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Doubly nonlinear equations for the 1-Laplacian

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Abstract. This paper is concerned with the Neumann problem for a class of doubly nonlinear equations for the 1-Laplacian,

$$\frac{\partial v}{\partial t} - \Delta_1 u \ni 0 \text{ in } (0,\infty)\times\Omega, \quad v \in \gamma(u),$$

and initial data in $L^1(\Omega)$, where Ω is a bounded smooth domain in $\mathbb{R}^N$ and γ is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$. We prove that, under certain assumptions on the graph γ, there is existence and uniqueness of solutions. Moreover, we proof that these solutions coincide with the ones of the Neumann problem for the total variational flow. We show that such assumptions are necessary.

1. Introduction

Consider the doubly nonlinear diffusion problem:

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = \operatorname{div}(\alpha(\nabla u(t,x))), & \text{in } (0,\infty)\times\Omega, \\ v \in \gamma(u), & \text{in } (0,\infty)\times\Omega, \\ v(0,x) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

completed with boundary conditions, being Ω a bounded domain in $\mathbb{R}^N$, γ a maximal monotone graph (possibly multivalued) in $\mathbb{R}\times\mathbb{R}$ and $\alpha : \mathbb{R}^N \to \mathbb{R}^N$. Typical examples are $\alpha(\xi) = \alpha_p(\xi) := |\xi|^{p-2}\xi$, $p > 1$, and $\gamma(r) = |r|^{m-1}r$, $m > 0$. In these particular cases, for $p = 2$ and $m = 1$ the equation reduces to the classical *heat equation*, while for $0 < m < 1$ it is the *porous medium equation* (see, e.g., [26]) and the *p-Laplacian diffusion equation* for $p > 1$ and $m = 1$. In a general framework, case $0 < m < p-1$ is known as a doubly nonlinear equation with *slow diffusion*, while the case $m > p-1$ is named a *fast diffusion equation* (see, e.g., [22]). Therefore, owing to the choice of α and the graph γ, this equation may arise a variety of different situations and it

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possess a wide spectrum of applications, for instance, in fluid dynamics, soil science and filtration, see [11] and [25]. Observe that, for $p = 2$, other typical examples are

$$\gamma(r) = \begin{cases} r & \text{if } r < 0, \\ 0 & \text{if } 0 \le r \le 1, \\ r - 1 & \text{if } r > 1, \end{cases}$$

for a *Stefan type problem*, or

$$\gamma(r) = \begin{cases} 0 & \text{if } r < 0, \\ [0,1] & \text{if } r = 0, \\ 1 & \text{if } r > 0. \end{cases}$$

for a *Hele-Shaw-type problem*.

From a mathematical point of view, there is an extensive literature related to problem (1.1). Existence, uniqueness, regularity and asymptotic behavior of solutions are treated under different restrictions on γ and α, and we refer some literature: [1,2,16,21,23,24,26] and the literature therein.

Our main aim is to deal with existence and uniqueness for the limit case $p = 1$ for the function α_p, that is, $\alpha_1(\xi) := \frac{\xi}{|\xi|}$, γ a maximal monotone graph and homogeneous Neumann boundary conditions. More precisely, by means of Crandall–Liggett's theorem we obtain existence and uniqueness of *entropy solution* (see Definition 4.5) of the doubly nonlinear problem

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta_1 u \ni 0 & \text{in } (0,\infty) \times \Omega, \\ v = \gamma(u) & \text{in } (0,\infty) \times \Omega, \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ v(0,x) = v_0(x) \; x \in \Omega, \end{cases} \tag{1.2}$$

under the condition

$$\begin{cases} \gamma \text{ is a non-decreasing continuous function such that } \gamma(0) = 0 \text{ and} \\ \mathrm{Rang}(\gamma) = \mathbb{R}. \end{cases} \tag{1.3}$$

For this purpose, first of all we deal with the following *elliptic problem*

$$\begin{cases} v - \Delta_1 u \ni f & \text{in } \Omega, \\ v \in \gamma(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases}$$

In Theorems 3.8 and 3.9, we prove the existence of solutions under the condition

$$\begin{cases} \gamma \text{ is a maximal monotone graph such that } \gamma(0) \ni 0 \text{ and} \\ \mathrm{Rang}(\gamma) = \mathbb{R}, \end{cases} \tag{1.4}$$

and we prove uniqueness for continuous γ in Theorem 3.7, that is, under assumption (1.3). Note that (1.3) implies (1.4). Moreover, we see that for non-continuous maximal monotone graphs there is non-uniqueness (Example 3.6). We also show that condition $\mathrm{Rang}(\gamma) = \mathbb{R}$ is necessary for the existence of solutions (Example 3.12).

Remark 1.1. On account of our approach to solve problem (1.2) and the above comments, condition (1.3) is natural for the study of such evolution problem. □

In [5] (see also [6]), it was studied the well-posedness of the Neumann problem

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta_1 v \ni 0 & \text{in } (0,\infty) \times \Omega, \\ \frac{\partial v}{\partial \eta} = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ v(0,x) = v_0(x) \ x \in \Omega, \end{cases} \tag{1.5}$$

by means of the Nonlinear Semigroup Theory. For that purpose, the following operator $\mathcal{A}$, defined in $L^1(\Omega) \times L^1(\Omega)$, was introduced to give mathematical sense to the formal expression of $\Delta_1 v := \mathrm{div}\left(\frac{\nabla v}{|\nabla v|}\right)$ (jointly with the homogenous Neumann boundary conditions).

Definition 1.2.

$$(v,w) \in \mathcal{A} \iff v \in L^1(\Omega),\ T_k(v) \in BV(\Omega) \text{ for all } k > 0,$$
$$\text{and there exists } \mathbf{z} \in X_1(\Omega),\ \|\mathbf{z}\|_\infty \le 1,\ \text{ such that}$$

$$w = -\,\mathrm{div}(\mathbf{z}) \quad \text{in } \mathcal{D}'(\Omega),$$
$$\int_\Omega (\mathbf{z}, DT_k(v)) = \int_\Omega |DT_k(v)| \quad \forall k > 0,$$

and

$$[\mathbf{z}, \nu] = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega,$$

(see notation in Sects. 2.1 and 2.2).

Moreover, it was shown that $\mathcal{A}$ is the closure in $L^1(\Omega) \times L^1(\Omega)$ of the subdifferential of the energy functional $\Phi : L^2(\Omega) \to (-\infty, +\infty]$ defined by

$$\Phi(v) = \begin{cases} \int_\Omega |Dv| & \text{if } v \in BV(\Omega) \cap L^2(\Omega), \\ +\infty & \text{if } v \in L^2(\Omega) \setminus BV(\Omega). \end{cases}$$

Since Φ is a proper convex and lower semi-continuous function, then $\partial\Phi$ is a maximal monotone operator with dense domain, generating a contraction semigroup in $L^2(\Omega)$ that solves problem (1.5) for L^2-data. Entropy solutions for L^1-data v_0 were introduced to characterize mild solutions of the abstract Cauchy problem

$$\begin{cases} v_t + \mathcal{A}(v) \ni 0, \\ \\ v(0) = v_0, \end{cases} \tag{1.6}$$

given by the Crandall–Liggett's semigroup generation theorem ([19]).

Remark 1.3. We show that the solutions of (1.2) are given by the solutions of (1.5) (Theorem 4.4). This is a non-trivial result; we first need to prove directly existence and uniqueness of solutions of problem (1.2). Observe that, at the level of *elliptic problems*, we first prove Theorem 3.9 and afterward we can prove Theorem 3.11.

The fact that solutions of (1.5) are solutions of (1.2) gives a kind of invariance property for the diffusion evolution problem via the 1-Laplacian, i.e., changing variables,

the solutions of $w_t - \Delta_1 w \ni 0$ and the solutions of $w_t - \Delta_1\gamma^{-1}(w) \ni 0$ are the same provided that γ satisfies (1.3).

Observe that, written in this way, γ^{-1} can be a non-continuous maximal monotone graph, hence not necessarily Lipschitz-continuous. When γ^{-1} is an increasing and Lipschitz-continuous function, solutions of (1.5) are solutions of (1.2), see Proposition 3.10 at the level of the *elliptic problems*. □

2. Preliminaries

2.1. Functions of bounded variation

We will denote by $\mathcal{M}(\Omega)$ the set of all Lebesgue measurable functions in Ω.

The natural energy space to study problem (1.2) is the space of functions of bounded variation. For further information concerning functions of bounded variation, we refer to [4] and [20]. Recall that if Ω is an open subset of $\mathbb{R}^N$, a function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du in Ω is defined by the formula

$$|Du|(\Omega) = \sup\left\{ \int_\Omega u \operatorname{div}(\phi) \,:\, \phi \in C_0^\infty(\Omega, \mathbb{R}^N),\ \|\phi\| \le 1 \right\}.$$

The space $BV(\Omega)$ is endowed with norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega).$$

Recall that an $\mathcal{L}^N$-measurable subset E of $\mathbb{R}^N$ has *finite perimeter* if $\chi_E \in BV(\mathbb{R}^N)$. The perimeter of E is defined by $\operatorname{Per}(E) = |D\chi_E|(\mathbb{R}^N)$.

2.2. A generalized Green's formula

Let Ω be an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary. Following [10], for $1 \le p \le \infty$ let

$$X_p(\Omega) = \{\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N) : \mathrm{div}(\mathbf{z}) \in L^p(\Omega)\}.$$

If $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$, we define the functional $(\mathbf{z}, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle(\mathbf{z}, Dw), \varphi\rangle = -\int_\Omega w\,\varphi\,\mathrm{div}(\mathbf{z})\,dx - \int_\Omega w\,\mathbf{z}\cdot\nabla\varphi\,dx.$$

Then, $(\mathbf{z}, Dw)$ is a Radon measure in Ω,

$$\int_\Omega (\mathbf{z}, Dw) = \int_\Omega \mathbf{z}\cdot\nabla w\,dx \qquad \forall\, w \in W^{1,1}(\Omega) \cap L^{p'}(\Omega)$$

and

$$\left|\int_B (\mathbf{z}, Dw)\right| \le \int_B |(\mathbf{z}, Dw)| \le \|\mathbf{z}\|_\infty \int_B |Dw| \tag{2.1}$$

for any Borel set $B \subseteq \Omega$.

Moreover, when $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$, we have the following integration by parts formula

$$\int_\Omega w\,\mathrm{div}\,(\mathbf{z})\,dx + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu] w\,d\mathcal{H}^{N-1}, \tag{2.2}$$

where $[\mathbf{z}, \nu]$ is the weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ (see [10]).

By (2.1), the measures $(\mathbf{z}, Du)$ and $|(\mathbf{z}, Du)|$ are absolutely continuous with respect to the measure $|Du|$ in Ω.

Thus, there is a density function

$$\theta(\mathbf{z}, Dw, \cdot) = \frac{d(\mathbf{z}, Dw)}{d|Dw|} \in L^\infty(\Omega, |Dw|),$$

satisfying

$$|\theta(\mathbf{z}, Dw, x)| \le 1 \text{ for } |Dw|\text{-a.e. } x \in \Omega.$$

The function $\theta(\mathbf{z}, Dw, \cdot)$ is called the Radon–Nikodým derivative of $(\mathbf{z}, Dw)$ with respect to $|Dw|$. Moreover, the following results hold.

Proposition 2.1. ([10], Chain rule for $(\mathbf{z}, D(\cdot))$) *Let Ω be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$, and, for $1 \le p \le N$ and p' its conjugate exponent, let $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$. Then, for every Lipschitz-continuous, monotonically increasing function $T : \mathbb{R} \to \mathbb{R}$, one has that*

$$\theta(\mathbf{z}, D(T \circ w), x) = \theta(\mathbf{z}, Dw, x) \quad \textit{for } |Dw|\textit{-a.e. } x \in \Omega.$$

We shall denote

$$\operatorname{sign}_0(r) := \begin{cases} 1 & \text{if } r > 0, \\ 0, & \text{if } r = 0, \\ -1, & \text{if } r < 0. \end{cases} \qquad \operatorname{sign}(r) := \begin{cases} 1 & \text{if } r > 0, \\ [-1, 1], & \text{if } r = 0, \\ -1, & \text{if } r < 0, \end{cases}$$

and $\operatorname{sign}^+(r) := (\operatorname{sign}(r))^+$, and $T_k(r) := [k - (k - |r|)^+]\operatorname{sign}_0(r)$, $k \geq 0$.

Remark 2.2. Let us point out that although T_k is only non-decreasing, we also have the following result

$$\theta(\mathbf{z}, D(T_k u), x) = \theta(\mathbf{z}, Du, x) \qquad \text{for } |DT_k|\text{-a.e. } x \in \Omega.$$

□

2.3. Accretive operators and nonlinear semigroups

An operator A on X is a possibly nonlinear and multivalued mapping $A : X \to 2^X$. It is standard to identify an operator A on X with its graph

$$A := \left\{ (u, v) \in X \times X \,\middle|\, v \in Au \right\} \text{ in } X \times X$$

and so, one sees A as a subset of $X \times X$. The set $D(A) := \{u \in X \mid Au \neq \emptyset\}$ is called the *domain* of A, and $R(A) := \bigcup_{u \in D(A)} Au$ the *range* of A.

Definition 2.3. An operator A on X is called *m-accretive operator* on X if A is *accretive*, that is, for every $(u, v), (\hat{u}, \hat{v}) \in A$ and every $\lambda > 0$,

$$\|u - \hat{u}\|_X \leq \|u - \hat{u} + \lambda(v - \hat{v})\|_X$$

and if for all $\lambda > 0$ the *range condition*

$$R(I + \lambda A) = X$$

holds.

Note that A is accretive if the resolvent $J_\lambda := (I + \lambda A)^{-1}$ are contractions for all $\lambda > 0$. The *Yosida approximation* of A is defined as

$$A_\lambda := \frac{1}{\lambda}(I - J_\lambda), \quad \text{for } \lambda > 0.$$

We have that

$$A_\lambda : D(J_\lambda) \to X \text{ is Lipschitz-continuous with Lipschitz constant } \frac{2}{\lambda}.$$

Moreover,

$$A_\lambda u \in A J_\lambda u, \quad u = J_\lambda u + \lambda A_\lambda u \quad \text{and} \quad \|A_\lambda u\| \leq \inf\{\|v\| \,:\, v \in Au\}.$$

In the case that the Banach space is $L^1(\Omega)$, with $\Omega \subset \mathbb{R}^N$ an open set and norm

$$\|u\|_1 := \int_\Omega |u(x)|dx, \quad u \in L^1(\Omega),$$

it is well known (see [14]) that

an operator A on $L^1(\Omega)$ is accretive $\iff$ for every $(u, v), (\hat{u}, \hat{v}) \in A$ there exists $\xi \in \operatorname{sign}(u - \hat{u})$ such that $\int_\Omega (v - \hat{v})\xi dx \geq 0$.

Definition 2.4. We say that an operator A on $L^1(\Omega)$ is *T-accretive* if for every (u, v), $(\hat{u}, \hat{v}) \in A$ and every $\lambda > 0$,

$$\|(u - \hat{u})^+\|_1 \leq \|(u - \hat{u} + \lambda(v - \hat{v}))^+\|_1.$$

If A is an m-accretive operator on a Banach space X, then by the classical existence theory (see, e.g., [14, Theorem 6.5], or [12, Corollary 4.2]), the first-order Cauchy problem

$$\begin{cases} \dfrac{du}{dt} + A(u(t)) \ni g(t) \text{ on } (0, T), \\ u(0) = u_0, \end{cases} \tag{2.3}$$

is well-posed for every $u_0 \in \overline{D(A)}^X$, and $g \in L^1(0, T; X)$ in the following *mild sense*.

Definition 2.5. For given $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0, T; X)$, a function $u \in C([0, T]; X)$ is called a *mild solution* of Cauchy problem (2.3) if $u(0) = u_0$ and for every $\varepsilon > 0$, there is a partition $0 = t_0 < t_1 < \cdots < t_N = T$ and a step function

$$u_{\varepsilon,N}(t) = u_0\, \chi_{\{t=0\}}(t) + \sum_{i=1}^N u_i\, \chi_{(t_{i-1},t_i]}(t), \quad t \in [0, T],$$

satisfying

- $t_i - t_{i-1} < \varepsilon \quad$ for all $i = 1, \ldots, N$,
- $\displaystyle\sum_{N=1}^N \int_{t_{i-1}}^{t_i} \|g(t) - \overline{g}_i\|_X\, dt < \varepsilon, \quad$ where $\displaystyle\overline{g}_i := \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} g(t)\, dt$,
- $\displaystyle\frac{u_i - u_{i-1}}{t_i - t_{i-1}} + Au_i \ni \overline{g}_i \quad$ for all $i = 1, \ldots, N$,

and

$$\sup_{t\in[0,T]} \|u(t) - u_{\varepsilon,N}(t)\|_X < \varepsilon.$$

In the case $g = 0$, the unique mild solution is given by the Crandall–Liggett's exponential formula

$$u(t) = e^{-tA}u_0 := \lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-n} u_0.$$

Mild solutions are limits of step functions which are not necessarily differentiable in time. This leads to the notion of *strong solution* of the Cauchy problem (2.3).

Definition 2.6. For given $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0, T; X)$, a function $u \in C([0, T]; X) \cap W^{1,1}_{\rm loc}((0, T); X)$ is called a *strong solution* of the Cauchy problem (2.3) if $u(0) = u_0$ and, for a.e. $t \in (0, T)$, $u(t) \in D(A)$ and $Au(t) \ni g(t) - \frac{du}{dt}(t)$.

We now recall a Bénilan–Crandall relation between functions $u, v \in L^1(\Omega, \nu)$. Denote by J_0 and P_0 the following sets of functions:

$$J_0 := \{j : \mathbb{R} \to [0, +\infty] : \ j \text{ is convex, lower semi-continuous and } j(0) = 0\},$$
$$P_0 := \left\{\rho \in C^\infty(\mathbb{R}) : \ 0 \le \rho' \le 1, \ \text{supp}(\rho') \text{ is compact and } 0 \notin \text{supp}(\rho)\right\}.$$

Assume that $\nu(\Omega) < +\infty$ and let $u, v \in L^1(\Omega, \nu)$. The following relation between u and v is defined in [15]:

$$u \ll v \ \text{ if } \int_\Omega j(u)\, d\nu \le \int_\Omega j(v)\, d\nu \ \text{ for every } j \in J_0.$$

Moreover, the following equivalences are proved in [15, Proposition 2.2]:

$$\int_\Omega v\rho(u)d\nu \ge 0 \quad \forall \rho \in P_0 \iff u \ll u + \lambda v \quad \forall \lambda > 0,$$
$$\int_\Omega v\rho(u)d\nu \ge 0 \quad \forall \rho \in P_0 \iff \int_{\{u<-h\}} v d\nu \le 0 \quad \& \quad 0 \le \int_{\{u>h\}} v d\nu \quad \forall h > 0.$$

The following result is given in [15]

Proposition 2.7. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set.*

(i) For any $u, v \in L^1(\Omega)$, if $\int_\Omega up(u)dx \le \int_\Omega vp(u)dx$ for all $p \in P_0$, then $u \ll v$.
(ii) If $u, v \in L^1(\Omega)$ and $u \ll v$, then $\|u\|_p \le \|v\|_p$ for all $1 \le p \le \infty$.
(iii) If $v \in L^1(\Omega)$, then $\{u \in L^1(\Omega) : u \ll v\}$ is a weakly compact subset of $L^1(\Omega)$.
(iv) If $u_n, u \in L^1(\Omega)$ satisfy $u_n \ll u$ and $u_n \to u$ weakly in $L^1(\Omega)$, then $u_n \to u$ in $L^1(\Omega)$.

Let $\gamma \subset \mathbb{R} \times \mathbb{R}$ be a *maximal monotone graph*. We denote by $\gamma^0(r)$ the element of $\gamma(r)$ of minimal absolute value. Then, for the Yosida approximations of γ we have that ([18, Proposition 2.6])

$$\text{for } r \in D(\gamma): \quad \lim_{\lambda\downarrow 0}\gamma_\lambda(r) = \gamma^0(r) \quad \text{and} \quad |\gamma_\lambda(r)| \uparrow |\gamma^0(r)| \ \text{ as } \lambda \downarrow 0.$$

3. The elliptic problem

From [4, Theorem 2], given $f \in L^1(\Omega)$ there exists a unique entropy solution v of the *elliptic problem*

$$\begin{cases} v - \Delta_1 v \ni f & \text{in } \Omega, \\ \frac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases}$$

defined as follows: $v \in L^1(\Omega)$ with $T_k(v) \in BV(\Omega)$ for all $k > 0$ and such that there exists $\mathbf{z} \in X_1(\Omega)$ with $\|\mathbf{z}\|_\infty \le 1$,

$$-\mathrm{div}(\mathbf{z}) = f - v \quad \text{in } \mathcal{D}'(\Omega),$$
$$(\mathbf{z}, DT_k(v)) = |DT_k(v)| \quad \text{as measures for all } k > 0$$

and

$$[\mathbf{z}, \nu] = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

Let γ be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \gamma(0)$. Following such concept, we give the following concept of entropy solution of the following *elliptic problem*

$$(S_f^\gamma) \quad \begin{cases} v - \Delta_1 u \ni f & \text{in } \Omega, \\ v \in \gamma(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases}$$

Definition 3.1. For $f \in L^1(\Omega)$, we say that v is an *entropy solution* of problem (S_f^γ) if $v \in L^1(\Omega)$ and there exist $u \in \mathcal{M}(\Omega)$ with $T_k(u) \in BV(\Omega)$ for all $k > 0$ and $\mathbf{z} \in X_1(\Omega)$ with $\|\mathbf{z}\|_\infty \le 1$ such that

$$v \in \gamma(u) \quad \text{a.e. in } \Omega, \tag{3.1}$$
$$-\mathrm{div}(\mathbf{z}) = f - v \quad \text{in } \mathcal{D}'(\Omega),$$
$$(\mathbf{z}, DT_k(u)) = |DT_k(u)| \quad \text{as measures for all } k > 0,$$
$$[\mathbf{z}, \nu] = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{3.2}$$

For data in $f \in L^\infty(\Omega)$, we also define the following concept of *weak solution* of problem (S_f^γ).

Definition 3.2. For $f \in L^\infty(\Omega)$, we say that v is a *weak solution* of problem (S_f^γ) if $v \in L^\infty(\Omega)$ and there exist $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{z} \in X_1(\Omega)$ with $\|\mathbf{z}\|_\infty \le 1$ such that

$$v \in \gamma(u) \quad \text{a.e. in } \Omega,$$
$$-\mathrm{div}(\mathbf{z}) = f - v \quad \text{in } \mathcal{D}'(\Omega),$$
$$(\mathbf{z}, Du) = |Du| \quad \text{as measures},$$
$$[\mathbf{z}, \nu] = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{3.3}$$

We have that every weak solution is an entropy solution.

Working as in [6, Lemma 2.4], it is easy to see the two following results.

Lemma 3.3. *For $f \in L^\infty(\Omega)$, the following assertions are equivalent:*

(a) v is weak solution of problem (S_f^γ);

(b) there exist $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $z \in X_\infty(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$ satisfying (3.1), (3.2) and

$$\int_\Omega (\varphi - u)(f - v)\, dx \leq \int_\Omega z \cdot \nabla\varphi\, dx - \int_\Omega |Du|, \qquad \forall \varphi \in W^{1,1}(\Omega) \cap L^\infty(\Omega); \tag{3.4}$$

(c) there exist $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $z \in X_\infty(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$ satisfying (3.1), (3.2) and

$$\int_\Omega (\varphi - u)(f - v)\, dx \leq \int_\Omega (z, D\varphi) - \int_\Omega |Du|, \qquad \forall \varphi \in BV(\Omega) \cap L^\infty(\Omega); \tag{3.5}$$

(d) there exist $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $z \in X_\infty(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$ satisfying (3.1), (3.2) and

$$\int_\Omega \varphi(f - v)\, dx = \int_\Omega (\mathbf{z}, D\varphi), \ \forall \varphi \in BV(\Omega) \cap L^\infty(\Omega).$$

Lemma 3.4. *For $f \in L^1(\Omega)$, the following assertions are equivalent:*

(a) v is an entropy solution of problem (S_f^γ);

(b) there exist $u \in \mathcal{M}(\Omega)$ with $T_k(u) \in BV(\Omega)$ for all $k > 0$ and $\mathbf{z} \in X_1(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$ satisfying (3.1), (3.2) and

$$\int_\Omega (\varphi - T_k(u))(f - v)\, dx \leq \int_\Omega z \cdot \nabla\varphi\, dx - \int_\Omega |DT_k(u)|, \qquad \forall \varphi \in W^{1,1}(\Omega) \cap L^\infty(\Omega);$$

(c) there exist $u \in \mathcal{M}(\Omega)$ with $T_k(u) \in BV(\Omega)$ for all $k > 0$ and $\mathbf{z} \in X_1(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$ satisfying (3.1), (3.2) and

$$\int_\Omega (\varphi - T_k(u))(f - v)\, dx \leq \int_\Omega (z, D\varphi) - \int_\Omega |DT_k(u)|, \qquad \forall \varphi \in L^\infty(\Omega) \cap BV(\Omega); \tag{3.6}$$

As can be verified in the above lemma, the notion of entropy solution for the 1-Laplacian defined here is analogous to the concept of entropy solution for the p-Laplacian $(1 < p < N)$ introduced in the pioneering paper [13].

Remark 3.5. Let v be an entropy solution of problem (S_f^γ). Then, there exist $u \in \mathcal{M}(\Omega)$ with $T_k(u) \in BV(\Omega)$ for all $k > 0$ and $\mathbf{z} \in X_1(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$ such that

$v \in \gamma(u)$ and (3.6) holds true. Then, if we take $T_k(u) \pm 1$ as test function in (3.6), it follows that

$$\int_\Omega f(x)dx = \int_\Omega v(x)dx.$$

Therefore, denoting

$$\gamma^- := \inf \mathrm{Ran}(\gamma) \quad \text{and} \quad \gamma^+ := \sup \mathrm{Ran}(\gamma),$$

the following condition must be satisfied

$$\gamma^- \mathcal{L}^N(\Omega) \leq \int_\Omega f(x)dx \leq \gamma^+ \mathcal{L}^N(\Omega).$$

Therefore, in the case $\mathrm{Rang}(\gamma) = \mathbb{R}$ this is always true for any $f \in L^1(\Omega)$. □

It is worthy to mention that if γ is a multivalued maximal monotone graph, the corresponding problem (S_f^γ) has more than one weak solution, as we show in the next example.

Example 3.6. Let γ be a multivalued graph such that

$$\gamma(0) = [0, 1].$$

Consider $\Omega :=]-1, 1[$ and $f(x) := \frac{1}{2}$ for all $x \in]-1, 1[$. We define $\mathbf{z} :]-1, 1[\to \mathbb{R}$ as

$$\mathbf{z}(x) := \begin{cases} 0, & \text{if } x \in \left]-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right[, \\ \frac{1}{2}x + \frac{1}{4}, & \text{if } x \in \left[-\frac{1}{2}, -\frac{1}{4}\right] \\ -\frac{1}{2}x, & \text{if } x \in \left[-\frac{1}{4}, \frac{1}{4}\right], \\ \frac{1}{2}x - \frac{1}{4}, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right]. \end{cases}$$

Then, $\|\mathbf{z}\|_\infty \leq 1$, $[\mathbf{z}, \nu] = 0$ and

$$v(x) := \mathrm{div}\, \mathbf{z}(x) + f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in \left]-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right[, \\ 1, & \text{if } x \in \left[-\frac{1}{2}, -\frac{1}{4}\right] \\ 0, & \text{if } x \in \left[-\frac{1}{4}, \frac{1}{4}\right], \\ 1, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right]. \end{cases}$$

Clearly, $v \in \gamma(0)$. Therefore, v is a weak solution of problem (S_f^γ). Now, taking $\mathbf{z} = 0$, it follows that f is also a weak solution of problem (S_f^γ). In particular, for the Stefan type problem, there is not uniqueness of weak solution of problem (S_f^γ).

Due to the above example, we need to impose some restriction to the maximal monotone graph γ in order to get uniqueness of entropy solution of problem $(S^{\gamma}_{f_i})$. In this direction, we have the following result for graphs satisfying (1.3) without the range condition.

Theorem 3.7. *Assume that $\gamma : D(\gamma) \subset \mathbb{R} \to \mathbb{R}$ is a non-decreasing continuous function with $\gamma(0) = 0$. Given $f_i \in L^1(\Omega)$ and v_i entropy solutions of $(S^{\gamma}_{f_i})$, for $i = 1, 2$, then*

$$\|(v_1 - v_2)^+\|_1 \le \|(f_1 - f_2)^+\|_1. \tag{3.7}$$

In particular,

$$\|v_1 - v_2\|_1 \le \|f_1 - f_2\|_1. \tag{3.8}$$

Proof. For $i = 1, 2$, we have that there exists $u_i \in L^1(\Omega)$ with $T_k(u_i) \in BV(\Omega)$ for all $k > 0$ and there exists $\mathbf{z}_i \in X_1(\Omega)$ with $\|\mathbf{z}_i\|_\infty \le 1$ such that $v_i = \gamma(u_i)$ and

$$\begin{aligned} -\operatorname{div}(\mathbf{z}_i) &= f_i - v_i \quad \text{in } \mathcal{D}'(\Omega), \\ (\mathbf{z}_i, DT_k(u_i)) &= |DT_k(u_i)| \quad \text{as measures for all } k > 0, \\ [\mathbf{z}_i, \nu] &= 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \end{aligned} \tag{3.9}$$

Let p_ϵ be a smooth strictly monotone approximation of the sign function. Then, applying integration by parts formula (2.2), we have

$$\begin{aligned} \int_\Omega (f_i - v_i) p_\epsilon(T_k(u_1) - T_k(u_2)) dx &= -\int_\Omega \operatorname{div}(\mathbf{z}_i) p_\epsilon(T_k(u_1) - T_k(u_2)) dx \\ &= \int_\Omega (\mathbf{z}_i, Dp_\epsilon(T_k(u_1) - T_k(u_2))). \end{aligned}$$

Thus,

$$\begin{aligned} \int_\Omega (v_1 - v_2) p_\epsilon(T_k(u_1) - T_k(u_2)) dx &= -\int_\Omega (\mathbf{z}_1 - \mathbf{z}_2, Dp_\epsilon(T_k(u_1) - T_k(u_2))) \\ &\quad + \int_\Omega (f_1 - f_2) p_\epsilon(T_k(u_1) - T_k(u_2)) dx \\ &\le -\int_\Omega (\mathbf{z}_1 - \mathbf{z}_2, Dp_\epsilon(T_k(u_1) - T_k(u_2))) \\ &\quad + \|f_1 - f_2\|_1 \end{aligned}$$

Now, from (3.9) and (2.1), we have

$$\int_B (\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2))) \ge 0, \quad \text{for all Borel set } B \subset \Omega.$$

This implies that

$$\theta(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2)), x) \ge 0 \quad |D(T_k(u_1) - T_k(u_2))|\text{-a.e.}$$

Since, according to Proposition 2.1, we have

$$\theta(\mathbf{z}_1 - \mathbf{z}_2, Dp_\epsilon(T_k(u_1) - T_k(u_2)), x) = \theta(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - T_k(u_2)), x)$$

$|D(T_k(u_1) - T_k(u_2))|$-a.e. and $|Dp_\epsilon(T_k(u_1) - T_k(u_2))|$-a.e., we get

$$\int_\Omega (\mathbf{z}_1 - \mathbf{z}_2, Dp_\epsilon(T_k(u_1) - T_k(u_2))) \\ = \int_\Omega \theta(\mathbf{z}_1 - \mathbf{z}_2, Dp_\epsilon(T_k(u_1) - T_k(u_2)), x) d|Dp_\epsilon(T_k(u_1) - T_k(u_2))| \geq 0.$$

Therefore,

$$\int_\Omega (v_1 - v_2) p_\epsilon(T_k(u_1) - T_k(u_2)) dx \leq \|f_1 - f_2\|_1.$$

Taking limit as $k \to \infty$, we get

$$\int_\Omega (v_1 - v_2) p_\epsilon(u_1 - u_2) dx \leq \|f_1 - f_2\|_1.$$

Taking now limit as $\epsilon \to 0^+$, we have that there exists $\xi(x) \in \operatorname{sign}((u_1(x) - u_2(x))$ $\mathcal{L}^N$-a.e. $x \in \Omega$ such that

$$\int_\Omega (v_1 - v_2)\xi(x) dx \leq \|f_1 - f_2\|_1.$$

Now, since $v_i = \gamma(u_i)$, $i = 1, 2$, and γ is non-decreasing and $\gamma(0) = 0$, we have $\xi(x) \in \operatorname{sign}((v_1(x) - v_2(x))$, if $v_1(x) \neq v_2(x)$. Hence, since γ is continuous, which for a maximal monotone graph is equivalent to say that $\gamma(r)$ is always univalued for any $r \in D(\gamma)$,

$$\|v_1 - v_2\|_1 = \int_{\{v_1 \neq v_2\}} (v_1 - v_2)\xi(x) dx \leq \|f_1 - f_2\|_1,$$

and (3.8) holds.

The proof of (3.7) is similar but using a smooth monotone approximation of the sign^+. □

Let us now prove existence for problem (S_f^γ) for graphs satisfying condition (1.4).

Theorem 3.8. *Let γ be a graph satisfying (1.4) and $f \in C_c^\infty(\Omega)$. Then, there exists v_f weak solution of problem (S_f^γ) with $v_f \ll f$.*

Moreover, for any $\tilde{f} \in C_c^\infty(\Omega)$, it follows

$$\|(v_f - v_{\tilde{f}})^+\|_1 \leq \|(f - \tilde{f})^+\|_1,$$

for the weak solutions constructed here.

Proof. Given $f \in C_c^\infty(\Omega)$, we must find $v \in L^\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$ with

$$v \in \gamma(u) \quad \text{a.e. in } \Omega,$$

and $\mathbf{z} \in X_1(\Omega)$, with $\|\mathbf{z}\|_\infty \leq 1$, satisfying (3.2) and (3.4).

By [8, Theorem 3.9 (i)], for any $p > 1$, there exist $u_p \in W^{1,p}(\Omega)$ and $v_p \in \gamma(u_p) \in L^1(\Omega)$ such that

$$\int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi dx + \int_\Omega v_p \varphi dx = \int_\Omega f \varphi dx, \tag{3.10}$$

for all $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover,

$$v_p \ll f, \quad \text{for all } p > 1, \tag{3.11}$$

and, since $\mathrm{Rang}(\gamma) = \mathbb{R}$,

$$\|u_p\|_\infty \leq M_1, \quad \text{for all } p > 1.$$

Taking $\varphi = u_p \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ as a test function and taking into account that $u_p v_p \geq 0$ it follows that

$$\int_\Omega |\nabla u_p|^p = \int_\Omega u_p (f - v_p) \leq \int_\Omega f u_p \leq M_2. \tag{3.12}$$

Therefore, by Hölder inequality we get

$$\int_\Omega |\nabla u_p| \leq M_3.$$

Thus, there exists $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that, up to a subsequence (no relabeled),

$$u_p \to u \ \text{ in } L^q(\Omega), \ \text{ for } 1 \leq q < 1^* := \frac{N}{N-1}. \tag{3.13}$$

Moreover, inequality (3.12) allows to establish the following statements (see [5]): There exists a bounded vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ with $||\mathbf{z}||_\infty \leq 1$ such that

$$|\nabla u_p|^{p-2} \nabla u_p \rightharpoonup \mathbf{z}, \ \text{ in } L^r(\Omega; \mathbb{R}^N), \ \text{ for all } 1 \leq r < \infty, \tag{3.14}$$

as $p \to 1^+$. In particular,

$$-\mathrm{div}(\mathbf{z}) = f - v, \quad \text{in } \mathcal{D}'(\Omega).$$

On the other hand, by (3.11) we obtain that

$$v_p \rightharpoonup v, \quad \text{in } L^q(\Omega), \ 1 \leq q < \infty, \tag{3.15}$$

being $v \ll f$. This result, in addition to (3.13), implies that

$$v \in \gamma(u) \quad \text{a.e. in } \Omega.$$

In order to show that v is a weak solution of problem (S^γ_f), for each $\varphi \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ we consider the sequence $\{\varphi_n\} \subset \mathcal{C}^\infty(\bar{\Omega})$ such that $\varphi_n \to \varphi$ in $W^{1,1}(\Omega)$. Now, taking $\varphi_n - u_p$ as a test function in (3.10) and taking limits it follows

$$\begin{aligned} \lim_{p\to 1^+} \int_\Omega |\nabla u_p|^p = & \lim_{p\to 1^+} \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \nabla \varphi_n \\ & - \lim_{p\to 1^+} \int_\Omega (f - v_p)(\varphi_n - u_p). \end{aligned} \tag{3.16}$$

By (3.13) and (3.15), we get

$$\int_\Omega (f - v_p)(\varphi_n - u_p) \to \int_\Omega (f - v)(\varphi_n - u),$$

and by (3.14)

$$\int_\Omega |\nabla u_p|^{p-2} \nabla u_p \nabla \varphi_n \to \int_\Omega \mathbf{z} \nabla \varphi_n.$$

In addition, using Young's inequality and the weak lower semi-continuity of the total variation, we obtain

$$\begin{aligned} \lim_{p\to 1^+} \int_\Omega |\nabla u_p|^p & \geq \lim_{p\to 1^+} \left(p \int_\Omega |\nabla u_p| - (p-1)|\Omega| \right) \\ & \geq \lim_{p\to 1^+} \inf \left(\int_\Omega |\nabla u_p| - (p-1)|\Omega| \right) \\ & = \int_\Omega |Du|. \end{aligned}$$

Therefore, expression (3.16) yields

$$\int_\Omega |Du| \leq \int_\Omega \mathbf{z} \nabla \varphi_n - \int_\Omega (f - v)(\varphi_n - u).$$

Finally, taking limits as $n \to \infty$ we obtain inequality (b) from Lemma 3.3, which means that v is a weak solution of problem (S^γ_f).

The second part is a consequence of [8, Theorem 3.9 (ii)] and the above construction. □

Theorem 3.9. *Assume that γ satisfies condition (1.4). Then, for any $f \in L^1(\Omega)$ there exists an entropy solution of problem (S^γ_f).*

Proof. Given $f \in L^1(\Omega)$, let $f_n \in \mathcal{C}^\infty_c(\Omega)$ be such that $f_n \to f$ in $L^1(\Omega)$. For any $n \in \mathbb{N}$, by Theorem 3.8 there exists a weak solution v_n of problem $(S^\gamma_{f_n})$ such that

$v_n \ll f_n$. Thus, there exists $u_n \in BV(\Omega) \cap L^\infty(\Omega)$ and there exists $\mathbf{z}_n \in X_1$ with $\|\mathbf{z}_n\|_\infty \le 1$ such that

$$\begin{aligned} v_n \in \gamma(u_n) \quad \text{a.e. in } \Omega, \\ -\text{div}(\mathbf{z_n}) = f_n - v_n, \quad \text{in } \mathcal{D}'(\Omega), \end{aligned} \tag{3.17}$$

and

$$\int_\Omega (f_n - v_n)(\varphi - u_n) + \|Du_n\| \le \int_\Omega (\mathbf{z_n}, D\varphi), \quad \forall \varphi \in BV(\Omega) \cap L^\infty(\Omega). \tag{3.18}$$

Taking $\varphi = u_n - T_k(u_n)$ in (3.18), we have

$$-\int_\Omega (f_n - v_n) T_k(u_n) + \int_\Omega |Du_n| \le \int_\Omega (\mathbf{z}_n, Du_n) - \int_\Omega |DT_k(u_n)|.$$

Then, by (2.1) and since $v_n T_k(u_n) \ge 0$, we get

$$\int_\Omega |DT_k(u_n)| \le \int_\Omega f_n T_k(u_n) \le k\|f\|_1.$$

Then, by the compact embedding, taking subsequences and using a diagonal process, we have

$$T_k(u_n) \to \sigma_k, \quad n \to \infty, \text{ in } L^q(\Omega) \text{ and a.e. for } 1 \le q < 1^*,$$

with

$$|\sigma_k| \le k.$$

Let us see now that (remark that this argument is not needed if $[0, +\infty[\subset D(\gamma)$, similarly for the argument with the negative part)

$$\mathcal{L}^N(\{x \in \Omega : \sigma_k^+(x) = k\}) \to 0, \quad \text{as } k \to \infty. \tag{3.19}$$

In fact, since γ^0 is lower semi-continuous and $\text{Rang}(\gamma^0) = \mathbb{R}$, by applying Fatou's lemma it follows that

$$\begin{aligned} \mathcal{L}^N(\{x \in \Omega : \sigma_k^+(x) = k\}) &= \int_{\{x\in\Omega:\sigma_k^+(x)=k\}} \frac{\gamma^0(\sigma_k^+(x))}{\gamma^0(k)} \le \frac{1}{\gamma^0(k)} \liminf_{n\to\infty} \int_\Omega \gamma^0(T_k(u_n)^+) \\ &\le \frac{1}{\gamma^0(k)} \liminf_{n\to\infty} \int_\Omega v_n^+ \le \frac{1}{\gamma^0(k)} \|f\|_1 \to 0, \quad k \to \infty. \end{aligned}$$

Similarly, it is shown that

$$\mathcal{L}^N(\{x \in \Omega : \sigma_k^-(x) = k\}) \to 0, \quad k \to \infty. \tag{3.20}$$

By (3.19) and (3.20), if we define

$$u(x) := \sigma_k(x) \quad \text{on } \{x \in \Omega : \ |\sigma_k(x)| = k\},$$

we have that u is measurable and

$$u_n \quad \text{converges to } u \text{ a.e. in } \Omega.$$

Now, by using the second part in Theorem 3.8, we get

$$\|v_n - v_m\|_1 \le \|f_n - f_m\|_1 \quad \text{for all } n, m \in \mathbb{N}.$$

Therefore,

$$v_n \to v \quad \text{in} \ \ L^1(\Omega), \tag{3.21}$$

and

$$v \in \gamma(u) \quad \text{a.e. in } \Omega.$$

On the other hand, since $\mathbf{z}_n \in X_1(\Omega)$ with $\|\mathbf{z}_n\|_\infty \le 1$, we may assume that

$$\mathbf{z}_n \to \mathbf{z} \quad \text{in the weak}^* \text{ topology of} \ \ L^\infty(\Omega, \mathbb{R}^N). \tag{3.22}$$

Then, from (3.17),

$$-\mathrm{div}(\mathbf{z}) = f - v, \quad \text{in} \ \ \mathcal{D}'(\Omega).$$

Given now $\varphi \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and taking $\varphi + u_n - T_k(u_n)$ as test function in (3.18), we obtain

$$\begin{aligned}\int_\Omega (f_n - v_n)(\varphi - T_k(u_n)) + \|Du_n\| &\le \int_\Omega (\mathbf{z}_n, D(\varphi + u_n - T_k(u_n)) \\ &\le \int_\Omega \mathbf{z}_n \cdot \nabla\varphi \, dx + \int_\Omega (\mathbf{z}_n, D(u_n - T_k(u_n)) \\ &\le \int_\Omega \mathbf{z}_n \cdot \nabla\varphi \, dx + \int_\Omega |D(u_n - T_k(u_n))|.\end{aligned}$$

Thus, applying [5, Lemma 3], we arrive to

$$\int_\Omega (f_n - v_n)(\varphi - T_k(u_n)) + \|DT_k(u_n)\| \le \int_\Omega \mathbf{z}_n \cdot \nabla\varphi \, dx.$$

Then, taking limit as $n \to \infty$ and having in mind (3.21), (3.22) and the lower semi-continuity of the total variation, we get

$$\int_\Omega (f - v)(\varphi - T_k(u)) + \|DT_k(u)\| \le \int_\Omega \mathbf{z} \cdot \nabla\varphi \, dx. \qquad \square$$

Let us now prove that, under assumption (1.3), the unique solution of problem (S_f^γ) coincides by the unique solution of (S_f^{Id}). Let us first see an easy situation.

Proposition 3.10. *Let γ^{-1} be an increasing and Lipschitz-continuous function with $\gamma(0) = 0$ and Rang$(\gamma) = \mathbb{R}$. Let $v \in BV(\Omega) \cap L^\infty(\Omega)$ be the unique weak solution of (S_f^{Id}) for $f \in L^\infty(\Omega)$. Then, v is also a weak solution of problem (S_f^γ).*

Proof. By setting $u := \gamma^{-1}(v)$ (which is well defined since Rang$(\gamma) = \mathbb{R}$), we have

$$u \in BV(\Omega) \cap L^\infty(\Omega)$$

and

$$|Du| \ll |Dv|$$

(see [4, Theorems 3.101 and 3.99]). Now, by Proposition 2.1 and (3.3) it follows

$$\theta(z, Du, .) = \theta(z, D\gamma^{-1}(v), .) = \theta(z, Dv, .) = 1 \quad |Dv|\text{-a.e., hence } |Du|\text{-a.e.;}$$

consequently,

$$(\mathbf{z}, Du) = |Du| \quad \text{as measures.}$$

Therefore, v is a weak solution of problem (S_f^γ). □

Theorem 3.11. *Under condition (1.3), the entropy solution of problem (S_f^γ) is given by the entropy solution of problem (S_f^{Id}), i.e., of problem*

$$\begin{cases} v - \Delta_1 v \ni f & \text{in } \Omega, \\ \frac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. By Theorem 3.7, it is enough to prove it for data $f \in C_c^\infty(\Omega)$. So, our aim is to see that the (weak) solution v to

$$\begin{cases} v - \Delta_1 v \ni f & \text{in } \Omega, \\ \frac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.23}$$

is the (weak) solution of

$$\begin{cases} v - \Delta_1 u \ni f & \text{in } \Omega, \\ v = \gamma(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.24}$$

Let $\tilde{\gamma}_n(r) = \gamma_{1/n}(r) + \frac{1}{n}r$, where $\gamma_{1/n}$ is the Yosida approximation of γ (and where $\frac{1}{n}r$ can be deleted if the graph of γ has not flat zones). Then, $\tilde{\gamma}_n(r)$ is a Lipschitz-continuous and increasing function also satisfying Rang$(\tilde{\gamma}_n) = \mathbb{R}$. Therefore, from

Theorem 3.8, there exist $v_n \in L^\infty(\Omega)$, $u_n \in BV(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{z}_n \in X_1(\Omega)$ with $\|\mathbf{z}_n\|_\infty \leq 1$ such that

$$\begin{aligned} v_n &= \tilde{\gamma}_n(u_n) \quad \text{in } \Omega, \\ v_n &\ll f, \\ -\operatorname{div}(\mathbf{z}_n) &= f - v_n \quad \text{in } \mathcal{D}'(\Omega), \qquad (3.25) \\ (\mathbf{z}_n, Du_n) &= |Du_n| \quad \text{as measures}, \\ [\mathbf{z}_n, \nu] &= 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \qquad (3.26) \end{aligned}$$

Moreover by (3.5), we have

$$\int_\Omega (\varphi - u_n)(f - v_n)\,dx \leq \int_\Omega (\mathbf{z}_n, D\varphi) - \int_\Omega |Du_n|, \qquad \forall \varphi \in BV(\Omega) \cap L^\infty(\Omega). \tag{3.27}$$

Since $v_n := \tilde{\gamma}_n(u_n)$, we have

$$v_n \in BV(\Omega)$$

and

$$|Dv_n| \ll |Du_n|.$$

By (3.26) and Proposition 2.1, we have

$$\theta(z_n, Dv_n, .) = \theta(z_n, D\tilde{\gamma}_n(u_n), .) = \theta(z_n, Du_n, .) = 1 \quad |Du_n|\text{-a.e., hence } |Dv_n|\text{-a.e.},$$

consequently,

$$(\mathbf{z}_n, Dv_n) = |Dv_n| \quad \text{as measures},$$

and, therefore, we get that v_n is a (weak) solution of problem (3.23) with vector field $\mathbf{z}_n$. Therefore, by uniqueness of problem (3.23), we have

$$v_n = v.$$

And, by (3.27), we have

$$\int_\Omega (\varphi - u_n)(f - v)\,dx \leq \int_\Omega (\mathbf{z}_n, D\varphi) - \int_\Omega |Du_n|, \qquad \forall \varphi \in BV(\Omega) \cap L^\infty(\Omega). \tag{3.28}$$

Now, since $\tilde{\gamma}_n(u_n) = v \ll f$, we get

$$\|\gamma_1(u_n)\|_q^q \leq \|\gamma_{1/n}(u_n)\|_q^q \leq \|\tilde{\gamma}_n(u_n)\|_q^q \leq \|f\|_q^q,$$

for $q \in [1, \infty]$. In particular, $\|u_n\|_\infty \leq C_1 := \max\{-\gamma_1^{-1}(-\|f\|_\infty), \gamma_1^{-1}(\|f\|_\infty)\}$, for all $n \in \mathbb{N}$.

Finally, taking $\varphi = 0$ as a test function in (3.28) we obtain

$$\int |Du_n| \le \int u_n(f - v) \le C_2, \quad \text{for all } n \in \mathbb{N},$$

so that $\{u_n\}$ is bounded in $BV(\Omega)$. It follows that there exists $u \in BV(\Omega)$ such that up to a subsequence (no relabeled)

$$u_n \to u \text{ in } L^m(\Omega), \text{ for } 1 \le m < \frac{N}{N-1},$$

and

$$u_n(x) \to u(x) \text{ for almost every } x \in \Omega.$$

This implies that $v = \gamma(u)$.

On the other hand, since $\mathbf{z}_n \in X_1(\Omega)$ with $\|\mathbf{z}_n\|_\infty \le 1$, we may assume that

$$\mathbf{z}_n \to \mathbf{z} \quad \text{in the weak* topology of} \quad L^\infty(\Omega, \mathbb{R}^N).$$

In particular, from (3.25)

$$-\text{div}(\mathbf{z}) = f - v, \quad \text{in } \mathcal{D}'(\Omega).$$

Then, by [6, Proposition C.12] and having in mind the lower semi-continuity of the total variation, taking limits in (3.28) as $n \to \infty$, we get

$$\int_\Omega (\varphi - u)(f - v)\, dx \le \int_\Omega (\mathbf{z}, D\varphi) - \int_\Omega |Du|, \ \forall \varphi \in BV(\Omega) \cap L^\infty(\Omega).$$

Therefore, by Lemma 3.3, we have that v is a solution of (3.24). □

In the next example will be see that the condition $\text{Rang}(\gamma) = \mathbb{R}$ is necessary in the above theorem.

Example 3.12. Let, for $n \in \mathbb{N}$, $\beta_n(r) = \frac{1}{n}\arctan(r)$, and let $f \in L^1(\Omega)$, $\int_\Omega f = 0$. If $(S_f^{\beta_n})$ has a solution for all $n \in \mathbb{N}$, then there exists $u_n \in \mathcal{M}(\Omega)$, $T_k(u_n) \in BV(\Omega)$ for all $k > 0$, $\beta_n(u_n) \in L^1(\Omega)$, and there exists $\mathbf{z}_n \in X_1(\Omega)$ with $\|\mathbf{z}_n\|_\infty \le 1$, such that

$$-\,\text{div}(\mathbf{z}_n) = f - \beta_n(u_n) \quad \text{in } \mathcal{D}'(\Omega). \tag{3.29}$$

Now, since

$$-\frac{\pi}{2n} \le \beta_n(u_n) \le \frac{\pi}{2n},$$

taking limits in (3.29) we find $\mathbf{z} \in X_1(\Omega)$, $\|\mathbf{z}\|_\infty \le 1$, such that

$$-\,\text{div}(\mathbf{z}) = f \quad \text{in } \mathcal{D}'(\Omega), \tag{3.30}$$

and we get a contradiction with the well-known fact that there exist $f \in L^1(\Omega)$, $\int_\Omega f = 0$, such that the above equation has not solution in $X_1(\Omega)$ (see, for instance, [17]). Nevertheless, let us see, with an easy example, that there are L^∞-functions for which (3.30) has no solution in $X_1(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$. In fact, (3.30) implies that

$$\left| \int_\Omega f(x)\varphi(x)dx \right| \leq \sum_{i=1}^N \int_\Omega \left| \frac{\partial \varphi}{\partial x_i}(x) \right| dx \quad \forall \varphi \in W_0^{1,1}(\Omega). \tag{3.31}$$

Take $\Omega = B_1(0)$ the ball in $\mathbb{R}^N$ centered at 0 of radius 1, and, for $k > 0$,

$$f(x) = \begin{cases} -k, & |x| \leq 1/2, \\ \frac{k}{2^N - 1}, & 1/2 < |x| < 1, \end{cases}$$

which satisfies $\int_\Omega f = 0$. Take now $\varphi(x) = 1 - |x|$, which belongs to $W_0^{1,1}(\Omega)$. Then, on the one hand,

$$\sum_{i=1}^N \int_\Omega \left| \frac{\partial \varphi}{\partial x_i}(x) \right| dx \leq N|\Omega|,$$

and, on the other hand, since $\int_\Omega f = 0$,

$$\left| \int_\Omega f(x)\varphi(x)dx \right| = \left| \int_\Omega |x| f(x)dx \right| = \frac{N|\Omega|k}{2(N+1)(2^N - 1)},$$

which, for $k > 2(N+1)(2^N - 1)$, contradicts (3.31). □

4. The evolution problem

In this section, we study the evolution problem (1.2).

We do this through the Nonlinear Semigroup Theory, and therefore, we introduce an operator $\mathcal{B}$ in $L^1(\Omega)$ that allows to rewrite problem (1.2) as the abstract Cauchy problem

$$\begin{cases} \dfrac{dv}{dt} + \mathcal{B}(v(t)) \ni 0 \text{ on } (0, T), \\ v(0) = v_0. \end{cases} \tag{4.1}$$

Definition 4.1. $(v, w) \in \mathcal{B}$ if and only if $v, w \in L^1(\Omega)$ and there exist $u \in \mathcal{M}(\Omega)$ such that $T_k(u) \in BV(\Omega)$ for all $k > 0$ and $v \in \gamma(u)$ a.e. in Ω, and there exists $\mathbf{z} \in X_1(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$, satisfying:

$$-\text{div}(\mathbf{z}) = w \quad \text{in } \mathcal{D}'(\Omega)$$

and

$$\int_\Omega (\varphi - T_k(u))w dx \leq \int_\Omega \mathbf{z} \cdot \nabla\varphi dx - \int_\Omega |DT_k(u)|, \quad \forall \varphi \in BV(\Omega) \cap L^\infty(\Omega).$$

Note that for $f \in L^1(\Omega)$, we have that

$$(I + \mathcal{B})^{-1} f = v \iff v \text{ is an entropy solution of problem } (S_f^\gamma).$$

By Theorems 3.7 and 3.9, we have the following result.

Theorem 4.2. *Under condition (1.3), $\mathcal{B}$ is a T-accretive and m-accretive operator on $L^1(\Omega)$.*

As a consequence of the above result, by Crandall–Liggett's theorem, it follows that, for every initial data $v_0 \in \overline{D(\mathcal{B})}^{L^1(\Omega)}$, the abstract Cauchy problem (4.1) has a unique mild solution $v(t)$ given by the exponential formula

$$v(t) = e^{-t\mathcal{B}} v_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}\mathcal{B} \right)^{-n} v_0.$$

In [5] (see also [6]), it is shown that the operator $\mathcal{A}$ given in Definition 1.2 is an m-completely accretive operator in $L^1(\Omega)$ and that for every initial data $v_0 \in L^1(\Omega)$ the mild solution

$$v(t) = e^{-t\mathcal{A}} v_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}\mathcal{A} \right)^{-n} v_0$$

is a strong solution.

Theorem 4.3. *The following equality holds*

$$\overline{D(\mathcal{B})}^{L^1(\Omega)} = L^1(\Omega).$$

Proof. It is enough to prove that $C_c^\infty(\Omega) \subset \overline{D(\mathcal{B})}^{L^1(\Omega)}$. So, take $f \in C_c^\infty(\Omega)$ and take $f_n = \left(I + \frac{1}{n}\mathcal{B}\right)^{-1} f$. Observe that $f_n \in D(\mathcal{B})$ and it is a (weak) solution of

$$(S_f^\gamma) \begin{cases} v - \frac{1}{n}\Delta_1 u \ni f & \text{in } \Omega, \\ v \in \gamma(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by Theorem 3.8 and Lemma 3.3 (d), $f_n \in L^\infty(\Omega)$, $f_n \ll f$, and there exist $u_n \in BV(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{z}_n \in X_1(\Omega)$ with $\|\mathbf{z}\|_\infty \le 1$ satisfying (3.1), (3.2) and

$$\int_\Omega \varphi(f - f_n)\, dx = \frac{1}{n} \int_\Omega (\mathbf{z}_n, D\varphi), \ \forall \varphi \in BV(\Omega) \cap L^\infty(\Omega).$$

Now, for $\varphi \in BV(\Omega) \cap L^\infty(\Omega)$, since

$$\lim_n \frac{1}{n} \int_\Omega (\mathbf{z}_n, D\varphi) = 0,$$

we have that

$$\lim_n \int_\Omega \varphi(f - f_n)\, dx = 0.$$

Now, since $f_n \ll f$, by (iv) in Proposition 2.7 we get

$$\lim_n f_n = f \quad \text{in } L^1(\Omega).$$

□

Then, as a consequence of Theorem 3.11, we obtain the following result:

Theorem 4.4. *Under condition (1.3) and for every initial data $v_0 \in L^1(\Omega)$, the abstract Cauchy problem (4.1) has a unique strong solution $v(t)$. Moreover, this solution coincides with the unique strong solution of problem (1.6).*

We introduce the following concept of solution of problem (1.2).

Definition 4.5. A measurable function $v : (0, T) \times \Omega \to \mathbb{R}$ is an *entropy solution* of (1.2) in $(0, T) \times \Omega$ if $v \in C([0, T], L^1(\Omega)) \cap W^{1,1}_{loc}(0, T; L^1(\Omega))$, $v(0) = v_0$, and, for almost all $t \in (0, T)$, there exists $u(t) \in \mathcal{M}(\Omega)$ with $T_k(u(t)) \in BV(\Omega)$ for all $k > 0$, and there exists $\mathbf{z}(t) \in L^\infty(\Omega)$ with $\|\mathbf{z}(t)\|_\infty \leq 1$, such that

$$v(t, x) \in \gamma(u(t, x)) \quad \text{a.e. } x \in \Omega,$$
$$v_t(t) = \mathrm{div}(\mathbf{z}(t)) \quad \text{in } \mathcal{D}'(\Omega)$$

and

$$\int_\Omega (T_k(u(t)) - w)\, v_t(t)\, dx \leq \int_\Omega \mathbf{z}(t) \cdot \nabla w\, dx - \int_\Omega |DT_k(u(t))|$$

for every $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$.

As a consequence of the above result, we have the following existence and uniqueness result.

Theorem 4.6. *Under condition (1.3) and for every initial data $v_0 \in L^1(\Omega)$, there exists a unique entropy solution of (1.2) in $(0, T) \times \Omega$ for every $T > 0$ such that $v(0) = v_0$. Moreover, if $v(t)$ and $\hat{v}(t)$ are entropy solutions corresponding to initial data v_0 and $\hat{v}_0$, respectively, then*

$$\|(v(t) - \hat{v}(t))^+\|_1 \leq \|(v_0 - \hat{v}_0)^+\|_1 \quad \textit{for all } t \geq 0.$$

In particular,

$$\|v(t) - \hat{v}(t)\|_1 \leq \|v_0 - \hat{v}_0\|_1 \quad \textit{for all } t \geq 0.$$

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