsilon(x,t_0+\frac{\varepsilon}{2}))w^\varepsilon(x,t_0)| \\
&\qquad + |w^\varepsilon(x,t_0+\frac{\varepsilon}{2})\left(w^\varepsilon(x,t_0) - w^\varepsilon\left(x,t_0+\frac{\varepsilon}{2}\right)\right)|\big],
\end{aligned}
$$

where C is the Lipschitz constant of F. After integrating and using Hölder's inequality, we obtain

$$
\begin{aligned}
\mathcal{I}_1 &= \int_B \big[w(x,t_0)F(w^\varepsilon)(x,t_0) - w^\varepsilon(x,t_0+\frac{\varepsilon}{2})F(w^\varepsilon)(x,t_0+\varepsilon)\big]\,\mathrm{d}x \\
&\le C\Big[\|w(\bullet,t_0) - w^\varepsilon\left(\bullet,t_0+\frac{\varepsilon}{2}\right)\|_{L^2(B)}\|w^\varepsilon(\bullet,t_0)\|_{L^2(B)} \\
&\quad + \|w^\varepsilon\left(\bullet,t_0+\frac{\varepsilon}{2}\right)\|_{L^2(B)}\|w^\varepsilon(\bullet,t_0) - w^\varepsilon(\bullet,t_0+\frac{\varepsilon}{2})\|_{L^2(B)}\Big].
\end{aligned}
$$

Since $w^\varepsilon \in C((T_0+\varepsilon_0, T_1-\varepsilon_0); L^2(B))$, uniformly, we have

$$
\lim_{\varepsilon\to 0}\|w^\varepsilon(\bullet,t_0+\frac{\varepsilon}{2})\|_{L^2(B)} = \|w(\bullet,t_0)\|_{L^2(B)}.
$$

Observe that

$$
\begin{aligned}
&\int_B |w(x,t_0) - w^\varepsilon(x,t_0+\frac{\varepsilon}{2})|^2\,\mathrm{d}x \\
&= \int_B \Big|\int_{-\frac12}^{\frac12} \zeta(-\sigma)\left[w(x,t_0) - w\left(x,t_0+\frac{\varepsilon}{2}+\varepsilon\sigma\right)\right]\,\mathrm{d}\sigma\Big|^2\,\mathrm{d}x \\
&\le \int_B \int_{-\frac12}^{\frac12} \big|\zeta(-\sigma)[w(x,t_0) - w(x,t_0+\frac{\varepsilon}{2}+\varepsilon\sigma)]\big|^2\,\mathrm{d}\sigma\,\mathrm{d}x \\
&= \int_{-\frac12}^{\frac12}\int_B \big|\zeta(-\sigma)[w(x,t_0) - w(x,t_0+\frac{\varepsilon}{2}+\varepsilon\sigma)]\big|^2\,\mathrm{d}x\,\mathrm{d}\sigma \\
&\le \int_{-\frac12}^{\frac12}\int_B \big|w(x,t_0) - w(x,t_0+\frac{\varepsilon}{2}+\varepsilon\sigma)\big|^2\,\mathrm{d}x\,\mathrm{d}\sigma \\
&\le \sup_{0\le t\le\varepsilon}\int_B \big|w(x,t_0) - w(x,t_0+t)\big|^2\,\mathrm{d}x
\end{aligned} \tag{6.3}
$$

which tends to zero since w is in $C([T_0,T_1], L^2(B))$. In a similar way, one can argue that

$$
\lim_{\varepsilon\to 0}\|w^\varepsilon(\bullet,t_0) - w^\varepsilon(\bullet,t_0+\frac{\varepsilon}{2})\|_{L^2(B)} = 0. \tag{6.4}
$$

Using the triangle inequality, we get

$$
\begin{aligned}
&\|w^{\varepsilon}(\cdot, t_0) - w^{\varepsilon}\left(\cdot, t_0 + \frac{\varepsilon}{2}\right)\|_{L^2(B)} \\
&\quad \le \|w^{\varepsilon}(\cdot, t_0) - w(\cdot, t_0)\|_{L^2(B)} + \|w(\cdot, t_0) - w^{\varepsilon}(\cdot, t_0 + \frac{\varepsilon}{2})\|_{L^2(B)},
\end{aligned}
$$

using a computation similar to (6.3), we obtain

$$
\|w^{\varepsilon}(\cdot, t_0) - w(\cdot, t_0)\|_{L^2(B)} \le \sup_{-\frac{\varepsilon}{2} \le t \le \frac{\varepsilon}{2}} \|w(\cdot, t_0 + t) - w(\cdot, t_0)\|_{L^2(B)}
$$

and

$$
\|w(\cdot, t_0) - w^{\varepsilon}(\cdot, t_0 + \frac{\varepsilon}{2})\|_{L^2(B)} \le \sup_{0 \le t \le \varepsilon} \|w(\cdot, t_0 + t) - w(\cdot, t_0)\|_{L^2(B)}.
$$

These two expressions converge to zero, since $w \in C([T_0, T_1], L^2(B))$ and $(t_0 - \varepsilon, t_1 + \varepsilon) \Subset (T_0, T_1)$. This shows that $\mathcal{I}_1$ converges to zero. In a similar way, one can argue that $\mathcal{I}_2$ tends to zero. For the term $\mathcal{I}_3$, we have

$$
\begin{aligned}
&\Big|\int_{t_0}^{t_1} \int_B \Big[(F(w^{\varepsilon}))^{\varepsilon} f - F(w) f\Big] \, \mathrm{d}x \, \mathrm{d}t\Big| \\
&\quad = \Big|\int_{t_0}^{t_1} \int_B \big(F(w^{\varepsilon}))^{\varepsilon} - F(w^{\varepsilon})\big) f + \big(F(w^{\varepsilon}) - F(w)\big) f \, \mathrm{d}x \, \mathrm{d}t\Big| \\
&\quad \le \int_{t_0}^{t_1} \int_B |(F(w^{\varepsilon}))^{\varepsilon} - F(w^{\varepsilon})||f(x,t)| + C|w^{\varepsilon} - w||f(x,t)| \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

The sequence w^{ε} is bounded in $L^{p_s^{\star}, p}(B \times (t_0, t_1))$; therefore, it has a weakly convergent subsequence. Using the pointwise convergence of w^{ε} to w, we get the weak convergence of $w^{\varepsilon} - w$ to zero. By the assumptions on q, r together with Hölder's inequality (2.11), $f(x,t)$ belongs to the dual space $L^{(p_s^{\star})', p'}(B \times (t_0, t_1))$. Therefore,

$$
\lim_{\varepsilon \to 0} \int_{t_0}^{t_1} \int_B |w^{\varepsilon} - w||f(x,t)| \, \mathrm{d}x \, \mathrm{d}t = 0.
$$

On the other hand,

$$
\begin{aligned}
&\int_{t_0}^{t_1} \int_B |(F(w^{\varepsilon}))^{\varepsilon} - F(w^{\varepsilon})||f(x,t)| \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_{t_0}^{t_1} \int_B \Big|\int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta(-\sigma)(F(w^{\varepsilon}(x, t + \varepsilon\sigma))) - F(w^{\varepsilon}(x,t)) \, \mathrm{d}\sigma\Big| |f(x,t)| \, \mathrm{d}x \, \mathrm{d}t \\
&\le \int_{t_0}^{t_1} \int_B \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta(-\sigma)|(F(w^{\varepsilon}(x, t + \varepsilon\sigma)) - F(w^{\varepsilon}(x,t))||f(x,t)| \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t \\
&\le C \int_{t_0}^{t_1} \int_B \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta(-\sigma)|(w^{\varepsilon}(x, t + \varepsilon\sigma) - w^{\varepsilon}(x,t)||f(x,t)| \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}
$$

$$\leq C \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{t_0}^{t_1} \int_B |(w^\varepsilon(x, t+\varepsilon\sigma) - w^\varepsilon(x,t)||f(x,t)| \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\sigma$$

$$\leq C \int_{-\frac{1}{2}}^{\frac{1}{2}} \big\| \, \|w^\varepsilon(x, t+\varepsilon\sigma) - w^\varepsilon(x,t)\|_{L^{p_s^\star}(B)} \big\|_{L^p(t_0,t_1)} \|f\|_{L^{(p_s^\star)',p'}(B\times(t_0,t_1))} \,\mathrm{d}\sigma.$$

Recall that the shift operator,

$$T(a)(g) := \|g(t+a)\|_{L^p((t_0,t_1))}$$

for a function $g \in L^p(t_0 - \varepsilon_0, t_1 + \varepsilon_0)$ is continuous for $-\varepsilon_0 \leq a \leq \varepsilon_0$. Hence, we get

$$\begin{aligned}\lim_{\varepsilon\to 0} \|w^\varepsilon(x,t)\|_{L^{p_s^\star,p}(B\times(t_0,t_1))} &= \lim_{\varepsilon\to 0} \|w^\varepsilon(x,t+\varepsilon\sigma)\|_{L^{p_s^\star,p}(B\times(t_0,t_1))} \\ &= \|w\|_{L^{p_s^\star,p}(B\times(t_0,t_1))}.\end{aligned}$$

Upon passing to a subsequence $w^\varepsilon(x, t+\varepsilon\sigma)$ and $w^\varepsilon(x,t)$ converge weakly in $L^{p_s^\star,p}(B\times(t_0,t_1))$, since they converge to $w(x,t)$ pointwise, we get the weak convergence

$$w^\varepsilon(x,t) \rightharpoonup w(x,t) \quad \text{and} \quad w^\varepsilon(x,t+\varepsilon\sigma) \rightharpoonup w(x,t) \quad \text{in} \quad L^{p_s^\star,p}(B\times(t_0,t_1)).$$

Combined with the convergence of the norms, this implies the strong convergence in the norm; in particular, we have

$$\big\| \, \|w^\varepsilon(x,t+\varepsilon\sigma) - w^\varepsilon(x,t)\|_{L^{p_s^\star}(B)} \big\|_{L^p(t_0,t_1)} \to 0.$$

Now we turn our attention to the terms on the left-hand side of (6.2). The terms $\Sigma_u(\varepsilon)$ and $\Sigma_v(\varepsilon)$ converge to zero. To show this, we start with the following computation, borrowed from [7, Lemma 3.3]. Using a suitable change of variables in (6.1) and recalling $\varphi = F(w^\varepsilon)$, we can also write

$$\begin{aligned}\Sigma_u(\varepsilon) = &-\int_B \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} u(x, t_0 - \varepsilon\rho + \frac{\varepsilon}{2})\,\zeta(\rho)\,\mathrm{d}\rho\right) F(w^\varepsilon)\left(x, t_0 + \frac{\varepsilon}{2}\right) \mathrm{d}x \\ &+ \int_B \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{\rho} u(x, \varepsilon\,\rho + t_0 - \varepsilon\,\sigma)\,\zeta'(\sigma)\,\mathrm{d}\sigma\right) F(w^\varepsilon)(x, \varepsilon\,\rho + t_0)\,\mathrm{d}\rho\,\mathrm{d}x \\ &+ \int_B \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} u(x, t_1 - \varepsilon\rho - \frac{\varepsilon}{2})\,\zeta(\rho)\,\mathrm{d}\rho\right) F(w^\varepsilon)\left(x, t_1 - \frac{\varepsilon}{2}\right) \mathrm{d}x \\ &- \int_B \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{\frac{1}{2}}^{\rho} u(x, \varepsilon\,\rho + T_1 - \varepsilon\,\sigma)\,\zeta'(\sigma)\,\mathrm{d}\sigma\right) F(w^\varepsilon)(x, \varepsilon\,\rho + T_1)\,\mathrm{d}\rho\,\mathrm{d}x \\ := &\ \Sigma_u^1(\varepsilon) + \Sigma_u^2(\varepsilon) + \Sigma_u^3(\varepsilon) + \Sigma_u^4(\varepsilon).\end{aligned} \tag{6.5}$$

In a similar way to the argument for convergence of $\mathcal{I}_1$, we can see that

$$\lim_{\varepsilon\to 0} \Sigma_u^1(\varepsilon) = -\int_B u(x,t_0) F(w)(x,t_0)\,\mathrm{d}x.$$

We spell out the details of the arguments for convergence of $\Sigma_u^2(\varepsilon)$.

$$
\begin{aligned}
&\left|\Sigma_u^2(\varepsilon) - \int_B u(x,t_0)F(w)(x,t_0)\,\mathrm{d}x\right| \\
&= \left|\int_B \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{\rho} (u(x,\varepsilon\rho + t_0 - \varepsilon\sigma) - u(x,t_0))\,\zeta'(\sigma)\,\mathrm{d}\sigma\right) F(w^\varepsilon)(x,\varepsilon\rho + t_0)\,\mathrm{d}\rho\,\mathrm{d}x\right. \\
&\quad \left. + \int_B \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{\rho} u(x,t_0)\zeta'(\sigma)\,\mathrm{d}\sigma\right)\left(F(w^\varepsilon)(x,\varepsilon\rho + t_0) - F(w)(x,t_0)\right)\mathrm{d}\rho\,\mathrm{d}x\right| \\
&\le \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\rho} \left(\int_B \left|u(x,\varepsilon\rho + t_0 - \varepsilon\sigma) - u(x,t_0))F(w^\varepsilon)(x,\varepsilon\rho + t_0)\zeta'(\sigma)\right|\mathrm{d}x\right)\mathrm{d}\sigma\,\mathrm{d}\rho \\
&\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_B |\zeta(\rho)| \left|u(x,t_0)\left(F(w^\varepsilon)(x,\varepsilon\rho + t_0) - F(w)(x,t_0)\right)\right|\mathrm{d}x\,\mathrm{d}\rho \\
&\le 8\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\rho} \|u(\bullet,\varepsilon\rho + t_0 - \varepsilon\sigma) - u(\bullet,t_0)\|_{L^2(B)} \|F(w^\varepsilon)(\bullet,\varepsilon\rho + t_0)\|_{L^2(B)}\,\mathrm{d}\sigma\,\mathrm{d}\rho \\
&\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}} \|u(\bullet,t_0)\|_{L^2(B)} \|F(w^\varepsilon)(\bullet,\varepsilon\rho + t_0) - F(w)(\bullet,t_0)\|_{L^2(B)}\,\mathrm{d}\rho \\
&\le 8 \sup_{t_0 \le t \le t_0+\varepsilon} \|u(\bullet,t_0+t) - u(\bullet,t_0)\|_{L^2(B)} \sup_{t_0-\frac{\varepsilon}{2} \le t \le t_0+\frac{\varepsilon}{2}} \|F(w^\varepsilon)(\bullet,t_0+t)\|_{L^2(B)} \\
&\quad + C\|u(\bullet,t_0)\|_{L^2(B)} \sup_{t_0-\frac{\varepsilon}{2} \le t \le t_0+\frac{\varepsilon}{2}} \|w^\varepsilon(\bullet,t_0+t) - w(\bullet,t_0)\|_{L^2(B)},
\end{aligned}
$$

where C is the Lipschitz constant of F. We have used $|\zeta| \le 1$ and $|\zeta'| \le 8$ in the computation. Since $u \in C([T_0, T_1]; L^2(B))$, we get

$$
\lim_{\varepsilon \to 0} \sup_{t_0 \le t \le t_0+\varepsilon} \|u(\bullet,t_0+t) - u(\bullet,t_0)\|_{L^2(B)} = 0.
$$

Using a computation similar to (6.3), we obtain

$$
\begin{aligned}
&\sup_{t_0-\frac{\varepsilon}{2} \le t \le t_0+\frac{\varepsilon}{2}} \|w^\varepsilon(\bullet,t_0+t) - w(\bullet,t_0)\|_{L^2(B)} \\
&\quad \le \sup_{t_0-\varepsilon \le t \le t_0+\varepsilon} \|w(\bullet,t_0+t) - w(\bullet,t_0)\|_{L^2(B)}.
\end{aligned}
$$

This converges to zero since $w \in C([T_0, T_1]; L^2(B))$, and $(t_0 - \varepsilon, t_1 + \varepsilon) \Subset (T_0, T_1)$ due to the choice of ε. In conclusion,

$$
\lim_{\varepsilon \to 0} \Sigma_u^1(\varepsilon) + \Sigma_u^2(\varepsilon) = 0.
$$

In a similar way, one can argue that

$$
\lim_{\varepsilon \to 0} \Sigma_u^3(\varepsilon) + \Sigma_u^4(\varepsilon) = 0.
$$

Hence, $\lim_{\varepsilon \to 0} \Sigma_u(\varepsilon) = 0$. The treatment of $\Sigma_v(\varepsilon)$ is similar.

The term

$$\int_B \mathcal{F}(u^\varepsilon - v^\varepsilon)\,\mathrm{d}x\Big]_{t_0+\frac{\varepsilon}{2}}^{t_1-\frac{\varepsilon}{2}} = \int_B \mathcal{F}(w^\varepsilon)\left(x, t_1 - \frac{\varepsilon}{2}\right)\,\mathrm{d}x - \int_B \mathcal{F}(w^\varepsilon)\left(x, t_0 + \frac{\varepsilon}{2}\right)\,\mathrm{d}x$$

converges to

$$\int_B \mathcal{F}(w)(x, t_1)\,\mathrm{d}x - \int_B \mathcal{F}(w)(x, t_0)\,\mathrm{d}x.$$

To show this, we consider two cases.

Case A: F is bounded. In this case, $\mathcal{F}$ is globally Lipschitz, that is, $|\mathcal{F}(a)-\mathcal{F}(b)| \leq C|a-b|$; therefore,

$$\begin{aligned}
&\Big|\int_B \mathcal{F}\left(w^\varepsilon\left(x, t_0 + \frac{\varepsilon}{2}\right)\right) - \mathcal{F}(w(x, t_0))\,\mathrm{d}x\Big| \\
&\leq \int_B C|w^\varepsilon(x, t_0 + \frac{\varepsilon}{2}) - w(x, t_0)|\,\mathrm{d}x \\
&\leq C|B|^{\frac{1}{2}}\|w^\varepsilon\left(\bullet, t_0 + \frac{\varepsilon}{2}\right) - w(\bullet, t_0)\|_{L^2(B)}\,\mathrm{d}x,
\end{aligned}$$

which converges to zero as was explained before, see (6.4).

Case B: In this case, we have $\mathcal{F}(a) = a^2$. Therefore,

$$\begin{aligned}
&\Big|\int_B \mathcal{F}\Big(w^\varepsilon(x, t_0 + \frac{\varepsilon}{2}\Big)\Big) - \mathcal{F}(w(x, t_0))\,\mathrm{d}x\Big| \\
&\leq \int_B |w^\varepsilon\left(x, t_0 + \frac{\varepsilon}{2}\right)\Big)^2 - w(x, t_0)^2|\,\mathrm{d}x \\
&\leq \int_B |w^\varepsilon\left(x, t_0 + \frac{\varepsilon}{2}\right)\Big) - w(x, t_0)||w^\varepsilon\left(x, t_0 + \frac{\varepsilon}{2}\right)\Big) - w(x, t_0)|\,\mathrm{d}x \\
&\leq \|w^\varepsilon\left(\bullet, t_0 + \frac{\varepsilon}{2}\right)\Big) - w(\bullet, t_0)\|_{L^2(B)}\|w^\varepsilon\left(\bullet, t_0 + \frac{\varepsilon}{2}\right)\Big) + w(\bullet, t_0)\|_{L^2(B)}
\end{aligned}$$

and since $w \in C([T_0, T_1]; L^2(B))$, with an argument similar to the treatment of $\mathcal{I}_1$, as we let ε go to zero this term converges to zero.

Now we discuss the convergence of the nonlocal term. Our treatment is similar to the argument in [7, Appendix B]. The aim is to show that the following converges to zero.

$$\begin{aligned}
\int_{t_0}^{t_1}\iint_{\mathbb{R}^n\times\mathbb{R}^n} &(J_p(u(x)-u(y)) - J_p(v(x)-v(y))) \times \big[(F(w^\varepsilon(x,t)))^\varepsilon - F(w(x,t) \\
&- \big((F(w^\varepsilon(y,t)))^\varepsilon - F(w(y,t))\big)\big]\,\mathrm{d}\mu\,\mathrm{d}t.
\end{aligned}$$

We split it into the two parts

$$\begin{aligned}
&\int_{t_0}^{t_1} \iint_{B_2\times B_2} (J_p(u(x)-u(y)) - J_p(v(x)-v(y))) \times \big[(F(w^\varepsilon(x,t)))^\varepsilon - F(w(x,t) \\
&\quad - \big((F(w^\varepsilon(y,t)))^\varepsilon - F(w(y,t))\big)\big] \,\mathrm{d}\mu \,\mathrm{d}t \\
&\quad + 2\int_{t_0}^{t_1} \iint_{B\times(\mathbb{R}^n\setminus B_2)} (J_p(u(x)-u(y)) - J_p(v(x)-v(y))) \\
&\quad \times \big[(F(w^\varepsilon(x,t)))^\varepsilon - F(w(x,t)\big] \,\mathrm{d}\mu \,\mathrm{d}t \\
&:= \Theta_1(\varepsilon) + 2\Theta_2(\varepsilon).
\end{aligned}$$

Here we have used the boundary condition $u = v (w = 0)$ for $y \in \mathbb{R}^n\setminus B$. Since $|F(a) - F(b)| \le C|a-b|$, we have

$$\int_{t_0}^{t_1} \|(F(w^\varepsilon))^\varepsilon\|^p_{W^{s,p}(B_2)} \,\mathrm{d}t \le C\int_{t_0}^{t_1} \|w^\varepsilon\|^p_{W^{s,p}(B_2)} \,\mathrm{d}t.$$

After passing to a subsequence this sequence converges weakly in $L^p((t_0,t_1); W^{s,p}(B_2))$ to $F(w(x,t))$ or in another words

$$\frac{(F(w^\varepsilon(x,t))^\varepsilon - (F(w^\varepsilon(y,t)))^\varepsilon}{|x-y|^{\frac{n}{p}+s}}$$

converges weakly in $L^p\big((t_0,t_1); L^p(B_2\times B_2)\big)$, and since

$$\frac{J_p(u(x)-u(y)) - J_p(v(x)-v(y))}{|x-y|^{\frac{n}{p'}+(p-1)s}}$$

belongs to $L^{p'}\big((t_0,t_1); L^{p'}(B_2\times B_2)\big)$, we get the desired convergence for $\Theta_1(\varepsilon)$. Now for $\Theta_2(\varepsilon)$ consider

$$\mathcal{G}(x,t) := \int_{\mathbb{R}^n\setminus B_2} \frac{J_p(u(x)-u(y)) - J_p(v(x)-v(y))}{|x-y|^{n+sp}} \,\mathrm{d}y.$$

Then for almost every $x \in B$,

$$\begin{aligned}
|\mathcal{G}(x,t)| &\le C(n,s,p)\int_{\mathbb{R}^n\setminus B_2} \frac{|u(x,t)|^{p-1} + |u(y,t)|^{p-1} + |v(x,t)|^{p-1} + |v(y,t)|^{p-1}}{|y|^{n+sp}} \,\mathrm{d}y \\
&\le C\big(2\mathrm{Tail}_{p-1,sp}(u(\bullet,t);0,2)^{p-1} + |u(x,t)|^{p-1} + |v(x,t)|^{p-1}\big).
\end{aligned} \tag{6.6}$$

The terms $|u(x,t)|^{p-1}$ and $|v(x,t)|^{p-1}$ belongs to $L^{p'}\big((t_0,t_1); L^{p'}(B)\big)$ since $u, v \in L^p((t_0,t_1); L^p(B))$. The tail term its independent of x and belongs to $L^{p'}(t_0,t_1)$ by the assumption

$$\int_{T_0}^{T_1} \big(\mathrm{Tail}_{p-1,sp}(u(\bullet,t);0,2)\big)^{p'} \,\mathrm{d}t \le \infty.$$

Thus, $\mathcal{G}(x,t) \in L^{p'}([T_0, T_1]; L^{p'}(B_2))$ and as before after extracting a subsequence:

$$F(w^\varepsilon(x,t))^\varepsilon \rightharpoonup F(w(x,t) \quad \text{in } L^p([t_0, t_1]; L^p(B)).$$

This shows that

$$\Theta_2(\varepsilon) = \int_{t_0}^{t_1} \int_B \mathcal{G}(x,t)\big(F(w^\varepsilon(x,t))^\varepsilon - F(w(x,t)\big)$$

converges to zero.

Finally, we let ε_0 go to zero to get the desired result for $[T_0, T_1]$. We need to show that the following converge to zero as ε_0 tends to 0.

$$\begin{aligned}
\mathcal{J}_1 &:= \int_B \mathcal{F}(w(x,T_0)) - \mathcal{F}(w(x, T_0 + \varepsilon_0))\, \mathrm{d}x,\\
\mathcal{J}_2 &:= \int_B \mathcal{F}(w(x,T_1)) - \mathcal{F}(w(x, T_1 - \varepsilon_0))\, \mathrm{d}x,\\
\mathcal{J}_3 &:= \int_{T_0}^{T_0+\varepsilon_0} \int_B F(w(x,t)) f(x,t)\, \mathrm{d}x\, \mathrm{d}t,\\
\mathcal{J}_4 &:= \int_{T_1-\varepsilon_0}^{T_1} \int_B F(w(x,t)) f(x,t)\, \mathrm{d}x\, \mathrm{d}t,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}_1 := \int_{T_0}^{T_0+\varepsilon_0} \iint_{\mathbb{R}^n\times\mathbb{R}^n} &\Big(\frac{J_p(u(x,t) - u(u,t)) - J_p(v(x,t) - v(y,t))}{|x-y|^{n+sp}}\Big)\\
&\times\Big(F(w(x,t)) - F(w(y,t))\Big)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}_2 := \int_{T_1-\varepsilon_0}^{T_1} \iint_{\mathbb{R}^n\times\mathbb{R}^n} &\Big(\frac{J_p(u(x,t) - u(u,t)) - J_p(v(x,t) - v(y,t))}{|x-y|^{n+sp}}\Big)\\
&\times\Big(F(w(x,t)) - F(w(y,t))\Big)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t.
\end{aligned}$$

The arguments will be reminiscent of the ideas in the previous part.

We start with $\mathcal{J}_2$, in the case of a bounded F, $\mathcal{F}$ is globally Lipschitz and we have

$$\begin{aligned}
\big|\mathcal{J}_2\big| &\le \int_B \big|\mathcal{F}(w(x,T_1)) - \mathcal{F}(w(x,T_1-\varepsilon_0))\big|\, \mathrm{d}x \le C \int_B |w(x,T_1) - w(x, T_1 - \varepsilon_0)|\, \mathrm{d}x\\
&\le C|B|^{\frac{1}{2}} \|w(\bullet, T_1) - w(\bullet, T_1 - \varepsilon_0)\|_{L^2(B)}.
\end{aligned}$$

This converges to 0 since $w \in C([T_0, T_1]; L^2(B))$, in the case of $F(a) = a$, we have

$$\begin{aligned}
\big|\mathcal{J}_2\big| &\le \int_B \big|\mathcal{F}(w(x,T_1)) - \mathcal{F}(w(x,T_1-\varepsilon_0))\big|\, \mathrm{d}x \le \int_B \big|w(x,T_1)^2 - w(x,T_1-\varepsilon_0)^2\big|\, \mathrm{d}x\\
&\le \int_B \big|w(x,T_1) - w(x,T_1-\varepsilon_0)\big|\big|w(x,T_1) + w(x,T_1-\varepsilon_0)\big|\, \mathrm{d}x\\
&\le \|w(\bullet,T_1) - w(\bullet,T_1-\varepsilon_0)\|_{L^2(B)} \|w(\bullet,T_1) + w(\bullet,T_1-\varepsilon_0)\|_{L^2(B)}.
\end{aligned}$$

Again since $w \in C([T_0, T_1]; L^2(B))$, this term converges to 0.

$\mathcal{J}_1$ can be treated in a similar way. For the term $\mathcal{J}_4$, using $|F(a)| \le C|a|$ we get

$$\left|\mathcal{J}_4\right| \le C \int_{T_1-\varepsilon_0}^{T_1} \int_B |w(x,t)||f(x,t)| \,\mathrm{d}x \,\mathrm{d}t.$$

Since $w \in L^{p_s^\star, p}(B \times [T_0, T_1])$ and $f \in L^{(p_s^\star)', p'}(B \times [T_0, T_1])$, using Hölder's inequality (2.11), one can see that

$$w(x,t) f(x,t) \in L^1(B \times [T_0, T_1]).$$

Now using the absolute continuity of the integral for integrable functions, we can conclude that $\mathcal{J}_4$ converges to 0. The reasoning for convergence of $\mathcal{J}_3$ is similar.

Now we turn our attention to the nonlocal terms.

$$\begin{aligned}\mathcal{N}_2 = & \int_{T_1-\varepsilon_0}^{T_1} \iint_{B_2 \times B_2} \Big(\frac{J_p(u(x,t)-u(y,t)) - J_p(v(x,t)-v(y,t))}{|x-y|^{n+sp}}\Big) \\ & \times \Big(F(w(x,t)) - F(w(y,t))\Big) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \\ & + 2 \int_{T_1-\varepsilon_0}^{T_1} \iint_{B \times (\mathbb{R}^n \setminus B_2)} \frac{J_p(u(x,t)-u(y,t)) - J_p(v(x,t)-v(y,t))}{|x-y|^{n+sp}} \\ & F(w(x,t)) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t := \Theta_1 + 2\Theta_2.\end{aligned}$$

First, we treat Θ_1. Notice that since $u, v \in L^p([T_0, T_1]; W^{s,p}(B_2))$ we have

$$\frac{J_p(u(x,t)-u(y,t)) - J_p(v(x,t)-v(y,t))}{|x-y|^{\frac{n}{p'}+(p-1)s}} \in L^{p'}([T_0, T_1]; L^{p'}(B_2 \times B_2)]),$$

and using Lipschitz continuity of F and the fact that $w \in L^p([T_0, T_1]; W^{s,p}(B_2))$, we have

$$\frac{F(w(x,t)) - F(w(y,t))}{|x-y|^{\frac{n}{p}+s}} \in L^p([T_0, T_1]; L^p(B_2 \times B_2)).$$

This implies that the integrand involved in Θ_1 belongs to $L^1([T_0, T_1]; L^1(B_2 \times B_2))$. And similar to the treatment of $\mathcal{J}_4$, since the volume of the integration region is shrinking to 0, Θ_1 converges to 0. To deal with Θ_2, notice that

$$F(w(x,t)) \in L^p([T_0, T_1]; L^p(B))$$

and define

$$\mathcal{G}(x,t) := \int_{\mathbb{R}^n \setminus B_2} \frac{J_p(u(x,t)-u(y,t)) - J_p(v(x,t)-v(y,t))}{|x-y|^{n+sp}} \,\mathrm{d}y.$$

We can estimate this integration in terms of the tail, that is,

$$\left|\mathcal{G}(x,t)\right| \le C(n,s,p)\Big(\mathrm{Tail}_{p-1,sp}(u(\bullet,t); 0, 2) + |u(x,t)|^{p-1} + |v(x,t)|^{p-1}\Big),$$

see, for example, (6.6). Therefore, $\mathcal{G}(x,t) \in L^{p'}([T_0, T_1]; L^{p'}(B))$. Hence, using Hölder's inequality

$$\mathcal{G}(x,t)F(x,t) \in L^1([T_0, T_1]; L^1(B)).$$

This concludes the result. $\mathcal{N}_1$ can be treated in an exactly similar manner. □

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