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Large time behavior of signed fractional porous media equations on bounded domains

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Abstract. Following the methodology of Brasco (Adv Math 394:108029, 2022), we study the long-time behavior for the signed fractional porous medium equation in open bounded sets with smooth boundary. Homogeneous exterior Dirichlet boundary conditions are considered. We prove that if the initial datum has sufficiently small energy, then the solution, once suitably rescaled, converges to a nontrivial constant sign solution of a sublinear fractional Lane–Emden equation. Furthermore, we give a nonlocal sufficient energetic criterion on the initial datum, which is important to identify the exact limit profile, namely the positive solution or the negative one.

Contents

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1. Introduction

In this paper, we will achieve some stabilization results for solutions to an initial boundary value problem for the *Fractional Porous Medium Equation* (FPME for short), of the form

$$\begin{cases} \partial_t u = -(-\Delta)^s(|u|^{m-1}u), & \text{in } Q := \Omega \times (0,+\infty), \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega \times (0,+\infty), \\ u(\cdot,0) = u_0, & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here we consider the porous medium regime, i.e., $m > 1$, we assume $0 < s < 1$, and Ω is a bounded open set of $\mathbb{R}^N$. A broad theory has been developed for this problem under several aspects (existence, uniqueness, regularity etc.), see for instance [2,3,5–7]. The main result of this paper concerns solutions emanating from initial data u_0 for which the energy functional

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y - \frac{m}{(m+1)(m-1)}\int_\Omega |\varphi|^{\frac{m+1}{m}}\,\mathrm{d}x$$

does not exceed its first excited level when the choice $\varphi = |u_0|^{m-1}u_0$ is made. By following the methodology of [13], in which the local case was considered, we compute the large time asymptotic profile of such solutions in this non-local framework. As in the local case, sign-changing initial data are included in the analysis: irrespective of their sign, if their energy is small enough then they give rise to solutions that in the large time limit are asymptotic to functions with a spatial profile arising in the energy minimization.

For a precise statement, we need to introduce, for all $q \in (1,2)$ and $\alpha \in (0,+\infty)$, the functional defined on the fractional Sobolev space $\mathcal{D}_0^{s,2}(\Omega)$ (see Sect. 2 for definitions) by

$$\mathscr{F}^s_{q,\alpha}(\varphi) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y - \frac{\alpha}{q}\int_\Omega |\varphi|^q\,\mathrm{d}x, \tag{1.2}$$

whose critical points, by definition, are the weak solutions of the Lane-Emden equation

$$(-\Delta)^s\varphi = \alpha\,|\varphi|^{q-2}\varphi, \qquad \text{in } \Omega, \tag{1.3}$$

with homogeneous Dirichlet boundary conditions. It is known [17] that the minimal energy

$$\Lambda_1 = \min\{\mathscr{F}^s_{q,\alpha}(\varphi) \,:\, \varphi \in \mathcal{D}_0^{s,2}(\Omega)\}, \tag{1.4}$$

is achieved by a solution with constant sign, that it is unique (up to the sign). Also, we set

$$\Phi(u) = |u|^{m-1}u$$

and we observe that $\Phi^{-1}(\varphi) = |\varphi|^{q-2}\varphi$ where $q = (m+1)/m$. Now, following [13], we define the *second critical energy level*, or *first excited level*, as it follows

$$\Lambda_2 = \inf \left\{ \Lambda > \Lambda_1 \,:\, \Lambda \text{ is a critical value of } \mathscr{F}^s_{q,\alpha} \right\}.$$

It turns out that there is a gap between this value and (1.4), i.e., we have $\Lambda_1 < \Lambda_2$ if the domain is smooth (see Corollary 3.1). The solution to problem (1.1) has the following stabilization property.

Theorem 1.1. *Let $m > 1$, $0 < s < 1$, and let Ω be a bounded open set in $\mathbb{R}^N$ with $C^{1,1}$ boundary. Given $u_0 \in L^{m+1}(\Omega)$, with $\Phi(u_0) \in \mathcal{D}^{s,2}_0(\Omega)$ and*

$$\mathscr{F}^s_{q,\alpha}(\Phi(u_0)) < \Lambda_2, \text{ where } q = \frac{m+1}{m} \text{ and } \alpha = \frac{1}{m-1},$$

and let u be the weak solution of the fractional porous media equation (1.1) with initial datum u_0. Then,

$$\lim_{t\to\infty} \|t^\alpha u(\cdot,t) - U\|_{L^{m+1}(\Omega)} = 0,$$

where $\Phi(U) \in \{w_\Omega, -w_\Omega\}$ and w_Ω is the positive minimiser of $\mathscr{F}^s_{q,\alpha}$ on $\mathcal{D}^{s,2}_0(\Omega)$.

Besides on Ω, the function w_Ω that achieves the minimum in (1.4) depends on m (through α and q) and on s; we refer to the material in Sect. 3 for its existence and uniqueness, that are however well known. Recall that, in the case of nonnegative data u_0, it is also well-known (see [3,5]) that the solution u stabilizes toward the so-called *Friendly Giant*, following the denomination due to Dahlberg and Kenig for the standard porous medium equation [15] (see also [21, Sec. 5.9]), namely

$$S(x,t) = t^{-\alpha} w_\Omega(x)^{q-1}.$$

That is a separate variable solution taking $+\infty$ as initial value. In particular, in [3, 5] various interesting results are shown, related to the finer problem of the sharp convergence rate of the relative error, a question that was also faced in the classical paper [1].

As said, Theorem 1.1 can be proved via the approach used in [13] to deal with the local problem, thanks to a Lyapunov-type property of the energy functional (1.2): namely, that

$$t \longmapsto \mathscr{F}^s_{q,\alpha}(\Phi(v(\cdot,t))) \text{ is non-increasing,}$$

whenever v is an energy solution (see Sect. 4 for precise definitions) of the initial boundary value problem for the rescaled equation

$$\partial_t v + (-\Delta)^s \Phi(v) = \alpha v. \tag{1.5}$$

This property is inferred, in this paper, from an entropy–entropy dissipation estimate in Sect. 4. In order to prove it, we produce solutions by the classical Euler implicit time discretization scheme. That has the advantage of providing a discrete version of the desired inequality in which we can pass to the limit. We prefer this approach to considering solutions to a uniformly parabolic approximation, as done in [13] for the local problem, mainly because that would require $C^{1,\alpha}$ estimates for the non-local operators that are obtained by regularizing the *signed* FPME; incidentally, we mention that strong results of this type can be found in [2] in the case of *non-negative* solutions.

We recall here in brief the use of the Lyapunov property for the proof of Theorem 1.1. Given a solution u of (1.1), the equation (1.5) for the function $v(x,t) = e^{\alpha t} u(x, e^t - 1)$ describes a system that evolves, irrespective of the starting conditions, to fixed points, *i.e.*, states of the form $\Phi^{-1}(w)$ with w being a critical point of the energy functional. Because of the isolation of the energy minimizing solutions $\pm w_\Omega$, proved in [17], this and the Lyapunov property imply that the disconnected set $\{\pm\Phi^{-1}(w_\Omega)\}$ has a non-empty basin of attraction, including all initial states with energy smaller than the first excited level. (The isolation property implies some restriction on boundary regularity: assumptions weaker than those made in the our main statement are also feasible, but that is not the object of this paper.)

Eventually, by the compactness of the relevant Sobolev embedding, by energy coercivity, and by the Lyapunov property, orbits are relatively compact; thus, the ω-limit is connected and the only possible cluster point of the orbit emanating from an initial state u_0 below the energy threshold is either $\Phi^{-1}(w_\Omega) = w_\Omega^{q-1}$ or $\Phi^{-1}(-w_\Omega) = -w_\Omega^{q-1}$.

Yet, meeting the threshold requirement in Theorem 1.1 implies no restriction on the sign that u_0 should take in Ω, nonetheless: indeed, sign-changing initial data with energy as small as required in Theorem 1.1 exist, see Proposition 3.2 and Corollary 3.1. The relevance in energy of the nodal sets, instead, enters in predicting which one of the two possible limit profiles the orbit will accumulate to. The following Proposition, which is the non-local counterpart of an analogous result of [13], quantifies this idea.

Proposition 1.1. *Under the assumption of Theorem 1.1, we have* $U = \Phi^{-1}(w_\Omega)$ *if either*

$$\mathscr{F}^s_{q,\alpha}\big(\Phi(u_0^-)\big) > 0 \text{ and } \mathscr{F}^s_{q,\alpha}(\Phi(u_0)) < \Lambda_2 \tag{1.6a}$$

or

$$\mathscr{F}^s_{q,\alpha}\big(\Phi(u_0^-)\big) \leq 0 \text{ and } \mathscr{F}^s_{q,\alpha}\big(\Phi(u_0^+)\big) + 2\iint \frac{\Phi(u_0)^+(x)\Phi(u_0)^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y < \Lambda_2. \tag{1.6b}$$

We observe that Assumption (1.6b) is consistent with the analogous one made in the local case in [13, Proposition 1.4]. In that respect, note that the double integral disappears in the limit as $s \to 1$, if renormalized by a degenerating factor (we refer to Remark 5.1 below for more details).

The proof of Proposition 1.1 is by contradiction and makes use of a *hidden convexity* of Gagliardo's seminorm. This property, under the assumptions (1.6), allows one to "prolong in the past" the orbits $v(\cdot, t)$ of (1.5) that stabilize toward $-w_\Omega^{q-1}$ by a trajectory defined for negative times, connecting the initial datum u_0 to w_Ω^{q-1}, with an energy control. It turns out, also in view again of the Lyapunov property, that this would contradict the mountain pass-type description

$$\inf_{\gamma} \max_{\varphi\in \mathrm{Im}(\gamma)} \mathscr{F}^s_{q,\alpha}(\varphi)$$

of an excited level. In order to see this, in Sect. 3 we formulate this variational principle in a way that differs from standards in that the admissible γ, joining w_Ω and $-w_\Omega$, are only required to be continuous with values in the class of real valued measurable functions on Ω endowed with the topology of the convergence in measure, rather than the strong topology of Sobolev spaces.

2. Notations and assumptions

Throughout this paper, we assume Ω to be an open bounded set, we take $s \in (0, 1)$, and we let $m > 1$. Then, we denote by $\mathcal{D}^{s,2}_0(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$[\varphi]_s = \left\{ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \right\}^{\frac{1}{2}}. \tag{2.1}$$

Remark 2.1. Since by assumption Ω supports a Poincaré-type inequality, we have

$$\mathcal{D}^{s,2}_0(\Omega) = \left\{ u \in L^p(\mathbb{R}^N) : [u]_s < +\infty \text{ and } u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}$$

Also, $\mathcal{D}^{s,2}_0(\Omega)$ coincides with the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{L^p} + [\cdot]_s$ (sometimes denoted by $\widetilde{W}^{s,2}_0(\Omega)$, see [12]). For more details on this functional-analytic setting, we refer to the treatise [16, Chap. 3].

For every $\varphi \in C^2(\Omega)$, we take

$$(-\Delta)^s\varphi(x) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}}\,dy \tag{2.2}$$

as the definition of the s-laplacian of u at point x. As s is fixed, we are not interested in multiplying the principal value integral by any renormalization factor.

By $L^0(\Omega)$, we shall denote the space of all (equivalence classes of) real valued measurable functions on Ω, endowed with the topology of the convergence in measure.

For all $1 < q < 2$ let us define

$$\lambda_1(\Omega, q, s) = \inf_{\varphi\in\mathcal{D}^{s,2}_0(\Omega)} \left\{ [\varphi]_s^2 : \int_\Omega |\varphi|^q\,\mathrm{d}x = 1 \right\}. \tag{2.3}$$

Given $1 < q < 2$ and $\alpha > 0$, we consider the functional

$$\mathscr{F}^s_{q,\alpha}(\varphi) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y - \frac{\alpha}{q}\int_\Omega |\varphi|^q\,\mathrm{d}x, \tag{2.4}$$

for all $\varphi \in \mathcal{D}^{s,2}_0(\Omega)$.

3. Elliptic toolkit

The critical points of $\mathscr{F}^s_{q,\alpha}$ are the weak solutions $\varphi \in \mathcal{D}^{s,2}_0(\Omega)$ of the Lane-Emden type equation

$$(-\Delta)^s \varphi = \alpha |\varphi|^{q-2}\varphi, \qquad \text{in } \Omega, \tag{3.1}$$

which means

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \alpha \int_\Omega |\varphi|^{q-2}\varphi\psi\,\mathrm{d}x, \quad \text{for all } \psi \in \mathcal{D}^{s,2}_0(\Omega). \tag{3.2}$$

Lemma 3.1. *Let* $1 < q < 2$ *and* $\alpha > 0$. *Then, the functional* $\mathscr{F}^s_{q,\alpha}$ *is coercive on* $\mathcal{D}^{s,2}_0(\Omega)$, i.e.,

$$\mathscr{F}^s_{q,\alpha}(\varphi) \geq \tfrac{1}{4}[\varphi]^2_s - C, \tag{3.3}$$

for all $\varphi \in \mathcal{D}^{s,2}_0(\Omega)$, *where C is a constant depending on* Ω,q,s, *only.*

Proof. Since $q < 2$, by Young's inequality we have

$$\mathscr{F}^s_{q,\alpha}(\varphi) \geq \tfrac{1}{2}[\varphi]^2_s - \tfrac{\lambda_1(\Omega,q,s)}{4}\left[\int_\Omega |\varphi|^q\,\mathrm{d}x\right]^{\frac{2}{q}} - \tfrac{2-q}{2q}\alpha^{\frac{2}{2-q}}\left[\frac{\lambda_1(\Omega,q,s)}{2}\right]^{-\frac{q}{2-q}}$$

and then using (2.3) gives (3.3). □

3.1. The ground state level

We collect some properties of the minimal energy level, defined as

$$\Lambda_1 = \inf_{\varphi\in\mathcal{D}^{s,2}_0(\Omega)} \mathscr{F}^s_{q,\alpha}(\varphi)\,. \tag{3.4}$$

Lemma 3.2. *Let* $\alpha > 0$ *and* $1 < q < 2$. *Then,*

(i) *the energy functional* $\mathscr{F}^s_{q,\alpha}$ *achieves the minimum in (3.4);*
(ii) *the minimum* Λ_1 *of* $\mathscr{F}^s_{q,\alpha}$ *is a strictly negative number;*
(iii) $\mathscr{F}^s_{q,\alpha}$ *has exactly two minimisers* w *and* $-w$, *where* w *is a strictly positive function;*

Proof. By (3.3), assertion (i) follows by the compactness of the embedding of $\mathcal{D}^{s,2}_0(\Omega)$ into $L^q(\Omega)$.

Then, given any nonzero $\varphi \in \mathcal{D}^{s,2}_0(\Omega)$, in view of (2.4) we have $\mathscr{F}^s_{q,\alpha}(t\varphi) < 0$ for t small enough, which implies (ii).

As for (iii), we argue as in [10, Proposition 2.3] and we let φ be a minimiser. Then $|\varphi|$ is also a minimiser, because $\mathscr{F}^s_{q,\alpha}(|\varphi|) \leq \mathscr{F}^s_{q,\alpha}(\varphi)$ by the elementary inequality

$(|a|-|b|)^2 \le (a-b)^2$ with $a=\varphi(x)$, $b=\varphi(y)$, and thence

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(|\varphi|(x)-|\varphi|(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \alpha\int_\Omega |\varphi|^{q-2}\varphi\psi\,\mathrm{d}x, \quad \text{for all } \psi\in\mathcal{D}_0^{s,2}(\Omega).$$

Summing the latter to (3.2) gives that the positive part $(u+|u|)/2$ of u is a non-negative weak supersolution of (3.1). Then, it must be either identically zero or strictly positive by the strong maximum principle for the fractional Laplacian, see e.g. [18, Lemma 6] or [17, Proposition 7.1]. As a consequence, minimisers are non-negative solutions of (3.1). By [17, Proposition 3.4] (see also Remark 4.1 therein), non-negative weak solutions of (3.1) are unique, and then we deduce (iii). □

3.2. Higher energies

We now consider general critical energy levels, *i.e.*, values of the energy functional $\mathscr{F}^s_{q,\alpha}$ at its critical points (not necessarily minimizers), and we prove some related basic properties.

Lemma 3.3. *Let* $1<q<2$ *and* $\alpha>0$. *Then*

(i) $\mathscr{F}^s_{q,\alpha}$ *satisfies the Palais-Smale condition;*
(ii) $\mathscr{F}^s_{q,\alpha}$ *has the mountain pass structure;*
(iii) *its critical levels form a compact subset of* $[\Lambda_1, 0]$.

Proof. The first statement is the compactness in $\mathcal{D}_0^{s,2}(\Omega)$ of every Palais-Smale sequence. Then, we let $(\varphi_n)_{n\in\mathbb{N}}$ be one. Without loss of generality, that amounts to assuming that

$$\sup_{n\in\mathbb{N}} \mathscr{F}^s_{q,\alpha}(\varphi_n) < +\infty \tag{3.5a}$$

and that, for all $\psi\in\mathcal{D}_0^{s,2}(\Omega)$ with $\|\psi\|_{\mathcal{D}_0^{s,2}(\Omega)}=1$, we have

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi_n(x)-\varphi_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}} - \alpha\int_\Omega |\varphi_n|^{q-2}\varphi_n\psi\,\mathrm{d}x\right| \le \frac{1}{n}. \tag{3.5b}$$

By Lemma 3.1, Eq. (3.5a) implies that $(\varphi_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}_0^{s,2}(\Omega)$. Thus, thanks to the compactness of the embedding into $L^q(\Omega)$, a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ (that we do not relabel) converges to some limit φ weakly in $\mathcal{D}_0^{s,2}(\Omega)$ and strongly in $L^q(\Omega)$.

The function φ is a weak solution of (3.1), thanks to (3.5b) and to the fact that

$$\lim_{n\to\infty}\int_\Omega |\varphi_n|^{q-2}\varphi_n\psi\,\mathrm{d}x = \int_\Omega |\varphi|^{q-2}\varphi\psi\,\mathrm{d}x,$$

which holds because $\psi\in L^q(\Omega)$ and $\||\varphi_n|^{q-2}\varphi_n - |\varphi|^{q-2}\varphi\|_{L^{q/(q-1)}(\Omega)}\to 0$, as $n\to\infty$. In turn, this latter assertion follows from the convergence to φ of the sequence $(\varphi_n)_n$

in $L^q(\Omega)$: to see this, one can use the Hölder continuity of the function $\tau \mapsto |\tau|^{q-2}\tau$ for $q < 2$.

Then, we can choose $\psi = \varphi$ in (3.2), which gives

$$\mathscr{F}^s_{q,\alpha}(\varphi) = \left(\frac{1}{2} - \frac{1}{q}\right)[\varphi]_s^2 = \alpha\left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega |\varphi|^q \,\mathrm{d}x. \tag{3.6}$$

Then,

$$\begin{aligned}[\varphi]_s^2 &= \alpha\int_\Omega |\varphi|^q \,\mathrm{d}x = \lim_{n\to\infty}\alpha\int_\Omega |\varphi_n|^q \,\mathrm{d}x \geq \limsup_{n\to\infty}\left([\varphi_n]_s^2 - \tfrac{1}{n}[\varphi_n]_s\right)\\ &= \limsup_{n\to\infty}[\varphi_n]_s^2.\end{aligned}$$

Here, we used (3.6) for the first equality, the convergence in $L^q(\Omega)$ for the second one, Eq. (3.5b) for the inequality, and Lemma 3.1 together with (3.5a) for the last equality. Hence, by the sequential weak lower semicontinuity of $[\,\cdot\,]_s^2$ the convergence of the sequence is also strong in $\mathcal{D}_0^{s,2}(\Omega)$.

We have proved statement (i) and we consider now (ii), which means

$$\inf\left\{\mathscr{F}^s_{q,\alpha}(\varphi)\,:\,\min\left\{\|\varphi - w\|_{\mathcal{D}_0^{s,2}(\Omega)}, \|\varphi + w\|_{\mathcal{D}_0^{s,2}(\Omega)}\right\} \geq \|w\|_{\mathcal{D}_0^{s,2}(\Omega)}\right\} > \Lambda_1,$$

where w is the positive solution of (3.4). The contrapositive statement is that $\mathscr{F}^s_{q,\alpha}(\varphi_j) \to \Lambda_1$ along a sequence all whose elements are far, in $\mathcal{D}_0^{s,2}(\Omega)$, both from w and from $-w$ at least half as much as the distance between w and $-w$, in contradiction with the strong convergence either to w or to $-w$, that follows by coercivity (see Lemma 3.1).

So, (ii) is true and we are left to prove (iii). To do so, we observe that if $\varphi \in \mathcal{D}_0^{s,2}(\Omega)$ is a critical point of $\mathscr{F}^s_{q,\alpha}$ then choosing $\psi = \varphi$ in (3.2) yields (3.6). Thus, all critical levels belong to a bounded set, contained in $[\Lambda_1, 0]$. To prove that their collection is closed, we write (3.6) with φ replaced by φ_n, an arbitrary sequence of critical points with energy accumulating to a limit value Λ. Then, by the compactness of the embedding of $\mathcal{D}_0^{s,2}(\Omega)$ into $L^q(\Omega)$, the limit φ of the sequence in $\mathcal{D}_0^{s,2}(\Omega)$ satisfies Eq. (3.1), and so $\mathscr{F}^s_{q,\alpha}(\varphi) = \Lambda$ by construction. □

3.3. Mountain pass level

In the following, we recall how to construct a mountain pass energy level by considering paths of bounded energy that are continuous with respect to the topology of the convergence in measure.

Proposition 3.1. *Set*

$$\mathfrak{Z} = \Big\{z \in C\Big([0,+\infty); L^0(\Omega)\Big) \cap L^\infty\Big([0,+\infty); \mathcal{D}_0^{s,2}(\Omega)\Big)\,:\, z(0) = w \text{ and } \lim_{t\to+\infty} z(t) = -w \text{ in } L^0(\Omega)\Big\}$$

Then, for all $1 < q < 2$ *and* $\alpha > 0$,

$$\Lambda^* = \inf_{z \in \mathfrak{Z}} \sup_{t \in [0,+\infty)} \mathscr{F}^s_{q,\alpha}(z(\cdot,t))$$

is a critical value of $\mathscr{F}^s_{q,\alpha}$.

Proof. We first notice that $\mathfrak{Z}$ can been replaced with the class

$$\mathfrak{Z}_q = \Big\{ z \in C\Big([0,+\infty); L^q(\Omega)\Big) \cap L^\infty\Big([0,+\infty); \mathcal{D}^{s,2}_0(\Omega)\Big) : z(0) = w, \lim_{t \to +\infty} \|z(t) + w\|_{L^q(\Omega)} = 0 \Big\}.$$

Indeed $\mathfrak{Z}_q \subset \mathfrak{Z}$ because the convergence in L^q implies that in measure. For the reverse inclusion, we take $z \in \mathfrak{Z}$. Then $[z(t)]^2_s \le C$, for some constant $C > 0$, for all $t \in [0,+\infty) \setminus \mathcal{N}$, where $\mathcal{N}$ is a negligible set in $[0,+\infty)$. If $\bar{t} \in \mathcal{N}$ and $(t_n) \subset [0,+\infty) \setminus \mathcal{N}$ converges to $\bar{t}$. Since by Fatou Lemma the nonlocal energy $[\,\cdot\,]^2_s$ is lower semicontinuous with respect to the convergence in measure and $z(t_n) \to z(\bar{t})$ in measure, we have $[z(\bar{t})]^2_s \le C$ as well. Hence, z is equibounded in $\mathcal{D}^{s,2}_0(\Omega)$ and, by the compact embedding in $L^q(\Omega)$, it follows that from every arbitrary sequence t_n converging to any given $t_0 \ge 0$ we may extract another one along which z converges in $L^q(\Omega)$ to a limit, that must always be $z(t_0)$ because $z(t_n) \to z(t_0)$ in measure. Hence, by the Urysohn property of $L^q(\Omega)$ convergence, we see that z is continuous with values in $L^q(\Omega)$.

Next, one sees that

$$\inf_{z \in \mathfrak{Z}_q} \sup_{t \in [0,+\infty)} \mathscr{F}^s_{q,\alpha}(z(\cdot,t)) = \inf_{\gamma \in \mathfrak{G}} \sup_{t \in [0,1]} \mathscr{F}^s_{q,\alpha}(\gamma(t)),$$

where

$$\mathfrak{G} = \Big\{ \gamma \in C\Big([0,1]; L^q(\Omega)\Big) \cap L^\infty\Big([0,1]; \mathcal{D}^{s,2}_0(\Omega)\Big) : \gamma(0) = w, \ \gamma(1) = -w \Big\}.$$

This can be seen by repeating verbatim the passages in Part 3 of the proof of [13, Theorem 4.2], and we skip the details here.

The third step is then to prove that

$$\inf_{\gamma \in \mathfrak{G}} \sup_{t \in [0,1]} \mathscr{F}^s_{q,\alpha}(\gamma(t)) = \inf_{\sigma \in \mathfrak{S}} \sup_{t \in [0,1]} \mathscr{F}^s_{q,\alpha}(\gamma(t)),$$

where

$$\mathfrak{S} = \Big\{ \gamma \in C\Big([0,1]; \mathcal{D}^{s,2}_0(\Omega)\Big) : \gamma(0) = w, \ \gamma(1) = -w \Big\}.$$

We prove this claim by arguing as in Part 2 of the proof of [13, Theorem 4.2]. More precisely, we fix $\varepsilon > 0$ and we take $\gamma_\varepsilon \in C([0,1]; L^q(\Omega))$, such that $\gamma_\varepsilon(0) = w$, $\gamma_\varepsilon(1) = -w$, and

$$\sup_{t \in [0,1]} \mathscr{F}^s_{q,\alpha}(\gamma_\varepsilon(t)) < \inf_{\sigma \in \mathfrak{G}} \sup_{t \in [0,1]} \mathscr{F}^s_{q,\alpha}(\gamma(t)) + \varepsilon. \tag{3.7}$$

Thus, if we fix $\delta > 0$, by uniform continuity there exists $\eta > 0$ such that if $|t-s| < \eta$, we have

$$\|\gamma_\varepsilon(t) - \gamma_\varepsilon(s)\|_{L^q(\Omega)} < \delta.$$

Now we take a partition $\{t_0, \ldots, t_k\}$ of $[0, 1]$ such that

$$t_0 = 0, \qquad t_k = 1, \qquad |t_i - t_{i+1}| < \eta, \text{ for every } i = 0, \ldots, k-1,$$

then we define the new curve $\theta_\varepsilon : [0, 1] \to \mathcal{D}_0^{s,2}(\Omega)$, which is given by the piecewise affine interpolation of the points $\gamma_\varepsilon(t_0), \gamma_\varepsilon(t_1), \ldots, \gamma_\varepsilon(t_k)$, namely

$$\theta_\varepsilon(t) = \left(1 - \frac{t-t_i}{t_{i+1}-t_i}\right) \gamma_\varepsilon(t_i) + \frac{t-t_i}{t_{i+1}-t_i}\, \gamma_\varepsilon(t_{i+1}), \qquad \text{for every } t \in [t_i, t_{i+1}].$$

Then we take $\delta > 0$ and we find a finite increasing sequence of real numbers $t_i \in [0, 1]$ such that for each interval $[t_{i-1}, t_i]$ we have $\|\gamma_\varepsilon(t) - \gamma(s)\|_{L^q(\Omega)} \le \delta$ for all s, t in that interval. Let $\theta_{\varepsilon,\delta}$ denote the piecewise affine interpolation of the finite sequence of points $\gamma(t_i)$. We set

$$\tau = \frac{t-t_i}{t_{i+1}-t_i}.$$

Then, by the standard convexity of the squared seminorm we have

$$[\theta_{\varepsilon,\delta}((1-\tau)t_{i-1} + \tau t_i)]_s^2 \le (1-\tau)[\gamma_\varepsilon(t_{i-1})]_s^2 + \tau[\gamma_\varepsilon(t_i)]_s^2.$$

Thus,

$$\mathscr{F}_{q,\alpha}^s\big(\theta_{\varepsilon,\delta}((1-\tau)t_{i-1} + \tau t_i)\big) \le (1-\tau)\mathscr{F}_{q,\alpha}^s\big(\gamma_\varepsilon(t_{i-1})\big) + \tau \mathscr{F}_{q,\alpha}^s(\gamma_\varepsilon(t_i)) + \frac{\alpha}{q}(\mathcal{R}_1 - \mathcal{R}_2),$$

where

$$\mathcal{R}_1 = (1-\tau)\int_\Omega |\gamma_\varepsilon(t_{i-1})|^q \,\mathrm{d}x + \tau \int_\Omega |\gamma_\varepsilon(t_i)|^q \,\mathrm{d}x,$$
$$\mathcal{R}_2 = \int_\Omega |(1-\tau)\gamma_\varepsilon(t_{i-1}) + \tau\gamma_\varepsilon(t_i)|^q \,\mathrm{d}x.$$

Now, by using (3.7) we get

$$\mathscr{F}_{q,\alpha}^s\big(\theta_{\varepsilon,\delta}((1-\tau)t_{i-1} + \tau t_i)\big) \le \inf_{\sigma\in\mathfrak{S}} \sup_{t\in[0,1]} \mathscr{F}_{q,\alpha}^s(\gamma(t)) + \varepsilon + \frac{\alpha}{q}(\mathcal{R}_1 - \mathcal{R}_2), \tag{3.8}$$

Also, since the infimum in (2.3) is positive, by the coercivity estimate of Lemma 3.1 and by (3.7),

$$\left(\int_\Omega |\gamma_\varepsilon(t_i)|^q \,\mathrm{d}x\right)^{\frac{q-1}{q}} \le \lambda_1(\Omega, s, q)^{\frac{q-1}{2}} \left[\inf_{\sigma\in\mathfrak{S}} \sup_{t\in[0,1]} \mathscr{F}_{q,\alpha}^s(\gamma(t)) + \varepsilon + C(\Omega, s, q)\right].$$

Hence, we can infer the estimate $\mathcal{R}_1 - \mathcal{R}_2 \le C\delta$, with a constant C depending only on the data, as done in [13]. Inserting this estimate in (3.8) we arrive at that

$$\sup_{t\in[0,1]} \mathscr{F}^s_{q,\alpha}\big(\theta_{\varepsilon,\delta}(t)\big) \le \inf_{\sigma\in\mathfrak{G}} \sup_{t\in[0,1]} \mathscr{F}^s_{q,\alpha}(\gamma(t)) + \varepsilon + C\delta$$

Since the piecewise affine path $\theta_{\varepsilon,\delta}$ is obviously continuous with values in $\mathcal{D}^{s,2}_0(\Omega)$ and δ, ε were arbitrary, we deduce that

$$\inf_{\theta\in\mathfrak{S}} \sup_{t\in[0,1]} \mathscr{F}^s_{q,\alpha}(\theta(t)) \le \inf_{\sigma\in\mathfrak{G}} \sup_{t\in[0,1]} \mathscr{F}^s_{q,\alpha}(\gamma(t))$$

The opposite inequality also holds, because $\mathfrak{S} \subset \mathfrak{G}$, and that ends the third step of the proof.

By combining the previous three steps, we see that

$$\Lambda^* = \inf_{\theta\in\mathfrak{S}} \sup_{t\in[0,1]} \mathscr{F}^s_{q,\alpha}(\theta(t))$$

and in order to conclude it suffices to prove that the right hand side defines a critical level. Since by Lemma 3.3 the functional $\mathscr{F}^s_{q,\alpha}$ has a mountain pass structure, the desired conclusion is therefore a general consequence of [20, Chap. §2, Theorem 6.1]. □

3.4. First excited level and spectral gap.

We set

$$\Lambda_2 = \inf\left\{\Lambda > \Lambda_1 \,:\, \Lambda \text{ is a critical value of } \mathscr{F}^s_{q,\alpha}\right\} \tag{3.9}$$

We point out that the set in the right-hand side is never empty because it always contains 0, which is the critical value associated with the critical point $u \equiv 0$. Also, its infimum is in fact a minimum because the critical levels form a closed set, by Lemma 3.3.

We first notice that if a gap exists between Λ_1 and Λ_2 then it gives room to energy levels of sign-changing functions.

Proposition 3.2. *Let $1 < q < 2$ and $\alpha > 0$, and assume that $\Lambda_2 > \Lambda_1$. Then there exists a sign-changing function $\varphi \in \mathcal{D}^{s,2}_0(\Omega)$ with $\Lambda_1 < \mathscr{F}^s_{q,\alpha}(\varphi) < \Lambda_2$.*

Proof. As in [13, Proposition 3.5], φ will be given by the separate contributions of the least energy solution w_ε in $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ and of a function supported in a ball $B_{r_\varepsilon}(x_\varepsilon)$ contained in $\Omega \setminus \Omega_\varepsilon$.

We fix $\eta_0 \in (0, \Lambda_2 - \Lambda_1)$. Arguing as done in [13] we see that for an appropriate $\varepsilon_0 > 0$ we have

$$\mathscr{F}^s_{q,\alpha}(w_\varepsilon) = \inf\left\{\mathscr{F}^s_{q,\alpha}(u) \,:\, u \in \mathcal{D}^{s,2}_0(\Omega_\varepsilon)\right\} \le \Lambda_2 - \eta_0, \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{3.10}$$

Then, we fix $\psi \in C_0^\infty(B_1)$, $\psi \not\equiv 0$, and for all $\varepsilon \in (0, \varepsilon_0)$ we define $\psi_\varepsilon(x) = r_\varepsilon^s \psi(\frac{x-x_\varepsilon}{r_\varepsilon})$, which implies

$$\mathscr{F}^s_{q,\alpha}(\psi_\varepsilon) = \frac{r_\varepsilon^2}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\psi_\varepsilon(x) - \psi_\varepsilon(y))^2}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y - \frac{\alpha r_\varepsilon^{N+q}}{q} \int_{B_1} |\psi_\varepsilon|^2 \, \mathrm{d}x. \tag{3.11}$$

Also, we have the general identity

$$\mathscr{F}^s_{q,\alpha}(w_\varepsilon - \psi_\varepsilon) = \mathscr{F}^s_{q,\alpha}(w_\varepsilon) + \mathscr{F}^s_{q,\alpha}(\psi_\varepsilon) + \int_{\Omega_\varepsilon} \int_{B_{r_\varepsilon}(x_\varepsilon)} \frac{w_\varepsilon(x)\psi_\varepsilon(y)}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y. \tag{3.12}$$

Now we make the choice $r_\varepsilon = \varepsilon^2$, so that

$$\int_{\Omega_\varepsilon} \int_{B_{\varepsilon^2}(x_\varepsilon)} \frac{w_\varepsilon(x)\psi_\varepsilon(y)}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y = O(\varepsilon^N), \qquad \text{as } \varepsilon \to 0^+. \tag{3.13}$$

With that choice, from (3.11) we also get $\mathscr{F}^s_{q,\alpha}(\psi_\varepsilon) = O(\varepsilon^{2N})$, as $\varepsilon \to 0^+$. By pairing this and (3.13), from the identity (3.12) we deduce that $\mathscr{F}^s_{q,\alpha}(w_\varepsilon - \psi_\varepsilon) = \mathscr{F}^s_{q,\alpha}(w_\varepsilon) + O(\varepsilon^N)$, as $\varepsilon \to 0^+$. Hence and from (3.10) we infer that the function $\varphi_\varepsilon = w_\varepsilon - \psi_\varepsilon$ has energy $\mathscr{F}^s_{q,\alpha}(\varphi_\varepsilon) < \Lambda_2$ for $\varepsilon \in (0, \varepsilon_0)$ small enough. On the other hand, $\varphi \in \mathcal{D}_0^{s,2}(\Omega)$ and so $\mathscr{F}^s_{q,\alpha}(\varphi_\varepsilon) \geq \Lambda_1$, and the inequality must be strict because of assertion (iii) in Lemma 3.2. □

Now we recall a general consequence of the uniqueness of (positive) energy minimizing functions: if there is a spectral gap, then the mountain pass does not collapse on the global minimum.

Proposition 3.3. *Under the assumptions of Proposition 3.1, we have* $\Lambda^* \geq \Lambda_2$.

Proof. Either $\Lambda_2 = \Lambda_1$, and then there is nothing to prove, or else $\Lambda_2 > \Lambda_1$. In arguing by contradiction, we then assume $\Lambda_2 > \Lambda_1$ and $\Lambda^* = \Lambda_1$. Then we can find a sequence of admissible paths z_j for the definition of Λ^* such that $\sup_{t \geq 0} \mathscr{F}^s_{q,\alpha}\big(z_j(\cdot, t)\big) < \Lambda_1 + 2^{-j}$. Since each $t \mapsto z_j(\cdot, t)$ is continuous with values in $L^0(\Omega)$, there exists two sets $B, B' \subset \mathcal{D}_0^{s,2}(\Omega)$ that are disjoint and open with respect to the (metrizable) topology of the convergence in measure, with $w \in B$ and $-w \in B'$, such that for every j we find $t_j > 0$ with $\mathscr{F}^s_{q,\alpha}\big(z_j(\cdot, t_j)\big) < \Lambda_1 + 2^{-j}$. Then, the functions $z_j(\cdot, t_j)$ would form a minimizing sequence for $\mathscr{F}^s_{q,\alpha}$, in contradiction with the fact that they all are bounded away both from the minimizers $w, -w$ in $L^0(\Omega)$. □

We end this section by recalling an important consequence of the stability of energy minimizing solution of the elliptic problem, which in turn implies the fundamental gap, proved in [17] for the non-local problem following the method used in [11] for the local one. The following statement differs little from the original one in [17], which simply states the isolation with respect to the $L^1(\Omega)$ topology instead of the topology of the convergence in measure.

Theorem 3.1. *Let $1 < q < 2$ and $\alpha > 0$, and let w be the positive minimizer of $\mathscr{F}^s_{q,\alpha}$. Assume that Ω is a bounded $C^{1,1}$ open set. Then, the set $\{w, -w\}$ is bounded away in $L^0(\Omega)$ from any other critical point of $\mathscr{F}^s_{q,\alpha}$.*

Proof. We argue by contradiction and we assume that a sequence of critical points u_j of $\mathscr{F}^s_{q,\alpha}$ converges in measure to w. Since critical energies are negative (see statement (iii) in Lemma 3.3) and the energy is coercive (see Lemma 3.1), the sequence belongs to a bounded subset of $\mathcal{D}^{s,2}_0(\Omega)$. Then, by the compactness of the embedding into $L^q(\Omega)$, the sequence converges to w also in $L^q(\Omega)$, which contradicts the conclusion of [17, Proposition 6.1]. □

Arguing similarly as in [13, Lemma 3.4] one can observe that if $\{\varphi_n\}$ is minimizing sequence of $\mathscr{F}^s_{q,\alpha}$, then it converges, up to subsequence, either to the positive minimizer w or to $-w$. We have then the following direct consequence of Theorem 3.1, namely the fundamental gap between the first and the second critical energy level of the functional $\mathscr{F}^s_{q,\alpha}$. The proof is similar to the relevant one in [13, Proposition 3.5].

Corollary 3.1. *Let $1 < q < 2$ and $\alpha > 0$. If Ω is a bounded $C^{1,1}$ open set, then $\Lambda_2 > \Lambda_1$.*

4. The (rescaled) parabolic problem

Given a solution u of (1.1), the rescaled function $v(x,t) = e^{\alpha t} u(x, e^t - 1)$ solves the following Dirichlet initial boundary value problem

$$\begin{cases} \partial_t v = -(-\Delta)^s \Phi(v) + \alpha v, & \text{in } Q := \Omega \times (0,+\infty), \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega \times (0,+\infty), \\ v(\cdot,0) = u_0, & \text{in } \Omega. \end{cases} \tag{4.1}$$

We recall here the definition of weak solutions for problem (4.1).

Definition 4.1. Given $T \in (0,+\infty]$ and given $u_0 \in L^{m+1}(\Omega)$, with $\Phi(u_0) \in \mathcal{D}^{s,2}_0(\Omega)$, a function

$$v \in C([0,T]; L^{m+1}(\Omega)), \text{ with } \Phi(v) \in L^2((0,T); \mathcal{D}^{s,2}_0(\Omega)), \tag{4.2}$$

is said to be a weak solution of (4.1) in $Q_T = \Omega \times (0,T)$ if the integral equation

$$\begin{aligned} &-\iint_{Q_T} v \, \frac{\partial \eta}{\partial t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\Phi(v(x,t)) - \Phi(v(y,t)))(\eta(x,t) - \eta(y,t))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &= \alpha \iint_{Q_T} v\, \eta \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.3}$$

holds for all $\eta \in C^\infty_c(Q_T)$.

Remark 4.1. Under the a priori assumption (4.2) made in Definition 4.1, weak solutions are often called *energy weak solution* in the literature [14]. Also, as done in [14], it is possible to prove that weak solutions are *strong*, with a number of implications (e.g., L^1 contractivity, comparison principles, and more), but in this paper we can limit our attention to (energy) weak solutions. We mention here that a complete basic theory for the *weak-dual solutions* to (1.1) is given in [4].

4.1. Well-posedness and entropy–entropy dissipation inequality

A crucial ingredient for the proof of our main result is the inequality proved in the following theorem, where we set

$$g(v) = |v|^{\frac{m-1}{2}} v.$$

As mentioned in the introduction, the main approach consists first in establishing a discrete version of the energy inequality (4.4) below: passing to the limit in the energy estimate will produce its continuous version, satisfied by the unique weak solution to the rescaled problem (4.1).

Theorem 4.1. *Let $u_0 \in L^{m+1}(\Omega)$, with $\Phi(u_0) \in \mathcal{D}_0^{s,2}(\Omega)$. Then, there exists a unique weak solution of (4.1) in Q with $v(0) = u_0$. Moreover, $\partial_t g(v) \in L^2(Q)$ and the estimate*

$$\mathscr{F}^s_{\frac{m+1}{m},\alpha}(\Phi(v(\cdot,t))) + C_0 \int_0^t \int_\Omega |\partial_t g(v)|^2 \, \mathrm{d}x \, \mathrm{d}y \leq \mathscr{F}^s_{\frac{m+1}{m},\alpha}(\Phi(u_0)) \tag{4.4}$$

holds for all $t > 0$, for some positive constant $C_0 = C_0(m)$, depending only on m.

Proof. We fix $T > 0$. We shall prove the existence of a weak solution $v \in C([0,T]; L^{m+1}(\Omega))$ of (4.1), with $v(0) = u_0$, such that the estimate (4.4) is valid for all $0 \leq t \leq T$. Uniqueness is well understood for equations of this type, see [2–7]. The uniqueness method by Oleĭnik, Kalašnikov, and Čžou used in [14, Theorem 6.1] for weak energy solutions in the case $\Omega = \mathbb{R}^N$ for the equation without forcing term can be adapted straightforwardly to the case under consideration.

More precisely, one takes the difference between (4.3) and the same equation with another weak solution $\widetilde{v}$ of (4.1). Then, with the choice

$$\eta(x,t) = e^{-\alpha t} \int_t^T e^{-\alpha\sigma} z(\cdot,\sigma)\, \mathrm{d}\sigma, \qquad \text{where } z(\cdot,t) = \Phi(v(\cdot,t)) - \Phi(\widetilde{v}(\cdot,t)),$$

one arrives at

$$\iint_{Q_T} (v-\widetilde{v})(\Phi(v) - \Phi(\widetilde{v}))\, e^{-2\alpha t}\, \mathrm{d}x\, \mathrm{d}t \, - \frac{1}{2}\int_0^T \partial_t \left[\int_t^T e^{-\alpha\sigma} z(\cdot,\sigma)\,\mathrm{d}\sigma\right]_s^2 \mathrm{d}t = 0,$$

which, after manipulating the second term, reads as

$$\iint_{Q_T} (v-\widetilde{v})(\Phi(v) - \Phi(\widetilde{v}))\, e^{-2\alpha t}\, \mathrm{d}x\, \mathrm{d}t \, + \tfrac{1}{2}\left[\int_0^T e^{-\alpha\sigma} z(\cdot,\sigma)\,\mathrm{d}\sigma\right]_s^2 = 0\,.$$

Therefore,

$$\iint_{Q_T} (v-\widetilde{v})(\Phi(v)-\Phi(\widetilde{v}))\, e^{-2\alpha t}\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

The integrand in the latter is always non-negative and hence $v=\widetilde{v}$ a.e. in Q_T.

To prove the existence of a solution with the energy estimate, in order to avoid difficult issues concerning the regularity of signed solutions to fractional nonlinear parabolic problems, we proceed by using the classical Euler time-discretization scheme. We fix $h>0$, we set $v_0 := u_0$ and for all integers k from 1 to the integer part $\lfloor T/h \rfloor$ of T/h we define recursively v_k as a solution of

$$\min\left\{\mathscr{F}^s_{\frac{m+1}{m},\alpha}(\Phi(v)) + \frac{1}{h}\delta(v\,,\,v_{k-1}) \,:\, v\in\mathcal{D}^{s,2}_0(\Omega)\right\}, \tag{4.5}$$

where for all $f\in L^{m+1}(\Omega)$ we set

$$\delta(v,f) = \frac{1}{m+1}\int_\Omega (|f|^{m+1}-|v|^{m+1})\,\mathrm{d}x + \int_\Omega \Phi(v)(v-f)\,\mathrm{d}x. \tag{4.6}$$

Note that $\delta(v,f)\ge 0$, with equality only if $v=f$, by the strict convexity of $\tau\mapsto |\tau|^{m+1}$. Hence and from Lemma 3.1, and from the compactness of the embedding of $\mathcal{D}^{s,2}_0(\Omega)$ into $L^{\frac{m+1}{m}}(\Omega)$, we see that minimizing sequences for (4.5) always admit subsequences along which $\Phi(v^{(j)})$ converges, as $j\to\infty$, to a limit w_k weakly in $\mathcal{D}^{s,2}_0(\Omega)$ and strongly in $L^{\frac{m+1}{m}}(\Omega)$. Then, $v_k=\Phi^{-1}(w_k)$ solves (4.5) by lower semicontinuity: indeed, by setting $w=\Phi(v)$ the objective in (4.5) takes the form

$$\frac{1}{2}[w]_s^2 + \left(\frac{1}{h}-\alpha\right)\frac{m}{m+1}\int_\Omega |w|^{\frac{m+1}{m}}\,\mathrm{d}x - \frac{1}{h}\int_\Omega wf\,\mathrm{d}x + \frac{1}{h}\frac{1}{m+1}\int_\Omega |f|^{m+1}\,\mathrm{d}x,$$

which clearly is convex for h small enough.

Incidentally, we mention that formally the Euler-Lagrange equation for (4.5) is

$$(-\Delta)^s\Phi(v_k) + \frac{1}{h}(v_k - v_{k-1}) = \alpha v_k, \tag{4.7}$$

i.e., a discretized version of (4.1). Indeed, to solve (4.5) we may equivalently search out a minimizer w of the previous functional and once w is found, set $v=\Phi^{-1}(w)$. The necessary minimality condition for w reads

$$\iint \frac{(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \left(\tfrac{1}{h}-\alpha\right)\int_\Omega |w|^{\frac{1-m}{m}} w\varphi\,\mathrm{d}x = \frac{1}{h}\int_\Omega f\varphi\,\mathrm{d}x,$$

for all $\varphi\in\mathcal{D}^{s,2}_0(\Omega)$. Hence, by changing back variables with $w=\Phi(v)$, we get

$$\iint \frac{(\Phi(v)(x)-\Phi(v)(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,dy + \frac{1}{h}\int_\Omega (v-f)\varphi\,\mathrm{d}x = \alpha\int_\Omega v\varphi\,\mathrm{d}x,$$

for all $\varphi\in\mathcal{D}^{s,2}_0(\Omega)$, which is precisely the weak formulation of (4.7) with $f=v_{k-1}$.

Then, we construct $\bar{v}_h : Q_T \to \mathbb{R}$ by setting

$$\bar{v}_h(x,t) = v_{\lfloor t/h \rfloor}(x), \tag{4.8}$$

for all $(x,t) \in Q_T$. The minimality of v_k for the problem (4.5) implies

$$\mathscr{F}^s_{\frac{m+1}{m},\alpha}(\Phi(v_k)) + \frac{1}{h}\delta(v_k\,,\,v_{k-1}) \le \mathscr{F}^s_{\frac{m+1}{m},\alpha}(\Phi(v_{k-1}))\,. \tag{4.9}$$

By Lemma A.1 with $a = u_{k-1}$ and $b = u_k$, there is a constant $C_0(m)$ depending only on m with

$$\frac{1}{h}\delta(v_k\,,\,v_{k-1}) \ge C_0(m)h \int_\Omega \left|\frac{g(v_k) - g(v_{k-1})}{h}\right|^2 \mathrm{d}x. \tag{4.10}$$

Besides (4.8), we define $\bar{v}_h$ for negative times by setting $\bar{v}_h(\cdot,t) = u_0$, for all $t < 0$, and we denote by $\widehat{G}_h$ the backward Steklov average of $g \circ \bar{v}_h$, namely

$$\widehat{G}_h(x,t) = \frac{1}{h}\int_{t-h}^t g(\bar{v}_h(x,\tau))\,\mathrm{d}\tau, \qquad \text{for all } (x,t) \in Q_T.$$

Then, by inserting (4.10) in (4.9) and summing up, we arrive at the energy estimate

$$\mathscr{F}^s_{\frac{m+1}{m},\alpha}(\Phi(\bar{v}_h(\cdot,t))) + C_0(m)\iint_{Q_t}\left|\partial_t \widehat{G}_h(x,\tau)\right|^2 \mathrm{d}x\,\mathrm{d}\tau \le \mathscr{F}^s_{\frac{m+1}{m},\alpha}(\Phi(u_0))\,,$$
$$\text{for all } 0 \le t \le T. \tag{4.11}$$

Incidentally, by Fubini theorem and Jensen's inequality we have

$$\begin{aligned}\iint_{Q_T}\left(\widehat{G}_h - g(\bar{v}_h)\right)^2 \mathrm{d}x\,\mathrm{d}t &\le \frac{1}{h}\int_0^T\int_{t-h}^t\int_\Omega \left(g(\bar{v}_h(x,\tau) - g(\bar{v}_h(x,t))\right)^2 \mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \\ &= \sum_{k=1}^{\lfloor T/h \rfloor}\int_\Omega \left(g(u_k) - g(u_{k-1})\right)^2.\end{aligned}$$

Thus, owing to the definition of $\bar{v}_h$, and by using (4.10) backward, we see that

$$\iint_{Q_T}\left(\widehat{G}_h - g(\bar{v}_h)\right)^2 \mathrm{d}x\,\mathrm{d}t \le C_0^{-1}h\left[\mathscr{F}^s_{\frac{m+1}{m},\alpha}(\Phi(u_0)) + C\right] \tag{4.12}$$

where $C > 0$ is the constant of Lemma 3.1.

We will make use of Eq. (4.12) below. But before doing so, we first aim at proving that

$$\{\widehat{G}_h : h > 0\} \text{ is relatively compact in } C([0,T];L^2(\Omega)). \tag{4.13}$$

To do so, we argue similarly as done in the proof of [13, Proposition 5.3, Step 4]. We first note that

$$\int_\Omega |\widehat{G}_h(x,t_1) - \widehat{G}_h(x,t_2)|^2\,\mathrm{d}x \le (t_2 - t_1)\int_{t_1}^{t_2}\int_\Omega |\partial_t \widehat{G}_h(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau$$

and, thanks to (4.11) and Lemma 3.1, we also obtain

$$(t_2 - t_1) \int_{t_1}^{t_2} \int_\Omega |\partial_t \widehat{G}_h(x,\tau)|^2 \,\mathrm{d}x\,\mathrm{d}\tau \le C_0^{-1} |t_2 - t_1| \Big[\mathscr{F}^s_{\frac{m+1}{m},\alpha}(\Phi(u_0)) + C \Big].$$

Putting the last two estimates together entails that

$$\{\widehat{G}_h : h > 0\} \text{ is an equicontinuous family of } L^2(\Omega)\text{-valued curves.} \tag{4.14}$$

Then, we fix $0 \le t \le T$ and $z \in \mathbb{R}^N$, and we use Jensen's inequality to get

$$\begin{aligned} &\int_{\mathbb{R}^N} \left|\widehat{G}_h(x+z,t) - \widehat{G}_h(x,t)\right|^{\frac{4m}{m+1}} \,\mathrm{d}x \\ &\quad \le \frac{1}{h} \int_{t-h}^t \int_{\mathbb{R}^N} |g(\bar{v}_h(x+z,\tau)) - g(\bar{v}_h(x,\tau))|^{\frac{4m}{m+1}} \,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned}$$

In view of the elementary inequality $|g(A) - g(B)|^{\frac{4m}{m+1}} \le C_1(m) |\Phi(A) - \Phi(B)|^2$ and of [12, Lemma A.1]

$$\int_{\mathbb{R}^N} |g(\bar{v}_h(x+z,\tau)) - g(\bar{v}_h(x,\tau))|^{\frac{4m}{m+1}} \,\mathrm{d}x \le C_2(m,N) |z|^{2s} [\Phi(\bar{v}_h(\cdot,\tau))]_s^2, \quad \text{for all } t-h < \tau < t.$$

Inserting this inequality into the previous one and using Hölder inequality to handle the result gives

$$\begin{aligned} &\int_{\mathbb{R}^N} \left|\widehat{G}_h(x+z,t) - \widehat{G}_h(x,t)\right|^2 \,\mathrm{d}x \\ &\quad \le |z|^{\frac{(m+1)s}{m}} \cdot \left[C_3(m,N)\, |\Omega|^{\frac{m-1}{2m}} \,\frac{1}{h} \int_{t-h}^t [\Phi(\bar{v}_h(\cdot,\tau))]^{\frac{m+1}{m}}_{W_0^{s,2}(\Omega)} \,\mathrm{d}\tau \right]. \end{aligned}$$

Also, by Lemma 3.1 and by Eq. (4.9) we have

$$\int_{t-h}^t [\Phi(\bar{v}_h(\cdot,\tau))]_s^{\frac{m+1}{m}} \,\mathrm{d}\tau \le h \cdot C_4(s,m,\Omega) \left[\mathscr{F}^s_{\frac{m+1}{m},\alpha}(\Phi(u_0)) + 1 \right]^{\frac{m+1}{2m}}.$$

Recalling that $t \ge 0$ and $z \in \mathbb{R}^N$ were arbitrary, the last two inequalities entail that

$$\limsup_{z\to 0} \sup_{h>0} \int_{\mathbb{R}^N} \left|\widehat{G}_h(x+z,t) - \widehat{G}_h(x,t)\right|^2 \,\mathrm{d}x = 0, \qquad \text{for all } 0 \le t \le T. \tag{4.15a}$$

Since $g(\sigma)^2 = |\Phi(\sigma)|^{\frac{m+1}{m}}$, by the fractional Sobolev embedding into $L^{\frac{m+1}{m}}(\Omega)$ we have

$$\int_\Omega |\widehat{G}_h(x,t)|^2 \,\mathrm{d}x \le \frac{1}{h} \int_{t-h}^t \int_\Omega |g(\bar{v}_h(x,\tau))|^2 \,\mathrm{d}x\,\mathrm{d}\tau \le \frac{C}{h} \int_{t-h}^t [\Phi(\bar{v}_h(\cdot,\tau))]^2 \,\mathrm{d}\tau$$

and thence, by arguing as done above, we infer that

$$\sup_{h>0}\int_{\Omega}|\widehat{G}_h(x,t)|^2\,\mathrm{d}x<+\infty,\qquad \text{for all } 0\le t\le T. \tag{4.15b}$$

By Fréchet-Kolmogorov theorem, (4.15) implies that

$$\{\widehat{G}_h(\cdot,t)\,:\,h>0\} \text{ is relatively compact in } L^2(\Omega), \text{ for all } 0\le t\le T. \tag{4.16}$$

Thanks to the vector-valued extension of Ascoli–Arzelà theorem [19, Lemma 1], from (4.14) and (4.16) we can infer (4.13).

Then, there exist a sequence $h_j\to 0^+$ and a function v, with $g(v)\in C([0,T];L^2(\Omega))$, such that

$$\widehat{G}_{h_j}\to g(v)\qquad \text{in } C([0,T];L^2(\Omega)) \tag{4.17a}$$

Clearly, the convergence (4.17a) is strong in $L^2(Q_T)$, too. Hence, recalling (4.12), we deduce that

$$g(\bar v_{h_j})\to g(v)\qquad \text{strongly in } L^2([0,T];L^2(\Omega)). \tag{4.17b}$$

Then, by Ineq. (A.2) in [13, Lemma A.1], it follows that

$$\bar v_{h_j}\to v\qquad \text{strongly in } L^{m+1}([0,T];L^{m+1}(\Omega)). \tag{4.17c}$$

Also, by possibly passing to a subsequence, from Lemma 3.1 and Eq. (4.9) we may infer that

$$\Phi(\bar v_{h_j})\to\Phi(v)\qquad \text{weakly in } L^2(0,T;\mathcal{D}^{s,2}_0(\Omega)). \tag{4.17d}$$

Moreover, we note that $(\partial_t\widehat{G}_{h_j})_j$ is a bounded sequence in $L^2(Q_T)$ because of estimate (4.11). Thus, in view of (4.17a), up to passing to a further subsequence, we may write that

$$\partial_t\widehat{G}_{h_j}\to\partial_t g(v)\qquad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{4.17e}$$

By lower semicontinuity, (4.11) and (4.17) imply (4.4). Since $\widehat{G}_{h_j}(0)=g(u_0)$ for all j, by (4.17a) we also have that $v(0)=u_0$. Then, in order to conclude we are left to prove that v is a weak solution of (4.1) in Q_T. To see this, we first deduce (4.2) from (4.17a), thanks to [13, Lemma A.1]. To prove that (4.3) holds too, we use the Euler-Lagrange equation (4.7) for v_k and (4.8) to get

$$\begin{aligned}
&\iint_{Q_T}\frac{\bar v_{h_j}(x,t)-\bar v_{h_j}(x,t-h_j)}{h_j}\,\eta(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Phi(\bar v_{h_j}(x,t))-\Phi(\bar v_{h_j}(y,t))}{|x-y|^{N+2s}}(\eta(x,t)-\eta(y,t))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&=\alpha\iint_{Q_T}\bar v_{h_j}\eta,
\end{aligned}$$

for all $\eta \in C_c^\infty(Q_T)$, and changing variables yields

$$-\iint_{Q_T} \bar{v}_{h_j}\, \partial_t \widehat{\eta}^{h_j} + \int_0^T \iint_{\mathbb{R}^{2N}} \frac{\Phi(\bar{v}_{h_j}(x,t)) - \Phi(\bar{v}_{h_j}(y,t))}{|x-y|^{N+2s}} (\eta(x,t) - \eta(y,t))\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t$$
$$= \alpha \iint_{Q_T} \bar{v}_{h_j} \eta.$$

Thanks to (4.17), by passing to the limit in the latter we obtain Eq. (4.3) and we conclude. □

4.2. Stabilization

We characterize the cluster points of large time asymptotic profiles of weak solutions, understood as in the following definition.

Definition 4.2. Let $u_0 \in L^{m+1}(\Omega)$, with $\Phi(u_0) \in \mathcal{D}_0^{s,2}(\Omega)$. Then, the *$\omega$-limit emanating from u_0* is the set

$$\omega(u_0) = \Big\{ f \in L^{m+1}(\Omega) : \text{ there exists } (t_j)_j \nearrow +\infty \text{ with } \lim_{j\to\infty} \|v(\cdot, t_j) - u_0\|_{L^{m+1}(\Omega)} = 0 \Big\}$$

where $v \in C([0,\infty); L^{m+1}(\Omega))$ is the weak solution of (4.1) with $v(0) = u_0$.

The structure of $\omega(u_0)$ is easier to understand under the assumptions

$$\partial_t g(v) \in L^2([T_0, +\infty), L^2(\Omega)), \qquad \Phi(v) \in L^\infty([T_0, +\infty), \mathcal{D}_0^{s,2}(\Omega)) \quad (4.18)$$

on the weak solution v of (4.1) with initial datum u_0, for an appropriate time $T_0 > 0$. These assumptions are the non-local counterpart of those considered in [13] for the local case.

Theorem 4.2. *Let v be the weak solution of (4.1) and assume that there exists $T_0 > 0$ for which (4.18) holds. Then, for every $U \in \omega(u_0)$, the function $\Phi(U)$ belongs to $\mathcal{D}_0^{s,2}(\Omega)$ and is a weak solution of (3.1).*

Proof. By repeating verbatim the proof of [13, Theorem 5.2], we can see that the assumptions (4.18) imply the first statement and, also, we arrive at

$$\lim_{j\to\infty} \|v_j - U\|_{L^{m+1}(\mathcal{Q})} = 0, \quad \text{where } \mathcal{Q} = \Omega \times (-1, 1), \quad (4.19)$$

and, for all $j \in \mathbb{N}$, we set $v_j(x,t) = v(x, t + t_j)$, for all $(x,t) \in \mathcal{Q}$.

In order to prove also that $\Phi(U)$ is a weak solution of (3.1), we follow [13], again: we take $\rho \in C_0^\infty(-1,1)$ and $\psi \in C_0^\infty(\Omega)$, and we test Eq. (4.3) with $\eta(x,t) =$

$\rho(t-t_j)\psi(x)$, so as to get

$$
\begin{aligned}
&-\int_{-1+t_j}^{1+t_j}\int_\Omega v\psi\rho'(t-t_j)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad+\int_{-1+t_j}^{1+t_j}\iint_{\mathbb{R}^{2N}}\frac{(\Phi(v(x,t))-\Phi(v(y,t)))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\rho(t-t_j)\,\mathrm{d}t \\
&=\alpha\int_{-1+t_j}^{1+t_j}\int_\Omega v\psi\rho(t-t_j)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

A change of variable in the time integral yields

$$
\begin{aligned}
-\iint_Q v_j\psi\rho'\,\mathrm{d}x\,\mathrm{d}t+\int_{-1}^{1}\iint_{\mathbb{R}^{2N}}\frac{(\Phi(v_j(x,t))-\Phi(v_j(y,t)))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\,\rho(t)\,\mathrm{d}t \\
=\alpha\iint_Q v_j\psi\rho\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

In view of the definition of the s-laplacian of the smooth function ψ, the latter implies

$$
-\iint_Q v_j\psi\rho'\,\mathrm{d}x\,\mathrm{d}t+\iint_Q\Phi(v_j)(-\Delta)^s\psi\;\rho\,\mathrm{d}t=\alpha\iint_Q v_j\psi\rho\,\mathrm{d}x\,\mathrm{d}t.
$$

Owing to (4.19), taking the limits yields

$$
-\int_{-1}^{1}\left(\int_\Omega U\psi\,\mathrm{d}x\right)\rho'\,\mathrm{d}t+\int_{-1}^{1}\left(\int_\Omega\Phi(U)(-\Delta)^s\psi\,\mathrm{d}x\right)\rho\,\mathrm{d}t=\alpha\int_{-1}^{1}\left(\int_\Omega U\psi\,\mathrm{d}x\right)\rho\,\mathrm{d}t.
$$

Since ρ vanishes at the endpoints of the interval $(-1,1)$, it follows that

$$
\left[\int_\Omega\left(\Phi(U)(-\Delta)^s\psi-\alpha U\psi\right)\,\mathrm{d}x\right]\cdot\int_{-1}^{1}\rho\,\mathrm{d}t=0.
$$

As ρ can be any element of $C_0^\infty(-1,1)$, we can choose is so as to make the time integral different from zero. Thus, recalling again the definition of $(-\Delta)^s\psi$ we deduce that (3.2) holds with $u=\Phi(U)$, as desired. □

5. Paths of controlled energy

Proposition 5.1. *Let $m>1$ and $\alpha>0$, let $u_0\in L^{m+1}(\Omega)$, with $\Phi(u_0)\in\mathcal{D}_0^{s,2}(\Omega)$. Assume that either (1.6a) or (1.6b) holds, and set $q=(m+1)/m$. Then, there exists $\theta\in C([0,1]\,;\,\mathcal{D}_0^{s,2}(\Omega))$ for which*

(i) $\theta(\cdot,0)$ *is the positive minimizer w of $\mathscr{F}_{q,\alpha}^s$,*
(ii) $\theta(\cdot,1)=\Phi(u_0)$*, and*
(iii) $\mathscr{F}_{q,\alpha}^s(\theta(\cdot,t))<\Lambda_2$ *for all $t\in(0,1)$.*

Proof. In order to construct the desired function θ, we first consider a special path γ in $\mathcal{D}_0^{s,2}(\Omega)$, connecting w and the positive part $\Phi(u_0)^+$ of $\Phi(u_0)$. This is done by setting

$$\gamma(\tau) = \left[(1-\tau)w^q + \tau|\Phi(u_0)^+|^q\right]^{\frac{1}{q}}, \qquad \text{for every } \tau \in [0,1].$$

By [9, Proposition 4.1], $\tau \mapsto [\gamma(\tau)]_{s,\Omega}^2$ is convex. In particular, it is continuous and it follows that γ is continuous with values in $\mathcal{D}_0^{s,2}(\Omega)$. Also, recalling (2.4), we have

$$\mathscr{F}_{q,\alpha}^s(\gamma(\tau)) \le (1-\tau)\mathscr{F}_{q,\alpha}^s(w) + \tau\mathscr{F}_{q,\alpha}^s\big(\Phi(u_0)^+\big), \quad \text{for all } \tau \in [0,1]. \tag{5.1}$$

Under either of the assumptions (1.6a) and (1.6b) that implies

$$\mathscr{F}_{q,\alpha}^s(\gamma(\tau)) < \Lambda_2, \qquad \text{for all } \tau \in [0,1]. \tag{5.2}$$

Then, we consider the segment in $\mathcal{D}_0^{s,2}(\Omega)$ with endpoints $\Phi(u_0)^+$ and $\Phi(u_0)$. The linear parametrization of such segment, defined by $\sigma(\tau) = \Phi(u_0)^+ - \tau\Phi(u_0)^-$ is obviously continuous with values in $\mathcal{D}_0^{s,2}(\Omega)$. Also, we have

$$\mathscr{F}_{q,\alpha}^s(\sigma(\tau)) = h(\tau) + C\tau + K$$

where

$$C = 2\iint \frac{\Phi(u_0)^+(x)\Phi(u_0)^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y, \qquad K = \mathscr{F}_{q,\alpha}^s\big(\Phi(u_0^+)\big),$$

and $h(\tau)$ is essentially the function considered in Appendix to [13] in the local case. Namely,

$$h(\tau) = A\tau^2 - B\tau^q, \qquad \text{where } A = \frac{1}{2}[\Phi(u_0)^-]_{s,\Omega}^2 \ \text{ and } B = \frac{\alpha}{q}\int_\Omega |\Phi(u_0)^-|^q\,\mathrm{d}x.$$

If $\tau_0 := \left(\frac{qB}{2A}\right)^{\frac{1}{2-q}} \ge 1$, then by direct inspection $h'(\tau) < 0$ for $\tau \in (0,1)$, and hence

$$\mathscr{F}_{q,\alpha}^s(\sigma(\tau)) \le C + K = \mathscr{F}_{q,\alpha}^s\big(\Phi(u_0^+)\big) + 2\iint \frac{\Phi(u_0)^+(x)\Phi(u_0)^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{5.3}$$

Otherwise, $qB < 2A$ and that implies $h'(1) > 0$. Also, for $\tau_0 \le \tau \le 1$ we have $h'' \le -2Aq < 0$, so that $h'(\tau) \ge h'(1) > 0$ for $\tau \in [\tau_0, 1]$. If instead $0 \le \tau < \tau_0$ then $h'(\tau) < 0$, because of the definition of τ_0. Therefore, for all $\tau \in [0,1]$ the inequality

$$\begin{aligned}\mathscr{F}_{q,\alpha}^s(\sigma(\tau)) &\le \max\{h(0) + C + K\,,\, h(1) + C + K\} \\ &= \max\left\{\mathscr{F}_{q,\alpha}^s\big(\Phi(u_0^+)\big) + 2\iint \frac{\Phi(u_0)^+(x)\Phi(u_0)^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\,,\, \mathscr{F}_{q,\alpha}^s(\Phi(u_0))\right\}\end{aligned}$$

holds regardless of the value of τ_0. Under either of the assumptions in (1.6), that entails

$$\mathscr{F}^s_{q,\alpha}(\sigma(\tau)) < \Lambda_2, \qquad \text{for all } \tau \in [0,1]. \tag{5.4}$$

By construction, setting

$$\theta(t) = \begin{cases} \gamma(2t), & \text{if } 0 \le t < \frac{1}{2}, \\ \sigma(2(t-\frac{1}{2})), & \text{if } \frac{1}{2} \le t < 1 \end{cases}$$

defines a continuous function from [0, 1] to $\mathcal{D}^{s,2}_0(\Omega)$ for which the assertions (i) and (ii) are true. As for (iii), that is a consequence of the inequalities (5.2) and (5.4). □

Remark 5.1. If $\varphi \in W^{1,2}_0(\Omega)$ then

$$\lim_{s\nearrow 1}(1-s)\iint \frac{\varphi^+(x)\varphi^-(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = 0.$$

That is a consequence of the known fact, see *e.g.* [16, Corollary 3.20], that

$$\lim_{s\nearrow 1}(1-s)[\varphi]^2_s = C(n)\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x,$$

and of the locality of Sobolev seminorm in the right-hand side of the latter. With $\varphi = \Phi(u_0)$, we see that the double integral in (1.6b) vanishes in the limit up to multiplying it by the factor $(1-s)$.

6. Proofs of the main results

6.1. Proof of Theorem 1.1

Weak solutions can be defined for (1.1) similarly as done in Definition 4.1 for the rescaled problem (4.1). By setting

$$v(x,t) = e^{\alpha t}u(x, e^t - 1)$$

the desired conclusion becomes that the weak solution $v(\cdot,t)$ of (4.1) with $v(0) = u_0$ converges, as $t \to +\infty$, either to $\Phi^{-1}(w)$ or to $-\Phi^{-1}(w)$ in $L^{m+1}(\Omega)$, where w is the positive minimiser of the functional $\mathscr{F}^s_{q,\alpha}$ defined by (2.4), that is unique by Lemma 3.2. By Theorem 4.1, v is uniquely determined and the estimate (4.4) holds. Therefore, by the compactness of the embedding $\mathcal{D}^{s,2}_0(\Omega)$ into $L^q(\Omega)$, it follows that the orbit $\{v(\cdot,t) : t > 0\}$ is precompact in $L^{m+1}(\Omega)$. Then, the omega-limit $\omega(u_0)$ is connected, and so in order to get the desired conclusion it suffices to prove that

$$\omega(u_0) \subseteq \{\Phi^{-1}(w), -\Phi^{-1}(w)\}. \tag{6.1}$$

Then, we take $U \in \omega(u_0)$. From (4.4) we can infer (4.18) and so, by Theorem 4.2, $\Phi(U)$ is a critical point of $\mathscr{F}^s_{q,\alpha}$. Therefore, it is enough to make sure that

$$\mathscr{F}^s_{q,\alpha}(\Phi(U)) < \Lambda_2 \tag{6.2}$$

because by Corollary 3.1 that entails that $\Phi(U)$ is either w or $-w$, which in turn gives (6.1).

We are left to prove (6.2). To do so, we take a sequence $t_j \nearrow +\infty$ with $v(\cdot, t_j) \to U$ in $L^{m+1}(\Omega)$. We raise the Lipschitz estimate $|\Phi(b) - \Phi(a)| \le m(|a| \vee |b|)^{m-1}|b-a|$, with $a = U(x)$ and $b = v(x, t_j)$, to the power $q = \frac{m+1}{m}$, we integrate the result over Ω, and hence we arrive at

$$\limsup_{j\to\infty} \|\Phi(v(\cdot, t_j)) - \Phi(U)\|_{L^q(\Omega)} \le m\Big[\|\Phi(U)\|_{L^q(\Omega)} + \sup_{t>0} \|v(\cdot, t)\|_{L^{m+1}(\Omega)}\Big] \lim_{j\to\infty} \|v(\cdot, t_j) - U\|_{L^{m+1}(\Omega)},$$

where we also used Hölder inequality with exponents $m/(m-1)$ and m. Hence, $\Phi(v(\cdot, t_j))$ converges to $\Phi(U)$ in measure. Then, by Fatou's Lemma,

$$\mathscr{F}^s_{q,\alpha}(\Phi(U)) \le \liminf_{j\to\infty} \mathscr{F}^s_{q,\alpha}\left(\Phi(v(\cdot, t_j))\right).$$

On the other hand, by Theorem 4.1,

$$\mathscr{F}^s_{q,\alpha}(\Phi(v(\cdot, t))) \le \mathscr{F}^s_{q,\alpha}(\Phi(u_0)), \qquad \text{for all } t > 0.$$

By assumption, we have

$$\mathscr{F}^s_{q,\alpha}(\Phi(u_0)) < \Lambda_2$$

and (6.2) follows by pairing this strict inequality with the previous two ones. □

Remark 6.1. It would be interesting to upgrade the L^{m+1} convergence in (1.1) to the *uniform* (resp., *local uniform*) convergence. Once C^α regularity up to the boundary (resp., interior C^α regularity) is available, which is still not the case for the sign changing solutions, it would be sufficient to reproduce the argument of [21, Chapter 20, page 526]. In the case of nonnegative solutions, a complete satisfactory answer to this question is given in [2].

6.2. Proof of Proposition 1.1

Assume that one of the two conditions in (1.6) holds and set $q = (m+1)/m$. Then, by Proposition 5.1 there exists $\theta \in C([0, 1]; \mathcal{D}^{s,2}_0(\Omega))$ such that $\theta(\cdot, 0)$ equals the positive minimizer w of $\mathscr{F}^s_{q,\alpha}$, moreover $\theta(\cdot, 1) = \Phi(u_0)$ and

$$\mathscr{F}^s_{q,\alpha}(\Phi(\theta(\cdot, t))) < \Lambda_2, \qquad \text{for all } t \in [0, 1]. \tag{6.3}$$

Now, we set

$$z(\cdot, t) = \begin{cases} \theta(\cdot, t), & \text{if } 0 \le t \le 1 \\ \Phi(v(\cdot, t-1)), & \text{if } t > 1, \end{cases}$$

where v is the unique solution of (4.1) with $v(0) = u_0$. Then, in view of Theorem 4.1 and of Proposition 5.1, we have

$$\mathscr{F}^s_{q,\alpha}(z(\cdot,t)) < \Lambda_2 \tag{6.4}$$

Hence, by coercivity (see Lemma 3.1) we deduce that the trajectory $z(\cdot,t)$ is contained in a bounded subset of $\mathcal{D}^{s,2}_0(\Omega)$. Since $t \mapsto z(\cdot,t)$ is continuous from $[0,1]$ to $\mathcal{D}^{s,2}_0(\Omega)$, by the compactess of the embedding into $L^q(\Omega)$ it is continuous as a function with values in $L^q(\Omega)$, as well; also, it is easily seen that the continuity of $t \mapsto v(\cdot,t-1)$ from $[1,+\infty)$ to $L^{m+1}(\Omega)$ implies that of $t \mapsto z(\cdot,t) = \Phi(v(\cdot,t-1))$ from $[1,+\infty)$ to $L^q(\Omega)$. Therefore, z belongs both to $L^\infty((0,+\infty);\mathcal{D}^{s,2}_0(\Omega))$ and to $C([0,+\infty);L^0(\Omega))$.

Now, we argue by contradiction, and we assume $v(\cdot,t)$ to converge in measure to $-w$. Then, so does $z(\cdot,t)$ and it follows that z is then eligible for the mountain pass formula of Proposition 3.1. Thus,

$$\Lambda^* = \inf_{z\in\mathfrak{Z}} \sup_{t\in[0,+\infty)} \mathscr{F}^s_{q,\alpha}(z(\cdot,t)) < \Lambda_2.$$

in contradiction with Proposition 3.3. $\square$

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Appendix: An elementary inequality

Lemma A.1. *Let* $m > 1$*, set* $f(t) = \frac{1}{m+1}|t|^{m+1}$, $g(t) = |t|^{\frac{m-1}{2}}t$ *and we let* $a, b \in \mathbb{R}$*. Then*

$$f(a) - f(b) - f'(b)(a-b) \geq C_0(m)|g(b) - g(a)|^2,$$

where $C_0(m) = (m+1)^{-3}$.

Proof. We claim that

$$\frac{1}{2}|a|^{m+1} + \frac{1}{2}|b|^{m+1} \geq \left|\frac{a+b}{2}\right|^{m+1} + \frac{1}{8}\max\left\{|a|^{m-1}, |b|^{m-1}\right\}|a-b|^2. \tag{A.1}$$

Thence, since - by strict convexity - we also have

$$\left|\frac{a+b}{2}\right|^{m+1} \geq |b|^{m+1} + (m+1)|b|^{m-1}b\,\frac{a-b}{2},$$

we would arrive at

$$|a|^{m+1} \geq |b|^{m+1} + (m+1)|b|^{m-1}b\,(a-b) + \frac{1}{4}\max\left\{|a|^{m-1}, |b|^{m-1}\right\}|a-b|^2.$$

By Lagrange mean value theorem applied to the function $g(v) = |v|^{\frac{m-1}{2}}v$, we also have

$$|g(a) - g(b)|^2 \leq \frac{(m+1)^2}{4}\max\left\{|a|^{m-1}, |b|^{m-1}\right\}|a-b|^2,$$

and because of the definition of f we would get the conclusion by combining the last two inequalities.

Then, we are left to prove (A.1). To do so, using Cauchy integral remainder theorem we write

$$f(a) = f\left(\tfrac{a+b}{2}\right) + \tfrac{1}{2}f'\left(\tfrac{a+b}{2}\right)(a-b) + \tfrac{1}{4}\int_0^1 f''\left(\lambda a + (1-\lambda)\tfrac{a+b}{2}\right)(a-b)^2(1-\lambda)\,\mathrm{d}\lambda,$$

$$f(b) = f\left(\tfrac{a+b}{2}\right) + \tfrac{1}{2}f'\left(\tfrac{a+b}{2}\right)(a-b) + \tfrac{1}{4}\int_0^1 f''\left(\lambda b + (1-\lambda)\tfrac{a+b}{2}\right)(b-a)^2(1-\lambda)\,\mathrm{d}\lambda.$$

Since $f''(t) = m|t|^{m-1}$, if follows that

$$\frac{1}{2} f(a) + \frac{1}{2} f(b) \geq f\left(\tfrac{a+b}{2}\right) + \frac{m}{8}(a-b)^2 \int_0^1 \left|\lambda a + (1-\lambda)\tfrac{a+b}{2}\right|^{m-1} (1-\lambda)\, \mathrm{d}\lambda + \frac{m}{8}(a-b)^2 \int_0^1 \left|\lambda b + (1-\lambda)\tfrac{a+b}{2}\right|^{m-1} (1-\lambda)\, \mathrm{d}\lambda.$$

We assume with no restriction that $|a| \geq |b|$. Hence, by the triangle inequality we see that

$$\begin{aligned}\int_0^1 \left|\lambda a + (1-\lambda)\tfrac{a+b}{2}\right|^{m-1} (1-\lambda)\, \mathrm{d}\lambda &= \int_0^1 \left|\tfrac{1+\lambda}{2} a + \tfrac{1-\lambda}{2} b\right|^{m-1} (1-\lambda)\, \mathrm{d}\lambda \\ &\geq \int_0^1 \left(\tfrac{1+\lambda}{2} a - \tfrac{1-\lambda}{2} b\right)^{m-1} (1-\lambda)\, \mathrm{d}\lambda \\ &\geq |a|^{m-1} \int_0^1 \lambda^{m-1}(1-\lambda)\, \mathrm{d}\lambda = \frac{|a|^{m-1}}{m(m+1)},\end{aligned}$$

and $|a|^{m+1} = \max\{|a|^{m-1}, |b|^{m-1}\}$ by assumption. Then, by inserting the latter in the previous inequality we get (A.1), as desired. □

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