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Conservation laws and Hamilton–Jacobi equations with space inhomogeneity

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Abstract. Conservation laws with an x-dependent flux and Hamilton–Jacobi equations with an x-dependent Hamiltonian are considered within the same set of assumptions. Uniqueness and stability estimates are obtained only requiring sufficient smoothness of the flux/Hamiltonian. Existence is proved without any convexity assumptions under a mild coercivity hypothesis. The correspondence between the semigroups generated by these equations is fully detailed. With respect to the classical Kružkov approach to conservation laws, we relax the definition of solution and avoid any restriction on the growth of the flux. A key role is played by the construction of sufficiently many entropy stationary solutions in $\mathbf{L}^\infty$ that provide global bounds in time and space.

1. Introduction

This paper provides a framework where Cauchy problems for x-dependent scalar conservation laws, such as

$$\begin{cases} \partial_t u + \partial_x H(x, u) = 0 & (t, x) \in \,]0, T[\times \mathbb{R} \\ u(0, x) = u_o(x) & x \in \mathbb{R}, \end{cases} \tag{CL}$$

and Cauchy problems for x-dependent scalar Hamilton–Jacobi equations, such as

$$\begin{cases} \partial_t U + H(x, \partial_x U) = 0 & (t, x) \in \,]0, T[\times \mathbb{R} \\ U(0, x) = U_o(x) & x \in \mathbb{R}, \end{cases} \tag{HJ}$$

are globally well posed and a complete identification between the two problems is possible.

The well-posedness of both **(CL)** and **(HJ)** is here proved under the *same* assumptions on the function H, which is the *flux* of **(CL)** and the *Hamiltonian* of **(HJ)**. These assumptions define a framework included neither in the one outlined by Kružkov in his classical work [27] devoted to **(CL)**, nor in the usual assumptions on **(HJ)** found in the literature, e.g., [3,4,14,25]. The identification of **(CL)** with **(HJ)** is then formalized, extending to the non-homogeneous case [26, Theorem 1.1], see also [10, Proposition 2.3]. This deep analogy also stems out from the direct identification of the constants appearing in the various stability estimates for the 2 equations, compare, for instance, (2.13) with (2.18).

A key role is played below by the handcrafted construction of a family of stationary entropy solutions to **(CL)**, with a merely $\mathbf{L}^\infty$ regularity, that provides the necessary uniform bounds on the vanishing viscosity limits, see Theorem 2.9.

The framework we propose is based on these assumptions[1] on H:

$$\textbf{Smoothness}: \qquad H \in \mathbf{C}^3(\mathbb{R}^2; \mathbb{R}). \qquad \textbf{(C3)}$$

$$\textbf{Compact NonHomogeneity}: \qquad \begin{array}{l} \exists\, X > 0: \forall\, (x, u) \in \mathbb{R}^2 \\ \text{if } |x| > X \text{ then } \partial_x H(x, u) = 0; \end{array} \qquad \textbf{(CNH)}$$

$$\textbf{Uniform Coercivity}: \qquad \begin{array}{l} \forall\, h \in \mathbb{R} \quad \exists\, \mathcal{U}_h \in \mathbb{R}: \forall\, (x, u) \in \mathbb{R}^2 \\ \text{if } |H(x, u)| \leq h \text{ then } |u| \leq \mathcal{U}_h. \end{array} \qquad \textbf{(UC)}$$

$$\textbf{Weak Genuine NonLinearity}: \qquad \begin{array}{l} \text{for a.e. } x \in \mathbb{R} \text{ the set} \\ \{w \in \mathbb{R}: \partial^2_{ww} H(x, w) = 0\} \\ \text{has empty interior.} \end{array} \qquad \textbf{(WGNL)}$$

However, in all general *a priori* estimates and qualitative properties, exclusively condition **(C3)** is used. Here, both **(UC)** and **(WGNL)** are shown to be not necessary to prove the trace at zero condition [27, Formula (2.2)], the semigroup property, the $\mathbf{L}^1_{\text{loc}}$ continuity in time and the contraction property [27, Formula (3.1)] in the case of **(CL)**.

Condition **(CNH)** qualifies the non-homogeneity of H and is apparently not common in the current literature on **(CL)** and **(HJ)**. Our approach can be seen as somewhat related to [17, Section 5], where the space variable varies on a torus. Remarkably, X plays no quantitative role: it is required to exist, but its value is irrelevant. Thus, we expect **(CNH)** might possibly be relaxed.

Here, **(UC)** replaces the usual condition $\sup_{(x,u)\in\mathbb{R}^2} \left(-\partial^2_{xu} H(x, u)\right) < +\infty$, see (1.1), that was introduced by Kružkov back in [27, Formula (4.2)] and that has since become standard in any existence proof. Example 1.1 motivates the necessity to abandon it in the context of **(CL)**. Moreover, this growth condition does not have, apparently, a clear counterpart among the usual assumptions on **(HJ)**. Note, however, that several coercivity conditions appear in the context of **(HJ)**, see, for instance, [4, § 2.4.2]. In particular, in the convex case, **(UC)** directly ensures $\mathbf{L}^\infty$ bounds, as shown for instance in [41, Theorem 8.2.2]. Recall that also in [32,33] some regularity assumptions on the Hamiltonian are relaxed, but not those requiring a suitable growth.

When dealing with **(HJ)**, the convexity of H is a recurrent hypothesis, see, for instance, [3,4,13,25], since it connects Hamilton–Jacobi equations to optimal control problems. On the other hand, convexity is typically not required in basic well-posedness results on scalar conservation laws, see [16,27]. Here, differently from

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[1]In view of **(CNH)**, in **(UC)** and in **(WGNL)** it is sufficient to consider only $x \in [-X, X]$.

[3,4,15,16,41], no convexity assumption on the Hamiltonian in **(HJ)** is requested and, hence, characteristics are hardly of any help. Below we adopt **(WGNL)**, which essentially asks that for a.e. x there does not exist any (non-empty) open set where $u \mapsto H(x, u)$ is linear, but clearly allows also for infinitely many inflection points. Thus, for all x in a null set, $u \mapsto H(x, u)$ may well be locally affine. Refer to Remark 2.22 for a stability estimate on **(HJ)** allowed by **(WGNL)**.

Moreover, we neither pose any strict monotonicity assumptions on H as done, for instance, in [9] where, on the other side, H may well be only piecewise continuous in space and in time.

The classical reference for the well-posedness of general scalar balance laws is Kružkov's paper [27]. Kružkov's assumptions [27, § 4, p. 230] in the present notation take the form:

$$H \in \mathbf{C}^3(\mathbb{R}^2; \mathbb{R}),$$
$$\forall K \in \mathbb{R}_+ \quad \sup_{(x,u) \in \mathbb{R} \times [-K, K]} |\partial_u H(x, u)| < +\infty,$$
$$\sup_{x \in \mathbb{R}} |\partial_x H(x, 0)| < +\infty, \quad \sup_{(x,u) \in \mathbb{R}^2} \left(-\partial_{xu}^2 H(x, u) \right) < +\infty \qquad (1.1)$$

and the initial datum is required to satisfy $u_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$. Our assumptions are not contained in Kružkov's hypotheses. On the other hand, clearly, Kružkov result applies to general balance laws in several space dimensions.

Example 1.1. Fix positive constants X, V_1, V_2 and let $v \in \mathbf{C}^3(\mathbb{R};]0, +\infty[)$ be such that $v(x) = V_1$ for $x < -X$ and $v(x) = V_2$ for $x > X$. Define $H(x, u) := v(x) u (1 - u)$. Then, $\partial_t u + \partial_x H(x, u) = 0$ is the Lighthill–Whitham [29] and Richards [36] model for a flow of vehicles described by their density u along a rectilinear road with maximal speed smoothly varying from V_2, for $x > X$, to V_1, for $x < -X$.

This Hamiltonian H satisfies **(C3)**–**(CNH)**–**(UC)**–**(WGNL)** but does not satisfy the latter requirement in (1.1).

For completeness, we add that a standard truncation argument could be used to extend Kružkov result to Example 1.1, as soon as the initial datum attains values between the stationary solutions $u(t, x) = 0$ and $u(t, x) = 1$. Note, however, that the *a priori* estimates and qualitative properties in Sect. 2.1 as well as the construction of stationary solutions in Sect. 2.2 are in general preliminary to any truncation argument. Technically, it is essentially due to our adopting **(UC)** that we can avoid truncation arguments. Moreover, such an argument applies to **(CL)** but hinders our simultaneous treatment of **(CL)** and **(HJ)**. Thus, we provide an existence proof alternative to that by Kružkov and explicitly state the correspondence between **(CL)** and **(HJ)** in Sects. 2.3, 2.4 and 2.5.

To our knowledge, only few results in the literature focus on the **(CL)** $\leftrightarrow$ **(HJ)** connection. The homogeneous, x independent, stationary case is considered in the **BV** case in [26] (by means of wave front tracking), see also [8, § 6] for the case of

fractional equations. An extension to $\mathbf{L}^\infty$ is in the more recent [10] (where Dafermos' [15] theory of generalized characteristics play a key role). The stationary x dependent case is considered in [6] (using semigroups generated by accretive operators). Here, we deal with the non-stationary x dependent case, relying on vanishing viscosity approximations and on the compensated compactness machinery. In this connection, note that the techniques developed in [32,33] cannot be directly applied here, due to our need of passing to the limit also in the Hamiltonian.

Remark that in Kružkov's paper [27], the latter condition in (1.1) is essential to obtain uniform $\mathbf{L}^\infty$ and $\mathbf{BV}$ bounds on the sequence of viscous approximations in the case $u_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$. In our approach, which does not rely on (1.1), the $\mathbf{L}^\infty$ bound on viscous solutions depends on the fact that $u_o \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$. We thus need to devise new additional bounds, provided by the stationary solutions to (CL), see Sect. 2.2, which are specific to the non-viscous case, and allow to pass from data in $\mathbf{W}^{1,\infty}$ to data in $\mathbf{L}^\infty$ at the non-viscous level.

In the literature, a recurrent tool in existence proofs for (CL) is the (parabolic) *Maximum Principle*, see for instance [23, Theorem B.1, Formula (B.3)] or [24, § 3.2], which provides an *a priori* uniform bound on vanishing viscosity approximate solutions, which is an essential step in passing to the vanishing viscosity limit. More precisely, only in the homogeneous case where $\partial_x H \equiv 0$, the Maximum Principle ensures that

1. vanishing viscosity approximate solutions have a common $\mathbf{L}^\infty$ bound, and
2. this bound only depends on the $\mathbf{L}^\infty$ norm of the initial datum.

In the present—non-homogeneous—case, we replace (1) obtaining $\mathbf{L}^\infty$ bounds on vanishing approximate solutions by means of a, here suitably adapted, Bernstein method, see [39, § 6] for a general introduction. This requires a higher regularity of the initial datum and (2) above is irremediably lost.

However, in the homogeneous case, one also takes advantage of the fact that constants are stationary solutions, ensuring 2. easily. This fact fails in the non-homogeneous case. Below, we exhibit (sort of) *foliations* of $\mathbb{R} \times [\mathcal{U}, +\infty[$ and $\mathbb{R} \times]-\infty, -\mathcal{U}]$ (for a sufficiently large $\mathcal{U}$) consisting of graphs of stationary solutions to (CL), each contained in a level curve of H. Then, solutions to (CL) are well known to preserve the ordering [16, Formula (6.2.8)] and 2. follows. Note that these stationary solutions need to be merely $\mathbf{L}^\infty$. Therefore, in their construction, the choice of jumps deserves particular care to ensure that they turn out to be entropy admissible. In general, the solutions to (HJ) corresponding to stationary solutions to (CL) may well be non-stationary.

The differences between the construction below and the classical one by Kružkov [27] arise from the different choices of the assumptions but are not limited to that. Indeed, the two procedures differ in several key points. In [27], uniform $\mathbf{L}^\infty$ *"parabolic"* bounds on vanishing viscosity approximate solutions to (CL) are obtained and $\mathbf{L}^1$ compactness follows from Kolmogorov criterion. Here, the stationary solutions constructed as described above allow to obtain $\mathbf{L}^\infty$ *"hyperbolic"* bounds directly

on the solutions to **(CL)**, while it is an application of the compensated compactness machinery that ensures the existence of a limit, thanks to our modified (weakened) definition of solution. Under **(WGNL)**, also the kinetic approach in [30,34] is likely to allow for analogous results. Moreover, in [27] the term $-\partial_x H$ is essentially treated as a contribution to the source term. Here, we exploit the conservative form of **(CL)**, thus respecting the analogy between **(CL)** and **(HJ)**. Our weakening of Kružkov definition, motivated also by our use of compensated compactness, avoids any requirement on the trace at time 0+. It is of interest that this construction actually relies also on a sort of stability with respect to the flux H, where condition **(WGNL)** appears essential.

However, continuity in time, not proved in [27], is recovered in weak-$*$ $\mathbf{L}^{\infty}_{\mathbf{loc}}(\mathbb{R}; \mathbb{R})$ in Proposition 2.5 and in $\mathbf{L}^1_{\mathbf{loc}}$ in Theorem 2.6, always relying exclusively on condition **(C3)**. Differently from [7,42], condition **(WGNL)** plays here no role. Thus, in the present setting, the trace at 0+ condition [27, Formula (2.2)] can be omitted from the definition of solution to **(CL)** without any consequence.

Throughout this paper, we alternate considering **(CL)** and **(HJ)**, simultaneously gathering step by step results on the two problems. When H does not depend on the space variable x, [26, Theorem 1.1] and [10, Proposition 2.3] ensure the equivalence between **(CL)** and **(HJ)**. In the space homogeneous case, the correspondence between **(CL)** and **(HJ)** is exploited in [5,31] and it is particularly effective in the characterization of the initial data evolving into a given profile at a given time, see [10,28]. Below, we extend this equivalence to the x dependent case, while [11] is devoted to the inverse design problem in the x-dependent case. This correspondence may also suggest new properties of **(CL)** or **(HJ)**, proving them in the present framework, posing the question of an intrinsic proof in more general settings, see Remark 2.22. As a matter of fact, our original goal was the detailed description of the relation between **(CL)** and **(HJ)**, but such a correspondence requires the two Cauchy problems to be settled in the *same* framework.

In this paper, results are presented in the paragraphs in Sect. 2, while all proofs are collected in the corresponding paragraphs in Sect. 3.

Paragraph 2.1 presents the weakened definition of solution to **(CL)** and verifies that it still ensures uniqueness, the contraction property and continuity in time. Analogous results for **(HJ)** are proved independently. Proofs use neither **(CNH)**, nor **(UC)** nor **(WGNL)** and are deferred to § 3.1.

Paragraph 2.2, where **(UC)** is essential, is devoted to the construction of a family of stationary entropy solutions to **(CL)**. It has no counterpart referred to **(HJ)**, it is intrinsic to **(CL)**. The actual construction is in Sect. 3.2.

Paragraph 2.3 deals with the vanishing viscosity approximations to **(CL)** and to **(HJ)**. The interplay between the 2 problems is exploited: all proofs, deferred to Sect. 3.3, are obtained for only one of the two equations, a quick corollary allowing to pass to the other equation.

Paragraph 2.4 ensures that vanishing viscosity solutions converge, up to subsequences, in both cases of **(CL)** and **(HJ)**. The corresponding proofs in Sect. 3.4, where the **(CL)** case relies on the compensated compactness method.

Paragraph 2.5 collects the final results, showing the properties of the semigroups S^{CL} and S^{HJ} generated by **(CL)** and **(HJ)** and detailing how they correspond to each other. The proofs are in Sect. 3.5.

The main goal of this paper are the results in Paragraph 2.5.

2. Main results

Throughout this work, T denotes a strictly positive time or $+\infty$.

2.1. Definitions of solution, local contraction and uniqueness

In this paragraph, we let $u_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$ while we require exclusively **(C3)** on H. No genuine nonlinearity condition is assumed, not even **(WGNL)**, differently from [7,42] (that have different goals and motivations).

Concerning the notion of solution to **(CL)**, we modify that in the sense of Kružkov [27, Definition 1]. Indeed, in view of the compensated compactness technique used below, we do not require continuity in time in the sense of [27, Formula (2.2)]. On the contrary, full $\mathbf{L}^1_{\mathbf{loc}}$ continuity in time is here proved, merely on the basis of **(C3)**.

With reference to **(CL)**, the following quantity often recurs below, where $x, u, k \in \mathbb{R}$:

$$\Phi(x, u, k) := \operatorname{sgn}(u - k) \ (H(x, u) - H(x, k)). \tag{2.1}$$

Definition 2.1. A function $u \in \mathbf{L}^\infty([0, T] \times \mathbb{R}; \mathbb{R})$ is an entropy solution to **(CL)** if for all nonnegative test functions $\varphi \in \mathbf{C}^1_c([0, T[\times\mathbb{R}; \mathbb{R}^+)$ and for all $k \in \mathbb{R}$,

$$\int_0^T \int_\mathbb{R} (|u(t, x) - k| \, \partial_t\varphi(t, x) + \Phi(x, u(t, x), k) \, \partial_x\varphi(t, x)) \, \mathrm{d}x\mathrm{d}t$$
$$- \int_0^T \int_\mathbb{R} \operatorname{sgn}(u(t, x) - k) \, \partial_x H(x, k) \, \varphi(t, x)\mathrm{d}x\mathrm{d}t$$
$$+ \int_\mathbb{R} |u_o(x) - k| \, \varphi(0, x)\mathrm{d}x \geq 0. \tag{2.2}$$

In (2.2), the integral term on the last line allows to avoid requiring the existence of the strong trace at $0+$, as required in [27, Definition 1]. Hence, Definition 2.1 is more amenable to various limiting procedures. Nevertheless, [27, Definition 1] clearly implies Definition 2.1, while Theorem 2.6 ensures the global in time strong continuity and recovers *all* properties of the classical Kružkov definition, in particular the existence of the strong trace at $0+$. Hence, Definition 2.1 and [27, Definition 1] are indeed equivalent.

Remark 2.2. Using $k \geq \|u\|_{\mathbf{L}^{\infty}([0,T] \times \mathbb{R}; \mathbb{R})}$ and $k \leq -\|u\|_{\mathbf{L}^{\infty}([0,T] \times \mathbb{R}; \mathbb{R})}$ in (2.2) shows that solutions to (CL) in the sense of Definition 2.1 are also distributional solutions, in the sense that for all test function $\varphi \in \mathbf{C}_c^1([0, T[\times \mathbb{R}; \mathbb{R})$

$$\int_0^T \int_{\mathbb{R}} (u(t, x) \, \partial_t \varphi(t, x) + H(x, u(t, x)) \, \partial_x \varphi(t, x)) \, dx dt$$
$$+ \int_{\mathbb{R}} u_o(x) \, \varphi(0, x) dx = 0. \tag{2.3}$$

We recall what we mean by entropy–entropy flux pair for (CL).

Definition 2.3. Let $H \in \mathbf{C}^1(\mathbb{R}^2; \mathbb{R})$. A pair of functions (E, F) with $E \in \mathbf{Lip}(\mathbb{R}; \mathbb{R})$ and $F \in \mathbf{Lip}(\mathbb{R}^2; \mathbb{R})$ is an *entropy–entropy flux pair* with respect to H if for all $x \in \mathbb{R}$ and for a.e. $u \in \mathbb{R}$

$$\partial_u F(x, u) = E'(u) \, \partial_x H(x, u). \tag{2.4}$$

The classical Kružkov choice in (2.4) amounts to set, for $k \in \mathbb{R}$,

$$E(u) = |u - k| \quad \text{and} \quad F(x, u) = \text{sgn}(u - k) \, (H(x, u) - H(x, k)). \tag{2.5}$$

By (C3), we can substitute (2.4) by

$$F^k(x, u) := E(u) \, \partial_u H(x, u) - E(k) \, \partial_u H(x, k) - \int_k^u E(v) \, \partial_{uu}^2 H(x, v) dv, \tag{2.6}$$

where $k \in \mathbb{R}$, which applies also when E is merely in $\mathbf{C}^0(\mathbb{R}; \mathbb{R})$. As soon as E is Lipschitz continuous, any pair (E, F) satisfying (2.6) also satisfy Definition 2.3.

We now check that the present Definition 2.1 keeps ensuring the properties of the original Kružkov definition [27, Definition 1]. First, we deal with the choice of the admissible entropies.

Proposition 2.4. *Let H satisfy* (C3).
1. *Call u a solution to* (CL) *with initial datum $u_o \in \mathbf{L}^{\infty}(\mathbb{R}; \mathbb{R})$, according to Definition 2.1. Then, for any entropy–entropy flux pair (E, F) with respect to H in the sense of Definition 2.3, if E is convex and in $\mathbf{C}^1(\mathbb{R}; \mathbb{R})$ then*

$$\int_0^T \int_{\mathbb{R}} (E(u(t, x)) \, \partial_t \varphi(t, x) + F(x, u(t, x)) \, \partial_x \varphi(t, x)) \, dx dt$$
$$- \int_0^T \int_{\mathbb{R}} \left(E'(u(t, x)) \, \partial_x H(x, u(t, x)) - \partial_x F(x, u(t, x)) \right) \varphi(t, x) dx dt$$
$$+ \int_{\mathbb{R}} E(u_o(x)) \, \varphi(0, x) dx \geq 0 \tag{2.7}$$

for any test function $\varphi \in \mathbf{C}_c^1([0, T[\times \mathbb{R}; \mathbb{R}_+)$.
2. *If $u_o \in \mathbf{L}^{\infty}(\mathbb{R}; \mathbb{R})$, $u \in \mathbf{L}^{\infty}([0, T] \times \mathbb{R}, \mathbb{R})$ and* (2.7) *holds for any entropy–entropy flux pair (E, F) with respect to H in the sense of Definition 2.3, with E convex and in $\mathbf{C}^{\infty}(\mathbb{R}; \mathbb{R})$, then u solves* (CL) *in the sense of Definition 2.1.*

Note that (2.7) corresponds to

$$\partial_t E\left(u(t,x)\right) + \partial_x \left(F\left(x, u(t,x)\right)\right) + E'\left(u(t,x)\right)\ \partial_x H\left(x, u(t,x)\right)$$
$$- \partial_x F\left(x, u(t,x)\right) \le 0$$

in the sense of distributions.

As a first step, we prove that Definition 2.1 ensures the weak-$*$ $\mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}; \mathbb{R})$ time continuity.

Proposition 2.5. *Let H satisfy (C3). Fix the initial datum $u_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$. Assume that the Cauchy Problem (CL) admits the distributional solution u in the sense of Remark 2.2. Then, for all $a, b \in \mathbb{R}$ with $a < b$, setting*

$$K^{CL} := 2 \sup\left\{ |H(x, p)|\colon x \in [a, b], |p| \le \|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R})} \right\}. \tag{2.8}$$

we have for almost all $\bar{t}, t_1, t_2 \in [0, T]$

$$\left| \int_a^b \left(u(\bar{t}, x) - u_o(x) \right) \mathrm{d}x \right| \le K^{CL}\, \bar{t}; \tag{2.9}$$

$$\left| \int_a^b \left(u(t_2, x) - u(t_1, x) \right) \mathrm{d}x \right| \le K^{CL}\, |t_2 - t_1|. \tag{2.10}$$

Even without the nonlinearity condition (WGNL), we can single out a particular representative of any solution, so that we obtain the continuity in time in the (strong) $\mathbf{L}^1_{\mathbf{loc}}$ topology, the uniqueness of solutions and their stability with respect to initial data for *all* times. Indeed, the next theorem shows that (2.9) and (2.10) hold at *every* time and with the same K^{CL}, provided at all times suitable representative $u_*(t, \cdot)$ is carefully chosen.

Theorem 2.6. *Let H satisfy (C3).*

1. *Fix the initial datum $u_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$. Assume that the Cauchy problem (CL) admits the solution u in the sense of Definition 2.1 on $[0, T]$. Then, u admits a representative, say u_*, such that*
   (a) *For a.e. $x \in \mathbb{R}$, $u_*(0, x) = u_o(x)$.*
   (b) *For all $a, b \in \mathbb{R}$ with $a < b$ and for all $t_1, t_2 \in [0, T]$*

$$\left| \int_a^b \left(u_*(t_2, x) - u_*(t_1, x) \right) \mathrm{d}x \right| \le K^{CL}\, |t_2 - t_1|, \tag{2.11}$$

   with K^{CL} defined as in (2.8).
   (c) *For all $R \in \mathbb{R}_+$ and for all $\bar{t} \in [0, T]$*

$$\lim_{t \to \bar{t}} \int_{-R}^R \left| u_*(t, x) - u_*(\bar{t}, x) \right| \mathrm{d}x = 0. \tag{2.12}$$

2. *Fix the initial data $u_o, v_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$. Assume that the corresponding Cauchy problems* **(CL)** *admit the solutions u, v in the sense of Definition* 2.1 *on $[0, T]$. Define*

$$C := \max \left\{ \|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R})}, \|v\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R})} \right\},$$
$$L := \sup \{|\partial_u H(x, w)| : x \in \mathbb{R} \text{ and } |w| \leq C\}, \tag{2.13}$$

and assume $L < +\infty$. Then, all representatives u_* and v_* satisfying Item 1 above are such that for all $t \in [0, T]$ and for all $R > 0$

$$\int_{-R}^{R} |u_*(t, x) - v_*(t, x)|\mathrm{d}x \leq \int_{-R-Lt}^{R+Lt} |u_o(x) - v_o(x)|\mathrm{d}x, \tag{2.14}$$

$$\int_{-R}^{R} [u_*(t, x) - v_*(t, x)]^+\mathrm{d}x \leq \int_{-R-Lt}^{R+Lt} [u_o(x) - v_o(x)]^+\mathrm{d}x. \tag{2.15}$$

In particular,

$$\|u_*(t, \cdot) - v_*(t, \cdot)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \leq \|u_o - v_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}. \tag{2.16}$$

We convene that when $(u_o - v_o) \notin \mathbf{L}^1(\mathbb{R}; \mathbb{R})$ the right hand side above is $+\infty$ and (2.16) holds. Moreover, by (2.16), if $(u_o - v_o) \in \mathbf{L}^1(\mathbb{R}; \mathbb{R})$, then $(u^*(t, \cdot) - v^*(t, \cdot)) \in \mathbf{L}^1(\mathbb{R}; \mathbb{R})$ for all t.

Remark that Definition 2.1 implies that $C < +\infty$ in (2.14). Then, condition **(CNH)**, if assumed, ensures that L is finite.

Turning to the Hamilton–Jacobi equation **(HJ)**, recall the apparently entirely different framework of the standard Crandall–Lions definition of viscosity solutions.

Definition 2.7. ([13, Definition 5.3]) Let $U \in \mathbf{C}^0([0, T] \times \mathbb{R}, \mathbb{R})$ satisfy $U(0) = U_o$.

 (i) *U is a subsolution to* **(HJ)** *when for all test functions $\varphi \in \mathbf{C}^1(]0, T[\times\mathbb{R}; \mathbb{R})$ and for all $(t_o, x_o) \in]0, T[\times\mathbb{R}$, if $U - \varphi$ has a point of local maximum at the point (t_o, x_o), then $\partial_t \varphi(t_o, x_o) + H(x_o, \partial_x \varphi(t_o, x_o)) \leq 0$;*
 (ii) *U is a supersolution to* **(HJ)** *when for all test functions $\varphi \in \mathbf{C}^1(]0, T[\times\mathbb{R}; \mathbb{R})$ and for all $(t_o, x_o) \in]0, T[\times\mathbb{R}$, if $U - \varphi$ has a point of local minimum at the point (t_o, x_o), then $\partial_t \varphi(t_o, x_o) + H(x_o, \partial_x \varphi(t_o, x_o)) \geq 0$.*
(iii) *U is a viscosity solution to* **(HJ)** *if it is both a supersolution and a subsolution.*

Definition 2.7 ensures uniqueness, extending to the present framework classical results, such as those in [4,25].

Theorem 2.8. *Let H satisfy* **(C3)**.

1. *Fix the initial datum $U_o \in \mathbf{Lip}(\mathbb{R}; \mathbb{R})$. Assume the corresponding Cauchy problem* **(HJ)** *admits the function U as solution in the sense of Definition* 2.7, *Lipschitz continuous in space, uniformly in time on $[0, T]$. Define*

$$K^{HJ} := \sup \left\{ |H(x, p)| : x \in \mathbb{R}, |p| \leq \|\partial_x U\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R})} \right\}. \tag{2.17}$$

We have for all $t_1, t_2 \in [0, T]$

$$\|U(t_2) - U(t_1)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \le K^{HJ} |t_2 - t_1|.$$

2. *Fix the initial data $U_o, V_o \in \mathbf{Lip}(\mathbb{R}; \mathbb{R})$. Assume the corresponding Cauchy problems* **(HJ)** *admit the functions U, respectively, V, as subsolution, respectively, supersolution, Lipschitz continuous in space, uniformly in time on $[0, T]$. Define*

$$
\begin{aligned}
C &:= \max\left\{ \|\partial_x U\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R})}, \|\partial_x V\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R})} \right\}; \\
L &:= \sup\left\{ |\partial_u H(x, p)| \colon x \in \mathbb{R}, \ |p| \le C \right\}.
\end{aligned}
\tag{2.18}
$$

If $L < +\infty$, then, for all $t \in [0, T]$, for all $R > 0$

$$\max_{|x| \le R} (U(t, x) - V(t, x)) \le \max_{|x| \le R + Lt} (U_o(x) - V_o(x)).\tag{2.19}$$

Remark that the Lipschitz continuity assumptions in Item 2 of Theorem 2.8 precisely mean that $C < +\infty$. Requiring also condition **(CNH)**, then ensures that L is finite.

We underline the evident deep analogy between Theorem 2.6 referring to the conservation law **(CL)** and Theorem 2.8 referring to the Hamilton–Jacobi equation **(HJ)**. The definitions (2.13) and (2.18) are essentially identical. Note moreover that the factor 2 appearing in (2.8) and not in (2.17) is a mandatory consequence of the correspondence between the two equations formalized in Sect. 2.5.

2.2. A bounding family of stationary solutions

Essential to get the necessary global in time $\mathbf{L}^\infty$ bounds on the solutions to **(CL)** is Theorem 2.9. In the homogeneous case, a sufficient supply of stationary solutions is immediately provided by constant functions, which are clearly also entropic. Here, we need to find $\mathbf{L}^\infty$ solutions that, first, are entropic and, second, are sufficiently many to ensure the necessary $\mathbf{L}^\infty$ bounds, together with the order preserving property (2.15) in Theorem 2.6.

Theorem 2.9. *Let H satisfy* **(C3)–(CNH)–(UC)–(WGNL)**. *Then, for all $U > 0$,* **(CL)** *admits stationary entropy solutions $u_-, u_+ \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$, i.e., solutions in the sense of Definition 2.1, that satisfy*

$$u_-(x) \le -U \quad and \quad u_+(x) \ge U \quad for\ a.e.\ x \in \mathbb{R}.$$

The proof begins with a careful construction of piecewise $\mathbf{C}^1$ stationary entropic solutions by means of the Implicit Function Theorem and Sard's Lemma for a particular class of fluxes whose level sets enjoy suitable geometric properties. Then, compensated compactness allows to pass to the limit on the fluxes, essentially showing a stability of solutions with respect to the flux, thus getting back to the general case. In this connection, we recall that already in [1,2] stationary solutions are assigned a key role in selecting solutions.

In the correspondence between **(CL)** and **(HJ)**, the stationary solutions to **(CL)** constructed in Theorem 2.9 have as counterpart viscosity solutions to **(HJ)** that may well be non-stationary, see (2.28), and are Lipschitz continuous but, in general, not differentiable.

2.3. Vanishing viscosity approximations

We now proceed toward existence results both for **(CL)** and for **(HJ)**, obtained through *vanishing viscosity* approximations, under the assumptions **(C3)–(CNH)–(UC)**. Thus, we consider the Cauchy problems

$$\begin{cases} \partial_t u + \partial_x H(x, u) = \varepsilon \, \partial^2_{xx} u \\ u(x, 0) = u_o(x) \end{cases} \tag{2.20}$$

and

$$\begin{cases} \partial_t U + H(x, \partial_x U) = \varepsilon \, \partial^2_{xx} U \\ U(x, 0) = U_o(x). \end{cases} \tag{2.21}$$

As a first step, we specify what we mean by *classical solutions* to (2.20) and to (2.21).

Definition 2.10. Let I be an open real interval and $\varepsilon > 0$. A *classical solution* to (2.20) on $]0, T[\times I$ is a function

$$u \in \mathbf{C}^0([0, T] \times \overline{I}; \mathbb{R}) \text{ such that } \begin{array}{l} \forall t \in]0, T[\\ \forall x \in I \end{array} \quad \begin{array}{l} \text{the map } x \mapsto u(t, x) \text{ is } \mathbf{C}^2(I; \mathbb{R}), \\ \text{the map } t \mapsto u(t, x) \text{ is } \mathbf{C}^1(]0, T[; \mathbb{R}), \end{array} \tag{2.22}$$

satisfying $\partial_t u(t, x) + \partial_x H(x, u(t, x)) = \varepsilon \, \partial^2_{xx} u(t, x)$ for all $(t, x) \in]0, T[\times I$ and $u(0, x) = u_o(x)$ for all $x \in \overline{I}$.

A *classical solution* to (2.21) on $]0, T[\times \mathbb{R}$ is a function

$$U \in \mathbf{C}^0([0, T] \times \overline{I}; \mathbb{R}) \text{ such that } \begin{array}{l} \forall t \in]0, T[\\ \forall x \in I \end{array} \quad \begin{array}{l} \text{the map } x \mapsto U(t, x) \text{ is } \mathbf{C}^3(I; \mathbb{R}), \\ \text{the map } t \mapsto U(t, x) \text{ is } \mathbf{C}^1(]0, T[; \mathbb{R}), \end{array} \tag{2.23}$$

satisfying $\partial_t U(t, x) + H(x, \partial_x U(t, x)) = \varepsilon \, \partial^2_{xx} U(t, x)$ for all $(t, x) \in]0, T[\times I$ and $U(0, x) = U_o(x)$ for all $x \in \overline{I}$.

Note that (2.23) in Definition 2.10 requires 3 space derivatives in U, although the third derivative does not appear in (2.21).

We now prove that the Cauchy problems (2.20) and (2.21) are equivalent.

Theorem 2.11. *Call I a non-empty open real interval and fix $T > 0$. Let H satisfy* **(C3)** *and $\varepsilon > 0$. Fix $u_o \in \mathbf{W}^{1,\infty}(I; \mathbb{R})$ and $U_o \in \mathbf{C}^1(\mathbb{R}; \mathbb{R})$ such that $U'_o = u_o$. Then, the problems (2.20) and (2.21) are* equivalent *in the sense that:*

(1) *Assume u is a classical solution to (2.20) on I in the sense of Definition 2.10. Then, for any $x_o \in I$, the map $U : [0, T] \times I \to \mathbb{R}$ defined by*

$$U(t, x) := \int_{x_o}^{x} u(t, \xi) \mathrm{d}\xi + \int_0^t (-H(x_o, u(\tau, x_o)) + \varepsilon \, \partial_x u(\tau, x_o)) \, \mathrm{d}\tau + U_o(x_o) \tag{2.24}$$

is the solution to (2.21) on I in the sense of Definition 2.10.

(2) *Assume U is a classical solution to (2.21) on I in the sense of Definition 2.10. Then, the map $u : [0, T] \times I \to \mathbb{R}$ defined by*

$$u(t, x) := \partial_x U(t, x)$$

is a classical solution to (2.20) on I in the sense of Definition 2.10.

We first get *a priori* estimates on the solutions to (2.21) and then on those to (2.20).

Theorem 2.12. *Let H satisfy (C3)–(CNH)–(UC). Choose $U_o \in \mathbf{C}^1(\mathbb{R}; \mathbb{R})$ with $U_o' \in \mathbf{Lip}(\mathbb{R}; \mathbb{R})$. Then, there exists a constant M such that for any $\varepsilon > 0$ sufficiently small, for any $T \in \mathbb{R}_+$ and for any classical solution U to (2.21) defined on $[0, T] \times \mathbb{R}$ we have*

$$\|\partial_t U\|_{\mathbf{L}^\infty([0,T] \times \mathbb{R}; \mathbb{R})} + \|\partial_x U\|_{\mathbf{L}^\infty([0,T] \times \mathbb{R}; \mathbb{R})} \leq M. \tag{2.25}$$

Since T is arbitrary both in Theorem 2.11 and in Theorem 2.12 and moreover M in (2.25) is independent of T (and ε), both results apply also to the case $T = +\infty$.

Corollary 2.13. *Let H satisfy (C3)–(CNH)–(UC). Choose $u_o \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$. Then, there exists a constant M such that for any $\varepsilon > 0$ sufficiently small, for any $T \in \mathbb{R}_+$ and for any classical solution u to (2.20) defined on $[0, T] \times \mathbb{R}$ which is also bounded,*

$$\|u\|_{\mathbf{L}^\infty([0,T] \times \mathbb{R}; \mathbb{R})} \leq M, \tag{2.26}$$

the case $T = +\infty$ is not excluded.

Thanks to Theorem 2.11, applied with $I = \mathbb{R}$, the proof of Corollary 2.13 is a direct consequence of Theorem 2.12 and is hence omitted.

Theorem 2.14. *Let H satisfy (C3) and (CNH). Choose an initial datum $u_o \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$. Then, for all $\varepsilon > 0$ sufficiently small, the Cauchy problem (2.20) admits a classical solution in the sense of Definition 2.10 on $\mathbb{R}$ defined for all $t \in \mathbb{R}_+$.*

Corollary 2.15. *Let H satisfy (C3)–(CNH)–(UC). Choose $U_o \in \mathbf{C}^0(\mathbb{R}; \mathbb{R})$ with $U_o' \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$. Then, for all $\varepsilon > 0$ sufficiently small, the Cauchy problem (2.21) admits a classical solution in the sense of Definition 2.10 on $\mathbb{R}$ defined for all $t \in \mathbb{R}_+$.*

Thanks to Theorem 2.11, applied with $I = \mathbb{R}$, the proof of Corollary 2.15 is a direct consequence of Theorem 2.14 and is hence omitted.

2.4. Existence of vanishing viscosity limits

We now deal with the vanishing viscosity limit of the solutions constructed in the previous Paragraph. Differently from [27], we complete this step in the case of more regular initial data, i.e., in the case where Theorem 2.12 and Corollary 2.13 apply.

Theorem 2.16. *Let H satisfy* **(C3)**–**(CNH)**–**(UC)**. *Choose an initial datum* $U_o \in \mathbf{C}^1(\mathbb{R}; \mathbb{R})$ *with* $U_o' \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$. *Let* ε_n *be a sequence converging to* 0. *Then, the sequence* U_{ε_n} *of the corresponding classical solutions to* (2.21) *on* $\mathbb{R}$ *converges uniformly on all compact subsets of* $\mathbb{R}_+ \times \mathbb{R}$ *to a function* $U_* \in \mathbf{Lip}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$ *which is a viscosity solution to* **(HJ)**.

Striving to treat **(CL)** and **(HJ)** in parallel, the next statement mirrors the previous one.

Theorem 2.17. *Let H satisfy assumptions* **(C3)**–**(CNH)**–**(UC)**–**(WGNL)**. *Fix an initial datum* $u_o \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$. *Then, the classical solutions* u_ε *to* (2.20) *on* $\mathbb{R}$ *converge pointwise a.e. in* $\mathbb{R}_+ \times \mathbb{R}$ *to a function* $u \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$ *which is an entropy solution to* **(CL)**.

The proof, entirely different from that of Theorem 2.16, by means of **(WGNL)**, relies on an *ad hoc* adaptation of classical compensated compactness arguments, see [16, Chapter 17] or [38, Chapter 9].

2.5. The limit semigroups and their equivalence

Here, we complete all previous steps obtaining the main results, stated in terms of the existence of the semigroups generated by **(CL)** and **(HJ)**, their properties and their connection.

Theorem 2.18. *Let H satisfy* **(C3)**–**(CNH)**–**(UC)**–**(WGNL)**. *For all* $T > 0$ *and for any initial datum* $u_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$, *there exists a unique entropy solution in* $\mathbf{L}^\infty([0, T] \times \mathbb{R}; \mathbb{R})$ *in the sense of Definition* 2.1, *to* **(CL)** *on* $[0, T]$. *Moreover, the maximal in time solution u:*

1. *is globally defined in time, corresponding to* $T = +\infty$ *in Definition* 2.1.
2. *is globally bounded, in the sense that* $u \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$.
   There exists a unique semigroup $S^{CL} : \mathbb{R}_+ \times \mathbf{L}^\infty(\mathbb{R}; \mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$ *such that for all* u_o $(t, x) \mapsto (S_t^{CL} u_o)(x)$ *solves* **(CL)** *in the sense of Definition* 2.1 *and enjoys the properties:*

3.a *For all* $u_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$, *the map* $t \mapsto S_t^{CL} u_o$ *is Lipschitz continuous with respect to the weak-∗* $\mathbf{L}_{\mathbf{loc}}^\infty(\mathbb{R}; \mathbb{R})$ *topology in the sense that there exists a* $K > 0$ *such that for all* $a, b \in \mathbb{R}$ *with* $a < b$ *and for all* $t_1, t_2 \in \mathbb{R}_+$

$$\left| \int_a^b \left((S_{t_2}^{CL} u_o)(x) - (S_{t_1}^{CL} u_o)(x) \right) \mathrm{d}x \right| \leq K \, |t_2 - t_1|.$$

3.b *For all $u_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$, the map $t \mapsto S_t^{CL} u_o$ is continuous with respect to the $\mathbf{L}_{\mathbf{loc}}^1(\mathbb{R}; \mathbb{R})$ topology, in the sense that for all $\bar{t} \in \mathbb{R}_+$ and for all $R > 0$*

$$\lim_{t \to \bar{t}} \int_{-R}^{R} \left| (S_t^{CL} u_o)(x) - (S_{\bar{t}}^{CL} u_o)(x) \right| dx = 0.$$

4. *For all $u_o, v_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$, define L as in (2.13). Then, for all $t \in \mathbb{R}_+$ and for all $R > 0$,*

$$\int_{-R}^{R} \left| (S_t^{CL} u_o)(x) - (S_t^{CL} v_o)(x) \right| dx \le \int_{-R-Lt}^{R+Lt} |u_o(x) - v_o(x)| dx.$$

Thanks to **(CNH)**, K^{CL}, as defined in (2.8), can be chosen independent of a and b, resulting in the K in **3.a**. Bounds L and on $\|u\|_{\mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})}$ depending on $\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})}$ are provided in the proof, see Sect. 3.5.

Theorem 2.19. *Let H satisfy **(C3)–(CNH)–(UC)–(WGNL)**. For all $T > 0$ and for any initial datum $U_o \in \mathbf{Lip}(\mathbb{R}; \mathbb{R})$, there exists a unique viscosity solution $U \in \mathbf{Lip}([0, T] \times \mathbb{R}; \mathbb{R})$ in the sense of Definition 2.7, to **(HJ)** on $[0, T]$. Moreover, the maximal in time solution U*

1. *is globally defined in time, corresponding to $T = +\infty$ in Definition 2.7.*
2. *is globally Lipschitz continuous, in the sense that $U \in \mathbf{Lip}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$.*

*There exists a unique semigroup $S^{HJ} : \mathbb{R}_+ \times \mathbf{Lip}(\mathbb{R}; \mathbb{R}) \to \mathbf{Lip}(\mathbb{R}; \mathbb{R})$ such that for all U_o $(t, x) \mapsto (S_t^{HJ} U_o)(x)$ solves **(HJ)** in the sense of Definition 2.7 and enjoys the properties:*

3. *For all $U_o \in \mathbf{Lip}(\mathbb{R}; \mathbb{R})$, the map $t \mapsto S_t^{HJ} U_o$ is Lipschitz continuous in the $\mathbf{L}^\infty$ norm.*
4. *For all $U_o, V_o \in \mathbf{Lip}(\mathbb{R}; \mathbb{R})$, define L as in (2.18). Then, for all $t \in \mathbb{R}_+$ and for all $R > 0$,*

$$\max_{|x| \le R} \left((S_t^{HJ} U_o)(x) - (S_t^{HJ} V_o)(x) \right) \le \max_{|x| \le R+Lt} (U_o(x) - V_o(x)).$$

Theorem 2.20. *Let H satisfy assumptions **(C3)–(CNH)–(UC)–(WGNL)**. Let the data $u_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$ and $U_o \in \mathbf{Lip}(\mathbb{R}; \mathbb{R})$ be such that $U_o'(x) = u_o(x)$ for a.e. $x \in \mathbb{R}$. Then, problems **(CL)** and **(HJ)** are* equivalent *in the sense that for all $t \in \mathbb{R}_+$ and for a.e. $x \in \mathbb{R}$,*

$$\left(S_t^{CL} u_o \right)(x) = \partial_x \left(S_t^{HJ} U_o \right)(x) \tag{2.27}$$

Remark 2.21. In the same setting of Theorem 2.20, formally, as a consequence of (2.27), for a fixed $x_o \in \mathbb{R}$, we can write

$$\left(S_t^{HJ} U_o \right)(x) = \int_{x_o}^{x} (S_t^{CL} u_o)(\xi) d\xi - \int_0^t H\left(x_o, (S_\tau^{CL} u_o)(x_o) \right) d\tau + U_o(x_o).$$

$$\tag{2.28}$$

The latter integral on the right hand side in (2.28) is meaningful only under further regularity conditions, such as in the case H is convex in u, which ensures that $S_t^{CL} u_o \in$ **BV**$(\mathbb{R}; \mathbb{R})$.

We can rephrase the above relations with the following commutative diagrams.

$$
\begin{array}{ccc}
U_o & \longrightarrow & S_t^{HJ} U_o \\
\partial_x \downarrow & & \downarrow \quad (2.27) \\
u_o & \longrightarrow & S_t^{CL} u_o
\end{array}
\qquad\qquad
\begin{array}{ccc}
U_o & \longrightarrow & S_t^{HJ} U_o \\
\int^x \uparrow & & \uparrow \quad (2.28) \\
u_o & \longrightarrow & S_t^{CL} u_o
\end{array}
$$

Remark 2.22. The correspondence between **(CL)** and **(HJ)** is instrumental in the existence results. Qualitative properties were independently obtained. However, Theorems 2.18 and 2.19 still lack a complete identification, thus suggesting possible improvements. The correspondence above between solutions to **(CL)** and to **(HJ)** actually gives more information than what is provided by Item 4 in Theorem 2.19. Indeed, Item 4 in Theorem 2.18 implies that S_t^{HJ} is non-expansive with respect to $\mathbf{W}_{\mathbf{loc}}^{1,1}$, i.e.,

$$
\left\| S_t^{HJ} U_o - S_t^{HJ} V_o \right\|_{\mathbf{W}^{1,1}([-R,R];\mathbb{R})} \leq \|U_o - V_o\|_{\mathbf{W}^{1,1}([-R-Lt,R+Lt];\mathbb{R})},
$$

We do not know of a proof of this bound for **(HJ)** independent from **(CL)**.

3. Analytical proofs

Throughout, $\mathbb{1}_I$ denotes the characteristic function of the set I. $\mathcal{L}$ stands for the Lebesgue measure in $\mathbb{R}$ and we call *negligible* a set of Lebesgue measure 0. The positive part of a real number is $[x]^+ := (x + |x|)/2$. Throughout, we set

$$
\operatorname{sgn} x := \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ +1 & \text{if } x > 0. \end{cases} \tag{3.1}
$$

3.1. Definitions of solution, local contraction and uniqueness

Lemma 3.1. *Let $E \in \mathbf{C}^1(\mathbb{R}; \mathbb{R})$ be convex. For any $\varepsilon, r > 0$, there exist $n \in \mathbb{N}$; positive weights $w_0, w_1, \ldots, w_n \in \mathbb{R}$ and points $p_0, p_1, \ldots, p_n \in \mathbb{R}$ such that setting for all $u \in \mathbb{R}$*

$$
\eta(u) := \sum_{k=0}^{n} w_k \, |u - p_k| \quad \text{so that} \quad \eta'(u) = \sum_{k=0}^{n} w_k \, \operatorname{sgn}(u - p_k) \tag{3.2}
$$

we have

$$
\forall u \in [-r, r] \quad |E(u) - \eta(u)| \leq \varepsilon \quad \text{and} \quad \left| E'(u) - \eta'(u) \right| \leq \varepsilon. \tag{3.3}
$$

The expression on the right in (3.2) is relevant when $u = p_k$. Indeed, it allows to prove that the bound on the derivatives in (3.3) holds at *every* u and not only at a.e. u.

Proof of Lemma 3.1. Let δ be a modulus of uniform continuity of E' on the interval $[-r, r]$ corresponding to $\min\{\varepsilon, \varepsilon/(2r)\}$, so that

$$\forall x_1, x_2 \in [-r, r] \quad \text{if} \quad |x_1 - x_2| < \delta \quad \text{then} \quad |E'(x_1) - E'(x_2)| < \min\{\varepsilon, \varepsilon/(2r)\}.$$

Choose n in $\mathbb{N}$ such that $n \geq 2r/\delta$. Define the points p_k and the map $\alpha \colon \mathbb{R} \to \mathbb{R}$ by

$$p_k := -r + k \frac{2r}{n} \quad \text{for } k = 0, \ldots, n \quad \text{and} \quad \alpha(u) := \begin{cases} E'(p_0) & u \in \,]-\infty, p_0]; \\ E'(p_k) & u \in \,]p_k, p_{k+1}]; \\ E'(p_n) & u \in \,]p_n, +\infty[. \end{cases}$$

Note that α is non-decreasing, since E' is. Set for $u \in [-r, r]$, $\tilde{\eta}(u) := E(-r) + \int_{-r}^{u} \alpha(v) \mathrm{d}v$ so that the condition on the left in (3.3) is satisfied by $\tilde{\eta}$, as well as the one on the right for $u \neq p_k$. Requiring the weights $w_0, \ldots, w_n$ to solve the $(n+1) \times (n+1)$ linear system

$$\sum_{k=0}^{n} |p_k - p_i| w_k = \tilde{\eta}(p_i) \quad i = 0, \ldots, n.$$

ensures that $\tilde{\eta} = \eta$ as defined in (3.2) for $u \in [a, b]$. The matrix of the above system is

$$A = \frac{2r}{n} \begin{bmatrix} 0 & 1 & 2 & 3 & \cdots & n \\ 1 & 0 & 1 & 2 & \cdots & n-1 \\ 2 & 1 & 0 & 1 & \cdots & n-2 \\ 3 & 2 & 1 & 0 & \cdots & n-3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n-1 & n-2 & n-3 & \cdots & 0 \end{bmatrix} \quad \text{i.e.,} \quad a_{ij} = \frac{2r}{n} |i - j| \text{ for } i, j = 1, \ldots, n+1$$

and straightforward calculations show that its determinant is $(-1)^n r \, 2^n$. Hence, this matrix is invertible, so that the weights $w_0, \ldots, w_n$ are uniquely defined. Moreover, differentiating $\tilde{\eta}$ we get $\tilde{\eta}'(p_k+) - \tilde{\eta}'(p_k-) = 2 \, w_k$. Since $\tilde{\eta}'$ is non-decreasing, we have that $w_k \geq 0$. We are left to prove that the expression for η' in (3.2) satisfy (3.3) also at $u = p_k$. Since $w_k \geq 0$, by the choice (3.1) and by the construction above, we have $E'(p_k) - \varepsilon \leq \eta'(p_k-) \leq \eta'(p_k) \leq \eta'(p_k+) \leq E'(p_k) + \varepsilon$. Possibly erasing the terms vanishing because $w_k = 0$, the proof is completed. $\qquad \square$

Proof of Proposition 2.4. **Claim 1: Proof of Item 1.**

Fix a positive ε and an entropy–entropy flux pair (E, F) in the sense of Definition 2.3. Call η the map (3.2) constructed in Lemma 3.1 corresponding to ε and $r := \|u\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})}$. Using (3.1), we use the following representative of η' and of a flux related to η, by (2.1):

$$\eta'(u) := \sum_{k=0}^{n} w_k \, \mathrm{sgn}(u - p_k) \quad \text{and} \quad q(x, u) := \sum_{k=1}^{n} w_k \, \Phi(x, u, p_k).$$

Choose a test function $\varphi \in \mathbf{C}_c^1([0, T[\times\mathbb{R}; \mathbb{R}_+)$ and let Y be such that spt $\varphi \subseteq [0, T] \times [-Y, Y]$. By the linearity in the entropy/entropy flux and by the positivity of the weights,

$$
\begin{aligned}
0 \leq &\int_0^T \int_{\mathbb{R}} (\eta(u(t, x)) \, \partial_t \varphi(t, x) + q(x, u(t, x)) \, \partial_x \varphi(t, x)) \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^T \int_{\mathbb{R}} \left(\sum_{k=1}^n w_k \, \mathrm{sgn}(u(t, x) - p_k) \, \partial_x H(x, p_k) \right) \varphi(t, x) \mathrm{d}x \, \mathrm{d}t \\
&+ \int_{\mathbb{R}} \eta(u_o(x)) \, \varphi(0, x) \mathrm{d}x \\
= &\int_0^T \int_{\mathbb{R}} (\eta(u(t, x)) \, \partial_t \varphi(t, x) + q(x, u(t, x)) \, \partial_x \varphi(t, x)) \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^T \int_{\mathbb{R}} \left(\eta'(u(t, x)) \, \partial_x H(x, u(t, x)) - \partial_x q(x, u(t, x)) \right) \varphi(t, x) \mathrm{d}x \, \mathrm{d}t \\
&+ \int_{\mathbb{R}} \eta(u_o(x)) \, \varphi(0, x) \mathrm{d}x \\
= &\int_0^T \int_{\mathbb{R}} \left(\eta(u(t, x)) \, \partial_t \varphi(t, x) - \eta'(u(t, x)) \, \partial_x H(x, u(t, x)) \varphi(t, x) \right) \mathrm{d}x \, \mathrm{d}t
\end{aligned}
$$
(3.4)

$$
+ \int_0^T \int_{\mathbb{R}} (q(x, u(t, x)) \, \partial_x \varphi(t, x) + \partial_x q(x, u(t, x)) \, \varphi(t, x)) \, \mathrm{d}x \, \mathrm{d}t \tag{3.5}
$$

$$
+ \int_{\mathbb{R}} \eta(u_o(x)) \, \varphi(0, x) \mathrm{d}x. \tag{3.6}
$$

Estimate the last three lines separately. To bound (3.4) use (3.3) (which holds on all $\mathbb{R}$):

$$
\begin{aligned}
[(3.4)] \leq &\int_0^T \int_{\mathbb{R}} \left(E(u(t, x)) \, \partial_t \varphi(t, x) - E'(u(t, x)) \, \partial_x H(x, u(t, x)) \, \varphi(t, x) \right) \mathrm{d}x \mathrm{d}t \\
&+ \left(\|\partial_t \varphi\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} + \|\varphi\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} \, \|\partial_x H\|_{\mathbf{L}^\infty([-Y,Y]\times[-r,r]; \mathbb{R})} \right) \varepsilon.
\end{aligned}
$$

To estimate the term (3.5), recall that from (2.6)

$$
\|\partial_u F - \partial_u q\|_{\mathbf{L}^\infty([-Y,Y]\times[-r,r]; \mathbb{R})} \leq \varepsilon \, \|\partial_u H\|_{\mathbf{L}^\infty([-Y,Y]\times[-r,r]; \mathbb{R})}. \tag{3.7}
$$

Using (2.4), thanks to $H \in \mathbf{C}^2(\mathbb{R}^2; \mathbb{R})$, write

$$
q(x, u) = q(x, 0) + \int_0^u \partial_u q(x, w) \, \mathrm{d}w = q(x, 0) + \int_0^u \eta'(w) \, \partial_u H(x, w) \, \mathrm{d}w
$$

$$
\partial_x q(x, u) = \partial_x q(x, 0) + \int_0^u \eta'(w) \, \partial_{xu}^2 H(x, w) \, \mathrm{d}w
$$

so that also using (3.2) and (3.7)

$$[(3.5)] = \int_0^T \int_{\mathbb{R}} (q(x,0)\, \partial_x \varphi(t,x) + \partial_x q(x,0)\, \varphi(t,x))\, \mathrm{d}x\, \mathrm{d}t$$

$$+ \int_0^T \int_{\mathbb{R}} \int_0^{u(t,x)} \left(\eta'(w)\, \partial_u H(x,w)\, \mathrm{d}w \right) \partial_x \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t$$

$$+ \int_0^T \int_{\mathbb{R}} \int_0^{u(t,x)} \left(\eta'(w)\, \partial_{xu}^2 H(x,w)\, \mathrm{d}w \right) \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t$$

$$\leq \underbrace{\int_0^T \int_{\mathbb{R}} \partial_x (q(x,0)\, \varphi(t,x))\, \mathrm{d}x\, \mathrm{d}t}_{=0}$$

$$+ \int_0^T \int_{\mathbb{R}} \int_0^{u(t,x)} \left(E'(w)\, \partial_u H(x,w)\, \mathrm{d}w \right) \partial_x \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t$$

$$+ \int_0^T \int_{\mathbb{R}} \int_0^{u(t,x)} \left(E'(w)\, \partial_{xu}^2 H(x,w)\, \mathrm{d}w \right) \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t$$

$$+ \|\partial_u H\|_{\mathbf{L}^\infty([-Y,Y] \times [-r,r]; \mathbb{R})} \|\partial_x \varphi\|_{\mathbf{L}^1(\mathbb{R}^2, \mathbb{R})}\, \varepsilon\, r$$

$$+ \|\partial_{xu}^2 H\|_{\mathbf{L}^\infty([-Y,Y] \times [-r,r]; \mathbb{R})} \|\varphi\|_{\mathbf{L}^1(\mathbb{R}^2, \mathbb{R})}\, \varepsilon\, r$$

$$= \int_0^T \int_{\mathbb{R}} \int_0^{u(t,x)} \partial_u F(x,w)\, \partial_x \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t$$

$$+ \int_0^T \int_{\mathbb{R}} \int_0^{u(t,x)} \partial_{ux}^2 F(x,w)\, \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t$$

$$+ \|\partial_u H\|_{\mathbf{L}^\infty([-Y,Y] \times [-r,r]; \mathbb{R})} \|\partial_x \varphi\|_{\mathbf{L}^1(\mathbb{R}^2, \mathbb{R})}\, \varepsilon\, r$$

$$+ \|\partial_{xu}^2 H\|_{\mathbf{L}^\infty([-Y,Y] \times [-r,r]; \mathbb{R})} \|\varphi\|_{\mathbf{L}^1(\mathbb{R}^2, \mathbb{R})}\, \varepsilon\, r$$

$$= \int_0^T \int_{\mathbb{R}} (F(x,u(t,x))\, \partial_x \varphi(t,x) + \partial_x F(x,u(t,x))\, \varphi(t,x))\, \mathrm{d}x\, \mathrm{d}t$$

$$- \underbrace{\int_0^T \int_{\mathbb{R}} \partial_x (F(x,0)\, \varphi(t,x))\, \mathrm{d}x\, \mathrm{d}t}_{=0} + \|\partial_u H\|_{\mathbf{L}^\infty([-Y,Y] \times [-r,r]; \mathbb{R})} \|\partial_x \varphi\|_{\mathbf{L}^1(\mathbb{R}^2, \mathbb{R})}\, \varepsilon\, r$$

$$+ \|\partial_{xu}^2 H\|_{\mathbf{L}^\infty([-Y,Y] \times [-r,r]; \mathbb{R})} \|\varphi\|_{\mathbf{L}^1(\mathbb{R}^2, \mathbb{R})}\, \varepsilon\, r$$

$$\xrightarrow[\varepsilon \to 0]{} \int_0^T \int_{\mathbb{R}} (F(x,u(t,x))\, \partial_x \varphi(t,x) + \partial_x F(x,u(t,x))\, \varphi(t,x))\, \mathrm{d}x\, \mathrm{d}t.$$

Passing to (3.6), use (3.3) to compute

$$[(3.6)] = \int_{\mathbb{R}} (\eta(u_o(x)) - E(u_o(x)))\, \varphi(0,x)\mathrm{d}x + \int_{\mathbb{R}} E(u_o(x))\, \varphi(0,x)\mathrm{d}x$$

$$\leq \int_{\mathbb{R}} E(u_o(x))\, \varphi(0,x)\mathrm{d}x + \|\varphi(0,\cdot)\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})}\, \varepsilon.$$

Adding the resulting estimates, we obtain

$$0 \leq \int_0^T \int_{\mathbb{R}} \left(E\left(u(t,x) \right) \, \partial_t \varphi(t,x) + F\left(x, u(t,x) \right) \, \partial_x \varphi(t,x) \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^T \int_{\mathbb{R}} \left(E'\left(u(t,x) \right) \, \partial_x H\left(x, u(t,x) \right) - \partial_x F\left(x, u(t,x) \right) \right) \varphi(t,x) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_{\mathbb{R}} E\left(u_o(x) \right) \, \varphi(0,x) \mathrm{d}x + \mathcal{O}(1)\,\varepsilon$$

where $\mathcal{O}(1)$ depends only on φ and on H. The proof of Claim 1 is completed.   ✓

Claim 2: Proof of Item 2.

Fix a regularizing kernel $\rho \in \mathbf{C}_c^\infty(\mathbb{R}; \mathbb{R})$ such that $\rho \geq 0$, $\rho(0) = 0$, spt $\rho \subseteq [-1,1]$, $\rho(-x) = \rho(x)$ for all $x \in \mathbb{R}$ and $\int_{\mathbb{R}} \rho = 1$. For any positive ε, let $\rho_\varepsilon(x) = (1/\varepsilon)\,\rho(x/\varepsilon)$. Fix $k \in \mathbb{R}$. Let E and F be as in (2.5). Recalling (2.6), define

$$E_\varepsilon(u) := \int_{\mathbb{R}} |w - k| \, \rho_\varepsilon(u - w) \mathrm{d}w \,, \tag{3.8}$$

$$F_\varepsilon(x,u) := E_\varepsilon(u) \, \partial_u H(x,u) - E_\varepsilon(k) \, \partial_u H(x,k) - \int_k^u E_\varepsilon(v) \, \partial_{uu}^2 H(x,v) \mathrm{d}v \,. \tag{3.9}$$

Clearly, E_ε is $\mathbf{C}^\infty$, F_ε is $\mathbf{C}^1$ and are an entropy–entropy flux pair in the sense of Definition 2.3, so that (2.4) holds. Moreover, since $E_\varepsilon(u) = \int_{\mathbb{R}} |u - w - k| \, \rho_\varepsilon(w) \mathrm{d}w$, $\rho \geq 0$ and the map $u \mapsto |u - w - k|$ is convex for $w \in \mathbb{R}$, for $\vartheta \in [0,1]$ and for $u_1, u_2 \in \mathbb{R}$ we have

$$E_\varepsilon\left(\vartheta u_1 + (1-\vartheta) u_2 \right) = \int_{\mathbb{R}} |(\vartheta u_1 + (1-\vartheta) u_2) - w - k| \, \rho_\varepsilon(w) \mathrm{d}w$$

$$\leq \int_{\mathbb{R}} \left(\vartheta |u_1 - w - k| + (1-\vartheta)|u_2 - w - k| \right) \rho_\varepsilon(w) \mathrm{d}w$$

$$= \vartheta \, E_\varepsilon(u_1) + (1 - \vartheta) \, E_\varepsilon(u_2),$$

hence E_ε is convex.

Use (2.7) and fix any test function $\varphi \in \mathbf{C}_c^1([0, T[\times\mathbb{R}; \mathbb{R}_+)$:

$$0 \leq \int_0^T \int_{\mathbb{R}} \left(E_\varepsilon\left(u(t,x) \right) \, \partial_t \varphi(t,x) + F_\varepsilon\left(x, u(t,x) \right) \, \partial_x \varphi(t,x) \right) \mathrm{d}x \mathrm{d}t \tag{3.10}$$

$$- \int_0^T \int_{\mathbb{R}} \left(E_\varepsilon'\left(u(t,x) \right) \, \partial_x H\left(x, u(t,x) \right) - \partial_x F_\varepsilon\left(x, u(t,x) \right) \right) \varphi(t,x) \mathrm{d}x \mathrm{d}t$$

$$\tag{3.11}$$

$$+ \int_{\mathbb{R}} E_\varepsilon\left(u_o(x) \right) \, \varphi(0,x) \mathrm{d}x \tag{3.12}$$

Note that (3.8) and (3.9) ensure the uniform convergence on compact sets of E_ε to E and of F_ε to F as $\varepsilon \to 0+$. Therefore, it is immediate to pass to the limit $\varepsilon \to 0+$ in (3.10) and (3.12). Indeed, with the notation (2.1),

$$\lim_{\varepsilon \to 0+} [(3.10)] = \int_0^T \int_{\mathbb{R}} (|u - k| \, \partial_t \varphi + \Phi(x, u, k) \, \partial_x \varphi) \, dx dt;$$

$$\lim_{\varepsilon \to 0+} [(3.12)] = \int_{\mathbb{R}} |u_o(x) - k| dx.$$

Consider now (3.11). Definition (3.9), (2.6) and **(C3)** ensure that $\partial_x F_\varepsilon$ converges uniformly on compact sets to $\partial_x F$. To deal with the term E'_ε, write

$$E_\varepsilon(u) = \int_{\mathbb{R}} |u - w - k| \, \rho_\varepsilon(w) dw$$

$$= \int_{-\infty}^{u-k} (u - w - k) \, \rho_\varepsilon(w) dw - \int_{u-k}^{+\infty} (u - w - k) \, \rho_\varepsilon(w) dw$$

so that

$$E'_\varepsilon(u) = \int_{-\infty}^{u-k} \rho_\varepsilon(w) dw - \int_{u-k}^{+\infty} \rho_\varepsilon(w) dw = \int_{\mathbb{R}} \operatorname{sgn}(u - w - k) \, \rho_\varepsilon(w) dw.$$

Since ρ_ε is even, we have that E'_ε converges pointwise *everywhere* to E' as $\varepsilon \to 0+$, with $|E'| \leq 1$. Thus, the Dominated Convergence Theorem [22, Theorem (12.24)] allows to pass to the limit also in (3.11):

$$\lim_{\varepsilon \to 0+} [(3.11)] = -\int_0^T \int_{\mathbb{R}} \Big(\operatorname{sgn}(u(t, x) - k) \, \partial_x H(x, u(t, x))$$

$$- \operatorname{sgn}(u(t, x) - k) (\partial_x H(x, u(t, x)) - \partial_x H(x, k)) \Big) \varphi(t, x) dx dt$$

$$= -\int_0^T \int_{\mathbb{R}} \operatorname{sgn}(u(t, x) - k) \, \partial_x H(x, k) \, dx dt.$$

Combining the obtained estimates of the limit $\varepsilon \to 0+$ of the terms (3.10)–(3.11)–(3.12) we get (2.2), completing the proof of Claim 2 and of Proposition 2.4. $\qquad\square$

Proof of Proposition 2.5. We adapt the arguments in [15, Lemma 3.2]. Therein, a similar result is obtained in a different setting: a source term is present, the flux is also time dependent but convex in u. Furthermore, the definition of solution in [15] requires the existence of both traces is required at any point for all time.

Proof of (2.10). Fix $a, b \in \mathbb{R}$ with $a < b$ and $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. For $\varepsilon \in]0, (b - a)/2[$, choose as φ in (2.3) the Lipschitz continuous map $\varphi_\varepsilon(t, x) := \chi_\varepsilon(t) \, \psi_\varepsilon(x)$ where

$$\chi_\varepsilon(t) := \begin{cases} 0 & t \in]-\infty, t_1[\\ (t - t_1)/\varepsilon & t \in [t_1, t_1 + \varepsilon[\\ 1 & t \in [t_1 + \varepsilon, t_2 - \varepsilon[\\ (t_2 - t)/\varepsilon & t \in [t_2 - \varepsilon, t_2[\\ 0 & t \in [t_2, +\infty[\end{cases}$$

$$\psi_\varepsilon(x) := \begin{cases} 0 & x \in]-\infty, a[\\ (x-a)/\varepsilon & x \in [a, a+\varepsilon[\\ 1 & x \in [a+\varepsilon, b-\varepsilon[\\ (b-x)/\varepsilon & x \in [b-\varepsilon, b[\\ 0 & x \in [b, +\infty[. \end{cases} \qquad (3.13)$$

By equality (2.3) in Remark 2.2, we obtain

$$\frac{1}{\varepsilon} \int_{t_1}^{t_1+\varepsilon} \int_{\mathbb{R}} u(t, x) \, \psi_\varepsilon(x) \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{\varepsilon} \int_{t_2-\varepsilon}^{t_2} \int_{\mathbb{R}} u(t, x) \, \psi_\varepsilon(x) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \frac{1}{\varepsilon} \int_0^T \int_a^{a+\varepsilon} H(x, u(t, x)) \, \chi_\varepsilon(t) \, \mathrm{d}x \, \mathrm{d}t$$

$$- \frac{1}{\varepsilon} \int_0^T \int_{b-\varepsilon}^b H(x, u(t, x)) \, \chi_\varepsilon(t) \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Recall the Definition (2.8) of K^{CL}, so that the first line above is estimated as follows:

$$\left| \frac{1}{\varepsilon} \int_{t_1}^{t_1+\varepsilon} \int_{\mathbb{R}} u(t, x) \, \psi_\varepsilon(x) \mathrm{d}x \mathrm{d}t - \frac{1}{\varepsilon} \int_{t_2-\varepsilon}^{t_2} \int_{\mathbb{R}} u(t, x) \, \psi_\varepsilon(x) \mathrm{d}x \mathrm{d}t \right|$$

$$\leq K^{CL} \int_0^T \chi_\varepsilon(t) \mathrm{d}t$$

$$\leq K^{CL} |t_2 - t_1|. \qquad (3.14)$$

To compute the limit as $\varepsilon \to 0$ of the left hand side in (3.14), observe first that

$$\frac{1}{\varepsilon} \left| \int_{t_1}^{t_1+\varepsilon} \int_{\mathbb{R}} u(t, x) \left(\psi_\varepsilon(x) - \mathbb{1}_{[a,b]}(x) \right) \mathrm{d}x \mathrm{d}t \right|$$

$$\leq \|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R})} \int_{\mathbb{R}} \left| \psi_\varepsilon(x) - \mathbb{1}_{[a,b]}(x) \right| \mathrm{d}x$$

$$\leq 2\varepsilon \, \|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R})}$$

$$\xrightarrow[\varepsilon\to 0]{} 0. \qquad (3.15)$$

An entirely similar procedure yields

$$\frac{1}{\varepsilon} \left| \int_{t_2-\varepsilon}^{t_2} \int_{\mathbb{R}} u(t, x) \left(\psi_\varepsilon(x) - \mathbb{1}_{[a,b]}(x) \right) \mathrm{d}x \mathrm{d}t \right| \xrightarrow[\varepsilon\to 0]{} 0. \qquad (3.16)$$

Recall that $u \in \mathbf{L}^\infty([0, T] \times \mathbb{R}; \mathbb{R})$, so that $u \in \mathbf{L}^1([0, T] \times [a, b]; \mathbb{R})$. By Fubini Theorem [22, Theorem 21.13], for almost all $t \in [0, T]$, the map $x \mapsto u(t, x)$ is in $\mathbf{L}^1([a, b]; \mathbb{R})$ and the map $t \mapsto \int_a^b u(t, x) \mathrm{d}x$ is in $\mathbf{L}^1([0, T]; \mathbb{R})$. Thus, if t_1 and t_2 are Lebesgue points [19, Chapter 1, § 7, Theorem 1.34] of $t \mapsto \int_a^b u(t, x) \mathrm{d}x$, we have

$$\frac{1}{\varepsilon} \int_{t_1}^{t_1+\varepsilon} \int_{\mathbb{R}} u(t,x) \, \mathbb{1}_{[a,b]}(x) \mathrm{d}x \mathrm{d}t \underset{\varepsilon \to 0}{\longrightarrow} \int_a^b u(t_1,x) \mathrm{d}x;$$

$$\frac{1}{\varepsilon} \int_{t_2-\varepsilon}^{t_2} \int_{\mathbb{R}} u(t,x) \, \mathbb{1}_{[a,b]}(x) \mathrm{d}x \mathrm{d}t \underset{\varepsilon \to 0}{\longrightarrow} \int_a^b u(t_2,x) \mathrm{d}x.$$

The latter relations, together with the limits (3.15) and (3.16), inserted in (3.14) complete the proof of (2.10). ✓

Proof of (2.9). Fix $a, b \in \mathbb{R}$ with $a < b$ and $\bar{t} \in \mathbb{R}_+$. For $\varepsilon \in]0, (b-a)/2[$, choose as φ in (2.3) the map $\varphi_\varepsilon(t,x) := \bar{\chi}_\varepsilon(t) \, \psi_\varepsilon(x)$ where

$$\bar{\chi}_\varepsilon(t) := \begin{cases} 1 & t \in]-\infty, \bar{t} - \varepsilon[\\ (\bar{t}-t)/\varepsilon & t \in [\bar{t} - \varepsilon, \bar{t}[\\ 0 & t \in [\bar{t}, +\infty[\end{cases}$$

and ψ_ε is as in (3.13). Repeat a procedure analogous to the one above choosing for $\bar{t}$ a Lebesgue point of the map $t \mapsto \int_a^b u(t,x) \mathrm{d}x$. The use of equality (2.3) in Remark 2.2 allows to let u_o appear explicitly.

The proof of Proposition 2.5 is completed. □

Proof of Theorem 2.6. Fix a representative u of a solution to **(CL)** in the sense of Definition 2.1.

Claim 1: There exists a u^* such that $u^* = u$ a.e. and u^* satisfies (a) and (b) in Item 1.

By (2.9)–(2.10), for all $a, b \in \mathbb{R}$ with $a < b$, there exists a negligible set $\mathcal{N}_{a,b} \subseteq [0, T]$ such that (2.10) holds for all $t_1, t_2 \in \mathbb{R}_+ \backslash \mathcal{N}_{a,b}$ and (2.9) holds for all $\bar{t} \in \mathbb{R}_+ \backslash \mathcal{N}_{a,b}$. Define

$$\mathcal{N} = \left\{ t \in [0, T] : \left\{ x \in \mathbb{R} : |u(t,x)| > \|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R})} \right\} \text{ is not negligible} \right\} \cup \bigcup_{a,b \in \mathbb{Q}: \, a<b} \mathcal{N}_{a,b}$$

which is also negligible by the definition of the $\mathbf{L}^\infty$ norm and by Fubini Theorem [22, Theorem 21.13] (set on the left) and by the choice of $\mathcal{N}_{a,b}$ (union on the right). Note that for all $\bar{t}, t_1, t_2 \in [0, T] \backslash \mathcal{N}$ and for all $a, b \in \mathbb{Q}$, u satisfies (2.9) and (2.10).

Fix now $a, b \in \mathbb{R}$ with $a < b$. Choose an increasing sequence a_n and a decreasing sequence b_n, both of rational numbers, such that $\lim_{n\to+\infty} a_n = a$, $\lim_{n\to+\infty} b_n = b$ and $a_n < b_n$. Then, $\left| \int_{a_n}^{b_n} (u(\bar{t},x) - u_o(x)) \, \mathrm{d}x \right|$ and $\left| \int_{a_n}^{b_n} (u(t_2,x) - u(t_1,x)) \, \mathrm{d}x \right|$ are uniformly bounded by the right hand sides in (2.9) and in (2.10). The Dominated Convergence Theorem [22, Theorem (12.24)] thus applies proving that u satisfies (2.9) and (2.10) for all $\bar{t}, t_1, t_2 \in [0, T] \backslash \mathcal{N}$ and also for all $a, b \in \mathbb{R}$.

Hence, for any real bounded interval I, $\left| \int_{\mathbb{R}} (u(t_2,x) - u(t_1,x)) \, \mathbb{1}_I(x) \mathrm{d}x \right| \leq C_I |t_2 - t_1|$, for a constant C_I depending on I. This bound then holds also for all piecewise constant functions and, by further approximations, we know that for all

$f \in \mathbf{L}^1(\mathbb{R}; \mathbb{R})$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $t_1, t_2 \in [0, T] \backslash \mathcal{N}$ and $|t_2 - t_1| < \delta$, then $\left| \int_{\mathbb{R}} (u(t_2, x) - u(t_1, x)) f(x) \mathrm{d}x \right| < \varepsilon$, thanks to the boundedness of $u(t, \cdot)$ uniform in $t \in [0, T] \setminus \mathcal{N}$. Hence, $u \colon [0, T] \backslash \mathcal{N} \to \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$ is uniformly continuous with respect to the weak-$*$ $\mathbf{L}^\infty_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$ topology.

Apply now Proposition A.1, which is possible since $\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$ is weakly-$*$ complete (as it follows, for instance, from Banach–Alaoglu Theorem [37, Theorem 3.15 and Theorem 3.18]), and obtain an extension $\bar{u}$ of u which is defined on all $[0, T]$, attains values in $\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$ and is continuous with respect to the weak-$*$ $\mathbf{L}^\infty_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$ topology.

The bound (2.9) also ensures that $\lim_{t \to 0+} \bar{u}(t) = u_o$ in the weak-$*$ topology of $\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$, so that $\bar{u}(0) = u_o$.

Define $u^* \colon [0, T] \times \mathbb{R} \to \mathbb{R}$ setting $u^*(t, x) = u(t, x)$ for all $t \in [0, T] \backslash \mathcal{N}$ and choose for $u^*(t)$ a precise representative, see [19, Chapter 1, § 7, Definition 1.26], of $\bar{u}(t)$ for $t \in \mathcal{N}$. Claim 1 is proved.   ✓

Fix a strictly convex entropy $E \in \mathbf{C}^1(\mathbb{R}; \mathbb{R})$. Choose a corresponding entropy flux F by means of (2.4). With reference to (2.7), introduce the function $G \in \mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})$

$$G(t, x) := E'\left(u_*(t, x)\right) \; \partial_x H\left(x, u_*(t, x)\right) - \partial_x F\left(x, u_*(t, x)\right) \qquad (3.17)$$

Fubini Theorem [22, Theorem 21.13] ensures that for any $\psi \in \mathbf{C}^1_c(\mathbb{R}; \mathbb{R}_+)$, the map $t \mapsto \int_{\mathbb{R}} E\left(u_*(t, x)\right) \psi(x) \mathrm{d}x$ is in $\mathbf{L}^1([0, T]; \mathbb{R})$. Call P_ψ the set of its Lebesgue points [19, Chapter 1, § 7, Theorem 1.34]. Call S the countable dense subset of $\mathbf{C}^1_c(\mathbb{R}; \mathbb{R})$ constructed in Proposition A.2. Denote for later use

$$P := \bigcap_{\gamma \in S} P_\gamma. \qquad (3.18)$$

Note that $[0, T] \setminus P$ has zero Lebesgue measure, since S is countable. For all $\psi \in \mathbf{C}^1_c(\mathbb{R}; \mathbb{R})$, each $t \in P$ is a Lebesgue point of $t \mapsto \int_{\mathbb{R}} E\left(u_*(t, x)\right) \psi(x) \mathrm{d}x$, by Proposition A.2.

Claim 2: For all $R > 0$, $\lim_{t \to 0+, \, t \in P} \int_{-R}^{R} |u_*(t, x) - u_o(x)| \mathrm{d}x = 0$.

By Item 1 in Proposition 2.4, for all $\varphi \in \mathbf{C}^1_c([0, T[\times \mathbb{R}; \mathbb{R}_+)$

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(E\left(u_*(t, x)\right) \partial_t \varphi(t, x) + F\left(x, u_*(t, x)\right) \partial_x \varphi(t, x) - G(t, x) \varphi(t, x) \right) \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_{\mathbb{R}} E\left(u_o(x)\right) \varphi(0, x) \mathrm{d}x \geq 0.$$

For $n \in \mathbb{N} \backslash \{0\}$ and $\tau > 0$, choose the test function $\varphi_{n, \tau} \in \mathbf{C}^1_c([0, T[\times \mathbb{R}; \mathbb{R}_+)$ defined by

$$\varphi_{n, \tau}(t, x) := \vartheta\left(n(t - \tau)\right) \psi(x) \quad \text{where} \quad \vartheta(\xi) := \begin{cases} 1 & \xi \leq 0 \\ 1 - \xi & \xi \in \;]0, 1[\quad \text{and} \; \psi \in \mathbf{C}^1_c(\mathbb{R}; \mathbb{R}_+). \\ 0 & \xi \geq 1 \end{cases}$$

Clearly, $\varphi_{n,\tau}(t, x) \xrightarrow[n\to+\infty]{} \mathbb{1}_{]-\infty,\tau]}(t)\,\psi(x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

Proceed now as in the Proof of Proposition 2.5. If $\tau \in \mathcal{P}_\psi$, then

$$
-\int_\mathbb{R} E\left(u_*(\tau, x)\right)\psi(x)\mathrm{d}x + \int_0^\tau \int_\mathbb{R} \left(F\left(x, u_*(t, x)\right)\psi'(x) - G(t, x)\,\psi(x)\right)\mathrm{d}x\mathrm{d}t
$$

$$
+ \int_\mathbb{R} E\left(u_o(x)\right)\psi(x)\mathrm{d}x \geq 0. \tag{3.19}
$$

Consider the linear functional $\mathcal{G}_\tau$ on $\mathbf{C}_c^1(\mathbb{R}; \mathbb{R})$ defined by

$$
\mathcal{G}_\tau\,\psi := -\int_\mathbb{R} E\left(u_*(t, x)\right)\psi(x)\mathrm{d}x
$$

$$
+ \int_0^\tau \int_\mathbb{R} \left(F\left(x, u_*(t, x)\right)\psi'(x) - G(t, x)\,\psi(x)\,\mathrm{d}x\mathrm{d}t\right.
$$

$$
+ \int_\mathbb{R} E\left(u_o(x)\right)\psi(x)\mathrm{d}x\,.
$$

By (3.19), for all $\tau \in P$ as defined in (3.18), we have that $\mathcal{G}_\tau\psi \geq 0$ for all $\psi \in \mathbf{C}_c^1(\mathbb{R}; \mathbb{R}_+)$.

Fix a positive R. Choose a sequence $\tau_n \in P$ with $\tau_n \xrightarrow[n\to+\infty]{} 0$. By [19, Chapter 1, § 9, Theorem 1.46], the sequence $u_*(\tau_n, \cdot)$ admits a subsequence $u_*(\tau_{n_k}, \cdot)$ and, for a.e. $x \in \mathbb{R}$, a Young measure [19, Chapter 1, § 9, Definition 1.34] ν_x, which is a Borel probability measure on $\left[-\|u_*\|_{\mathbf{L}^\infty(\mathbb{R}_+\times\mathbb{R};\mathbb{R})}, \|u_*\|_{\mathbf{L}^\infty(\mathbb{R}_+\times\mathbb{R};\mathbb{R})}\right]$ such that for all $\psi \in \mathbf{C}_c^1([-R, R]; \mathbb{R}_+)$

$$
\int_{-R}^R E\left(u_*(\tau_{n_k}, x)\right)\psi(x)\mathrm{d}x \xrightarrow[k\to+\infty]{} \int_{-R}^R \int_\mathbb{R} E(w)\,\mathrm{d}\nu_x(w)\,\psi(x)\mathrm{d}x.
$$

Since $\mathcal{G}_\tau\psi \geq 0$ and thanks to the Dominated Convergence Theorem [22, Theorem (12.24)], for all $\psi \in \mathbf{C}_c^1([-R, R]; \mathbb{R}_+)$ we have

$$
\int_{-R}^R \int_\mathbb{R} E(w)\,\mathrm{d}\nu_x(w)\,\psi(x)\mathrm{d}x \leq \int_{-R}^R E\left(u_o(x)\right)\psi(x)\mathrm{d}x.
$$

On the other hand, by Claim 1, $u_o(x) = \int_\mathbb{R} w\mathrm{d}\nu_x(w)$ for a.e. $x \in \mathbb{R}$, so that

$$
\int_\mathbb{R} E(w)\,\mathrm{d}\nu_x(w) \leq E\left(\int_\mathbb{R} w\,\mathrm{d}\nu_x(w)\right).
$$

The strict convexity of E yields the equality in Jensen [22, Exercise 30.34] hence for a.e. $x \in \mathbb{R}$, ν_x is the Dirac delta at $u_o(x)$, ensuring the pointwise convergence, up to a subsequence, see [38, Proposition 9.1.7]. The Dominated Convergence Theorem [22, Theorem (12.24)], can be applied since for all t and for a.e. x we have $|u_*(t, x)| \leq \|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R})}$ and implies that $u_*(\tau_{n_k}, \cdot) \xrightarrow[k\to+\infty]{} u_o$ in $\mathbf{L}^1([-R, R]; \mathbb{R})$. The choice of the τ_n is arbitrary, up to the set P, as is the choice of R. Hence, Claim 2 is proved. ✓

Claim 3: For all $R > 0$ and for all $t_1 \in P$, $\displaystyle\lim_{t_2\to t_1+,\,t_2\in P} \int_\mathbb{R} |u_*(t_2, x) - u_*(t_1, x)|\mathrm{d}x = 0$.

By Item 1 in Proposition 2.4, for all $\varphi \in \mathbf{C}_c^1([0, T[\times\mathbb{R}; \mathbb{R}_+)$

$$\int_0^{+\infty} \int_{\mathbb{R}} (E(u_*(t, x))\, \partial_t\varphi(t, x) + F(x, u_*(t, x))\, \partial_x\varphi(t, x) - G(t, x)\, \varphi(t, x))\, dx\, dt$$

$$+ \int_{\mathbb{R}} E(u_o(x))\, \varphi(0, x)dx \geq 0.$$

For $\varepsilon > 0$ and $t_2 > t_1 > 0$, choose the test function χ_ε as in (3.13) and define

$$\varphi_\varepsilon(t, x) := \chi_\varepsilon(t)\, \psi(x) \quad \text{with} \quad \psi \in \mathbf{C}_c^1(\mathbb{R}; \mathbb{R}),$$

so that $\varphi_\varepsilon(t, x) \xrightarrow[\varepsilon\to 0]{} \mathbb{1}_{[t_1, t_2]}(t)\, \psi(x)$.

Proceed now as in the Proof of Proposition 2.5 and as in Claim 2. If $t_1, t_2 \in P$ as defined in (3.18), then

$$-\int_{\mathbb{R}} E(u_*(t_2, x))\, \psi(x)dx + \int_{t_1}^{t_2} \int_{\mathbb{R}} \left(F(x, u_*(t, x))\, \psi'(x) - G(t, x)\, \psi(x) \right) dx\, dt$$

$$+ \int_{\mathbb{R}} E(u_*(t_1, x))\, \psi(x)dx \geq 0. \tag{3.20}$$

Proceed now exactly as in the previous Claim 2 to complete the proof of Claim 3. ✓

Claim 4: For all $\bar{t} \in P$, the map $(t, x) \mapsto u_*(\bar{t}+t, x)$ solves $\begin{cases} \partial_t u + \partial_x H(x, u) = 0 \\ u(0, x) = u_*(\bar{t}, x) \end{cases}$

in the sense of Definition 2.1 for $(t, x) \in [0, T - \bar{t}] \times \mathbb{R}$.

Define for $\varepsilon > 0$

$$\vartheta(\xi) := \begin{cases} 0 & \xi \leq 0 \\ \xi & \xi \in\,]0, 1[\\ 1 & \xi \geq 1 \end{cases} \quad \text{and} \quad \varphi_\varepsilon(t, x) := \vartheta\left(\frac{t - \bar{t}}{\varepsilon}\right) \psi(t, x) \text{ where } \psi \in \mathbf{C}_c^1([0, T[\times\mathbb{R}; \mathbb{R}_+).$$

Use φ_ε as a test function in (2.2) in Definition 2.1. Then,

$$0 \leq \int_0^T \int_{\mathbb{R}} (|u_*(t, x) - k|\, \partial_t\varphi_\varepsilon(t, x) + \Phi(x, u_*(t, x), k)\, \partial_x\varphi_\varepsilon(t, x))\, dx\, dt$$

$$- \int_0^T \int_{\mathbb{R}} \text{sgn}(u_*(t, x) - k)\, \partial_x H(x, k)\, \varphi_\varepsilon(t, x)dx\, dt$$

$$+ \int_{\mathbb{R}} |u_o(x) - k|\, \varphi_\varepsilon(0, x)dx$$

$$= \int_{\bar{t}}^T \vartheta\left(\frac{t - \bar{t}}{\varepsilon}\right) \int_{\mathbb{R}} (|u_*(t, x) - k|\, \partial_t\psi(t, x)$$

$$+ \Phi(x, u_*(t, x), k)\, \partial_x\psi(t, x))\, dx\, dt$$

$$- \int_{\bar{t}}^T \vartheta\left(\frac{t - \bar{t}}{\varepsilon}\right) \int_{\mathbb{R}} \text{sgn}(u_*(t, x) - k)\, \partial_x H(x, k)\, \psi(t, x)dx\, dt$$

$$+ \frac{1}{\varepsilon} \int_{\bar{t}}^{\bar{t}+\varepsilon} \int_{\mathbb{R}} |u_*(t, x) - k|\, \psi(t, x)dx\, dt$$

$$\xrightarrow[\varepsilon \to 0]{} \int_{\bar{t}}^{T} \int_{\mathbb{R}} \left(|u_*(t, x) - k| \, \partial_t \psi(t, x) + \Phi(x, u_*(t, x), k) \, \partial_x \psi(t, x) \right) \mathrm{d}x \mathrm{d}t$$

$$- \int_{\bar{t}}^{T} \int_{\mathbb{R}} \operatorname{sgn}(u_*(t, x) - k) \, \partial_x H(x, k) \, \psi(t, x) \mathrm{d}x \mathrm{d}t$$

$$+ \int_{\mathbb{R}} |u_*(\bar{t}, x) - k| \, \psi(\bar{t}, x) \mathrm{d}x,$$

where in the last line above we used Claim 3. Claim 4 is proved. ✓

Claim 5: (c) in Item 1 holds.

For any $R > 0$ define

$$\ell_R := \sup \left\{ |\partial_u H(x, w)| : |x| \le R + 1 \text{ and } |w| \le \|u_*\|_{\mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})} \right\}.$$

Fix $\bar{t} \in [0, T[$ and choose $t_1 \in [\bar{t} - 1/\ell_R, \bar{t}] \cap P, t_2 \in [t_1, t_1 + 1/\ell_R] \cap P$. By Claim 3 and Claim 4, the maps $(t, x) \mapsto u_*(t_1 + t, x)$ and $(t, x) \mapsto u_*(t_2 + t, x)$ solve

$$\begin{cases} \partial_t u + \partial_x H(x, u) = 0 & (t, x) \in]0, T - t_1[\times \mathbb{R} \\ u(0, x) = u_*(t_1, x) & x \in \mathbb{R}, \end{cases}$$
$$\begin{cases} \partial_t u + \partial_x H(x, u) = 0 & (t, x) \in]0, T - t_2[\times \mathbb{R} \\ u(0, x) = u_*(t_2, x) & x \in \mathbb{R}, \end{cases}$$

also in the sense of [27, Definition 1]. By [27, Theorem 1 and Theorem 3], which we can apply thanks to **(C3)**, for a.e. $s \in [0, t_1 - t_2 + 1/\ell_R]$

$$\int_{-R}^{R} |u_*(t_2 + s, x) - u_*(t_1 + s, x)| \mathrm{d}x \le \int_{-R-1+\ell_R s}^{R+1-\ell_R s} |u_*(t_2 + s, x) - u_*(t_1 + s, x)| \mathrm{d}x$$

$$\le \int_{-R-1}^{R+1} |u_*(t_2, x) - u_*(t_1, x)| \mathrm{d}x$$

$$\le \omega_R(t_2 - t_1) \tag{3.21}$$

where we set

$$\omega_R(\delta) := \operatorname*{ess\,sup}_{t \in [t_1, t_1 + \delta]} \int_{-R-1}^{R+1} |u_*(t, x) - u_*(t_1, x)| \mathrm{d}x$$

and recall that by Claim 3, $\lim_{\delta \to 0+} \omega_R(\delta) = 0$. Combine (3.21) with Claim 3 to obtain that for all $t_2, t_3 \in [t_1, t_1 + 1/\ell_R] \cap P$

$$\int_{-R}^{R} |u_*(t_3, x) - u_*(t_2, x)| \mathrm{d}x \le \omega_R(|t_3 - t_2|).$$

The above inequality shows that the map

$$[t_1, t_1 + 1/\ell_R] \cap P \to \mathbf{L}^1([-R, R]; \mathbb{R})$$
$$t \mapsto u_*(t, \cdot)$$

is uniformly continuous. Hence, it can be uniquely extended to a continuous map defined on all of $[t_1, t_1 + 1/\ell_R]$. Since Claim 1 ensures that u_* is continuous in the weak-$*$ $\mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}; \mathbb{R})$ topology, this extension coincides with u_*. Claim 5 follows because $\bar{t} \in]t_1, t_1 + 1/\ell_R[$.

Claim 6: Item 2 holds.

Let u^*, v^* be solutions to **(CL)** with data u_o and v_o, satisfying (c) in Item 1, proved in Claim 5. Then, u^* and v^* are also solutions to **(CL)** in the sense of [27, Definition 1]. By [27, Theorem 1 and Theorem 3], which we can apply thanks to **(C3)**, we have that if L in (2.13) is finite, for all $R > 0$ and for almost all $t \in [0, T]$ the following estimates hold:

$$\int_{-R}^{R} |u_*(t, x) - v(t, x)| \mathrm{d}x \le \int_{-R-Lt}^{R+Lt} |u_o(x) - v_o(x)| \mathrm{d}x;$$

$$\int_{-R}^{R} [u_*(t, x) - v(t, x)]^+ \mathrm{d}x \le \int_{-R-Lt}^{R+Lt} [u_o(x) - v_o(x)]^+ \mathrm{d}x,$$

Use the $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R})$ continuity to obtain the above inequalities for all $t \in [0, T]$, proving Claim 6 and thus completing the proof of Theorem 2.6. □

Proof of Item 2 in Theorem 2.8. We follow the general ideas in [4, Chapter 2]. Fix $\tau \in]0, T[$ and $R > 0$. Define

$$\Omega := \{(t, x) \in [0, \tau[\times \mathbb{R} : |x| < R + L(\tau - t)\} ; \qquad (3.22)$$

with L as in (2.18). Let C be as in (2.18), define $\tilde{H} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ so that

$$\tilde{H}(x, p) := \inf_{|q| \le C} (H(x, q) + L|p - q|) \qquad \text{for } (x, p) \in \mathbb{R} \times \mathbb{R}. \qquad (3.23)$$

Claim 1: $\tilde{H}(x, p) = H(x, p)$ whenever $|p| \le C$, with C defined in (2.18).

For all $(x, p) \in \mathbb{R} \times [-C, C]$, we have $\tilde{H}(x, p) \le H(x, p)$. By the Mean Value Theorem, for all $x \in \mathbb{R}$ and $p_1, p_2 \in [-C, C], |H(x, p_1) - H(x, p_2)| \le L |p_1 - p_2|$. For $q \in [-C, C], H(x, p) \le H(x, q) + L |q - p|$ and, by the Definition (3.23) of $\tilde{H}$, we have $H(x, p) \le \tilde{H}(x, p)$, proving Claim 1. ✓

Claim 2: For all $x \in \mathbb{R}$, the map $p \mapsto \tilde{H}(x, p)$ is Lipschitz continuous with Lipschitz constant L as defined in (2.18).

Fix $x, p_1, p_2 \in \mathbb{R}$. By (3.23), for all $q \in [-C, C]$, we have

$$\tilde{H}(x, p_1) \le H(x, q) + L |p_1 - q| \le H(x, q) + L |p_2 - q| + L |p_1 - p_2|$$

so that $\tilde{H}(x, p_1) - L |p_1 - p_2| \le \tilde{H}(x, q) + L |p_2 - q|$ implying $\tilde{H}(x, p_1) - L |p_1 - p_2| \le \tilde{H}(x, p_2)$ and therefore $\tilde{H}(x, p_1) - \tilde{H}(x, p_2) \le L |p_1 - p_2|$. The analogous inequality exchanging p_1 with p_2 is obtained similarly, proving Claim 2.✓

Claim 3: Let C, L be as in (2.18). Then, Formula (3.23) can be rewritten as

$$\tilde{H}(x, p) = \begin{cases} H(x, -C) - L (p + C) & \text{if } p \in] - \infty, -C[\\ H(x, p) & \text{if } p \in [-C, C] \\ H(x, C) + L (p - C) & \text{if } p \in]C, +\infty[, \end{cases} \qquad (3.24)$$

so that $\tilde{H}$ is continuous on $\mathbb{R} \times \mathbb{R}$.

First, by (3.23), note that for $p \geq C$, $\tilde{H}(x, p) \leq H(x, C) + L (p - C)$, while for $q \in [-C, C]$ the other inequality follows from

$$H(x, q) + L (p - q) = H(x, q) - H(x, C) + L (C - q) + H(x, C) + L (p - C)$$
$$\geq H(x, C) + L (p - C)$$

which, passing to the infimum over q, also proves the third line in (3.24). The first line is analogous and the middle one follows from Claim 1, completing the proof of Claim 3. ✓

Claim 4: Let U, V be as in Item 2 of Theorem 2.8. Then, they are a subsolution and a supersolution of $\partial_t w + \tilde{H}(x, \partial_x w) = 0$ in the sense of Definition 2.7.

Let φ be a $\mathbf{C}^1$ test function and assume that $U - \varphi$ admits a local maximum at $(t_o, x_o) \in]0, T[\times \mathbb{R}$. Then, for all x in a neighborhood of x_o,

$$U(t_o, x) - \varphi(t_o, x) \leq U(t_o, x_o) - \varphi(t_o, x_o)$$
$$\varphi(t_o, x_o) - \varphi(t_o, x) \leq U(t_o, x_o) - U(t_o, x)$$
$$\varphi(t_o, x_o) - \varphi(t_o, x) \leq C |x_o - x| \qquad \text{[By (2.18)]}$$
$$\text{sgn}(x_o - x) \frac{\varphi(t_o, x_o) - \varphi(t_o, x)}{x_o - x} \leq C.$$

Passing to the limits $x \to x_o\pm$, we get $|\partial_x \varphi(t_o, x_o)| \leq C$ hence, by Claim 1 and using the fact that U is a subsolution of **(HJ)**,

$$0 \geq \partial_t \varphi(t_o, x_o) + H (x_o, \partial_x \varphi(t_o, x_o)) = \partial_t \varphi(t_o, x_o) + \tilde{H} (x_o, \partial_x \varphi(t_o, x_o)).$$

To complete the proof of Claim 4, repeat the same procedure with the supersolution V. ✓

Choose $\chi \in \mathbf{C}^\infty(] - \infty, R[; \mathbb{R}_+)$ satisfying

$$\forall z \in] - \infty, 0] \ \chi(z) = 0 \quad \text{and} \quad \chi(z) \underset{z \to R-}{\longrightarrow} + \infty \qquad (3.25)$$
$$\forall z \in] - \infty, R[\ \chi'(z) \geq 0$$

and define, for $A > 0$,

$$\gamma : \Omega \to \mathbb{R} \quad \text{by} \quad \gamma(t, x) := \chi (|x| - L(\tau - t)),$$
$$U_A : \Omega \to \mathbb{R} \quad \text{by} \quad U_A(t, x) := U(t, x) - \frac{A}{\tau - t} - A \gamma(t, x). \qquad (3.26)$$

Claim 5: U_A is a strict subsolution of $\partial_t w + \tilde{H}(x, \partial_x w) = 0$ on Ω as defined in (3.22).

Let $\varphi \in \mathbf{C}^1(\Omega; \mathbb{R})$, $(t_o, x_o) \in \mathring{\Omega}$ such that $U_A - \varphi$ has a point of maximum at (t_o, x_o). Then, $\gamma \in \mathbf{C}^1(\Omega; \mathbb{R})$, since by the Definition (3.25) of χ, γ locally vanishes near $x = 0$

for $t < \tau$. The regularity of φ combined with that of $(t, x) \mapsto \dfrac{A}{\tau - t} + A\gamma(t, x)$, together with Claim 4, ensures that

$$\partial_t \varphi(t_o, x_o) + \frac{A}{(\tau - t_o)^2} + A \partial_t \gamma(t_o, x_o) + \tilde{H}(x_o, A \partial_x \gamma(t_o, x_o) + \partial_x \varphi(t_o, x_o)) \leq 0$$

$$\partial_t \varphi(t_o, x_o) + \tilde{H}(x_o, \partial_x \varphi(t_o, x_o)) + \frac{A}{(\tau - t_o)^2} + A \partial_t \gamma(t_o, x_o) - A L |\partial_x \gamma(t_o, x_o)| \leq 0$$

where Claim 2 was used. Recall that by (3.26)

$$\partial_t \gamma(t_o, x_o) = L \chi'(|x_o| - L(\tau - t_o)) \quad \text{and}$$
$$\partial_x \gamma(t_o, x_o) = \text{sgn}(x_o) \chi'(|x_o| - L(\tau - t_o))$$

so that

$$\partial_t \varphi(t_o, x_o) + \tilde{H}(x_o, \partial_x \varphi(t_o, x_o)) + A \chi'(|x_o| - L(\tau - t_o)) \underbrace{(L - L)}_{=0} + \frac{A}{(\tau - t_o)^2} \leq 0$$

$$\partial_t \varphi(t_o, x_o) + \tilde{H}(x_o, \partial_x \varphi(t_o, x_o)) + \frac{A}{(\tau - t_o)^2} \leq 0 \tag{3.27}$$

completing the proof of Claim 5.   ✓

Claim 6: Any convergent subsequence of a maximizing sequence of $U_A - V$ attains a limit in Ω.

For all $(t, x) \in \Omega$,

$$U_A(t, x) - V(t, x) \leq U(t, x) - V(t, x) \leq \|U\|_{\mathbf{L}^\infty(\overline{\Omega};\mathbb{R})} + \|V\|_{\mathbf{L}^\infty(\overline{\Omega};\mathbb{R})} < +\infty$$

by the compactness of $\overline{\Omega}$ and the continuity of U, V. Introduce a maximizing sequence $(t_n, x_n) \in \Omega$, so that $U_A(t_n, x_n) - V(t_n, x_n) \underset{n \to +\infty}{\longrightarrow} \sup_\Omega(U_A - V)$. Up to a subsequence, we have $(t_n, x_n) \underset{n \to +\infty}{\longrightarrow} (\bar{t}, \bar{x})$, for a suitable $(\bar{t}, \bar{x}) \in \overline{\Omega}$.

If $\bar{t} = \tau$, then (3.26) imply the bound

$$U_A(t_n, x_n) - V(t_n, x_n) \leq \|U\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|V\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} - \frac{A}{\tau - t_n}$$

that would imply $U_A(t_n, x_n) - V(t_n, x_n) \underset{n \to +\infty}{\longrightarrow} -\infty$, which is absurd.

If $|\bar{x}| = R + L(\tau - \bar{t})$, then, by (3.22), we have the bound

$$U_A(t_n, x_n) - V(t_n, x_n) \leq \|U\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|V\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} - A\gamma(t_n, x_n)$$

that would once again imply $U_A(t_n, x_n) - V(t_n, x_n) \underset{n \to +\infty}{\longrightarrow} -\infty$, which is not acceptable, since (t_n, x_n) is a maximizing sequence, completing the proof of Claim 6.   ✓

For all $\varepsilon > 0$, $(t, x) \in \Omega$ and $(s, y) \in \overline{\Omega}$, define

$$\psi_\varepsilon(t, x, s, y) := U_A(t, x) - V(s, y) - \frac{1}{2\varepsilon^2}(x - y)^2 - \frac{1}{2\varepsilon^2}(t - s)^2 \quad \text{and} \quad \begin{aligned} M_A &:= \max_{\Omega}(U_A - V) \\ M_{A,\varepsilon} &:= \sup_{\Omega \times \overline{\Omega}} \psi_\varepsilon \end{aligned}$$

Claim 7: For all $\varepsilon > 0$, there exist points $(t_\varepsilon, x_\varepsilon) \in \Omega$ and $(s_\varepsilon, y_\varepsilon) \in \overline{\Omega}$ such that $\psi_\varepsilon(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = \sup_{\Omega \times \overline{\Omega}} \psi_\varepsilon$.

This claim is proved by exactly the same technique used in Claim 6.   ✓

Using Claim 7, for any $\varepsilon > 0$ let $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$ be a point of maximum in $\Omega \times \overline{\Omega}$ of ψ_ε, so that $\psi_\varepsilon(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = M_{A,\varepsilon}$.

Claim 8: $\lim_{\varepsilon \to 0} M_{A,\varepsilon} = M_A$ and $\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon^2}\left((x_\varepsilon - y_\varepsilon)^2 + (t_\varepsilon - s_\varepsilon)^2\right) = 0$.

Since $U_A(t, x) - V(t, x) = \psi_\varepsilon(t, x, t, x)$ and $U_A \leq U$, we have

$$M_A \leq M_{A,\varepsilon} \leq \|U\|_{\mathbf{L}^\infty(\overline{\Omega};\mathbb{R})} + \|V\|_{\mathbf{L}^\infty(\overline{\Omega};\mathbb{R})} - \frac{1}{2\,\varepsilon^2}|x_\varepsilon - y_\varepsilon|^2 - \frac{1}{2\,\varepsilon^2}|t_\varepsilon - s_\varepsilon|^2$$

and therefore

$$|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2 \leq 2\left(\|U\|_{\mathbf{L}^\infty(\overline{\Omega};\mathbb{R})} + \|V\|_{\mathbf{L}^\infty(\overline{\Omega};\mathbb{R})} - M_A\right)\varepsilon^2 \xrightarrow[\varepsilon \to 0]{} 0.$$

Let ω_V be a modulus of continuity of V in (t, x) on $\overline{\Omega}$ and compute:

$$\begin{aligned} M_{A,\varepsilon} &= \psi_\varepsilon(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \\ &= U_A(t_\varepsilon, x_\varepsilon) - V(s_\varepsilon, y_\varepsilon) - \frac{1}{2\varepsilon^2}(x_\varepsilon - y_\varepsilon)^2 - \frac{1}{2\varepsilon^2}(t_\varepsilon - s_\varepsilon)^2 \quad (3.28) \\ &\leq (U_A(t_\varepsilon, x_\varepsilon) - V(t_\varepsilon, x_\varepsilon)) + (V(t_\varepsilon, x_\varepsilon) - V(s_\varepsilon, y_\varepsilon)) \\ &\leq M_A + \omega_V\left(|t_\varepsilon - s_\varepsilon| + |x_\varepsilon - y_\varepsilon|\right) \\ &\xrightarrow[\varepsilon \to 0]{} M_A, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (3.29) \end{aligned}$$

proving the first limit in Claim 8. To prove the second one, refine the computations (3.28)–(3.29) above as

$$\begin{aligned} \frac{1}{2\varepsilon^2}\left((x_\varepsilon - y_\varepsilon)^2 + (t_\varepsilon - s_\varepsilon)^2\right) &\leq M_A - M_{A,\varepsilon} + \omega_V\left(|t_\varepsilon - s_\varepsilon| + |x_\varepsilon - y_\varepsilon|\right) \\ &\leq \omega_V\left(|t_\varepsilon - s_\varepsilon| + |x_\varepsilon - y_\varepsilon|\right), \end{aligned}$$

completing the proof of Claim 8.   ✓

Claim 9: $\frac{1}{\varepsilon^2}|x_\varepsilon - y_\varepsilon| < C$.

For all y close to y_ε, we have

$$\psi_\varepsilon(t_\varepsilon, x_\varepsilon, s_\varepsilon, y) \leq \psi_\varepsilon(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$$

$$-V(s_\varepsilon, y) - \frac{1}{2\varepsilon^2}(x_\varepsilon - y)^2 \leq -V(s_\varepsilon, y_\varepsilon) - \frac{1}{2\varepsilon^2}(x_\varepsilon - y_\varepsilon)^2$$

$$\frac{1}{2\varepsilon^2}\,(y - y_\varepsilon)\,(2x_\varepsilon - y - y_\varepsilon) \le C\,|y - y_\varepsilon|$$

$$\frac{1}{\varepsilon^2}\,\mathrm{sgn}(y - y_\varepsilon)\left(x_\varepsilon - \frac{y + y_\varepsilon}{2}\right) \le C$$

and Claim 9 follows in the limits $y \to y_\varepsilon\pm$.   ✓

Claim 10: $\max_\Omega (U_A - V) = \max_{|x|<R} (U_A(0, x) - V(0, x))$.

By contradiction, assume that $\max_\Omega (U_A - V) > \max_{|x|<R} (U_A(0, x) - V(0, x))$.

Using Claim 9, we can introduce a sequence ε_n converging to 0, such that $\frac{1}{\varepsilon_n^2}\,(x_{\varepsilon_n} -$

$y_{\varepsilon_n}) \to \bar{p}$ for a suitable $\bar{p} \in [-C, C]$ and so that $t_{\varepsilon_n} \underset{n\to+\infty}{\longrightarrow} \bar{t}$ and $x_{\varepsilon_n} \underset{n\to+\infty}{\longrightarrow} \bar{x}$ for a

suitable $(\bar{t}, \bar{x}) \in \overline{\Omega}$. By Claim 8, we also have that $s_{\varepsilon_n} \underset{n\to+\infty}{\longrightarrow} \bar{t}$ and $y_{\varepsilon_n} \underset{n\to+\infty}{\longrightarrow} \bar{x}$. Then,

$$\left|M_{A,\varepsilon_n} - \left(U_A(t_{\varepsilon_n}, x_{\varepsilon_n}) - V(t_{\varepsilon_n}, x_{\varepsilon_n})\right)\right|$$

$$\le \omega_V\left(\left|t_{\varepsilon_n} - s_{\varepsilon_n}\right| + \left|x_{\varepsilon_n} - y_{\varepsilon_n}\right|\right) + \frac{1}{2\varepsilon_n^2}\left((x_{\varepsilon_n} - y_{\varepsilon_n})^2 + (t_{\varepsilon_n} - s_{\varepsilon_n})^2\right)$$

$$\underset{n\to+\infty}{\longrightarrow} 0$$

so that $U_A(t_{\varepsilon_n}, x_{\varepsilon_n}) - V(t_{\varepsilon_n}, x_{\varepsilon_n}) \underset{n\to+\infty}{\longrightarrow} M_A$. Claim 6 implies that $(\bar{t}, \bar{x}) \in \Omega$. Since
we are proceeding by contradiction, $\bar{t} > 0$ and for all n sufficiently large, also $t_{\varepsilon_n} > 0$,
so that $(t_{\varepsilon_n}, x_{\varepsilon_n}) \in \mathring{\Omega}$ and also $(s_{\varepsilon_n}, y_{\varepsilon_n}) \in \mathring{\Omega}$.

Let now n be sufficiently large and consider the maps

$$(t, x) \mapsto U_A(t, x) - \left(V(s_{\varepsilon_n}, y_{\varepsilon_n}) + \frac{1}{2\varepsilon_n^2}(x - y_{\varepsilon_n})^2 + \frac{1}{2\varepsilon_n^2}(t - s_{\varepsilon_n})^2\right);$$

$$(s, y) \mapsto V(s, y) - \left(U_A(t_{\varepsilon_n}, x_{\varepsilon_n}) + \frac{1}{2\varepsilon_n^2}(x_{\varepsilon_n} - y)^2 + \frac{1}{2\varepsilon_n^2}(t_{\varepsilon_n} - s)^2\right).$$

The former one admits a maximum at $(t_{\varepsilon_n}, x_{\varepsilon_n})$, while the latter admits a minimum at
$(s_{\varepsilon_n}, y_{\varepsilon_n})$. Since U_A is a subsolution and V is a supersolution, by (3.27) in the proof
of Claim 5 and Claim 4 we have

$$\frac{1}{\varepsilon_n^2}(t_{\varepsilon_n} - s_{\varepsilon_n}) + \tilde{H}\left(x_{\varepsilon_n}, \frac{1}{\varepsilon_n^2}(x_{\varepsilon_n} - y_{\varepsilon_n})\right) + \frac{A}{(\tau - t_{\varepsilon_n})^2} \le 0;$$

$$\frac{1}{\varepsilon_n^2}(t_{\varepsilon_n} - s_{\varepsilon_n}) + \tilde{H}\left(y_{\varepsilon_n}, \frac{1}{\varepsilon_n^2}(x_{\varepsilon_n} - y_{\varepsilon_n})\right) \ge 0.$$

Take the difference between the last lines above, let $n \to +\infty$ and we get the contra-
diction: $A/(\tau - \bar{t})^2 \le 0$, proving Claim 10.   ✓

Conclusion.

For all $(t, x) \in \Omega$, we have $U_A(t, x) - V(t, x) \le U(t, x) - V(t, x)$ so that

$$\max_{|x|\le R+LT} U_A(0, x) - V(0, x) \le \max_{|x|\le R+LT} (U_o(x) - V_o(x)).$$

Hence, using Claim 10, for fixed $(t, x) \in \Omega$,

$$U_A(t, x) - V(t, x) \leq \max_{|x| \leq R + L \, T} (U_o(x) - V_o(x))$$

$$U(t, x) - V(t, x) \leq \max_{|x| \leq R + L \, T} (U_o(x) - V_o(x)) + \frac{A}{\tau - t} + A \, \gamma(t, x)$$

and in the limit $A \to 0$ we have $U(t, x) - V(t, x) \leq \max_{|x| \leq R + L \, T} (U_o(x) - V_o(x))$. By the continuity of $U - V$, the latter inequality holds for all $(t, x) \in \overline{\Omega}$, completing the proof of Item 2 in Theorem 2.8. $\qquad\square$

Proof of Item 1 in Theorem 2.8. Fix $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$. Define $\hat{C} = \|\partial_x U\|_{\mathbf{L}^\infty([0, T] \times \mathbb{R}; \mathbb{R})}$, recall K^{HJ} from (2.17) and set

$$
\begin{array}{ll}
V : \mathbb{R}^2 \to \mathbb{R} & W : \mathbb{R}^2 \to \mathbb{R} \\
(t, x) \mapsto U(s, y) + K^{HJ}(t - s) + \hat{C}|x - y| & (t, x) \mapsto U(s, y) - K^{HJ}(t - s) - \hat{C}|x - y|.
\end{array}
$$
(3.30)

Claim 1: For all $x \in \mathbb{R}$, $U(s, x) \leq V(s, x)$ and V is a supersolution to (H,J) in the sense of Definition 2.7 on $\mathbb{R}^2$.

The bound $U(s, x) \leq V(s, x)$ follows from (3.30) and the Lipschitz continuity of U in x.

Let $\varphi \in \mathbf{C}^1(\mathbb{R}^2; \mathbb{R})$ and fix $(t, x) \in \mathbb{R}^2$ such that $V - \varphi$ has a point of minimum at (t, x). For all $\varepsilon \in \mathbb{R}$, if $|\varepsilon|$ is sufficiently small, then

$$V(t, x) - \varphi(t, x) \leq V(t + \varepsilon, x) - \varphi(t + \varepsilon, x)$$
$$\varphi(t + \varepsilon, x) - \varphi(t, x) \leq V(t + \varepsilon, x) - V(t, x) = K^{HJ} \varepsilon$$

so that letting $\varepsilon \to 0+$ we have $\partial_t \varphi(t, x) \leq K^{HJ}$, while letting $\varepsilon \to 0-$ we have $\partial_t \varphi(t, x) \geq K^{HJ}$. Hence, $\partial_t \varphi(t, x) = K^{HJ}$.

Again for $|\varepsilon|$ is sufficiently small,

$$V(t, x) - \varphi(t, x) \leq V(t, x + \varepsilon) - \varphi(t, x + \varepsilon)$$
$$\varphi(t, x + \varepsilon) - \varphi(t, x) \leq V(t, x + \varepsilon) - V(t, x)$$

so that letting $\varepsilon \to 0+$ we have $\partial_x \varphi(t, x) \leq \hat{C}$, while letting $\varepsilon \to 0-$ we have $\partial_x \varphi(t, x) \geq -\hat{C}$. Hence, $|\partial_x \varphi(t, x)| \leq \hat{C}$.

The definition of K^{HJ} ensures that $\partial_t \varphi(t, x) + H(x, \partial_x \varphi(t, x)) \geq 0$, proving Claim 1. $\qquad\checkmark$

Claim 2: For all $x \in \mathbb{R}$, $U(s, x) \geq W(s, x)$, W is a subsolution to (H,J) in the sense of Definition 2.7 on $\mathbb{R}^2$.

The proof of this claim is entirely analogous to that of the previous one. $\qquad\checkmark$

Conclusion.

We apply Item 2 in Theorem 2.8, which was proved above, on $[s, +\infty[\times \mathbb{R}$ to the couples of subsolution–supersolution (U, V) and (W, U) to get for all $(t, x) \in [s, +\infty[\times \mathbb{R}$

$$W(t, x) \leq U(t, x) \leq V(t, x)$$
$$|U(t, x) - U(s, y)| \leq K^{HJ}|t - s| + \hat{C}|x - y|$$

and by the arbitrariness of (s, y) we complete the proof of Item 1 in Theorem 2.8 $\square$

3.2. Existence of helpful stationary solution

Here, we prove Theorem 2.9, which yields, for all $U \in \mathbb{R}$, 2 stationary entropic solutions u_- and u_+ to **(CL)** such that $|u_\pm| > U$. We detail the case of u_+, that of u_- is similar. Further information and visualizations of the solutions constructed below, together with hints to their role as asymptotic states, can be found in [12].

Lemma 3.2. *Let H satisfy* **(C3)**–**(CNH)**–**(UC)**. *Fix $U > 0$. There exist $\bar{H} \in \mathbb{R}$, $V \in \mathbb{R}$ and real monotone sequences a_n, b_n with $\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} b_n = 0$ such that if*

$$\forall (x, u) \in \mathbb{R}^2 \quad H_n(x, u) := H(x, u) - a_n u - \frac{1}{2} b_n u^2, \qquad (3.31)$$

then:

1. *For all $n \in \mathbb{N}$, for all $(x, u) \in \mathbb{R}^2$, $H_n(x, u) = \bar{H}$ implies $\nabla H_n(x, u) \neq 0$.*
2. *For all $(x, u) \in \mathbb{R}^2$, $H(x, u) = \bar{H}$ implies $\nabla H(x, u) \neq 0$.*
3. *For all $n \in \mathbb{N}$, for all $(x, u) \in \mathbb{R}^2$, $|u| \leq U$ implies $|H_n(x, u)| < \bar{H}$ and $|H(x, u)| < \bar{H}$.*
4. *For all $n \in \mathbb{N}$, for all $(x, u) \in \mathbb{R}^2$, $u \geq V$ implies $|H_n(x, u)| > \bar{H}$ and $|H(x, u)| > \bar{H}$.*
5. *For all $n \in \mathbb{N}$, for all $(x, u) \in \mathbb{R}^2$, $H_n(x, u) = \bar{H}$ and $\partial_u H_n(x, u) = 0$ imply $\partial_{uu}^2 H_n(x, u) \neq 0$.*

Proof of Lemma 3.2. By **(UC)** we know that $|H(x, u)| \underset{u \to +\infty}{\longrightarrow} +\infty$. We assume that

$$\forall x \in \mathbb{R} \quad \lim_{u \to +\infty} H(x, u) = +\infty, \qquad (3.32)$$

the other case, namely $\lim_{u \to +\infty} H(x, u) = -\infty$, is entirely analogous.

Introduce the map $G \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\forall (x, u) \in \mathbb{R}^2 \quad G(x, u) := \left(\partial_u H(x, u) - u \, \partial_{uu}^2 H(x, u), \, \partial_{uu}^2 H(x, u) \right)$$

and note that, by **(C3)**, $G \in \mathbf{C}^1(\mathbb{R}^2; \mathbb{R}^2)$.

Claim 1: There exist increasing sequences a_n and b_n converging to 0 such that for all $n \in \mathbb{N}$, (a_n, b_n) is a regular value for G and $a_o > -1$, $b_o > -1$.

This claim follows from Sard's Lemma A.3 applied with $f = G, k = 1, n_1 = n_2 = 2$. Remark that here condition **(C3)** is fully exploited.   ✓

The assumption (3.31) allows to introduce

$$\mathcal{P} := \left\{ h \in \mathbb{R} \colon \exists\, n \in \mathbb{N},\, \exists\, (x, u) \in \mathbb{R}^2 \text{ such that } \begin{array}{c} H_n(x, u) = h \\ \partial_u H_n(x, u) = 0 \\ \partial^2_{uu} H_n(x, u) = 0 \end{array} \right\}, \qquad (3.33)$$

$$\mathcal{Y} := \left\{ h \in \mathbb{R} \colon \exists\, (x, u) \in \mathbb{R}^2 \text{ such that } \begin{array}{c} H(x, u) = h \\ \nabla H(x, u) = 0 \end{array} \right\}$$

$$\cup \left\{ h \in \mathbb{R} \colon \exists\, n \in \mathbb{N},\, \exists\, (x, u) \in \mathbb{R}^2 \text{ such that } \begin{array}{c} H_n(x, u) = h \\ \nabla H_n(x, u) = 0 \end{array} \right\}. \qquad (3.34)$$

Claim 2: $\mathcal{Y}$ is negligible and $\mathcal{P}$ is countable.

The former statement directly follows from Sard's Lemma A.3 applied first with $f = H$ then with $f = H_n$ and $k = 3, n_1 = 2, n_2 = 1$. Fix $n \in \mathbb{N}$ and define

$$\begin{aligned}
\mathcal{Q}_n &:= \left\{ (x, u) \in \mathbb{R} \colon \partial_u H_n(x, u) = 0 \text{ and } \partial^2_{uu} H_n(x, u) = 0 \right\} \\
&= \left\{ (x, u) \in \mathbb{R} \colon \partial_u H(x, u) - b_n u = a_n \text{ and } \partial^2_{uu} H(x, u) = b_n \right\} \\
&= \left\{ (x, u) \in \mathbb{R} \colon \partial_u H(x, u) - \partial^2_{uu} H(x, u)\, u = a_n \text{ and } \partial^2_{uu} H(x, u) = b_n \right\} \\
&= \left\{ (x, u) \in \mathbb{R} \colon G(x, u) = (a_n, b_n) \right\}.
\end{aligned}$$

Recall that (a_n, b_n) is a regular value for G, so we have that $\mathcal{Q}_n$ is discrete, hence countable. As a consequence, also $H_n(\mathcal{Q}_n)$ is countable.

This holds for all $n \in \mathbb{N}$, hence $\mathcal{P} = \bigcup_{n \in \mathbb{N}} H_n(\mathcal{Q}_n)$ is countable, proving Claim 2.✓

Define, using **(CNH)**,

$$H_1 := \sup_{(x,u) \in \mathbb{R} \times [-U, U]} |H(x, u)| = \max_{(x,u) \in [-X, X] \times [-U, U]} |H(x, u)|$$

and note that the set $]H_1 + U + \frac{1}{2} U^2, +\infty[\setminus (\mathcal{Y} \cup \mathcal{P})$ is not empty by Claim 2 and (3.32). Choose $\bar{H}$ in this set and with this choice, items 1, 2 and 5 hold by construction.

Claim 3: Item 3 holds.

Fix $n \in \mathbb{N}$ and $(x, u) \in \mathbb{R}^2$ such that $|u| \leq U$. Then, $|H(x, u)| \leq H_1 < \bar{H}$. Moreover, thanks to Claim 1 ensuring that $|a_n| \leq 1$ and $|b_n| \leq 1$,

$$|H_n(x, u)| \leq |H(x, u)| + U + \frac{1}{2} U^2 \leq H_1 + U + \frac{1}{2} U^2 < \bar{H}$$

proving Claim 3.   ✓

By **(UC)**, we have a $V \in \mathbb{R}$ such that for $(x, u) \in \mathbb{R}^2$ if $|u| \geq V$, then $|H(x, u)| \geq \bar{H} + 1 > \bar{H} > 0$.
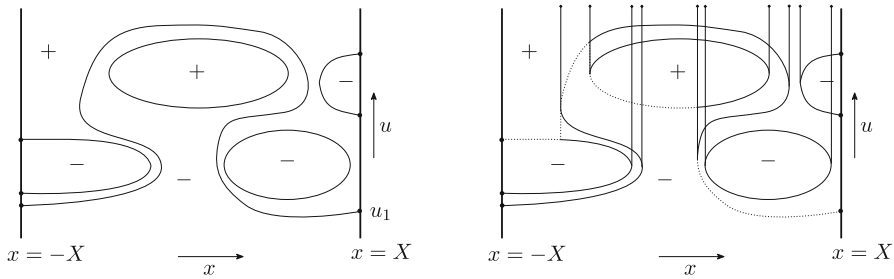
Figure 1. Left, the level set $H(x, u) = \bar{H}$, with $\pm$ denoting the regions where $H(x, u) \gtrless \bar{H}$. Right, the dashed line is the graph of the stationary entropic solution $x \mapsto u_+(x)$, which is inside this level set. The diamonds on top of the vertical lines indicate the positions of the points that, along the x axis, constitute the discrete set $\mathcal{X}$ defined in (3.40)

Claim 4: Item 4 holds.

Given this choice of V and assumption (3.32), we have that for $u \geq V$, $H(x, u) \geq 0$. Fix $(x, u) \in \mathbb{R}^2$ with $u \geq V$. We have $H(x, u) = |H(x, u)| > \bar{H}$ and since for all $n \in \mathbb{N}$, $a_n < 0$, $b_n < 0$, we also have $H_n(x, u) \geq H(x, u) > 0$. Claim 4 is proved, as is Lemma 3.2. $\qquad\square$

Lemma 3.3. *Let H satisfy* **(C3)–(CNH)–(UC)** *and moreover*

$$\forall x \in \mathbb{R} \qquad \lim_{u \to +\infty} H(x, u) = +\infty. \tag{3.35}$$

If U, V and $\bar{H}$ are positive real numbers such that

$$\forall (x, u) \in \mathbb{R}^2 \, u \in [0, U] \qquad \Longrightarrow H(x, u) < \bar{H}, \tag{3.36}$$

$$\forall (x, u) \in \mathbb{R}^2 \, u \geq V \qquad \Longrightarrow H(x, u) > \bar{H}, \tag{3.37}$$

$$\forall (x, u) \in \mathbb{R}^2 \, H(x, u) = \bar{H} \qquad \Longrightarrow \nabla H(x, u) \neq 0, \tag{3.38}$$

$$\forall (x, u) \in \mathbb{R}^2 \, \left.\begin{array}{r} H(x, u) = \bar{H} \\ \partial_u H(x, u) = 0 \end{array}\right\} \qquad \Longrightarrow \partial^2_{uu} H(x, u) \neq 0. \tag{3.39}$$

Then, there exists a stationary solution $u_+ \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R}^+)$, in the sense of Definition 2.1, to $\partial_t u + \partial_x H(x, u) = 0$ that satisfies $H(x, u_+(x)) = \bar{H}$ (so that u_+ attains values in $]U, V[$).

Proof of Lemma 3.3. In the construction below, we refer to Fig. 1.

Claim 1: There exists $u_1 > 0$ such that $H(X, u_1) = \bar{H}$ and $\partial_u H(X, u_1) > 0$.

Define

$$\mathcal{U} := \left\{ u \in [U, +\infty[: \text{ for all } v \in [U, u] \quad H(x, v) \leq \bar{H} \right\}.$$

Clearly, $U \in \mathcal{U}$ and V is an upper bound of $\mathcal{U}$. Define $u_1 := \sup \mathcal{U}$. By **(C3)**, $H(X, u_1) = \bar{H}$ and $\partial_u H(X, u_1) \geq 0$. By (3.38), $\nabla H(X, u_1) \neq 0$ while **(CNH)** ensures that $\partial_x H(X, u_1) = 0$. Hence, $\partial_u H(X, u_1) > 0$, proving Claim 1. ✓

Call $\pi_x \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the canonical projection $\pi_x(x, u) = x$. Introduce the set (corresponding to the diamonds in Fig. 1, right)

$$\mathcal{X} := \mathbb{R} \setminus \left\{ x \in \mathbb{R} \colon \text{if } u \in \mathbb{R}_+ \text{ is such that } H(x, u) = \bar{H} \text{ then } \partial_u H(x, u) \neq 0 \right\}$$
$$= \pi_x \left(\left\{ (x, u) \in \mathbb{R} \times \mathbb{R}_+ \colon H(x, u) = \bar{H} \text{ and } \partial_u H(x, u) = 0 \right\} \right). \tag{3.40}$$

Claim 2: $\mathcal{X}$ is finite.

The set $\left\{ (x, u) \in \mathbb{R} \times \mathbb{R}_+ \colon H(x, u) = \bar{H} \text{ and } \partial_u H(x, u) = 0 \right\}$ is closed by **(C3)**, contained in $[-X, X] \times [U, V]$ by the choice of $\bar{H}$ and consists of isolated points (apply the Inverse Function Theorem to $(x, u) \to \left(H(x, u) - \bar{H}, \partial_u H(x, u) \right)$ and then use (3.38) and (3.39)). Hence, it is finite and so is its projection on the x axis. The proof of Claim 2 follows. ✓

Define $y_* := \inf \mathcal{Y}$ where, denoting $\mathrm{co}(A)$ the convex hull of A and using the notation (2.1),

$$\mathcal{Y} := \left\{ y \in [-X, X] \colon \begin{array}{l} \exists u \text{ piecewise } \mathbf{C}^1, \ u \colon [y, X] \to \mathbb{R}_+ \text{ such that} \\ (i) \ \ u(X) = u_1 \\ (ii) \ \ H(x, u(x)) = \bar{H} \text{ for all } x \in [y, X] \\ (iii) \ \ \partial_u H(x, u(x)) \geq 0 \text{ for all } x \in [y, X] \\ (iv) \ \ \forall x \in [y, X] \ \forall k \in \mathrm{co}\{u(x-), u(x+)\} \ \Phi(x, u(x-), k) \geq 0 \end{array} \right\}. \tag{3.41}$$

Above, u piecewise $\mathbf{C}^1$ on $[y, X]$ means that that there exist finitely many pairwise disjoint open intervals I_ℓ such that $[y, X] = \bigcup \overline{I_\ell}$, $u_{|\overline{I_\ell}} \in \mathbf{C}^0(\overline{I_\ell}; \mathbb{R})$ and $u_{|I_\ell} \in \mathbf{C}^1(I_\ell; \mathbb{R})$.

Claim 3: $y_* \in \mathcal{Y}$.

The Implicit Function Theorem and Claim 1 ensure that $\mathcal{Y}$ contains a left neighborhood of X, so that $\mathcal{Y} \neq \emptyset$. Moreover, $\mathcal{Y} \subseteq [-X, X]$, so that $y_* = \inf \mathcal{Y}$ is finite.

If $\mathcal{X} = \emptyset$, define $\bar{y} := X$. Otherwise, note that there exists $\bar{y} \in \mathcal{Y}$ such that $\bar{y} < \min(\mathcal{X} \cap]y_*, X])$, since $\mathcal{X}$ is finite by Claim 2 and by the properties of the infimum. In both cases, there exists a map u satisfying *(i)*, *(ii)*, *(iii)* and *(iv)* in (3.41) defined on $[\bar{y}, X]$. An application of the Implicit Function Theorem, since $]y_*, \bar{y}] \cap \mathcal{X} = \emptyset$, allows to extend u down to y_* so that $u_{|[y_*, \bar{y}]}$ is $\mathbf{C}^1$. Hence, $y_* \in \mathcal{Y}$, proving Claim 3. ✓

Call u_+ the map corresponding to $y_* \in \mathcal{Y}$ as defined in (3.41) and set $u_* := u_+(y_*)$.

Claim 4: $y_* = -X$

Assume $y_* > -X$. Then, consider first the case $\partial_u H(y_*, u_*) \neq 0$. The Implicit Function Theorem ensures that u_+ can be extended toward left in a $\mathbf{C}^1$ way (so that the properties defining $\mathcal{Y}$ remain trivially satisfied), contradicting the above construction.

Consider now the case $\partial_u H(y_*, u_*) = 0$. Again, the Implicit Function Theorem and the assumptions (3.38) ensure the existence of $\varepsilon > 0$ and of a function ϑ such that $H(y, v) = \bar{H}$ with $y \in]y_* - \varepsilon, y_* + \varepsilon[$ and $v \in]u_* - \varepsilon, u_* + \varepsilon[$ is equivalent to $y = \vartheta(v)$. Direct computations show that $y_* = \vartheta(u_*)$, $0 = \vartheta'(u_*)$ and, by (3.39), $\vartheta''(u_*) \neq 0$. Moreover, $y = \vartheta(u_+(y))$ for $y > y_*$. Hence, $\vartheta''(u_*) > 0$.

There exists $\varepsilon_* > 0$ such that for all $u \in]u_* - \varepsilon_*, u_* + \varepsilon_*[\setminus\{u_*\}$, $\vartheta(u) > y_*$. Hence, for all $u \in]u_* - \varepsilon_*, u_* + \varepsilon_*[$, if $u \neq u_*$ then $H(y_*, u) \neq \bar{H}$.

Case 1: Suppose that $H(y_*, u) < \bar{H}$ for all $u \in]u_*, u_* + \varepsilon_*[$.
Introduce

$$\mathcal{V} := \left\{ u \in [u_*, +\infty[: \forall v \in [u_*, u]\ \ H(y_*, v) \leq \bar{H} \right\}. \tag{3.42}$$

$\mathcal{V} \neq \emptyset$ since $[u_*, u_* + \varepsilon_*[\subseteq \mathcal{V}$. By (3.37), $\mathcal{V}$ is bounded above by V and we can introduce $v_* := \sup \mathcal{V}$, which is finite. Note that for u near to v_*

$$H(y_*, u) \leq \bar{H} \text{ for } u < v_* \qquad H(y_*, v_*) = \bar{H} \qquad H(y_*, u) > \bar{H} \text{ for } u > v_*$$

showing that v_* is neither an isolated point of maximum nor an isolated point of minimum of $u \mapsto H(y_*, u)$. By (3.39), it then follows that $\partial_u H(y_*, v_*) \neq 0$ and, hence, $\partial_u H(y_*, v_*) > 0$. Apply now the Implicit Function Theorem on the level set $H(x, u) = \bar{H}$ in a neighborhood of (y_*, v_*), obtaining a map $u = \psi(x)$ defined on $]y_* - \eta, y_* + \eta[$. Define

$$
\begin{aligned}
u^\flat : [y_* - \eta, X] &\to &&\mathbb{R}^+ \\
x &\mapsto &&\begin{cases} \psi(x) & x \in [y_* - \eta, y_*[\\ u_+(x) & x \in [y_*, X] \end{cases}
\end{aligned}
$$

Clearly, $u^\flat$ is piecewise $\mathbf{C}^1$. Moreover, it satisfies *(i)*, *(ii)* and *(iii)* because u_* and ψ (thanks to the definition of v_* as the supremum of $\mathcal{V}$) satisfy them. Concerning *(iv)*: if $y < y_*$, simply note that ψ is $\mathbf{C}^1$; for $y > y_*$, u_+ satisfies *(iv)* and at $y = y_*$ we have $u^\flat(y_*+) = u_*$, $u^\flat(y_*-) = v_*$ and by the definition of v_*, $v_* > u_*$ and for all $k \in]u_*, v_*[$ by (3.42), $H(y_*, k) \leq \bar{H} = H(y_*, v_*)$. This implies $y_* - \eta \in \mathcal{Y}$, which contradicts the choice $y_* := \inf \mathcal{Y}$.

Case 2: Otherwise, since $u \mapsto H(y_*, u)$ is continuous, a connectedness argument ensures that $H(y_*, u) > \bar{H}$ for all $u \in]u_*, u_* + \varepsilon_*[$.
We have $\partial_u H(y_*, u_*) = 0$, so $\partial_{uu}^2 H(y_*, u_*) \geq 0$ and by (3.39), $\partial_{uu}^2 H(y_*, u_*) > 0$. Thus, for all $u \in]u_* - \varepsilon_*, u_*[$, $H(y_*, u) > \bar{H}$. We now proceed as in the case above. Introduce

$$\mathcal{V} := \left\{ u \in [0, u_*]: \forall v \in [u, u_*]\ \ H(y_*, v) \geq \bar{H} \right\}.$$

$\mathcal{V} \neq \emptyset$ since $]u_* - \varepsilon_*, u_*] \subseteq \mathcal{V}$. By (3.36), $\mathcal{V}$ is bounded below by U and we can introduce $v_* := \inf \mathcal{V}$, which is finite. Note that for u near to v_*

$$H(y_*, u) < \bar{H} \text{ for } u < v_* \qquad H(y_*, v_*) = \bar{H} \qquad H(y_*, u) \geq \bar{H} \text{ for } u > v_*$$

showing that v_* is neither an isolated point of maximum nor an isolated point of minimum of $u \mapsto H(y_*, u)$. By (3.39), it then follows that $\partial_u H(y_*, v_*) \neq 0$ and, hence, $\partial_u H(y_*, v_*) > 0$. Apply now the Implicit Function Theorem on the level set $H(x, u) = \bar{H}$ in a neighborhood of (y_*, v_*), obtaining a map $x \mapsto \psi(x)$ defined on $]y_* - \eta, y_* + \eta[$. Define

$$u^\flat : [y_* - \eta, X] \to \mathbb{R}^+$$
$$x \mapsto \begin{cases} \psi(x) & x \in [y_* - \eta, y_*[\\ u_+(x) & x \in [y_*, X] \end{cases}$$

Clearly, $u^\flat$ is piecewise $\mathbf{C}^1$. Moreover, it clearly satisfies *(i)*, *(ii)* and *(iii)* because u_* and ψ satisfy them. Concerning *(iv)*: for $y < y_*$, ψ is $\mathbf{C}^1$; for $y > y_*$, u_+ satisfies *(iv)* and at $y = y_*$ we have $u^\flat(y_*+) = u_*$, $u^\flat(y_*-) = v_*$ and by the definition of v_*, $v_* < u_*$ and for all $k \in]u_*, v_*[$, $H(y_*, k) > \bar{H} = H(y_*, v_*)$. This implies $y_* - \eta \in \mathcal{Y}$, which contradicts the choice $y_* := \inf \mathcal{Y}$. Claim 4 is proved ✓

Conclusion.

First, extend u_+ on $]-\infty, -X]$ setting it to be constant and, separately, on $[X, +\infty[$ also setting it to be constant. Note that u_+ is of class $\mathbf{C}^1$ both on a neighborhood of $-X$ and on a neighborhood of X, since by **(CNH)**, $\partial_x H(\pm X, u) = 0$ for all u and thanks to *(ii)* in (3.41).

Then, we verify that u_+ is a Kružkov (stationary) solution in the sense Definition 2.1. (Recall the notation introduced in (2.1)). Let $k \in \mathbb{R}$, $\varphi \in \mathbf{C}^1_c([0, T[\times\mathbb{R}; \mathbb{R}_+)$ and define:

$$A := \int_0^{+\infty} \int_{\mathbb{R}} |u_+(x) - k| \, \partial_t \varphi(t, x) \, \mathrm{d}x \, \mathrm{d}t \, ;$$

$$B(t) := \int_{\mathbb{R}} \Phi(x, u_+(x), k) \, \partial_x \varphi(t, x) \mathrm{d}x \, ;$$

$$C(t) := -\int_{\mathbb{R}} \mathrm{sgn}(u_+(x) - k) \, \partial_x H(x, k) \mathrm{d}x \, \varphi(t, x) \, ;$$

$$D := \int_{\mathbb{R}} |u_+(x) - k| \, \varphi(0, x) \mathrm{d}x \, .$$

We show that $A + \int_0^{+\infty} (B(t) + C(t)) \, \mathrm{d}t + D \geq 0$ considering the different terms separately.

$$A = \int_{\mathbb{R}} |u_+(x) - k| \int_0^{+\infty} \partial_t \varphi(t, x) \mathrm{d}t \, \mathrm{d}x = -\int_{\mathbb{R}} |u_+(x) - k| \, \varphi(0, x) \, \mathrm{d}x = -D.$$

Call $p_1, p_2, \ldots, p_n$ (with $p_i < p_{i+1}$) the points of jump in $x \mapsto u_+(x)$, they are finitely many by the Definition (3.41) of $\mathcal{Y}$ and that of u_+. For later use, let $p_0 := -X$ and $p_{n+1} := X$. We know that $u_+ \in \mathbf{C}^1(]p_i, p_{i+1}[; \mathbb{R}) \cap \mathbf{C}^0([p_i, p_{i+1}]; \mathbb{R})$ for $i = 0, \ldots, n$. When x is different from all $p_1, \ldots, p_n$ and, using [27, Lemma 3], compute

$$\frac{d}{dx}\Phi\,(x, u_+(x), k) = \partial_x\Phi\,(x, u_+(x), k) + \partial_u\Phi\,(x, u_+(x), k)\,\partial_x u_+(x)$$

$$= \operatorname{sgn}\,(u_+(x) - k)\,(\partial_x H\,(x, u_+(x)) - \partial_x H\,(x, k))$$

$$+ \operatorname{sgn}\,(u_+(x) - k)\,\partial_u H\,(x, u_+(x))\,\partial_x u_+(x)$$

$$= -\operatorname{sgn}\,(u_+(x) - k)\,\partial_x H\,(x, k) \tag{3.43}$$

since, by the definition of u_+, $H\,(x, u_+(x)) \equiv \bar{H}$. Fix $t \in \mathbb{R}_+$ and compute:

$$B(t) = \int_{-\infty}^{p_1}\Phi\,(x, u_+(x), k)\,\partial_x\varphi(t, x)dx + \sum_{i=1}^{n-1}\int_{p_i}^{p_{i+1}}\Phi\,(x, u_+(x), k)\,\partial_x\varphi(t, x)dx$$

$$+ \int_{p_n}^{+\infty}\Phi\,(x, u_+(x), k)\,\partial_x\varphi(t, x)\,dx$$

$$= \Phi\,(p_1, u_+(p_1-), k)\,\varphi(t, p_1) - \int_{-\infty}^{p_1}\frac{d}{dx}\Phi\,(x, u_+(x), k)\,\varphi(t, x)\,dx$$

$$+ \sum_{i=1}^{n-1}(\Phi\,(p_{i+1}, u_+(p_{i+1}-), k)\,\varphi(t, p_{i+1}) - \Phi\,(p_i, u_+(p_i+), k)\,\varphi(t, p_i))$$

$$- \sum_{i=1}^{n-1}\int_{p_i}^{p_{i+1}}\frac{d}{dx}\Phi\,(x, u_+(x), k)\,\varphi(t, x)\,dx$$

$$- \Phi\,(p_n, u_+(p_n+), k)\,\varphi(t, p_n) - \int_{p_n}^{+\infty}\frac{d}{dx}\Phi\,(x, u_+(x), k)\,\varphi(t, x)\,dx$$

$$= \sum_{i=1}^{n}(\Phi\,(p_i, u_+(p_i-), k) - \Phi\,(p_i, u_+(p_i+), k))\,\varphi(t, p_i)$$

$$+ \int_{\mathbb{R}}\operatorname{sgn}\,(u_+(x) - k)\,\partial_x H\,(x, k)\,\varphi(t, x)\,dx \qquad \text{[by (3.43)]}$$

$$= \sum_{i=1}^{n}(\Phi\,(p_i, u_+(p_i-), k) - \Phi\,(p_i, u_+(p_i+), k))\,\varphi(t, p_i) - C(t).$$

We thus obtain

$$A + \int_0^{+\infty}(B(t) + C(t))\,dt + D$$

$$= \int_0^{+\infty}\sum_{i=1}^{n}(\Phi\,(p_i, u_+(p_i-), k) - \Phi\,(p_i, u_+(p_i+), k))\,\varphi(t, p_i)dt$$

and we compute the generic i-th term of the latter sum as

$$\Phi\,(p_i, u_+(p_i-), k) - \Phi\,(p_i, u_+(p_i+), k)$$

$$= \operatorname{sgn}\,(u_+(p_i-) - k)\,(\bar{H} - H(p_i, k))$$

$$- \operatorname{sgn}\,(u_+(p_i+) - k)\,(\bar{H} - H(p_i, k)) \tag{3.44}$$

where we used $H(x, u_+(x)) = \bar{H}$ for all x. Clearly, if $k \notin \mathrm{co}\{u_+(p_i-), u_+(p_i+)\}$, the latter term vanishes. Assume $k \in \mathrm{co}\{u_+(p_i-), u_+(p_i+)\}$. Then, property *(iv)* in (3.41) ensures that $\mathrm{sgn}(u_+(p_i-) - k)\ (\bar{H} - H(p_i, k)) \geq 0$. On the other hand, being k between $u_+(p_i-)$ and $u_+(p_i+)$, $\mathrm{sgn}(u_+(p_i+) - k) = -\mathrm{sgn}(u_+(p_i-) - k)$, so that the difference (3.44) is nonnegative and so is the test function φ.

The proof of Lemma 3.3 is completed. $\qquad\square$

Lemma 3.4. *Let H satisfy* **(C3)–(CNH)–(UC)** *and moreover*

$$\forall x \in \mathbb{R} \qquad \lim_{u \to +\infty} H(x, u) = -\infty. \tag{3.45}$$

Let U and V be positive real numbers and $\bar{H}$ be negative such that

$$\forall (x, u) \in \mathbb{R}^2 \quad u \in [0, U] \qquad \Longrightarrow H(x, u) \quad > \bar{H}, \tag{3.46}$$

$$\forall (x, u) \in \mathbb{R}^2 \quad u \geq V \qquad \Longrightarrow H(x, u) \quad < \bar{H}, \tag{3.47}$$

$$\forall (x, u) \in \mathbb{R}^2 \quad H(x, u) = \bar{H} \qquad \Longrightarrow \nabla H(x, u) \quad \neq 0, \tag{3.48}$$

$$\forall (x, u) \in \mathbb{R}^2 \quad \left.\begin{array}{l} H(x, u) = \bar{H} \\ \partial_u H(x, u) = 0 \end{array}\right\} \quad \Longrightarrow \partial^2_{uu} H(x, u) \quad \neq 0. \tag{3.49}$$

Then, there exist a stationary solution $u_+ \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R}^+)$ to $\partial_t u + \partial_x H(x, u) = 0$ in the sense of Definition 2.1 that satisfies $H(x, u_+(x)) = \bar{H}$.

When (3.35) is replaced by (3.45), the above procedure can be repeated with essentially only technical modifications. We list below the various steps, omitting the details. We stress that it is critical that the case below be treated *"from left to right"*, i.e., from $-X$ to X, corresponding, with the terminology of the previous proof, to $y_* = \sup \mathcal{Y}$.

Proof of Lemma 3.4. Referring to the proof of Lemma 3.3, we only describe below the necessary modifications when (3.45) substitutes (3.35).

Claim 1 is modified to: There exists a real $u_1 > 0$ such that $H(-X, u_1) = \bar{H}$ and $\partial_u H(-X, u_1) < 0$.

Introduce the set

$$\mathcal{X} := \mathbb{R} \setminus \big\{ x \in \mathbb{R}: \text{ if } u \in \mathbb{R}_+ \text{ is such that } H(x, u) = \bar{H} \text{ then } \partial_u H(x, u) \neq 0 \big\}$$
$$= \pi_x \big(\{ (x, u) \in \mathbb{R} \times \mathbb{R}_+ : H(x, u) = \bar{H} \text{ and } \partial_u H(x, u) = 0 \} \big)$$

Claim 2 is modified to: $\mathcal{X}$ is finite.

Define $y_* = \sup \mathcal{Y}$, where, using the notation (2.1),

$$\mathcal{Y} := \left\{ y \in [-X, X]: \begin{array}{l} \exists u \text{ piecewise } \mathbf{C}^1,\ u: [-X, y] \to \mathbb{R}_+ \text{ such that} \\ (i)\quad u(-X) = u_1 \\ (ii)\quad H(x, u(x)) = \bar{H} \text{ for all } x \in [-X, y] \\ (iii)\quad \partial_u H(x, u(x)) \leq 0 \text{ for all } x \in [-X, y] \\ (iv)\quad \forall x \in [-X, y]\ \ \forall k \in \mathrm{co}\{u(x-), u(x+)\}\ \ \Phi(x, u(x-), k) \geq 0 \end{array} \right\}.$$

Claim 3 is modified to: $y_* \in \mathcal{Y}$.
Claim 4 is modified to: $y_* = X$.
Conclusion. No change is necessary.   □

Lemma 3.5. *Let* $H \in \mathbf{C}^2(\mathbb{R}^2; \mathbb{R})$ *and* **(CNH)** *hold. Let* u *be a stationary solution to* **(CL)** *in the sense of Definition* 2.1. *Then, for any* $\mathbf{C}^1$ *entropy–entropy flux pair* (E, F), *in the sense of Definition* 2.3, *with E convex, the entropy production distribution*

$$P: x \mapsto -\partial_x\left(F\left(x, u(x)\right)\right) - E'\left(u(x)\right) \partial_x H\left(x, u(x)\right) + \partial_x F\left(x, u(x)\right) \quad (3.50)$$

is a positive measure and satisfies for all $r \in \mathbb{R}_+$

$$\int_{-r}^{r} \mathrm{d}P(x) = F\left(-r, u(-r)\right) - F\left(r, u(r)\right)$$

$$+ \int_{-r}^{r} \left(E'\left(u(x)\right) \partial_x H\left(x, u(x)\right) - \partial_x F\left(x, u(x)\right)\right) \mathrm{d}x.$$

By Proposition 2.4, since u is stationary, the proof of Lemma 3.5 consists in integrating (3.50) against test functions that approximate the characteristic function of $[-r, r]$.

Proof of Theorem 2.9. Apply Lemma 3.2 to obtain $\bar{H}$, V and the sequence of Hamiltonians H_n. Both Lemmas 3.3 and 3.4 can be applied to each H_n, U, V, $\bar{H}$ and ensure the existence of a stationary solution u_n to $\partial_t u + \partial_x H_n(x, u) = 0$ in the sense of Definition 2.1, for each n. Moreover, $H_n\left(x, u_n(x)\right) = \bar{H}$ and u_n attains values in $]U, V[$.

Since: $a_n, b_n \in [-1, 1]$, both sequences vanish, **(CNH)** and (3.31) hold, we get

$$H_n \underset{n \to +\infty}{\longrightarrow} H \quad \text{in} \quad \mathbf{C}^3(\mathbb{R} \times [U, V]; \mathbb{R}). \tag{3.51}$$

Given an entropy $E \in \mathbf{C}^2(\mathbb{R}; \mathbb{R})$, we can introduce by means of (2.6) the corresponding flux

$$F_n(x, u) := \int_0^u E'(v) \, \partial_u H_n(x, v) \mathrm{d}v$$

$$= F(x, u) - a_n\left(E(u) - E(0)\right) - b_n \, E(u)\, u + b_n \int_0^u E(v) \mathrm{d}v. \tag{3.52}$$

Claim 1: For any $R > 0$ **and for any convex entropy** $E \in \mathbf{C}^2(\mathbb{R}; \mathbb{R})$, **define** F_n **by (3.52). Then,** $\{\partial_x\left(F_n(\cdot, u_n)\right) : n \in \mathbb{N}\}$ **is relatively compact in** $\mathbf{H}^{-1}([-R, R]; \mathbb{R})$.

We apply [38, Lemma 9.2.1], which we adapt here to the present (stationary) situation. By (3.51), using Proposition 2.4, straightforward computations yield:

$$\partial_x\left(F_n\left(x, u_n(x)\right)\right) = v_n(x) - \mu_n(x) \tag{3.53}$$

where

$$v_n(x) := - E'(u_n(x)) \, \partial_x H_n(x, u_n(x)) + \partial_x F_n(x, u_n(x))$$
$$\mu_n(x) := - \partial_t E(u_n(x)) - \partial_x (F_n(x, u_n(x)))$$
$$- E'(u_n(x)) \, \partial_x H_n(x, u_n(x)) + \partial_x F_n(x, u_n(x)).$$

The family (u_n) is in $\mathbf{L}^\infty([0, T] \times [-R, R]; [U, V])$, thus the family (v_n) is bounded in $\mathbf{L}^\infty([0, T] \times [-R, R]; \mathbb{R})$ by **(C3)** and it is also bounded in $\mathcal{M}([0, T] \times [-X, X]; \mathbb{R})$. The family (μ_n) is bounded in $\mathcal{M}([0, T] \times [-X, X]; \mathbb{R})$ by Lemma 3.5. Clearly, $\{\partial_x (F_n(\cdot, u_n)) : \varepsilon \in]0, \varepsilon_*[\}$ is bounded in $\mathbf{W}^{-1,\infty}([-R, R]; \mathbb{R})$. Thus, Murat Lemma [38, Lemma 9.2.1] completes the proof of Claim 1.   ✓

By [19, Chapter 1, § 9, Theorem 1.46], the sequence (u_n) admits a subsequence, which we keep denoting (u_n), and, for a.e. $x \in \mathbb{R}$, a Young measure [19, Chapter 1, § 9, Definition 1.34] ν_x, which is a Borel probability measure on $[U, V]$ and such that

$$\lim_{n \to +\infty} \int_{\mathbb{R}} g(u_n(x)) \, \varphi(x) \mathrm{d}x = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(w) \mathrm{d}\nu_x(w) \right) \varphi(x) \mathrm{d}x$$

for any $g \in \mathbf{C}^0(\mathbb{R}; \mathbb{R})$ and for any $\varphi \in \mathbf{L}^1(\mathbb{R}; \mathbb{R})$. Clearly, we also obtain that for any $\varphi \in \mathbf{L}^1([0, T] \times \mathbb{R}; \mathbb{R})$, we have

$$\lim_{n \to +\infty} \int_0^T \int_{\mathbb{R}} g(u_n(x)) \, \varphi(t, x) \, \mathrm{d}t \mathrm{d}x = \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(w) \, \mathrm{d}\nu_x(w) \right) \varphi(t, x) \mathrm{d}t \mathrm{d}x. \tag{3.54}$$

Claim 2: For any $G \in \mathbf{C}^0(\mathbb{R}^2; \mathbb{R})$ **such that** $G(x, u) = G(-X, u)$ **for all** $x \in {]} -\infty, -X]$ **and** $G(x, u) = G(X, u)$ **for all** $x \in [X, +\infty[$,

$$\lim_{n \to +\infty} \int_0^T \int_{\mathbb{R}} G(x, u_n(x)) \, \varphi(t, x) \mathrm{d}t \mathrm{d}x = \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} G(x, w) \, \mathrm{d}\nu_x(w) \right) \varphi(t, x) \mathrm{d}t \mathrm{d}x.$$

Recall that $u_n \in \mathbf{L}^\infty(\mathbb{R}; [U, V])$. In view of our later use of Fubini Theorem, we use Stone–Weierstrass Theorem [22, Corollary 7.31] so that for every $\delta > 0$ there exist a $\nu \in \mathbb{N}$ and functions $f_1, \ldots, f_\nu \in \mathbf{C}^0([-X, X]; \mathbb{R})$, $g_1, \ldots, g_\nu \in \mathbf{C}^0([U, V]; \mathbb{R})$ such that

$$\sup_{(x, w) \in [-X, X] \times [U, V]} \left| G(x, w) - \sum_{\ell=1}^{\nu} f_\ell(x) \, g_\ell(w) \right| < \delta. \tag{3.55}$$

Since G satisfies **(CNH)**, for $\ell = 1, \ldots, \nu$, introducing the functions

$$\bar{f}_\ell(x) := \begin{cases} f_\ell(-X) & \text{for } x < -X \\ f_\ell(x) & \text{for } x \in [-X, X] \\ f_\ell(X) & \text{for } x > X \end{cases}$$

we can extend the latter statement (3.55) to

$$\sup_{(x,w)\in\mathbb{R}\times[U,V]} \left| G(x,w) - \sum_{\ell=1}^{\nu} \bar{f}_\ell(x)\, g_\ell(w) \right| < \delta.$$

Recall that the support of ν_x is included in $[U, V]$ for a.e. x. Then,

$$\left| \int_0^T \int_{\mathbb{R}} G(x, u_n(x))\, \varphi(t, x)\mathrm{d}t\mathrm{d}x - \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} G(x, w)\mathrm{d}\nu_x(w) \right) \varphi(t, x)\mathrm{d}t\mathrm{d}x \right|$$

$$\leq \left| \int_0^T \int_{\mathbb{R}} \left(G(x, u_n(x)) - \sum_{\ell=1}^{\nu} \bar{f}_\ell(x)\, g_\ell(u_n(x)) \right) \varphi(t, x)\mathrm{d}t\mathrm{d}x \right|$$

$$+ \left| \int_0^T \int_{\mathbb{R}} \sum_{\ell=1}^{\nu} \bar{f}_\ell(x)\, g_\ell(u_n(x))\, \varphi(t, x)\mathrm{d}t\mathrm{d}x \right.$$

$$\left. - \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \sum_{\ell=1}^{\nu} \bar{f}_\ell(x)\, g_\ell(w)\mathrm{d}\nu_x(w) \right) \varphi(t, x)\mathrm{d}t\mathrm{d}x \right|$$

$$+ \left| \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} -G(x, w) + \sum_{\ell=1}^{L} \bar{f}_\ell(x)\, g_\ell(w)\mathrm{d}\nu_x(w) \right) \varphi(t, x)\mathrm{d}t\mathrm{d}x \right|$$

$$\leq 2\,\delta\, \|\varphi\|_{\mathbf{L}^1([0,T]\times\mathbb{R};\mathbb{R})}$$

$$+ \sum_{\ell=1}^{\nu} \left| \int_0^T \int_{\mathbb{R}} \left(g_\ell(u_n(x)) - \int_{\mathbb{R}} g_\ell(w)\mathrm{d}\nu_x(w) \right) \left(\bar{f}_\ell(x)\, \varphi(t, x) \right) \mathrm{d}t\mathrm{d}x \right|$$

and each term in the latter sum above converges to 0 by (3.54), since each $\bar{f}_\ell\, \varphi$ is in $\mathbf{L}^1([0, T] \times \mathbb{R}; \mathbb{R})$. Passing to the lim sup and using the arbitrariness of δ, Claim 2 is proved.   ✓

Claim 3: For any $G_n \in \mathbf{C}^0(\mathbb{R}^2; \mathbb{R})$ with $G_n(x, u) = G_n(-X, u)$ for all $x \in\,]-\infty, -X]$ and $G_n(x, u) = G_n(X, u)$ for all $x \in [X, +\infty[$, such that G_n converges to G uniformly on $\mathbb{R} \times [U, V]$,

$$\lim_{n\to+\infty} \int_0^T \int_{\mathbb{R}} G_n(x, u_n(x))\, \varphi(t, x)\mathrm{d}t\mathrm{d}x$$

$$= \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} G(x, w)\, \mathrm{d}\nu_x(w) \right) \varphi(t, x)\mathrm{d}t\mathrm{d}x. \tag{3.56}$$

The above assumptions ensure that G satisfies the hypotheses of Claim 2. Therefore,

$$\left| \int_0^T \int_{\mathbb{R}} \left(G_n(x, u_n(x))\, \varphi(t, x) - \left(\int_{\mathbb{R}} G(x, w)\, \mathrm{d}\nu_x(w) \right) \right) \mathrm{d}t\mathrm{d}x \right|$$

$$\leq \left| \int_0^T \int_{\mathbb{R}} \left(G_n(x, u_n(x))\, \varphi(t, x) - G(x, u_n(x))\, \varphi(t, x) \right) \mathrm{d}t\mathrm{d}x \right|$$

$$+ \left| \int_0^T \int_{\mathbb{R}} \left(G(x, u_n(x))\, \varphi(t, x) - \left(\int_{\mathbb{R}} G(x, w)\, \mathrm{d}\nu_x(w) \right) \right) \mathrm{d}t\mathrm{d}x \right|$$

$$\leq \|G_n - G\|_{\mathbf{L}^\infty([-X,X]\times[U,V];\mathbb{R})} \, \|\varphi\|_{\mathbf{L}^1([0,T]\times\mathbb{R};\mathbb{R})}$$
$$+ \left| \int_0^T \int_{\mathbb{R}} \left(G\,(x, u_n(x))\, \varphi(t, x) - \left(\int_{\mathbb{R}} G(x, w)\, \mathrm{d}\nu_x(w) \right) \right) \mathrm{d}t\,\mathrm{d}x \right|$$
$$\underset{n\to+\infty}{\longrightarrow} 0$$

where we used (3.51) and Claim 2, completing the proof of Claim 3.   ✓

Claim 4: For any entropy $E \in \mathbf{C}^2(\mathbb{R}; \mathbb{R})$, there exists a set $\Omega_E \subseteq \mathbb{R}$ such that $\mathbb{R} \setminus \Omega_E$ is negligible and for all $x \in \Omega_E$

$$\int_{\mathbb{R}} (w\, F(x, w) - E(w)\, H(x, w))\, \mathrm{d}\nu_x(w)$$
$$= \int_{\mathbb{R}} w\,\mathrm{d}\nu_x(w) \int_{\mathbb{R}} F(x, w)\,\mathrm{d}\nu_x(w) - \int_{\mathbb{R}} E(w)\,\mathrm{d}\nu_x(w) \int_{\mathbb{R}} H(x, w)\,\mathrm{d}\nu_x(w) \tag{3.57}$$

where F is an entropy flux corresponding to E with respect to H, according to Definition 2.3.

Consider the vector fields

$$V_n(t, x) := \begin{bmatrix} u_n(x) \\ H_n\,(x, u_n(x)) \end{bmatrix} \qquad W_n(t, x) := \begin{bmatrix} F_n\,(x, u_n(x)) \\ -E\,(x, u_n(x)) \end{bmatrix}$$

and assume preliminarily that E is convex. Call F_n the flux corresponding to E with respect to H_n as defined by (3.52).

Fix an arbitrary $R > 0$. In the present stationary situation, $\nabla \cdot V_n$ vanishes. Moreover, by Claim 1, $\nabla \wedge W_n$ lies in a relatively compact subset of $\mathbf{H}^{-1}([0, T] \times [-R, R]; \mathbb{R})$. By the div–curl Lemma [16, Theorem 17.2.1], we have

$$\lim_{n\to+\infty} (V_n \cdot W_n) = \left(\lim_{n\to+\infty} V_n \right) \cdot \left(\lim_{n\to+\infty} W_n \right). \tag{3.58}$$

More precisely, applying (3.56) to the sequences $G_n(x, u) = u\, F_n(x, u) - E(u)\, H_n(x, u)$, $G_n(x, u) = u$, $G_n(x, u) = H_n(x, u)$, $G_n(x, u) = F_n(x, u)$ and $G_n(x, u) = E(x, u)$, the following limits hold in the sense of distributions over $[0, T] \times [-R, R]$, the functions being understood in $\mathbf{L}^2([0, T] \times [-R, R]; \mathbb{R})$, so that their products are in $\mathbf{L}^1([0, T] \times [-R, R]; \mathbb{R})$:

$$\lim_{n\to+\infty} (V_n \cdot W_n) : (t, x) \mapsto \int_{\mathbb{R}} (w\, F(x, w) - E(w)\, H(x, w))\, \mathrm{d}\nu_x(w)$$

$$\lim_{n\to+\infty} V_n(t, x) : (t, x) \mapsto \begin{bmatrix} \int_{\mathbb{R}} w\,\mathrm{d}\nu_x(w) \\ \int_{\mathbb{R}} H(x, w)\,\mathrm{d}\nu_x(w) \end{bmatrix}$$

$$\lim_{n\to+\infty} W_n(t, x) : (t, x) \mapsto \begin{bmatrix} \int_{\mathbb{R}} F(x, w)\,\mathrm{d}\nu_x(w) \\ - \int_{\mathbb{R}} E(w)\,\mathrm{d}\nu_x(w) \end{bmatrix}$$

where F is an entropy flux corresponding to E with respect to H. Since R is arbitrary, equality (3.58) ensures that (3.57) is proved in the case of a convex entropy for all $(t, x) \in \hat{\Omega}_E$, for a set $\hat{\Omega}_E$ such that $([0, T] \times \mathbb{R}) \setminus \hat{\Omega}_E$ is negligible.

Note that equality (3.57) is independent of time and $([0, T] \times \mathbb{R}) \setminus \hat{\Omega}_E$ is negligible, hence we may assume that (3.57) holds for all $x \in \Omega_E$, where $\mathbb{R} \setminus \Omega_E$ is negligible. Claim 4 is proved in the case of a convex entropy.

Assume now that E is not necessarily convex. Then, we can introduce two convex functions E_+, E_- of class $\mathbf{C}^2(\mathbb{R}; \mathbb{R})$ such that

$$\forall\, w \in \mathbb{R} \quad \begin{array}{l} E''_+(w) := \max\{E''(w), 0\} \\ E''_-(w) := \max\{-E''(w), 0\} \end{array} \quad \text{and} \quad E(w) := E_+(w) - E_-(w)$$

These functions are not uniquely defined, since adding/subtracting affine functions of w does not alter the validity of the latter requirements. Repeating the argument above, for all $x \in \Omega_{E_+} \cap \Omega_{E_-}$, equality (3.57) holds also for the not necessarily convex entropy E, the set $\mathbb{R} \setminus (\Omega_{E_+} \cap \Omega_{E_-})$ being negligible. Claim 4 is proved.   ✓

Call $\mathcal{E}$ the countable set of all polynomials with rational coefficients and define

$$\Omega := \bigcap_{E \in \mathcal{E}} \Omega_E. \tag{3.59}$$

Claim 5: The set Ω is such that $\mathbb{R} \setminus \Omega$ is negligible and for all $E \in \mathbf{C}^0(\mathbb{R}; \mathbb{R})$ and for all $x \in \Omega$, equality (3.57) holds, where F^k is given by (2.6), for any $k \in \mathbb{R}$..

For any $E \in \mathcal{E}$ and for all $x \in \Omega$, by Claim 4 equality (3.57) holds, $\mathbb{R} \setminus \Omega$ being negligible.

Let now $E \in \mathbf{C}^0(\mathbb{R}; \mathbb{R})$ be fixed. By the classical Stone–Weierstrass Theorem [22, Corollary 7.31], there exists a sequence E_n in $\mathcal{E}$ converging to E uniformly on $[U, V]$. Clearly, the sequence of fluxes F^k_n corresponding to E_n defined by (2.6) converges to the flux F^k, also defined by (2.6). Since (3.57) holds in Ω for each pair (E_n, F^k_n), it also holds for (E, F^k). By the arbitrariness of E, Claim 5 is proved.   ✓

Define for all $x \in \mathbb{R}$

$$u(x) := \int_{\mathbb{R}} w \; \mathrm{d}\nu_x(w). \tag{3.60}$$

Claim 6: With reference to (3.59) and (3.60), for all $x \in \Omega$,

$$\int_{\mathbb{R}} H(x, w) \, \mathrm{d}\nu_x(w) = H(x, u(x)). \tag{3.61}$$

Set for $\xi \in \Omega$, $E(w) := |w - u(\xi)|$ so that by Definition 2.3 $F^{u(\xi)}(x, w) :=$ $\mathrm{sgn}\,(w - u(\xi))\,(H(x, w) - H(x, u(\xi)))$, see also (2.6). By Claim 5, using (2.1), we get that for all $x \in \Omega$

$$\int_{\mathbb{R}} (w\, \Phi\,(x, w, u(\xi)) - H(x, w)\, |w - u(\xi)|)\, \mathrm{d}\nu_x(w)$$

$$= u(x) \int_{\mathbb{R}} \Phi\,(x, w, u(\xi))\, \mathrm{d}\nu_x(w) - \int_{\mathbb{R}} H(x, w)\mathrm{d}\nu_x(w) \int_{\mathbb{R}} |w - u(\xi)|\mathrm{d}\nu_x(w).$$

Rearranging the terms, one gets

$$\int_{\mathbb{R}} [(w - u(x)) \ \Phi(x, w, u(\xi)) - H(x, w) |w - u(\xi)|] \, \mathrm{d}\nu_x(w)$$

$$= -\int_{\mathbb{R}} H(x, w) \mathrm{d}\nu_x(w) \int_{\mathbb{R}} |w - u(\xi)| \mathrm{d}\nu_x(w).$$

Choose $x = \xi$, use (2.1) to get $\left(\int_{\mathbb{R}} H(\xi, w) \, \mathrm{d}\nu_\xi(w) - H(\xi, u(\xi))\right) \int_{\mathbb{R}} |w - u(\xi)|$ $\mathrm{d}\nu_\xi(w) = 0$. Either the first factor vanishes, or ν_ξ is Dirac delta at $u(\xi)$. In both cases, using (3.60) and the arbitrariness of ξ, Claim 6 is proved. ✓

Claim 7: The sequence u_n converges to u, as defined in (3.60), a.e. in $\mathbb{R}$.

(The content of this step is heavily inspired by [24, Section 5.4]).

From Claim 5 and from (3.61) in Claim 6, we obtain that for all $x \in \Omega$, as defined in (3.59), and for all $E \in \mathbf{C}^0(\mathbb{R}; \mathbb{R})$

$$\int_{\mathbb{R}} [(w - u(x)) F(x, w) - (H(x, w) - H(x, u(x))) E(w)] \, \mathrm{d}\nu_x(w) = 0$$
(3.62)

where F is as in (2.6), for any k. For a.e. $x \in \mathbb{R}$, ν_x is a probability measure and the maps $w \mapsto w - u(x)$, $w \mapsto H(x, w) - H(x, u(x))$ are sufficiently regular to ensure that the set functions

$$\alpha_x(S) := \int_S (w - u(x)) \, \mathrm{d}\nu_x(w) \quad \text{and} \quad \beta_x(S)$$

$$:= \int_S (H(x, w) - H(x, u(x))) \, \mathrm{d}\nu_x(w)$$

(S being any Borel set) are finite Radon measures. Hence, the two maps

$$A_x(v) := \alpha_x(] - \infty, v]) \quad \text{and} \quad B_x(v) := \beta_x(] - \infty, v])$$
(3.63)

are in $\mathbf{BV}(\mathbb{R}; \mathbb{R})$. Since spt $\nu_x \subseteq [U, V]$, then $A_x(v)$ and $B_x(v)$ vanish for $v < U$ and attain a constant value for $v > V$. Moreover, (3.60) implies that $\alpha_x(\mathbb{R}) = 0$ while (3.61) in Claim 6 implies that $\beta_x(\mathbb{R}) = 0$. Therefore, for all $x \in \mathbb{R}$, both A_x and B_x are supported in $[U, V]$. An integration by parts, see [21, Theorem B] (in particular the remark at the bottom of [21, p. 422]), then ensures that from equality (3.62) we can deduce

$$\int_{\mathbb{R}} A_x(w) \ \partial_w F(x, w) \, \mathrm{d}w = \int_{\mathbb{R}} B_x(w) \ E'(w) \, \mathrm{d}w$$

and therefore

$$\int_{\mathbb{R}} E'(w) \ \partial_w H(x, w) \ A_x(w) \, \mathrm{d}w = \int_{\mathbb{R}} E'(w) \ B_x(w) \, \mathrm{d}w.$$

In the above equality, E can be any $\mathbf{C}^1$ function, E' can be any continuous function, hence

$$\partial_w H(x, w) \, A_x(w) = B_x(w) \qquad \text{for a.e. } (x, w) \in \mathbb{R} \times \mathbb{R}. \tag{3.64}$$

Furthermore, we have that

$$(H(x, w) - H(x, u(x))) \, A_x(w) = (w - u(x)) \, B_x(w) \qquad \text{for a.e. } (x, w) \in \mathbb{R} \times \mathbb{R}. \tag{3.65}$$

Indeed, the two sides have the same distributional derivative in w by (3.64) and (3.63), while they clearly coincide when $w = u(x)$. Inserting (3.64) in (3.65), we have

$$(H(x, w) - H(x, u(x))) \, A_x(w) = (w - u(x)) \, \partial_w H(x, w) \, A_x(w)$$

Call $[a, b]$ the minimal (with respect to set inclusion) interval containing the support of ν_x and assume by contradiction that $a < b$. Note that $A_x(w) \neq 0$ for $w \in]a, b[$. Indeed, by the definition of $A_x(w)$ and since ν_x is nonnegative, the map $w \mapsto A_x(w)$ vanishes for $w < a$, weakly decreases for $w \in]a, u(x)[$, weakly increases for $w \in]u(x), b[$, and then vanishes for $w > b$. The minimality of $[a, b]$ ensures that A_x is nonzero in both a right neighborhood of a and a left neighborhood of b. Simplifying, we thus obtain

$$(H(x, w) - H(x, u(x))) = (w - u(x)) \, \partial_w H(x, w) \quad \text{for all } w \in]a, b[\text{ and for a.e. } x \in \mathbb{R}.$$

The latter equality contradicts **(WGNL)** unless $a = b$, ensuring that, for a.e. $x \in \mathbb{R}$, ν_x is a Dirac measure, which in turn implies pointwise convergence up to a subsequence by (3.54), see [38, Proposition 9.1.7]. Claim 7 is proved. ✓

Conclusion.

By Claim 7, up to a subsequence, we have the pointwise a.e. convergence $u_n \to u$ as $n \to +\infty$. The $\mathbf{L}^\infty$ bound $u_n(x) \in [U, V]$ for a.e. $x \in \mathbb{R}$ allows to use the Dominated Convergence Theorem [22, Theorem (12.24)] in (2.7). By Proposition 2.4, we get that u is a weak entropy stationary solution (Definition 2.1) attaining values between U and V. This accomplishes the construction of u_+, that of u_- is entirely similar. The proof of Theorem 2.9 is completed. □

3.3. Vanishing viscosity approximations

Proof of Theorem 2.11. Let u be a classical solution to (2.20) on I. Clearly, U as defined by (2.24) satisfies (2.23), simple computations yield $U(0, x) = U_o(x)$ and

$$\partial_t U(t, x) + H(x, \partial_x U(t, x))$$
$$= \int_{x_o}^x \partial_t u(t, \xi) d\xi - H(x_o, u(t, x_o)) + \varepsilon \, \partial_x u(t, x_o) + H(x, u(t, x))$$

$$= \int_{x_o}^{x} \left(-\partial_x H\left(\xi, u(t, \xi)\right) + \varepsilon\, \partial_{xx}^2 u(t, \xi) \right) d\xi$$
$$- H\left(x_o, u(t, x_o)\right) + \varepsilon\, \partial_x u(t, x_o) + H\left(x, u(t, x)\right)$$
$$= -H\left(x, u(t, x)\right) + H\left(x_o, u(t, x_o)\right) + \varepsilon\, \partial_x u(t, x) - \varepsilon\, \partial_x u(t, x_o)$$
$$- H\left(x_o, u(t, x_o)\right) + \varepsilon\, \partial_x u(t, x_o) + H\left(x, u(t, x)\right)$$
$$= \varepsilon\, \partial_{xx}^2 U(t, x),$$

thus U is a classical solution to (2.21) on I, proving Item (1). Verifying Item (2) is immediate, completing the proof of Theorem 2.11. $\qquad\square$

Lemma 3.6. *Fix T, $\varepsilon > 0$. Let H satisfy* **(C3)** *and* **(CNH)**. *Assume there exist bounded classical solutions u^- on $]-\infty, -X[$ and u^+ on $]X, +\infty[$ to (2.20). Then, setting $\Omega_X^- = (\{0\}\times]-\infty, -X]) \cup ([0, T] \times \{-X\})$ and $\Omega_X^+ = (\{0\} \times [X, +\infty[) \cup ([0, T] \times \{X\})$,*

$$\sup\nolimits_{\Omega_X^-} \left|u^-\right| = \max\left\{\sup\nolimits_{x \le -X} |u_o(x)|,\ \sup\nolimits_{t \in [0, T]} |u(t, -X)|\right\};$$
$$\sup\nolimits_{\Omega_X^+} \left|u^+\right| = \max\left\{\sup\nolimits_{x \ge X} |u_o(x)|,\ \sup\nolimits_{t \in [0, T]} |u(t, X)|\right\}. \tag{3.66}$$

Proof of Lemma 3.6. We consider only the case of maxima of u^+ in $[X, +\infty[$, the same procedure applies to u^- in $]-\infty, -X[$, while straightforward sign changes apply to the case of a minimum. We follow the general lines of [23, Theorem B.1] and [24, Chapter III].

For $\eta \in]0, 1[$, define

$$v_\eta : [0, T] \times \quad\quad\quad [X, +\infty[\quad\quad\quad \to \mathbb{R}$$
$$(t, x) \mapsto u(t, x) - \eta\left(2\,\varepsilon\,t + \tfrac{1}{2}\,(\eta\,x)^2\right). \tag{3.67}$$

By the boundedness assumption on u^+, it follows that v_η attains its global maximum at a point $(t_\eta, x_\eta) \in [0, T] \times [X, +\infty[$. Three possible cases are in order.

Case 1: $t_\eta = 0$ **and** $x_\eta \ge X$.

For all $(t, x) \in [0, T] \times [X, +\infty[$ we have

$$v_\eta(t, x) \le v_\eta(t_\eta, x_\eta) = v_\eta(0, x_\eta) = u(0, x_\eta) - \frac{1}{2}\,\eta^3\,x_\eta^{\ 2} \le u(0, x_\eta) \le \sup_{\xi \ge X} u(0, \xi)$$

so that

$$u(t, x) = v_\eta(t, x) + \eta\left(2\,\varepsilon\,t + \frac{1}{2}(\eta\,x)^2\right)$$
$$\le \sup_{\xi \ge X} u(0, \xi) + \eta\left(2\,\varepsilon\,t + \frac{1}{2}(\eta\,x)^2\right). \tag{3.68}$$

$$\checkmark$$

Case 2: $t_\eta \in [0, T]$ **and** $x_\eta = X$.

For all $(t, x) \in [0, T] \times [X, +\infty[$ we have

$$v_\eta(t, x) \le v_\eta(t_\eta, x_\eta) = v_\eta(t_\eta, X) = u(t_\eta, x) - \eta \left(2\varepsilon t + \frac{1}{2} (\eta X)^2 \right)$$

$$\le u(t_\eta, X) \le \sup_{\tau \in [0,T]} u(\tau, X)$$

so that

$$u(t, x) = v_\eta(t, x) + \eta \left(2\varepsilon t + \frac{1}{2} (\eta x)^2 \right)$$

$$\le \sup_{\tau \in [0,T]} u(\tau, X) + \eta \left(2\varepsilon t + \frac{1}{2} (\eta x)^2 \right). \tag{3.69}$$

$\checkmark$

Case 3: $t_\eta \in]0, T]$ and $x_\eta > X$.

Then, by the choice of (t_η, x_η), $\partial_t v_\eta(t_\eta, x_\eta) \ge 0$, $\partial_x v_\eta(t_\eta, x_\eta) = 0$ and $\partial_{xx}^2 v_\eta(t_\eta, x_\eta) \le 0$. Equivalently, $\partial_t u(t_\eta, x_\eta) \ge 2\varepsilon \eta$, $\partial_x u(t_\eta, x_h) = \eta^3 x_\eta$ and $\partial_{xx}^2 u(t_\eta, x_\eta) \le \eta^3$. Hence, using **(CNH)**, $\partial_x H\left(x_\eta, u(t_\eta, x_\eta) \right) = 0$ and

$$\left(\partial_t u + \partial_x (H(x, u)) - \varepsilon \partial_{xx}^2 u \right)_{|t=t_\eta, x=x_\eta}$$

$$= \partial_t u(t_\eta, x_\eta) + \partial_x H\left(x_\eta, u(t_\eta, x_\eta) \right)$$

$$+ \partial_u H\left(x_\eta, u(t_\eta, x_\eta) \right) \partial_x u(t_\eta, x_\eta) - \varepsilon \partial_{xx}^2 u(t_\eta, x_\eta)$$

$$\ge 2\varepsilon \eta + \partial_u H\left(x_\eta, u(t_\eta, x_\eta) \right) \eta^3 x_\eta - \varepsilon \eta^3$$

$$\ge 2\varepsilon \eta - \eta^3 |x_\eta| \sup_{|v| \le \|u\|_{\mathbf{L}^\infty([0,T]\times]X,+\infty[;\mathbb{R})}} \left| \partial_u H(x_\eta, v) \right| - \varepsilon \eta^3. \tag{3.70}$$

To obtain a strictly positive lower bound for the right hand side (3.70), recall that $v_\eta(t_\eta, x_\eta) \ge v_\eta(X, 0)$ which, together with (3.67), implies that

$$\frac{1}{2} \eta^3 |x_\eta|^2 \le u(t_\eta, x_\eta) - u(X, 0) - 2\varepsilon t_\eta + \frac{1}{2} \eta^3 X^2$$

$$\le 2 \|u\|_{\mathbf{L}^\infty([0,T]\times]X,+\infty[;\mathbb{R})} + \frac{1}{2} \eta^3 X^2$$

whence

$$\eta^{3/2} |x_\eta| \le \sqrt{4 \|u\|_{\mathbf{L}^\infty([0,T]\times]X,+\infty[;\mathbb{R})} + \eta^3 X^2}. \tag{3.71}$$

Use now (3.71) in (3.70) and **(CNH)** to obtain

$$\left(\partial_t u + \partial_x H(x, u) - \varepsilon \partial_{xx}^2 u \right)_{|t=t_\eta, x=x_\eta}$$

$$\ge 2\varepsilon \eta - \eta^{3/2} \sqrt{4 \|u\|_{\mathbf{L}^\infty([0,T]\times]X,+\infty[;\mathbb{R})} + \eta^3 X^2}$$

$$\sup_{|v| \le \|u\|_{\mathbf{L}^\infty([0,T]\times]X,+\infty[;\mathbb{R})}} \left| \partial_u H(X, v) \right| - \varepsilon \eta^3$$

showing that $\left(\partial_t u + \partial_x H(x, u) - \varepsilon\, \partial^2_{xx} u\right)_{|t=t_\eta, x=x_\eta} > 0$ for all sufficiently small $\eta > 0$. This contradicts the choice of u, hence Case 3 is not to be considered.   ✓

From (3.68) and (3.69), we thus obtain that for all $(t, x) \in [0, T] \times [X, +\infty[$ and $\eta \in\,]0, 1[$,

$$u(t, x) \le \max\left\{\sup_{\xi \ge X} u(0, \xi), \quad \sup_{\tau \in [0,T]} u(\tau, X)\right\} + \eta\left(2\,\varepsilon\, t + \frac{1}{2}(\eta\, x)^2\right).$$

Passing to the limit $\eta \to 0$, we complete the proof of Lemma 3.6. $\square$

Corollary 3.7. *Fix $T, \varepsilon > 0$. Let* **(C3)** *and* **(CNH)** *hold. Choose a bounded initial datum $u_o \in \mathbf{C}^0(\mathbb{R}; \mathbb{R})$. Assume there exists a bounded classical solution u to* (2.20) *on $\mathbb{R}$. Then,*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |u(t, x)| = \max\left\{\sup_{x \in \mathbb{R}} |u_o(x)|, \quad \sup_{[0,T] \times [-X, X]} |u(t, x)|\right\}.$$

Proof. Define u^+, respectively, u^-, the restriction of u to $[0, T] \times\,]-\infty, -X[$, respectively, $[0, T] \times\,]X, +\infty[$. Apply Lemma 3.6 to complete the proof. $\square$

Corollary 3.8. *Fix $T, \varepsilon > 0$. Let H satisfy* **(C3)** *and* **(CNH)**. *Choose an initial datum $U_o \in \mathbf{C}^1(\mathbb{R}; \mathbb{R})$. Assume there exists a classical solution U to* (2.21) *on $\mathbb{R}$ which is also Lipschitz continuous. Then,*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |\partial_x U(t, x)| = \max\left\{\sup_{x \in \mathbb{R}} \left|U'_o(x)\right|, \quad \sup_{(t,x) \in [0,T] \times [-X, X]} |\partial_x U(t, x)|\right\}.$$

Proof. By Theorem 2.11, with $I = \mathbb{R}$, it is sufficient to apply Corollary 3.7 to $\partial_x U$. $\square$

Proof of Theorem 2.12. Define the ε *independent* quantity

$$A := \left\|U''_o\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} + \sup_{\substack{|\xi| \le X \\ |p| \le \|U'_o\|_{\mathbf{L}^\infty}}} |H(\xi, p)|. \tag{3.72}$$

Claim 1: The following bound on $\partial_t U$ holds uniformly in ε:

$$\|\partial_t U\|_{\mathbf{L}^\infty([0,T] \times \mathbb{R};\mathbb{R})} \le A. \tag{3.73}$$

The function $\varphi := \partial_t U$ is a classical solution to the linear parabolic Cauchy problem

$$\begin{cases} \partial_t \varphi + \partial_u H(x, \partial_x U)\, \partial_x \varphi = \varepsilon\, \partial^2_{xx} \varphi \\ \varphi(0, x) = \partial_t U(0, x). \end{cases}$$

The standard comparison principle, see, e.g., [18, Theorem 8, § 7.1.4], ensure for $t \in [0, T[$ the bound $\varphi(t, x) \in [\inf_{\xi \in \mathbb{R}} \varphi(0, \xi), \sup_{\xi \in \mathbb{R}} \varphi(0, \xi)]$ and, equivalently,

$$\partial_t U(t, x) \in [\inf_{\xi \in \mathbb{R}} \partial_t U(0, \xi), \sup_{\xi \in \mathbb{R}} \partial_t U(0, \xi)] \qquad \text{for all } (t, x) \in [0, T[\times \mathbb{R}.$$

$$(3.74)$$

Introduce

$$\psi^{\pm} : [0, T] \times \mathbb{R} \to \qquad \mathbb{R} \qquad\qquad (3.75)$$
$$(t, x) \quad \mapsto U_o(x) \pm A\, t.$$

so that $\psi^{\pm}(0, x) = U_o'(x)$. Moreover, since $\varepsilon \in]0, 1]$,

$$\partial_t \psi^+ + H(x, \partial_x \psi^+) - \varepsilon\, \partial_{xx}^2 \psi^+ = A + H\left(x, U_o'(x)\right) - \varepsilon\, U_o''(x)$$
$$\geq A + H\left(x, U_o'(x)\right) - \left| U_o''(x) \right| \geq 0$$
$$\partial_t \psi^- + H(x, \partial_x \psi^-) - \varepsilon\, \partial_{xx}^2 \psi^- = -A + H\left(x, U_o'(x)\right) - \varepsilon\, U_o''(x)$$
$$\leq -A + H\left(x, U_o'(x)\right) + \left| U_o''(x) \right| \leq 0$$

proving by (3.72) that ψ^+, respectively, ψ^- is a supersolution, respectively, a subsolution to (2.21), so that the standard comparison principle for regular functions, see for instance [35, Proposition 52.6], yields $\psi^- \leq U \leq \psi^+$. By (3.75), $-A \leq \frac{1}{t}\left(U(t, x) - U_o(x)\right) \leq A$ and in the limit $t \to 0+$ we obtain $\|\partial_t U(0)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \leq A$ which, together with (3.74) completes the proof of Claim 1. $\qquad\checkmark$

Claim 2: For all $\eta \in \mathbf{C}^2(\mathbb{R}; \mathbb{R})$ with $\eta'' > 0$, define $\omega(t, x) := \eta\left(\partial_x U(t, x)\right)$. If arg max $\omega \neq \emptyset$, then for any $(t^*, x^*) \in$ arg max ω with $t^* > 0$,

$$\left(H\left(x^*, \partial_x U(t^*, x^*)\right)\right)^2 \leq \frac{\left(\partial_t U(t^*, x^*)\right)^2}{1 - \varepsilon}$$
$$- \frac{\eta'\left(\partial_x U(t^*, x^*)\right)}{\eta''\left(\partial_x U(t^*, x^*)\right)} \partial_x H\left(x^*, \partial_x U(t^*, x^*)\right).$$

$$(3.76)$$

Since U is a solution to (2.21) in the sense of Definition 2.10, we can compute:

$$\partial_t \omega(t, x) = \eta'\left(\partial_x U(t, x)\right)\, \partial_{tx}^2 U(t, x)$$
$$\partial_x \omega(t, x) = \eta'\left(\partial_x U(t, x)\right)\, \partial_{xx}^2 U(t, x)$$
$$\partial_{xx}^2 \omega(t, x) = \eta''\left(\partial_x U(t, x)\right)\left(\partial_{xx}^2 U(t, x)\right)^2 + \eta'\left(\partial_x U(t, x)\right)\, \partial_{xxx}^3 U(t, x)$$
$$\partial_t \omega(t, x) = \varepsilon\left(\partial_{xx}^2 \omega(t, x) - \eta''\left(\partial_x U(t, x)\right)\left(\partial_{xx}^2 U(t, x)\right)^2\right)$$
$$- \eta'\left(\partial_x U(t, x)\right)\, \partial_x H\left(x, \partial_x U(t, x)\right) - \partial_u H\left(x, \partial_x U(t, x)\right)\, \partial_x \omega(t, x)$$

where we used (2.21) to get to the last equality. Therefore,

$$\partial_t \omega(t, x) + \partial_u H (x, \partial_x U(t, x)) \, \partial_x \omega(t, x) - \varepsilon \, \partial^2_{xx} \omega(t, x)$$
$$= -\eta' (\partial_x U(t, x)) \, \partial_x H (x, \partial_x U(t, x))$$
$$- \frac{1}{\varepsilon} \, \eta'' (\partial_x U(t, x)) \, (\partial_t U(t, x) + H (x, \partial_x U(t, x)))^2 \,.$$

Use the inequality $(a + b)^2 \geq (1 - \alpha) \, a^2 + \left(1 - \frac{1}{\alpha}\right) b^2$, that holds for $a, b \in \mathbb{R}$ and $\alpha > 0$ with $a = \partial_t U(t, x)$, $b = H (x, \partial_x U(t, x))$ and $\alpha = 1/(1 - \varepsilon)$ to get, by the convexity hypothesis on η,

$$\partial_t \omega(t, x) + \partial_u H (x, \partial_x U(t, x)) \, \partial_x \omega(t, x) - \varepsilon \, \partial^2_{xx} \omega(t, x)$$
$$\leq -\eta' (\partial_x U(t, x)) \, \partial_x H (x, \partial_x U(t, x))$$
$$- \frac{1}{\varepsilon} \, \eta'' (\partial_x U(t, x)) \left(-\frac{\varepsilon}{1 - \varepsilon} \, (\partial_t U(t, x))^2 + \varepsilon \, (H (x, \partial_x U(t, x)))^2\right).$$

Compute the above terms at (t^*, x^*), where $\partial_x \omega(t^*, x^*) = 0$, $\partial_t \omega(t^*, x^*) \geq 0$ and $\partial^2_{xx} \omega(t^*, x^*) \leq 0$ to obtain (3.76). Claim 2 is proved.   ✓

Claim 3: There exists a constant B such that for all $\varepsilon \in]0, 1/2[$ and for all $T \in \mathbb{R}_+$,

$$\|\partial_x U\|_{\mathbf{L}^\infty([0, T] \times \mathbb{R}; \mathbb{R})} \leq B. \tag{3.77}$$

By means of a function

$$r \in \mathbf{C}^1(\mathbb{R}; \mathbb{R}), \quad r \text{ even}, \quad r'(v) \geq 0 \text{ for } v \in \mathbb{R}_+ \quad \text{and} \quad r(v) \geq \sup_{\substack{x \in \mathbb{R} \\ |u| \leq |v|}} |\partial_x H(x, u)|. \tag{3.78}$$

define the maps ϑ and η on all $\mathbb{R}$ so that

$$\begin{cases} \vartheta'(v) = v \, (1 + r(v)) \\ \vartheta(0) = 0 \end{cases} \quad \text{and} \quad \eta(v) := \exp \vartheta(v). \tag{3.79}$$

Note that ϑ is even, hence also η is. We also have

$$\eta'(v) = \vartheta'(v) \, \exp \vartheta(v)$$
$$\eta''(v) = \left((\vartheta'(v))^2 + \vartheta''(v)\right) \exp \vartheta(v) \quad \text{and} \quad \vartheta''(v) = 1 + r(v) + v \, r'(v) \geq 1. \tag{3.80}$$

Hence, η satisfies the assumptions of Claim 2. By Corollary 3.8, we have 2 cases.

Case 1: $\sup_{(t, x) \in [0, T] \times \mathbb{R}} |\partial_x U(t, x)| > \sup_{x \in \mathbb{R}} |U'_o(x)|$.
   Then, by Corollary 3.8 and **(C3)**.

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}} |\partial_x U(t, x)| = \sup_{(t, x) \in [0, T] \times [-X, X]} |\partial_x U(t, x)| = \max_{(t, x) \in [0, T] \times [-X, X]} |\partial_x U(t, x)|.$$

Hence, $\arg\max_{[0,T]\times\mathbb{R}} |\partial_x U|$ is non-empty. If $(t^*, x^*) \in \arg\max_{[0,T]\times\mathbb{R}} |\partial_x U|$, then $t^* > 0$. Moreover, η is convex and even, so that (t^*, x^*) is also a point of maximum of ω, as defined in Claim 2.

By Claim 1, Claim 2 and (3.78)–(3.79)–(3.80), setting $v^* = \partial_x U(t^*, x^*)$ and, for $\varepsilon < 1/2$,

$$
\begin{aligned}
\left(H(x^*, v^*)\right)^2 &\leq \frac{A^2}{1-\varepsilon} - \frac{\vartheta'(v^*)}{(\vartheta'(v^*))^2 + \vartheta''(v^*)} \partial_x H(x^*, v^*) \\
&\leq 2A^2 + \left| \frac{\vartheta'(v^*)}{(\vartheta'(v^*))^2 + \vartheta''(v^*)} r(v^*) \right| \\
&\leq 2A^2 + \frac{|v^*|\,(1 + r(v^*))\,r(v^*)}{1 + (v^*\,(1 + r(v^*)))^2} \\
&\leq 2A^2 + \frac{|v^*|\,(1 + r(v^*))^2}{1 + (v^*\,(1 + r(v^*)))^2} \\
&\leq 2A^2 + \begin{cases} 1 & \text{for } |v^*| \geq 1 \\ (1 + r(1))^2 & \text{for } |v^*| \leq 1 \end{cases}
\end{aligned}
$$

and the latter bound above is uniform in ε and T, so that we obtained $(H(x^*, v^*))^2 \leq 2A^2 + \max\left\{1, (1 + r(1))^2\right\} \leq 2A^2 + (1 + r(1))^2$. Proceed as follows:

$$
\begin{aligned}
\left|\partial_x U(t^*, x^*)\right| &\leq \mathcal{U}_{\sqrt{2A^2 + (1 + r(1))^2}} && \text{[With the notation (UC)]} \\
\left|\partial_x U(t, x)\right| &\leq \mathcal{U}_{\sqrt{2A^2 + (1 + r(1))^2}} && \text{[By the choice of } (t^*, x^*)\text{]}
\end{aligned}
$$

Claim 3 is proved in Case 1 with

$$
B := \mathcal{U}_{\sqrt{2A^2 + (1 + r(1))^2}}. \tag{3.81}
$$

Case 2: $\sup_{(t,x)\in[0,T]\times\mathbb{R}} |\partial_x U(t, x)| = \sup_{x\in\mathbb{R}} |U_o'(x)|$.

By the Definition (3.72) of A, we have $\left|H\left(x, U_o'(x)\right)\right| \leq A \leq \sqrt{2A^2 + (1 + r(1))^2}$ for any $x \in \mathbb{R}$. Thus, by (UC) and by the Definition (3.81) of B, we have that $\left|U_o'(x)\right| \leq B$ for any $x \in \mathbb{R}$. Hence, finally, $\sup_{(t,x)\in[0,T]\times\mathbb{R}} |\partial_x U(t, x)| = \sup_{x\in\mathbb{R}} \left|U_o'(x)\right| \leq B$, proving Claim 3 also in Case 2 and completing the proof of Theorem 2.12. $\qquad\square$

Recall, see, e.g., [24, § 3.2], for $\varepsilon \in\,]0, 1[$ the heat kernel and its basic properties

$$
\begin{aligned}
\mathcal{H}_\varepsilon : \mathbb{R}_+ \times \mathbb{R} &\to \quad \mathbb{R} \\
(t, x) &\mapsto \frac{e^{-x^2/(4\varepsilon t)}}{\sqrt{4\pi\,\varepsilon\,t}}
\end{aligned}
\qquad
\begin{aligned}
&\int_{\mathbb{R}} \mathcal{H}_\varepsilon(t, x)\mathrm{d}x = 1; \\
&\int_{\mathbb{R}} |\partial_x \mathcal{H}_\varepsilon(t, x)|\mathrm{d}x = 1/\sqrt{\pi\,\varepsilon\,t}\,.
\end{aligned}
\tag{3.82}
$$

Below, we obtain the well-posedness of the parabolic approximations (2.20) and (2.21), first in the conservation law case.

Proof of Theorem 2.14. Throughout this proof, we keep ε fixed and omit it.

Claim 1: Problem (2.20) **admits a local solution in the sense of Definition** 2.10.

Let T be positive and introduce the linear map $\mathcal{F} \colon \mathcal{V} \to \mathcal{V}$ where

$$\mathcal{V} := \left\{ v \in \mathbf{C}^0([0, T] \times \mathbb{R}) \colon \sup_{(t,x) \in [0,T] \times \mathbb{R}} |v(t, x) - (\mathcal{H}_\varepsilon(t) * u_o)(x)| \leq 1 \right\}$$

$$(\mathcal{F}v)(t, x) := (\mathcal{H}_\varepsilon(t) * u_o)(x) - \int_0^t \int_{\mathbb{R}} \partial_x \mathcal{H}_\varepsilon(t - \tau, x - \xi) \, H(\xi, v(\tau, \xi)) \, \mathrm{d}\xi \mathrm{d}\tau.$$

(3.83)

We now choose T so that the Banach Fixed Point Theorem can be applied. Clearly, $\mathcal{V}$ is closed. It is also invariant with respect to $\mathcal{F}$. Indeed, using (3.82) one proves the continuity of $\mathcal{F}v$ and the estimate

$$|(\mathcal{F}v)(t, x) - (\mathcal{H}_\varepsilon(t) * u_o)(x)|$$

$$\leq \int_0^t \int_{\mathbb{R}} |\partial_x \mathcal{H}_\varepsilon(t - \tau, x - \xi) \, H(\xi, v(\tau, \xi))| \mathrm{d}\xi \mathrm{d}\tau$$

$$\leq \int_0^t \int_{\mathbb{R}} |\partial_x \mathcal{H}_\varepsilon(t - \tau, x - \xi)| \mathrm{d}\xi \mathrm{d}\tau \sup_{\substack{|\xi| \leq X \\ |w| \leq 1 + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}}} H(\xi, w)$$

$$= \int_0^t \frac{1}{\sqrt{\pi \varepsilon \tau}} \mathrm{d}\tau \sup_{\substack{|\xi| \leq X \\ |w| \leq 1 + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}}} H(\xi, w)$$

$$= \frac{2}{\sqrt{\pi \varepsilon}} \sqrt{T} \sup_{\substack{|\xi| \leq X \\ |w| \leq 1 + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}}} H(\xi, w).$$

Entirely similar estimates show that $\mathcal{F}$ is Lipschitz continuous:

$$|(\mathcal{F}v_2)(t, x) - (\mathcal{F}v_1)(t, x)|$$

$$\leq \int_0^t \int_{\mathbb{R}} |\partial_x \mathcal{H}_\varepsilon(t - \tau, x - \xi)| |H(\xi, v_2(\tau, \xi)) - H(\xi, v_1(\tau, \xi))| \mathrm{d}\xi \mathrm{d}\tau$$

$$\leq \frac{2}{\sqrt{\pi \varepsilon}} \sqrt{T} \sup_{\substack{|\xi| \leq X \\ |w| \leq 1 + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}}} |\partial_u H(\xi, w)| \; \|v_2 - v_1\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}.$$

Choosing T positive and such that

$$\frac{2\sqrt{T}}{\sqrt{\pi \varepsilon}} \sup_{\substack{|\xi| \leq X \\ |w| \leq 1 + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}}} H(\xi, w) \leq 1 \quad \text{and} \quad \frac{2\sqrt{T}}{\sqrt{\pi \varepsilon}} \sup_{\substack{|\xi| \leq X \\ |w| \leq 1 + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}}} |\partial_u H(\xi, w)| \leq \frac{1}{2},$$

(3.84)

an application of Banach Fixed Point Theorem ensures the existence of a map $u \in \mathcal{V}$ such that $u = \mathcal{F}u$, so that u solves (2.20), see, for instance, [23, Theorem B.1 and Lemma B.3] for a similar case. Claim 1 is proved. ✓

Below, we exploit the fact that (3.84) actually depends on u_o only through its $\mathbf{L}^\infty$ norm.

Claim 2: Problem (2.20) **admits a global solution.**

Introduce

$$T_m := \sup\{\tau > 0 \colon (2.20) \text{ admits a solution in the sense of Definition } 2.10 \text{ on }]0, \tau[\} .$$

By Claim 1, we know that T_m is well defined and that $T_m \geq T$ as defined in (3.84). We prove that $T_m = +\infty$ assuming that $T_m < +\infty$. Let C be the constant given by Corollary 2.13, which can be applied since u_o is actually required Lipschitz continuous. Fix $\tau > 0$ so that

$$\frac{2\sqrt{\tau}}{\sqrt{\pi}\,\varepsilon} \sup_{\xi \in \mathbb{R} |w| \leq C+1} |H(\xi, w)| \leq 1 \quad \text{and} \quad \frac{2\sqrt{\tau}}{\sqrt{\pi}\,\varepsilon} \sup_{\xi \in \mathbb{R} |w| \leq C+1} |\partial_u H(\xi, w)| \leq \frac{1}{2},$$

and note that $T_m \geq \tau$ by Step 1. Set $\tau_m = T_m - \tau/2$, so that $\tau_m \in]0, T_m[$. By the choice of T_m, there exists a solution u to (2.20) in the sense of Definition 2.10 on $]0, \tau_m[\times\mathbb{R}$ and by Corollary 2.13, $\|u\|_{\mathbf{L}^\infty([0,\tau_m]\times\mathbb{R};\mathbb{R})} \leq C$. Applying Claim 1, since $x \mapsto u(\tau_m, x) \in \mathbf{C}^2(\mathbb{R}; \mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$, we can construct a solution u_τ in the sense of Definition 2.10 to

$$\begin{cases} \partial_t v + \partial_x H(x, v) = \varepsilon\, \partial_{xx}^2 v \\ v(\tau_m, x) = u(\tau_m, x). \end{cases}$$

The concatenation

$$w(t, x) = \begin{cases} u(t, x) \text{ if } 0 \leq t \leq \tau_m \\ u_\tau(t, x) \text{ if } \tau_m < t \leq \tau_m + \tau \end{cases}$$

of classical solutions to (2.20) is, by construction, $\mathbf{C}^1$ in time. This implies that w solves (2.20) in the sense of Definition 2.10 on $]0, T_m + \tau/2[\times\mathbb{R}$, which contradicts the definition of T_m, completing the proof of Theorem 2.14. $\qquad\square$

3.4. Passing to the limit

Proof of Theorem 2.16. Passing to the convergence of vanishing viscosity approximations, the case of Hamilton–Jacobi equation is standard.

The existence of U_{ε_n} (for sufficiently large n) follows from Corollary 2.15. The bound (2.25) in Theorem 2.12 ensures that Ascoli–Arzelà Theorem [18, § C.7] can be applied on every compact subset of $\mathbb{R}_+ \times \mathbb{R}$. Use a diagonal argument to obtain U_* as the limit of a convergent subsequence. Clearly, U_* is Lipschitz continuous with the Lipschitz constant provided by (2.25). Proving that U_* satisfies Definition 2.7 is classical, we refer, for instance, to [4, Chapter 2] or [18, Chapter 10].

By Theorem 2.8, U_* is independent of the particular subsequence, hence the whole sequence U_{ε_n} converges to U_*. $\qquad\square$

Proof of Theorem 2.17. **Claim 1: The map $\varepsilon \mapsto \sqrt{\varepsilon} \, \partial_x u_\varepsilon$ is bounded in $\mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$.**

We now prove that for every positive T and R there is a constant $C_{T,R}$ such that $\left\| \sqrt{\varepsilon} \, \partial_x u_\varepsilon \right\|_{\mathbf{L}^2([0,T] \times [-R,R]; \mathbb{R})} \leq C_{T,R}$.

For all $\varphi \in \mathbf{C}^1_c(\mathbb{R}; \mathbb{R})$ and for all $t \in]0, T[$, by (2.20) we have

$$\int_\mathbb{R} \partial_t u_\varepsilon(t,x) \, \varphi(t,x) \, \mathrm{d}x + \varepsilon \int_\mathbb{R} \partial_x u_\varepsilon(t,x) \, \partial_x \varphi(t,x) \, \mathrm{d}x$$
$$= - \int_\mathbb{R} \partial_x \left(H\left(x, u_\varepsilon(t,x)\right)\right) \varphi(t,x) \, \mathrm{d}x. \tag{3.85}$$

Choose $\varphi(t,x) = u_\varepsilon(t,x) \, \psi_R(x)^2$ where $\psi_R \in \mathbf{C}^\infty_c(\mathbb{R}; \mathbb{R})$, $\psi_R(x) = 1$ for $x \in [-R, R]$, $\psi_R(x) = 0$ whenever $|x| > R+1$ and $\left| \psi'_R(x) \right| \leq 2$ for all $x \in \mathbb{R}$. By direct computations, using also (2.20), from (3.85) we get:

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\mathbb{R} \left(u_\varepsilon(t,x) \, \psi_R(x)\right)^2 \mathrm{d}x + \int_\mathbb{R} \left(\sqrt{\varepsilon} \, \partial_x \left(u_\varepsilon(t,x) \, \psi_R(x)\right)\right)^2 \mathrm{d}x$$
$$= \varepsilon \int_\mathbb{R} \left(u_\varepsilon(t,x) \, \psi'_R(x)\right)^2 \mathrm{d}x - \int_\mathbb{R} \partial_x \left(H\left(x, u_\varepsilon(t,x)\right)\right) u_\varepsilon(t,x) \, \psi_R{}^2(x) \mathrm{d}x$$

so that, integrating also over t on $[0, T]$ and using the definition of ψ_R, we have

$$\left(\left\| \sqrt{\varepsilon} \, \partial_x u_\varepsilon \right\|_{\mathbf{L}^2([0,T] \times [-R,R]; \mathbb{R})} \right)^2$$
$$\leq \int_0^T \int_\mathbb{R} \left(\sqrt{\varepsilon} \, \partial_x \left(u_\varepsilon(t,x) \, \psi_R(x)\right)\right)^2 \mathrm{d}x$$
$$= \frac{1}{2} \int_\mathbb{R} \left(u_o(x) \, \psi_R(x)\right)^2 \mathrm{d}x - \frac{1}{2} \int_\mathbb{R} \left(u_\varepsilon(T,x) \, \psi_R(x)\right)^2 \mathrm{d}x$$
$$+ \varepsilon \int_0^T \int_\mathbb{R} \left(u_\varepsilon(t,x) \, \psi'_R(x)\right)^2 \mathrm{d}x \mathrm{d}t$$
$$- \int_0^T \int_\mathbb{R} \partial_x \left(H\left(x, u_\varepsilon(t,x)\right)\right) u_\varepsilon(t,x) \, \psi_R{}^2(x) \mathrm{d}x \mathrm{d}t$$
$$\leq \frac{1}{2} \|u_o\|^2_{\mathbf{L}^2(\mathbb{R}; \mathbb{R})} + 8 \, \varepsilon \, M^2 - \int_0^T \int_\mathbb{R} \partial_x \left(H\left(x, u_\varepsilon(t,x)\right)\right) u_\varepsilon(t,x) \, \psi_R{}^2(x) \, \mathrm{d}x \, \mathrm{d}t, \tag{3.86}$$

where M is as in (2.26) from Corollary 2.13. To bound the latter term, introduce the function

$$f(t,x) := \int_0^{u_\varepsilon(t,x)} v \, \partial_u H(x, v) \mathrm{d}v,$$

defined for $(t,x) \in [0, T] \times \mathbb{R}$. Note that by Corollary 2.13

$$|f(t,x)| \leq \int_{-M}^M |v| \sup_{\substack{|\xi| \leq R+1 \\ |w| \leq M}} |\partial_u H(\xi, w)| \mathrm{d}v \leq M^2 \sup_{\substack{|\xi| \leq R+1 \\ |w| \leq M}} |\partial_u H(\xi, w)|. \tag{3.87}$$

Moreover,

$$\partial_x f(t, x) = u_\varepsilon(t, x) \, \partial_u H\left(x, u_\varepsilon(t, x)\right) \, \partial_x u_\varepsilon(t, x) + \int_0^{u_\varepsilon(t,x)} v \, \partial_{xu}^2 H(x, v) \, dv$$

$$= \partial_x \left(H\left(x, u_\varepsilon(t, x)\right)\right) u_\varepsilon(t, x) - \partial_x H\left(x, u_\varepsilon(t, x)\right) u_\varepsilon(t, x)$$

$$+ \int_0^{u_\varepsilon(t,x)} v \, \partial_{xu}^2 H(x, v) dv$$

hence

$$\partial_x \left(H\left(x, u_\varepsilon(t, x)\right)\right) u_\varepsilon(t, x) = \partial_x H\left(x, u_\varepsilon(t, x)\right) u_\varepsilon(t, x)$$

$$- \int_0^{u_\varepsilon(t,x)} v \, \partial_{xu}^2 H(x, v) \, dv + \partial_x f(t, x).$$

Multiply by $\psi_R{}^2(x)$, integrate over $[0, T] \times \mathbb{R}$ and take the absolute value:

$$\left| \int_0^T \int_{\mathbb{R}} \partial_x \left(H\left(x, u_\varepsilon(t, x)\right)\right) u_\varepsilon(t, x) \, \psi_R{}^2(x) dx dt \right|$$

$$\leq \left| \int_0^T \int_{\mathbb{R}} \partial_x H\left(x, u_\varepsilon(t, x)\right) u_\varepsilon(t, x) \, \psi_R{}^2(x) dx dt \right|$$

$$+ \left| \int_0^T \int_{\mathbb{R}} \int_0^{u_\varepsilon(t,x)} v \, \partial_{xu}^2 H(x, v) \, dv \, \psi_R{}^2(x) dx dt \right|$$

$$+ \left| \int_0^T \int_{\mathbb{R}} \partial_x f(t, x) \, \psi_R{}^2(x) dx dt \right|$$

$$\leq \int_0^T \int_{-R-1}^{R+1} \sup_{\substack{|\xi| \leq R+1 \\ |v| \leq M}} |\partial_x H(\xi, v)| \, M dx dt \tag{3.88}$$

$$+ \int_0^T \int_{-R-1}^{R+1} \int_{-M}^{M} |v| \sup_{\substack{|\xi| \leq R+1 \\ |v| \leq M}} \left| \partial_{xu}^2 H(\xi, v) \right| dv dx dt \tag{3.89}$$

$$+ \left| \int_0^T \int_{-R-1}^{R+1} \partial_x f(t, x) \, \psi_R{}^2(x) dx dt \right|, \tag{3.90}$$

where M is as in Corollary 2.13. The two summands on the lines (3.88)–(3.89) are both independent of ε. Concerning (3.90) above, integrate by parts and use (3.87) to obtain

$$\left| \int_0^T \int_{-R-1}^{R+1} \partial_x f(t, x) \, \psi_R{}^2(x) dx dt \right|$$

$$\leq 2 \left| \int_0^T \int_{-R-1}^{R+1} |f(t, x)| \, \psi_R(x) \left| \psi_R'(x) \right| dx dt \right|$$

$$\leq 2 T M^2 \sup_{\substack{|\xi| \leq R+1 \\ |w| \leq M}} |\partial_u H(\xi, w)| \int_{-R-1}^{R+1} \psi_R(x) \left| \psi_R'(x) \right| dx$$

which, again, is a quantity independent of both ε and u_ε. The latter bound inserted together with (3.88) in (3.86) provides the desired $\mathbf{L}_{\text{loc}}^2(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$ bound. Claim 1 is proved. ✓

Claim 2: For any T, $R > 0$ and for any entropy $E \in \mathbf{C}^2(\mathbb{R}; \mathbb{R})$, let F be a flux satisfying (2.4). Then, the set $\{\partial_t E(u_\varepsilon) + \partial_x (F(\cdot, u_\varepsilon)) : \varepsilon \in]0, \varepsilon_*[\}$ is relatively compact in $\mathbf{H}^{-1}([0, T] \times [-R, R]; \mathbb{R})$.

This Claim essentially amounts to an application of Murat Lemma [38, Lemma 9.2.1], which we adapt here to the present situation.

Using (2.20), straightforward computations yield:

$$\partial_t E(u_\varepsilon(t, x)) + \partial_x (F(x, u_\varepsilon(t, x))) = v_\varepsilon(t, x) + w_\varepsilon(t, x) \tag{3.91}$$

where

$$v_\varepsilon(t, x) := \hat{v}_\varepsilon(t, x) + \check{v}_\varepsilon(t, x)$$
$$\hat{v}_\varepsilon(t, x) := \partial_x F(x, u_\varepsilon(t, x)) - E'(u_\varepsilon(t, x)) \; \partial_x H(u_\varepsilon(t, x))$$
$$\check{v}_\varepsilon(t, x) := \varepsilon \; E''(u_\varepsilon(t, x)) \; (\partial_x u_\varepsilon(t, x))^2$$
$$w_\varepsilon(t, x) := \varepsilon \; \partial_{xx}^2 (E(u_\varepsilon(t, x))) (t, x)$$

We now verify the following 3 assumptions to apply Murat Lemma [38, Lemma 9.2.1]:
(1): $\{\partial_t E(u_\varepsilon) + \partial_x (F(\cdot, u_\varepsilon)) : \varepsilon \in]0, \varepsilon_*[\}$ is bounded in $\mathbf{W}^{-1,\infty}([0, T] \times [-R, R]; \mathbb{R})$,

Indeed, by Corollary 2.13, by the regularity of E and by (2.4), the ranges of both $\varepsilon \mapsto E(u_\varepsilon)$ and of $\varepsilon \mapsto F(\cdot, u_\varepsilon)$ are bounded in $\mathbf{L}^\infty([0, T] \times \mathbb{R}; \mathbb{R})$. Use the definition of weak derivative to complete the proof of **(1)**.
(2): $\{v_\varepsilon : \varepsilon \in]0, \varepsilon_*[\}$ bounded in the set of Radon measures $\mathcal{M}([0, T] \times [-R, R]); \mathbb{R})$,

Indeed, Corollary 2.13 shows that the range of $\varepsilon \mapsto \hat{v}_\varepsilon$ is bounded in $\mathbf{L}^\infty([0, T] \times [-R, R]; \mathbb{R})$ uniformly in ε. Hence, the range of $\varepsilon \mapsto \hat{v}_\varepsilon$ is bounded in $\mathbf{L}^1([0, T] \times [-R, R]; \mathbb{R})$, which implies the required boundedness in $\mathcal{M}([0, T] \times [-R, R]); \mathbb{R})$. Claim 1 ensures that the range of $\varepsilon \mapsto \varepsilon (\partial_x u_\varepsilon)^2$ is bounded in $\mathbf{L}^1([0, T] \times [-R, R]; \mathbb{R})$. This, together with the boundedness in $\mathbf{L}^\infty$ ensured by Corollary 2.13, proves that the range of $\varepsilon \mapsto \check{v}_\varepsilon$ is bounded in $\mathcal{M}([0, T] \times [-R, R]; \mathbb{R})$.
(3): $\{w_\varepsilon : \varepsilon \in]0, \varepsilon_*[\}$ relatively compact in $\mathbf{H}^{-1}([0, T] \times [-R, R]; \mathbb{R})$,

Indeed, by the $\mathbf{C}^2$ regularity of E, we have $|\varepsilon \; \partial_x (E(u_\varepsilon))| = \sqrt{\varepsilon} \; E'(u_\varepsilon) \; \sqrt{\varepsilon} \; |\partial_x u_\varepsilon|$ which converges to 0 in $\mathbf{L}_{\text{loc}}^2([0, T] \times \mathbb{R}; \mathbb{R})$ as $\varepsilon \to 0$ by Corollary 3.7 and by Claim 1 above. Hence, the range of $\varepsilon \mapsto w_\varepsilon$ is relatively compact in $\mathbf{H}^{-1}([0, T] \times [-R, R]; \mathbb{R})$

Murat Lemma [38, Lemma 9.2.1] thus applies and Claim 2 is proved. ✓

Introduce an arbitrary sequence ε_n converging to 0. By [19, Chapter 1, § 9, Theorem 1.46], we know there exists a Young measure [19, Chapter 1, § 9, Definition 1.34] $\nu_{t,x}$ corresponding to a subsequence ε_{n_k}, meaning that for each $(t, x) \in [0, T] \times \mathbb{R}$, $\nu_{t,x}$ is a Borel probability measure on $\mathbb{R}$ such that for any $g \in \mathbf{C}^0(\mathbb{R}; \mathbb{R})$ and for any $\varphi \in \mathbf{L}^1([0, T] \times \mathbb{R}; \mathbb{R})$, we have

$$\lim_{k\to+\infty} \int_0^T \int_{\mathbb{R}} g\left(u_{\varepsilon_{n_k}}(t,x)\right)\varphi(t,x)\mathrm{d}t\mathrm{d}x$$

$$= \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(w)\mathrm{d}\nu_{t,x}(w)\right)\varphi(t,x)\mathrm{d}t\mathrm{d}x. \tag{3.92}$$

Remark 3.9. Following a standard habit, to simplify the notation, in the sequel we write ε for ε_{n_k}, $\varepsilon \to 0$ for $k \to +\infty$ and, correspondingly, refer to u_ε as to a *sequence*.

As usual, we assume that ε is sufficiently small, say $\varepsilon \in]0, \varepsilon_*[$ for a suitable $\varepsilon_* > 0$.

Claim 3: For any $G \in \mathbf{C}^0(\mathbb{R}^2; \mathbb{R})$ such that $G(x, u) = G(-X, u)$, for all $(x, u) \in]-\infty, -X] \times \mathbb{R}$ and $G(x, u) = G(X, u)$ for all $(x, u) \in [X, +\infty[\times\mathbb{R}$ and for any $\varphi \in \mathbf{L}^1([0, T] \times \mathbb{R}; \mathbb{R})$,

$$\lim_{\varepsilon\to0} \int_0^T \int_{\mathbb{R}} G\left(x, u_\varepsilon(t,x)\right)\varphi(t,x)\mathrm{d}t\mathrm{d}x = \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} G(x,w)\,\mathrm{d}\nu_{t,x}(w)\right)\varphi(t,x)\mathrm{d}t\mathrm{d}x.$$

By Corollary 2.13, the sequence u_ε attains values in $[-M, M]$, with M as in (2.26). By Stone–Weierstrass Theorem [22, Corollary 7.31] for every $\delta > 0$ there exist an integer m and functions $f_1, \ldots, f_m \in \mathbf{C}^0([-X, X]; \mathbb{R})$, $g_1, \ldots, g_m \in \mathbf{C}^0([-M, M]; \mathbb{R})$ such that

$$\sup_{(x,w)\in[-X,X]\times[-M,M]} \left|H(x,w) - \sum_{\ell=1}^m f_\ell(x)\,g_\ell(w)\right| < \delta. \tag{3.93}$$

By **(CNH)**, introducing for $\ell = 1, \ldots, m$ the functions

$$F_\ell(w) := \begin{cases} f_\ell(-X) & \text{for } x < -X \\ f_\ell(x) & \text{for } x \in [-X, X] \\ f_\ell(X) & \text{for } x > X \end{cases}$$

we can extend the latter statement (3.93) to

$$\sup_{(x,w)\in\mathbb{R}\times[-M,M]} \left|H(x,w) - \sum_{\ell=1}^m F_\ell(x)\,g_\ell(w)\right| < \delta.$$

Hence, for any $\varphi \in \mathbf{L}^1([0, T] \times \mathbb{R}; \mathbb{R})$

$$\left|\int_0^T \int_{\mathbb{R}} H\left(x, u_\varepsilon(t,x)\right)\varphi(t,x)\mathrm{d}t\mathrm{d}x - \int_0^T \int_{\mathbb{R}}\left(\int_{\mathbb{R}} H(x,w)\mathrm{d}\nu_{t,x}(w)\right)\varphi(t,x)\mathrm{d}t\mathrm{d}x\right|$$

$$\leq \left|\int_0^T \int_{\mathbb{R}}\left(H\left(x, u_\varepsilon(t,x)\right) - \sum_{\ell=1}^m F_\ell(x)\,g_\ell\left(u_\varepsilon(t,x)\right)\right)\varphi(t,x)\mathrm{d}t\mathrm{d}x\right|$$

$$+ \left|\int_0^T \int_{\mathbb{R}}\sum_{\ell=1}^m F_\ell(x)\,g_\ell\left(u_\varepsilon(t,x)\right)\,\varphi(t,x)\mathrm{d}t\mathrm{d}x\right.$$

$$
\left. -\int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \sum_{\ell=1}^m F_\ell(x) \, g_\ell(w) \mathrm{d}\nu_{t,x}(w) \right) \varphi(t,x) \mathrm{d}t \mathrm{d}x \right|
$$

$$
+ \left| \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \left(-H(x,w) + \sum_{\ell=1}^m F_\ell(x) \, g_\ell(w) \right) \mathrm{d}\nu_{t,x}(w) \, \varphi(t,x) \, \mathrm{d}t \mathrm{d}x \right|
$$

$$
\leq 2 \, \delta \, \|\varphi\|_{\mathbf{L}^1([0,T]\times\mathbb{R};\mathbb{R})}
$$

$$
+ \sum_{\ell=1}^m \left| \int_0^T \int_{\mathbb{R}} \left(g_\ell\left(u_\varepsilon(t,x)\right) - \int_{\mathbb{R}} g_\ell(w) \mathrm{d}\nu_{t,x}(w) \right) \left(F_\ell(x) \, \varphi(t,x) \right) \mathrm{d}t \mathrm{d}x \right|
$$

where, to get to the last inequality, we used the inclusion spt $\nu_{t,x} \subseteq [-M, M]$. Moreover, each term in the latter sum above converges to 0 by (3.92), since each $F_\ell \, \varphi$ is in $\mathbf{L}^1([0,T] \times \mathbb{R}; \mathbb{R})$. Then, Claim 3 is proved.   ✓

Claim 4: For any entropy $E \in \mathbf{C}^2(\mathbb{R}; \mathbb{R})$, there exists a set $\Omega_E \subseteq \mathbb{R}_+ \times \mathbb{R}$ such that $(\mathbb{R}_+ \times \mathbb{R})\backslash\Omega_E$ is negligible and for all $(t,x) \in \Omega_E$

$$
\int_{\mathbb{R}} (w \, F(x,w) - E(w) \, H(x,w)) \, \mathrm{d}\nu_{t,x}(w)
$$

$$
= \int_{\mathbb{R}} w \mathrm{d}\nu_{t,x}(w) \int_{\mathbb{R}} F(x,w) \mathrm{d}\nu_{t,x}(w) - \int_{\mathbb{R}} E(w) \mathrm{d}\nu_{t,x}(w) \int_{\mathbb{R}} H(x,w) \mathrm{d}\nu_{t,x}(w)
$$

$$
\tag{3.94}
$$

where F is any entropy flux corresponding to E with respect to H, according to Definition 2.3.

(The content of this step closely follows Claim 5 in the proof of Theorem 2.9).

Consider the vector fields

$$
V_\varepsilon(t,x) = \begin{bmatrix} u_\varepsilon(t,x) \\ H\left(x, u_\varepsilon(t,x)\right) \end{bmatrix} \quad \text{and} \quad W_\varepsilon(t,x) = \begin{bmatrix} F\left(x, u_\varepsilon(t,x)\right) \\ -E\left(x, u_\varepsilon(t,x)\right) \end{bmatrix}
$$

where E is in $\mathbf{C}^2(\mathbb{R}; \mathbb{R})$ and F is a corresponding flux defined by (2.4). By Claim 2, $\nabla{\cdot}V_\varepsilon$ and $\nabla \wedge W_\varepsilon$ lie in sets that are relatively compact in $\mathbf{H}^{-1}([0,T] \times [-R, R]; \mathbb{R})$, for any $R > 0$. By the div–curl Lemma [16, Theorem 17.2.1], we have

$$
\lim_{\varepsilon \to 0} (V_\varepsilon \cdot W_\varepsilon) = (\lim_{\varepsilon \to 0} V_\varepsilon) \cdot (\lim_{\varepsilon \to 0} W_\varepsilon),
\tag{3.95}
$$

the limits in the right hand side are understood in the weak $\mathbf{L}^2([0,T] \times [-R, R]; \mathbb{R})$ sense, while the limit in the left hand side and the equality are understood in the sense of distribution. Moreover, since both sides of (3.95) are in $\mathbf{L}^1([0,T] \times [-R, R]; \mathbb{R})$, equality (3.95) also holds in $\mathbf{L}^1([0,T] \times [-R, R]; \mathbb{R})$. At the same time, using (3.92)

we have

$$\lim_{\varepsilon \to 0} (V_\varepsilon \cdot W_\varepsilon) \,:\, (t, x) \mapsto \int_{\mathbb{R}} (w\, F(x, w) - E(w)\, H(x, w))\, \mathrm{d}\nu_{t,x}(w),$$

$$\lim_{\varepsilon \to 0} V_\varepsilon(t, x) \,:\, (t, x) \mapsto \begin{bmatrix} \int_{\mathbb{R}} w \mathrm{d}\nu_{t,x}(w) \\ \int_{\mathbb{R}} H(x, w) \mathrm{d}\nu_{t,x}(w) \end{bmatrix},$$

$$\lim_{\varepsilon \to 0} W_\varepsilon(t, x) \,:\, (t, x) \mapsto \begin{bmatrix} \int_{\mathbb{R}} F(x, w) \mathrm{d}\nu_{t,x}(w) \\ - \int_{\mathbb{R}} E(w) \mathrm{d}\nu_{t,x}(w) \end{bmatrix}.$$

Hence, we may now intend (3.94) as an equality between representatives of $\mathbf{L}^1([0, T] \times [-R, R]; \mathbb{R})$ functions. Call Ω_E the set of points where (3.95) holds. By the arbitrariness of R, the complement of Ω_E is negligible. Claim 4 is proved.   ✓

Call now $\mathcal{E}$ the countable set of all polynomials with rational coefficients and define

$$\Omega := \bigcap_{E \in \mathcal{E}} \Omega_E. \tag{3.96}$$

Then, for any $E \in \mathcal{E}$ and for all $(t, x) \in \Omega$, (3.94) holds and the set $([0, T] \times \mathbb{R}) \backslash \Omega$ is negligible.

Claim 5: For all $E \in \mathbf{C}^0(\mathbb{R}; \mathbb{R})$ and for all $(t, x) \in \Omega$, equality (3.94) holds, where $F = F^k$ is given by (2.6), for any $k \in \mathbb{R}$.

For any $E \in \mathcal{E}$ and for all $(t, x) \in \Omega$, (3.94) holds, the set $([0, T] \times \mathbb{R}) \backslash \Omega$ being negligible.

Let now $E \in \mathbf{C}^0(\mathbb{R}; \mathbb{R})$ be fixed. By the classical Stone–Weierstrass Theorem [22, Corollary 7.31], there exists a sequence E_n in $\mathcal{E}$ converging to E uniformly on $[-M, M]$, with M as in (2.26). Clearly, the sequence of fluxes F_n^k corresponding to E_n defined by (2.6) converges uniformly to the flux F^k, also defined by (2.6). Since (3.94) holds in Ω for each pair (E_n, F_n^k), repeating the same argument as in the proof of Claim 3, one proves that it also holds for (E, F^k). By the arbitrariness of E, Claim 5 is proved.   ✓

Define the limit

$$u(t, x) = \int_{\mathbb{R}} w \, \mathrm{d}\nu_{t,x}(w) \tag{3.97}$$

Claim 6: With reference to (3.96) and (3.97), for all $(t, x) \in \Omega$,

$$\int_{\mathbb{R}} H(x, w) \, \mathrm{d}\nu_{t,x}(w) = H(x, u(t, x)) \tag{3.98}$$

Fix $(\tau, \xi) \in \Omega$, set $E(w) := |w - u(\tau, \xi)|$ so that by (2.6) $F^{u(\tau,\xi)}(x, w) := \mathrm{sgn}\,(w - u(\tau, \xi))\,(H(x, w) - H(x, u(\tau, \xi)))$. Inserting these expressions in (3.94) and using (3.97), we get that for all $(t, x) \in \Omega$

$$\int_{\mathbb{R}} (w \, \operatorname{sgn}(w - u(\tau, \xi)) \, (H(x, w) - H(x, u(\tau, \xi))) - H(x, w) \, |w - u(\tau, \xi)|) \, dv_{t,x}(w)$$

$$= u(t, x) \int_{\mathbb{R}} \operatorname{sgn}(w - u(\tau, \xi)) \, (H(x, w) - H(x, u(\tau, \xi))) \, dv_{t,x}(w)$$

$$- \int_{\mathbb{R}} H(x, w) dv_{t,x}(w) \int_{\mathbb{R}} |w - u(\tau, \xi)| dv_{t,x}(w)$$

Move the first term in the left hand side above to the right hand side to obtain

$$\int_{\mathbb{R}} (w - u(t, x)) \, \operatorname{sgn}(w - u(\tau, \xi)) \, (H(x, w) - H(x, u(\tau, \xi))) \, dv_{t,x}(w)$$

$$- \int_{\mathbb{R}} H(x, w) \, |w - u(\tau, \xi)| dv_{t,x}(w)$$

$$= - \int_{\mathbb{R}} H(x, w) dv_{t,x}(w) \int_{\mathbb{R}} |w - u(\tau, \xi)| dv_{t,x}(w).$$

Choosing $(t, x) = (\tau, \xi)$, there is a cancelation between the first two lines above, resulting in

$$\left(\int_{\mathbb{R}} H(\xi, w) \, dv_{\tau, \xi}(w) - H(\xi, u(\tau, \xi)) \right) \int_{\mathbb{R}} |w - u(\tau, \xi)| dv_{\tau, \xi}(w) = 0.$$

Hence, either the first factor vanishes, or $v_{\tau, \xi}$ is Dirac delta at $u(\tau, \xi)$. In both cases, Claim 6 is proved.   ✓

Claim 7: Up to a subsequence, the sequence u_ε converges to u, defined in (3.97), **a.e. in Ω defined in** (3.96).

(This step, similarly to Claim 7 in the proof of Theorem 2.9, is inspired by [24, Section 5.4]).

Recall (3.98) from Claim 6. From (3.94) using Claim 5, we get that for $(t, x) \in \Omega$,

$$\int_{\mathbb{R}} [(w - u(t, x)) F(x, w) - (H(x, w) - H(x, u(t, x))) E(w)] dv_{t,x}(w) = 0.$$
$$(3.99)$$

For a.e. $(t, x) \in [0, T] \times \mathbb{R}$, $v_{t,x}$ is a probability measure, the maps $w \mapsto (w - u(t, x)) F(x, w)$ and $w \mapsto (H(x, w) - H(x, u(t, x))) E(w)$ are sufficiently regular to ensure that the functions

$$\alpha_{t,x}(S) := \int_S (w - u(t, x)) \, dv_{t,x}(w) \quad \text{and}$$

$$\beta_{t,x}(S) := \int_S (H(x, w) - H(x, u(t, x))) \, dv_{t,x}(w)$$

(S being any Borel set) meet the requirements in the definition of finite Radon measures. Hence, the two maps

$$A_{t,x}(v) := \alpha_{t,x}(]\infty, v]) \quad \text{and} \quad B_{t,x}(v) := \beta_{t,x}(]\infty, v]) \qquad (3.100)$$

are in $\mathbf{BV}(\mathbb{R}; \mathbb{R})$. Since spt $\nu_{t,x} \subseteq [-M, M]$, then $A_{t,x}(v)$ and $B_{t,x}(v)$ vanish for $v < -M$ and attain a constant value for $v > M$. Moreover, (2.14) implies that $\alpha_{t,x}(\mathbb{R}) = 0$ while (3.98) in Claim 6 implies that $\beta_{t,x}(\mathbb{R}) = 0$. Therefore, for all $(t, x) \in \Omega$, both $A_{t,x}$ and $B_{t,x}$ are supported in $[-M, M]$. An integration by parts, see [21, Theorem B] (in particular the remark at the bottom of [21, p. 422]), then ensures that from equality (3.99) we can deduce

$$\int_{\mathbb{R}} A_{t,x}(w) \, \partial_w F(x, w) \, \mathrm{d}w = \int_{\mathbb{R}} B_{t,x}(w) \, E'(w) \, \mathrm{d}w.$$

By means of (2.4), we then have

$$\int_{\mathbb{R}} E'(w) \, \partial_w H(x, w) \, A_{t,x}(w) \, \mathrm{d}w = \int_{\mathbb{R}} E'(w) \, B_{t,x}(w) \, \mathrm{d}w.$$

The above equality holds for any continuous function E', hence for all $(t, x) \in \Omega$

$$\partial_w H(x, w) \, A_{t,x}(w) = B_{t,x}(w) \qquad \text{for a.e. } w \in \mathbb{R}. \tag{3.101}$$

A direct computation shows that $(H(x, w) - H(x, u(t, x))) \, A_{t,x}(w) = (w - u(t, x)) \, B_{t,x}(w)$ since the two sides have the same distributional derivative in w by (3.101) and the definitions (3.100) of $A_{t,x}$, $B_{t,x}$. Inserting (3.101) in the last equality, we have

$$(H(x, w) - H(x, u(t, x))) \, A_{t,x}(w)$$
$$= (w - u(t, x)) \, \partial_w H(x, w) \, A_{t,x}(w) \quad \text{for a.e. } w \in \mathbb{R}.$$

Call $[a, b]$ the minimal (with respect to set inclusion) interval containing the support of $\nu_{t,x}$. Note that $A_{t,x}(w) \neq 0$ for $w \in]a, b[$. Indeed, by the definition of $A_{t,x}(w)$ and since $\nu_{t,x}$ is nonnegative, the map $w \mapsto A_{t,x}(w)$ vanishes for $w < a$, weakly decreases for $w \in]a, u(t, x)[$, weakly increases for $w \in]u(t, x), b[$ and vanishes for $w > b$. At the same time, the minimality of $[a, b]$ ensures that $A_{t,x}$ is nonzero in both a right neighborhood of a and a left neighborhood of b. Simplifying, we thus obtain

$$H(x, w) - H(x, u(t, x)) = (w - u(t, x)) \, \partial_w H(x, w) \quad \text{for all } w \in]a, b[,$$

and differentiating this equality with respect to w we contradict **(WGNL)**, unless $a = b$, which in turn ensures that, for a.e. $(t, x) \in [0, T] \times \mathbb{R}$, $\nu_{t,x}$ is a Dirac measure. We thus have the pointwise a.e. convergence, up to a subsequence, of the vanishing viscosity solutions, see [38, Proposition 9.1.7]. Claim 7 is proved.   ✓

Claim 8: u solves **(CL)** in the sense of Definition 2.1.

Let (E, F) be an entropy–entropy flux pair in the sense of Definition 2.3, with E of class $\mathbf{C}^2$ and convex. Using (2.20), thanks to the regularity of u_ε, simple computations give

$$\partial_t E(u_\varepsilon) + \partial_x \left(F(x, u_\varepsilon) \right) + E'(u_\varepsilon) \, \partial_x H(x, u_\varepsilon) - \partial_x F(x, u_\varepsilon)$$
$$= \varepsilon_n \partial_{xx}^2 E(u_\varepsilon) - \varepsilon_n \, E''(u_\varepsilon) \, (\partial_x u_\varepsilon)^2$$

so that by the convexity of E

$$\partial_t E(u_\varepsilon) + \partial_x \left(F(x, u_\varepsilon) \right) + E'(u_\varepsilon) \, \partial_x H(x, u_\varepsilon) - \partial_x F(x, u_\varepsilon) \leq \varepsilon_n \partial_{xx}^2 E(u_\varepsilon).$$
$$(3.102)$$

Fix a test function $\varphi \in \mathbf{C}_c^2(\mathbb{R}^2; \mathbb{R}_+)$, multiply both sides in (3.102) by φ and integrate to get

$$\int_0^T \int_{\mathbb{R}} \left(E \left(u_\varepsilon(t, x) \right) \, \partial_t \varphi(t, x) + F \left(x, u_\varepsilon(t, x) \right) \, \partial_x \varphi(t, x) \right) \mathrm{d}x \mathrm{d}t$$
$$- \int_0^T \int_{\mathbb{R}} \left(E' \left(u_\varepsilon(t, x) \right) \, \partial_x H \left(x, u_\varepsilon(t, x) \right) - \partial_x F \left(x, u_\varepsilon(t, x) \right) \right) \varphi(t, x) \mathrm{d}x \mathrm{d}t$$
$$+ \int_{\mathbb{R}} E \left(u_o(x) \right) \, \varphi(0, x) \mathrm{d}x$$
$$\geq -\varepsilon \int_0^T \int_{\mathbb{R}} E(u_\varepsilon) \, \partial_{xx}^2 \varphi(t, x) \mathrm{d}x \mathrm{d}t.$$

By (2.26), we have the $\mathbf{L}^\infty$ boundedness of u_ε uniformly in ε. Using Claim 7 and the Dominated Convergence Theorem [22, Theorem (12.24)] we obtain (2.7) for any test function $\varphi \in \mathbf{C}_c^2(\mathbb{R}^2; \mathbb{R}_+)$. A standard approximation argument allows to relax (2.7) to any test function $\varphi \in \mathbf{C}_c^1(\mathbb{R}^2; \mathbb{R}_+)$. The proof of Claim 8 follows by Item 2 in Proposition 2.4.   ✓

Conclusion

By Claim 8, u solves (CL) in the sense of Definition 2.1 and thus its uniqueness follows from Theorem 2.6. Recall that the sequence ε_n, prior to the simplification in the notation in Remark 3.9, is an arbitrary sequence converging to 0. Above, we proved that there exists a subsequence ε_{m_k} such that the corresponding subsequence $u_{\varepsilon_{n_k}}$ converges to a limit u, independent of the choice of the initial sequence ε_n. The arbitrariness of the choice of ε_n ensures that u_ε, now understood as a continuous family, converges to u.

The proof of Theorem 2.17 is completed.   □

An alternative approach allowing to pass from weak to strong convergence might be adapted from [20, Items 2 and 3 in the proof of Theorem 4.1].

3.5. Properties of the limit semigroups

Proof of Theorem 2.18. Theorem 2.17 ensures the existence of a solution in the sense of Definition 2.1 globally in time, for all initial data in $\mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$, proving **1.** and **2.** for such data. The uniqueness of this solution follows from estimate (2.14) in Theorem 2.6.

Define pointwise $(S_t^{CL} u_o)(x) := u^*(t, x)$, where u^* is as in Theorem 2.6. We thus have the existence of a map S^{CL} defined on $\mathbb{R}_+ \times \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$ attaining values in (a precise representative in) $\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$, satisfying **3.a** and **3.b** for all $u_o \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$ and **4.** for all $u_o, v_o \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$, thanks to Theorem 2.6.

Fix an initial datum u_o in $\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$. Use Theorem 2.9 to find two stationary solutions $\check{u}$ and $\hat{u}$ such that for all $x \in \mathbb{R}$,

$$\check{u}(x) + 1 \le -\|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \le u_o(x) \le \|u_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \le \hat{u}(x) - 1.$$

Take a sequence $u_o^n \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$ converging to u_o in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R})$ and such that $u_o^n(x) \in [\check{u}(x), \hat{u}(x)]$ for all $x \in \mathbb{R}$.

By the contraction property (2.14), for all $t \in \mathbb{R}_+$, $S_t^{CL} u_o^n$ is a Cauchy sequence in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R})$. Define $S_t^{CL} u_o$ as this limit and note that (2.14) also shows that S^{CL} does not depend on the choice of the sequence (u_o^n).

Nevertheless, by (2.15), for all $t \in \mathbb{R}_+$, $(S_t^{CL} u_o^n)(x) \in [\check{u}(x), \hat{u}(x)]$ for a.e. $x \in \mathbb{R}$, so that $(t, x) \mapsto (S_t^{CL} u_o)(x)$ is in $\mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$.

Moreover, $(t, x) \mapsto (S_t^{CL} u_o^n)(x)$ is a converging sequence in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$. Up to the extraction of a subsequence, we have that the sequence $(t, x) \mapsto (S_t^{CL} u_o^n)(x)$ converges pointwise a.e. to $(t, x) \mapsto (S_t^{CL} u_o)(x)$. Since we have the $\mathbf{L}^\infty$ bound $(S_t^{CL} u_o^n)(x) \in [\check{u}(x), \hat{u}(x)]$ we can pass to the limit in (2.2), apply the Dominated Convergence Theorem [22, Theorem (12.24)] and obtain that $(t, x) \mapsto (S_t^{CL} u_o)(x)$ solves **(CL)** in the sense of Definition 2.1.

By this construction, we immediately have that the map $u(t, x) := (S_t^{CL} u_o)(x)$ satisfies **1.** and **2.**, while S^{CL} satisfies **3.a** and **4.**.

Fix $u_o \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$. Applying again Theorem 2.6, we see that the map $(t, x) \mapsto (S_t^{CL} u_o)(x)$ admits a representative that satisfies **3.b**. Since S^{CL} satisfies **3.a**, we can conclude that for all $t \in \mathbb{R}_+$ and for a.e. $x \in \mathbb{R}$ that $(S_t^{CL} u_o)(x)$ equals this representative. Hence, S^{CL} satisfies **3.b**.

To complete the proof, note that S^{CL} is a semigroup, thanks to the uniqueness and $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R})$ continuity proved in Theorem 2.6 and since $S_t^{CL} (\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})) \subseteq \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$. □

Proof of Theorem 2.19. Define for later use

$$\mathcal{D} := \left\{ W \in \mathbf{C}^1(\mathbb{R}; \mathbb{R}) \colon W' \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R}) \right\}.$$

Fix $U_o \in \mathcal{D}$. Then, Theorem 2.16 ensures the existence of a solution U to **(HJ)** in the sense of Definition 2.7 globally defined in time, satisfying **1.** and **2.** for these data. The uniqueness of this solution follows from (2.19) in Theorem 2.8.

Define now, for all $U_o \in \mathcal{D}$,

$$(S_t^{HJ} U_o)(x) := U(t, x).$$

Note that Theorem 2.16 also proves **3.**, while **4.** is a consequence of Item 2 in Theorem 2.8.

Note that S^{HJ} is not a semigroup on $\mathcal{D}$, since *a priori* $\mathcal{D}$ is not invariant.

Below, we use the semigroup S^{CL} as defined in Theorem 2.18.

Claim: For all $U_o \in \mathcal{D}$, set $u_o = U_o'$. Then, $(S_t^{CL} u_o)(x) = \partial_x (S_t^{HJ} U_o)(x)$ for a.e. $(t, x) \in [0, T] \times \mathbb{R}$.

Let $U_o \in \mathcal{D}$ and set $u_o = U_o'$, so that $u_o \in \mathbf{W}^{1,\infty}(\mathbb{R}; \mathbb{R})$. For any $\varepsilon > 0$, call u_ε the classical solution to (2.20) as provided by Theorem 2.14 and U_ε that of (2.21) as provided by Corollary 2.15. By Theorem 2.11, we have $u_\varepsilon = \partial_x U_\varepsilon$. Choose a sequence ε_n positive and vanishing to 0. Theorem 2.16 ensures that $(t, x) \mapsto U_{\varepsilon_n}(t, x)$ converges, as $n \to +\infty$, to $(t, x) \mapsto (S_t^{HJ} U_o)(x)$ uniformly on compact sets. Similarly, Theorem 2.17 (which uses **(WGNL)**) ensures that $(t, x) \mapsto u_{\varepsilon_n}(t, x)$ converges pointwise a.e. to $(t, x) \mapsto (S_t^{CL} u_o)(x)$. Hence, in distributional sense, $(S_t^{CL} u_o)(x) = \partial_x (S_t^{HJ} U_o)(x)$ and therefore this equality also holds a.e. in $\mathbb{R}_+ \times \mathbb{R}$, proving the Claim. ✓

Fix an initial datum U_o in **Lip**$(\mathbb{R}; \mathbb{R})$. Use Theorem 2.9 to find two stationary solutions $\check{u}$ and $\hat{u}$ to **(HJ)** such that for all $x \in \mathbb{R}$,

$$\check{u}(x) + 1 \le -\left\| U_o' \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \le U_o'(x) \le \left\| U_o' \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \le \hat{u}(x) - 1. \quad (3.103)$$

Take a sequence $U_o^n \in \mathcal{D}$ converging to U_o in $\mathbf{L}_{\mathbf{loc}}^\infty(\mathbb{R}; \mathbb{R})$ and such that $(U_o^n)'(x) \in [\check{u}(x), \hat{u}(x)]$ for a.e. $x \in \mathbb{R}$. Moreover, for a.e. $x \in \mathbb{R}$,

$$\begin{aligned} \partial_x \left(S_t^{HJ}(U_o^n) \right)(x) &= \left(S_t^{CL}(U_o^n)' \right)(x) && \text{[by the Claim above]} \\ &\in [\check{u}(x), \hat{u}(x)] && \text{[by (2.15) and Theorem 2.9]}. \end{aligned}$$

Define $\tilde{C} := \max \left\{ \left\| \hat{u} \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}, \left\| \check{u} \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \right\}$, so that for all $t \in \mathbb{R}_+$ and for all $n \in \mathbb{N}$,

$$\left\| \partial_x S_t^{HJ}(U_o^n) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \le \tilde{C}. \quad (3.104)$$

Similarly to (2.18), define

$$\tilde{L} := \sup \left\{ |\partial_x H(x, W)| : x \in \mathbb{R}, \ |W| \le \tilde{C} \right\}.$$

By Item 2 in Theorem 2.8, we have for all $R > 0$ and all $t \in \mathbb{R}_+$,

$$\max_{|x| \le R} \left| (S_t^{HJ} U_o^n)(x) - (S_t^{HJ} U_o^m)(x) \right| \le \max_{|x| \le R + \tilde{L} t} \left| U_o^n(x) - U_o^m(x) \right|,$$

hence $(t, x) \mapsto (S_t^{HJ} U_o^n)(x)$ is a Cauchy sequence in $\mathbf{L}_{\mathbf{loc}}^\infty(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$. Extend S^{HJ} to **Lip**$(\mathbb{R}; \mathbb{R})$ by $S_t^{HJ} U_o = \lim_{n \to +\infty} S_t^{HJ} U_o^n$ and note that, by (3.104), $\left\| \partial_x S_t^{HJ} U_o \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \le \tilde{C}$ for $t \in \mathbb{R}_+$.

This extension S^{HJ} satisfies **4.** by Item 2 in Theorem 2.8. Moreover, [13, Theorem 1.2] ensures that $(t, x) \mapsto (S_t^{HJ} U_o)(x)$ is a solution to **(HJ)** in the sense of Definition 2.7. As a consequence, **1.**, **2.** and the existence of the maximal solution follow. Its uniqueness is guaranteed by Item 2 in Theorem 2.8. The uniform

bound $\left\|\partial_x S_t^{HJ} U_o\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \leq \tilde{C}$ proves that the set $\mathbf{Lip}(\mathbb{R};\mathbb{R})$ is invariant, so that $S^{HJ}: \mathbb{R}_+ \times \mathbf{Lip}(\mathbb{R};\mathbb{R}) \to \mathbf{Lip}(\mathbb{R};\mathbb{R})$ is a semigroup.

Since Item 1 in Theorem 2.8 implies Item 3, the proof of Theorem 2.19 is completed.

$\square$

Proof of Theorem 2.20. Consider first the case $u_o \in \mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R})$, so that $U_o \in \mathbf{C}^1(\mathbb{R};\mathbb{R})$. For any $\varepsilon > 0$, call u_ε the classical solution to (2.20) as provided by Theorem 2.14 and U_ε that of (2.21) as provided by Corollary 2.15. By Theorem 2.11, we have $u_\varepsilon = \partial_x U_\varepsilon$. Choose a sequence ε_n positive and vanishing to 0. Theorem 2.16 ensures that $(t, x) \mapsto U_{\varepsilon_n}(t, x)$ converges to $(t, x) \mapsto (S_t^{HJ} U_o)(x)$ uniformly on compact sets. Similarly, Theorem 2.17 ensures that $(t, x) \mapsto u_{\varepsilon_n}(t, x)$ converges pointwise a.e. to $(t, x) \mapsto (S_t^{CL} u_o)(x)$. Hence, in distributional sense, $(S_t^{CL} u_o)(x) = \partial_x(S_t^{HJ} U_o)(x)$ and therefore this equality also holds a.e. in $\mathbb{R}_+ \times \mathbb{R}$. This proves (2.27) in the case $u_o \in \mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R})$.

Let now $u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$. Choose a sequence $u_o^n \in \mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R})$ converging to u_o in $\mathbf{L}_{\mathbf{loc}}^1(\mathbb{R};\mathbb{R})$. Define $U_o^n(x) = \int_0^x u_o^n(\xi)\mathrm{d}\xi$. Then, by Theorem 2.18 (which uses **(WGNL)**), also $(t, x) \mapsto (S_t^{CL} u_o^n)(x)$ converges to $(t, x) \mapsto (S_t^{CL} u_o)(x)$ in $\mathbf{L}_{\mathbf{loc}}^1(\mathbb{R};\mathbb{R})$. Similarly, $(t, x) \mapsto (S_t^{HJ} U_o^n)(x)$ converges to $(t, x) \mapsto (S_t^{HJ} U_o)(x)$ by (2.19) in Theorem 2.8 uniformly on compact sets. In particular, both sequences converge in distributional sense, proving (2.27) in the general case, thanks to the uniqueness of entropy solutions to **(CL)** (Theorem 2.18) and of viscosity solutions to **(HJ)** (Theorem 2.19).

$\square$

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A. Appendix

The next results are somewhat classical and we provide their proofs or full references for completeness.

Proposition A.1. (Extension Theorem in Locally Convex Spaces) *Let X be a locally convex sequentially complete topological vector space. Fix $f : A \to X$ with $A \subseteq \mathbb{R}$. If f is uniformly continuous, then there exists a unique continuous map $\overline{f} : \overline{A} \to X$ satisfying $\overline{f}_{|A} = f$.*

Proof. A sequence x_n in X is a *Cauchy sequence* if for every neighborhood U of the origin in X, there exists a $\nu \in \mathbb{N}$ such that if $n, m \in \mathbb{N}$ with $m, n > \nu$, then $x_n - x_m \in U$. Moreover, f *uniformly continuous* means that for every neighborhood U of the origin in X, there exists a $\delta > 0$ such that if $t_1, t_2 \in A$ are such that $|t_2 - t_1| < \delta$, then $f(t_2) - f(t_1) \in U$.

Let $t_* \in \overline{A} \setminus A$. Then, there exists a sequence t_n in A such that $\lim_{n \to +\infty} t_n = t_*$. Let U be a neighborhood of the origin in X. By the uniform continuity of f, there exists an $\varepsilon > 0$ such that $f(t'') - f(t') \in U$ whenever $|t'' - t'| < \varepsilon$. Thus, there exists a $\nu \in \mathbb{N}$ such that if $n, m \in \mathbb{N}$ with $m, n > \nu$, then $|t_m - t_n| < \varepsilon$ and hence $f(t_m) - f(t_n) \in U$, proving that $f(t_n)$ is a Cauchy sequence converging to, say, $x_* \in X$. Define $\overline{f}(t_*) = x_*$.

Any other sequence t'_n in A converging to t_* is such that the sequence defined by $\tau_{2n} = t_n$ and $\tau_{2n+1} = t'_n$ is a Cauchy sequence and the argument above ensures that $\lim_{n \to +\infty} f(\tau_n) = x_*$, proving that $\overline{f}$ is well defined.

To prove the continuity of $\overline{f}$, let t_n be now a sequence in $\overline{A}$ converging to t_*. Then, there exists a sequence τ_n in A such that $|\tau_n - t_n| < 1/n$. The argument above ensures that $\lim_{n \to +\infty} f(\tau_n) = \overline{f}(t_*)$, so that by the uniform continuity of f and the definition of $\overline{f}$ we also have $\lim_{n \to +\infty} \overline{f}(t_n) = \overline{f}(t_*)$, ensuring the sequential continuity of $\overline{f}$.

Fix an open neighborhood U of 0 in X and apply [37, Theorem 1.10] with $K = \{0\}$ and $C = X \setminus U$ to obtain an open neighborhood V of 0 such that $(K+V) \cap (C+V) = \emptyset$. Since $V = K + V$ and $C + V$ is open, then $\overline{V} \cap (C + V) = \emptyset$. At the same time $C \subseteq (C + V)$, thus $\overline{V} \subseteq U$. By the uniform continuity of f, there exists a positive δ such that for all $t_1, t_2 \in A$ with $|t_2 - t_1| < \delta$, we have $(f(t_2) - f(t_1)) \in V$. The

sequential closedness of $\overline{V}$ and the sequential continuity of $\overline{f}$ proved above ensures that for all $t_1, t_2 \in \overline{A}$ with $|t_2 - t_1| < \delta$, we have $\left(\overline{f}(t_2) - \overline{f}(t_1) \right) \in \overline{V} \subseteq U$, completing the proof. $\qquad\square$

Proposition A.2. (The set $\mathbf{C}_c^1$ is separable.) *There exists a countable set $S \subset \mathbf{C}_c^1(\mathbb{R}; \mathbb{R})$ with the following property: for any $\psi \in \mathbf{C}_c^1(\mathbb{R}; \mathbb{R})$ there exists a compact set $\mathcal{K} \subset \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a map $\sigma \in S$ satisfying $\operatorname{spt} \sigma \subseteq \mathcal{K}$ and $\|\sigma - \psi\|_{\mathbf{C}^1(\mathbb{R};\mathbb{R})} \leq \varepsilon$.*

Proof. For $n \in \mathbb{N}\setminus\{0\}$, call ϑ_n a function in $\mathbf{C}^\infty(\mathbb{R}; \mathbb{R})$ such that

$$
\begin{aligned}
x \in [-n, n] &\implies \vartheta_n(x) = 1, \\
x \in [-n-1, -n] &\implies \vartheta_n(x) \in [0, 1] \text{ and } \vartheta_n'(x) \in [0, 2] \\
x \in [n, n+1] &\implies \vartheta_n(x) \in [0, 1] \text{ and } \vartheta_n'(x) \in [-2, 0] \\
x \in \mathbb{R} \setminus [-n-1, n+1] &\implies \vartheta_n(x) = 0.
\end{aligned}
$$

For every polynomial p with rational coefficients, define

$$
\chi_{n,p}(x) := \vartheta_n(x) \int_{-\infty}^{x} \vartheta_n(\xi)\, p(\xi) \mathrm{d}\xi.
$$

Clearly, $\chi_{n,p} \in \mathbf{C}_c^1(\mathbb{R}; \mathbb{R})$ and the set of all such $\chi_{n,p}$ is countable. It is also dense in $\mathbf{C}_c^1(\mathbb{R}; \mathbb{R})$. Indeed, choose $\varphi \in \mathbf{C}_c^1(\mathbb{R}; \mathbb{R})$ so that $\operatorname{spt} \varphi \subseteq [-n, n]$. By Stone–Weierstrass Theorem [22, Corollary 7.31], for every $\varepsilon > 0$ there exists a polynomial p with rational coefficients such that $\sup_{\operatorname{spt} \varphi} |p - \varphi'| < \varepsilon/(6n+6)$. Then, set $\mathcal{K} = [-n-1, n+1]$ and estimate

$$
\begin{aligned}
\left| \varphi(x) - \chi_{n,p}(x) \right| &= \left| \varphi(x) - \vartheta_n(x) \int_{-\infty}^{x} \vartheta_n(\xi)\, p(\xi) \mathrm{d}\xi \right| \\
&= \vartheta_n(x) \left| \int_{-\infty}^{x} \left(\varphi'(\xi) - \vartheta_n(\xi)\, p(\xi) \right) \mathrm{d}\xi \right| \\
&\leq \begin{cases} 0 & x \in \mathbb{R} \setminus [-n-1, n+1], \\ \varepsilon/3 & x \in [-n-1, n+1]; \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\left| \varphi'(x) - \chi_{n,p}'(x) \right| &\leq \left| \varphi'(x) - \vartheta_n(x)\, \vartheta_n(x)\, p(x) \right| + \left| \vartheta_n'(x) \int_{-\infty}^{x} \vartheta_n(\xi)\, p(\xi) \mathrm{d}\xi \right| \\
&= (\vartheta_n(x))^2 \left| \varphi'(x) - p(x) \right| + \left| \vartheta_n'(x) \int_{-\infty}^{x} \vartheta_n(\xi)\, p(\xi) \mathrm{d}\xi \right| \\
&\leq \begin{cases} 0 & x \in \mathbb{R} \setminus [-n-1, n+1], \\ \varepsilon/3 & x \in [-n, n], \\ \varepsilon & x \in [-n-1, -n] \cup [n, n+1]. \end{cases}
\end{aligned}
\tag{A.1}
$$

To obtain the bound in the latter line above in the case $x \in]n, n+1]$, use $\int_{-\infty}^{n} \vartheta_n(\xi)\, p(\xi)\mathrm{d}\xi = \chi_{n,p}(n)$ and $\varphi(n) = 0$. Hence, using (A.1), for $x \in [n, n+1]$, we have

$$
(\vartheta_n(x))^2 \left| \varphi'(x) - p(x) \right| \leq \frac{\varepsilon}{6n+6};
$$

$$\left| \vartheta_n'(x) \int_{-\infty}^{x} \vartheta_n(\xi)\, p(\xi)\mathrm{d}\xi \right| \leq \left| \vartheta_n'(x) \int_{-\infty}^{n} \vartheta_n(\xi)\, p(\xi)\mathrm{d}\xi \right| + \left| \vartheta_n'(x) \int_{n}^{x} \vartheta_n(\xi)\, p(\xi)\mathrm{d}\xi \right|$$

$$\leq \frac{2\,\varepsilon}{3} + 2\,\frac{\varepsilon}{6n+6},$$

completing the proof. $\qquad\square$

Lemma A.3. (Sard's Lemma [40, Chapter II, Theorem 3.1])

Let $f \in \mathbf{C}^k(\mathbb{R}^{n_1}; \mathbb{R}^{n_2})$. *If* $k > \max\{n_1 - n_2, 0\}$, *then* $\mathcal{L}\big(f\big(\{x \in \mathbb{R}^{n_1} : Df(x)$ *is not surjective*$\}\big)\big) = 0$.

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