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Global existence of a nonlinear wave equation arising from Nordström's theory of gravitation

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Abstract. We show global existence of classical solutions for the nonlinear Nordström theory with a source term and a cosmological constant under the assumption that the source term is small in an appropriate norm, while in some cases no smallness assumption on the initial data is required. In this theory, the gravitational field is described by a single scalar function that satisfies a certain semi-linear wave equation. We consider spatial periodic deviation from the background metric, that is why we study the semi-linear wave equation on the three-dimensional torus $\mathbb{T}^3$ in the Sobolev spaces $H^m(\mathbb{T}^3)$. We apply two methods to achieve the existence of global solutions, the first one is by Fourier series, and in the second one, we write the semi-linear wave equation in a non-conventional way as a symmetric hyperbolic system. We also provide results concerning the asymptotic behavior of these solutions and, finally, a blow-up result if the conditions of our global existence theorems are not met.

Contents

1. Introduction

The purpose of the work presented here is to prove global existence and uniqueness of classical solutions and its asymptotic behavior of a semi-linear wave equation with

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damping terms. This wave equation arises in the context of the nonlinear Nordström theory of gravity, which we shall describe in what follows.

The first fully relativistic, consistent, theory of gravitation was a scalar theory developed by Nordström [20], where the gravitational field is described by a nonlinear hyperbolic equation for the scalar field ϕ. Although the theory is not in agreement with observations it provides, due to its nonlinearity, some interesting mathematical challenges. Surprisingly, this theory has never been mathematically investigated, although its linear version coupled to the Euler equations has been studied by Speck [25] and coupled to the Vlasov equation by Calogero [5] and others [1,8,9,13,31].

We follow here the geometric reformulation provided by Einstein–Fokker [11] and will use the Euler equations as a matter model. See also Straumann [27, Chap. 2.] for a modern representation of that theory. The basic idea of this theory is that the physical metric $g_{\alpha\beta}$ is related to the Minkowski metric $\eta_{\alpha\beta}$ by the following conformal transformation.

$$g_{\alpha\beta} = \phi^2 \eta_{\alpha\beta}, \tag{1.1}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. The matter is described by an energy-momentum tensor, which in the case of a perfect fluid takes the form

$$T^{\alpha\beta} = (\epsilon + p)\, u^\alpha u^\beta + p g^{\alpha\beta},$$

where ϵ denotes the energy density, p the pressure, and u^α is the unit timelike vector which satisfies

$$g_{\alpha\beta} u^\alpha u^\beta = -1.$$

The field equation by Einstein and Fokker takes the following form

$$R = T, \tag{1.2}$$

here we set the relevant constants to one, the Ricci scalar is denoted by R, and the trace of the fundamental energy tensor by $T = g_{\alpha\beta} T^{\alpha\beta}$. Using Eq. (1.1), the Ricci scalar takes the form

$$R = -6 \frac{\Box \phi}{\phi^3}, \qquad \Box \stackrel{\mathrm{def}}{=} \eta^{\alpha\beta} \partial_\alpha \partial_\beta. \tag{1.3}$$

While the Euler equations take the form

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{1.4}$$

where ∇_α is the covariant derivative associated with $g_{\alpha\beta}$. Combining Eqs. (1.2), (1.3), and (1.4), the Euler-Nordström system takes the following form

$$\begin{aligned} \Box \phi &= -\frac{1}{6} T \phi^3 \\ \nabla_\alpha T^{\alpha\beta} &= 0. \end{aligned} \tag{1.5a}$$

Remark 1 (Different form of the field equation). We want to point out that it is possible to consider a slightly different conformal transformation (see for example [5] or [25]), namely

$$g_{\alpha\beta} = e^{2\psi}\eta_{\alpha\beta},$$

which leads to an equivalent nonlinear wave equation

$$\Box\psi + (\nabla\psi)^2 = -\frac{1}{6}e^{2\psi}T.$$

1.1. The field equation with cosmological constant and the background solutions

In what follows we modify the field equation (1.5a) by adding a term that corresponds to the cosmological constant Λ in General Relativity in the following way,

$$\Box\phi = -\frac{1}{6}T\phi^3 - \Lambda\phi. \tag{1.6}$$

This choice is motivated by the properties of explicit solutions which are homogeneous and isotropic, namely that these properties are very similar to the ones of Euler–Einstein (see, e.g., [6, Chap. V], [22, Chap. 10]), and Euler–Poisson ([4]), which we will discuss below.

We denote an isotropic and homogeneous vacuum background solution by $\mathring{\phi}$, and for convenience we set $\varkappa^2 = \Lambda > 0$. Homogeneity implies that the function $\mathring{\phi}$ depends just on t, while the fact that the solution describes vacuum leads to the conclusion that $T \equiv 0$. Therefore, Eq. (1.6) reduces to

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathring{\phi} = -\varkappa^2\mathring{\phi}.$$

This differential equation has a general solution of the form $\mathring{\phi} = Ae^{\varkappa t} + Be^{-\varkappa t}$. Since we want that our solution has similar behavior to the so-called flat de Sitter solution in general relativity (see for example [6, Chap. V]), namely, that $\mathring{\phi}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathring{\phi}$ are positive, we chose

$$\mathring{\phi}(t) = e^{\varkappa t}$$

as the background solution. Considering also the part $Be^{-\varkappa t}$ would complicate the analysis but should not change the global behavior of the solutions, that is why we are neglecting this term.

We now study small deviations from the background solution $\mathring{\phi}$. So we make the following Ansatz

$$\phi = \mathring{\phi} + \Psi = e^{\varkappa t} + \Psi, \tag{1.7}$$

where Ψ denotes the deviation from the background. Then Ψ satisfies the following equation

$$\Box\phi = \Box(e^{\varkappa t} + \Psi) = -\varkappa^2 e^{\varkappa t} + \Box\Psi = -\frac{1}{6}T(e^{\varkappa t} + \Psi)^3 - \varkappa^2(e^{\varkappa t} + \Psi).$$

Thus, Ψ satisfies the initial value problem

$$\begin{cases} \Box\Psi = -\frac{1}{6}T(e^{\varkappa t}+\Psi)^3 - \varkappa^2\Psi \\ \Psi(0,x) = \Psi_0(x),\ \partial_t\Psi(0,x) = \Psi_1(x) \end{cases}. \tag{1.8}$$

Our goal is:

a. To show global existence of classical solutions for Eq. (1.8) demanding a small source term T and small initial data.
b. To show that for large t, the metric $\phi^2\eta_{\alpha\beta}$ approaches asymptotically the background metric $e^{2\varkappa t}\eta_{\alpha\beta}$, in the following sense,

$$\lim_{t\to\infty}\frac{\phi(t,x)}{\mathring{\phi}(t)} = \lim_{t\to\infty}\frac{e^{\varkappa t}+\Psi(t,x)}{e^{\varkappa t}} \approx 1.$$

Note that if Ψ is small, then $(e^{\varkappa t}+\Psi)^3 \sim e^{3\varkappa t}$, and this term growths very rapidly and might prevent that the solution exists for all time. So in order to achieve the desired asymptotic behavior of Ψ, expressed by Eq. (1.8), we multiply ϕ by $e^{-\varkappa t}$, then from equality (1.7) we conclude that $e^{-\varkappa t}\phi = 1 + e^{-\varkappa t}\Psi$, and therefore, we set

$$\Omega \stackrel{\text{def}}{=} e^{-\varkappa t}\Psi.$$

The resulting equation for Ω takes the form

$$\begin{aligned}
\partial_t\Omega &= \partial_t(e^{-\varkappa t}\Psi) = e^{-\varkappa t}\partial_t\Psi - \varkappa e^{-\varkappa t}\Psi = e^{-\varkappa t}\partial_t\Psi - \varkappa\Omega,\\
\partial_t^2\Omega &= e^{-\varkappa t}\partial_t^2\Psi - 2\varkappa e^{-\varkappa t}\partial_t\Psi + \varkappa^2 e^{-\varkappa t}\Psi = e^{-\varkappa t}\partial_t^2\Psi - 2\varkappa(\partial_t\Omega + \varkappa\Omega) + \varkappa^2\Omega\\
&= e^{-\varkappa t}\partial_t^2\Psi - 2\varkappa\partial_t\Omega - \varkappa^2\Omega,
\end{aligned}$$

or

$$-e^{-\varkappa t}\partial_t^2\Psi = -\partial_t^2\Omega - 2\varkappa\partial_t\Omega - \varkappa^2\Omega.$$

Thus, we have obtained

$$e^{-\varkappa t}\Box\Psi = \Box\Omega - 2\varkappa\partial_t\Omega - \varkappa^2\Omega = -\frac{1}{6}\widehat{T}e^{-\varkappa t}(e^{\varkappa t}+\Psi)^3 - \varkappa^2\Omega,$$

or

$$\Box\Omega - 2\varkappa\partial_t\Omega = -\frac{1}{6}T(t,x)e^{2\varkappa t}(1+\Omega)^3. \tag{1.9}$$

We wish to show the existence of global classical solutions for system (1.9) demanding a small source term $T(t,x)$. On the one hand, the term $e^{2\varkappa t}$ seems to hamper the proof of the desired global existence, but on the other hand, we have obtained a good dissipative term of the form $-2\varkappa\partial_t\Omega$. This is why we perform the transformation $\widetilde{T} = \widetilde{g}_{\alpha\beta}\widetilde{T}^{\alpha\beta} = e^{3\kappa t}T$, which also implies that the right-hand of the wave equation

(1.9) takes the form $-\frac{1}{6}e^{-\varkappa t}\widetilde{T}(1+\Omega)^3$. If Ω remains bounded, then the right-hand side will tend to zero. That is why we finally consider the following system

$$-\partial_t^2\Omega - 2\varkappa\partial_t\Omega + \Delta\Omega = -e^{-\varkappa t}a(t,x)(1+\Omega)^3 \tag{1.10a}$$

$$(\Omega(0,x), \partial_t\Omega(0,x)) = (f(x), g(x)), \tag{1.10b}$$

where we have denoted $\frac{1}{6}\widetilde{T}$ by $a(t,x)$.

Remark 2. (The scaling and the Euler equations) The above scaling of the trace of the energy-momentum tensor will change the Euler equations. That is why this scaling has to be taken into account for the coupled Euler-Nordström system, which we want to treat in a forthcoming paper. Moreover, it turns out that we also need to scale the metric and the velocity as follows: $\widetilde{g}_{\alpha\beta} = e^{-2\kappa t}g_{\alpha\beta} = e^{-2\kappa t}\phi^2\eta_{\alpha\beta}$, and $\widetilde{u}^\alpha = e^{\varkappa t}u^\alpha$, which is compatible with the scaling $\widetilde{T} = e^{3\kappa t}T$.

In what follows we will not consider the Euler-Nordström system but instead consider the fluid as a given source of the field equations, and therefore, we will consider the right-hand side of Eq. (1.2) as a given function of (t,x). A similar setting was considered by H. Friedrich for the Einstein vacuum equations with positive cosmological constant, in which he proved global existence of classical solutions for small initial data [14].

We point out that we require the deviation $e^{\varkappa t}\Omega = \Psi$ to be spatially periodic, and that is why we study the Cauchy problem (1.10a)–(1.10b) in the Sobolev spaces $H^m(\mathbb{T}^3)$. We shall decompose this space into two orthogonal components, namely, $H^m(\mathbb{T}^3) = \mathbb{R} \oplus \dot{H}^m(\mathbb{T}^3)$, where the second component consists of all functions with zero mean over the torus $\mathbb{T}^3$. The reason for this decomposition is that the homogeneous space $\dot{H}^m(\mathbb{T}^3)$ possesses some convenient features for our energy estimates and seems best suited for our setting. However, there is a technical difficulty in using these spaces, namely the presence of the nonlinear term $(1+\Omega)^3$, that cannot belong to the homogeneous spaces. We solve this problem by performing a projection of our variables into a part that belongs to these spaces, and another part that satisfies an ordinary differential equation, we refer to Sect. 2 for details. As we will see, in Sect. 2, these spaces possess some nice features, such as Proposition 1, that simplify the energy estimates which we shall use for proving our results.

Having set up the problem, we outline the structure of our paper and summarize our main results.

In Sect. 2, we introduce the necessary mathematical tools, such as homogeneous and non-homogeneous Sobolev spaces on the torus $\mathbb{T}^3$. Using Fourier series, in Sect. 3, we obtain, for small initial data and a small source term, global existence and uniqueness of these solutions in the $H^m(\mathbb{T}^3)$ spaces (see Theorem 2).

We then turn, in Sect. 4, to the theory of symmetric hyperbolic systems. We write the wave equation in a slightly unorthodox way as a symmetric hyperbolic system (see system (4.3)) and then prove global existence, uniqueness, and asymptotic decay for a small source term, but not necessarily small initial data, (see Theorem 3). The reason we consider the semi-linear wave equation (1.10a) in the framework of the theory of symmetric hyperbolic systems is that in the future we want to consider the coupled Euler-Nordström system, and we know already the Euler equations can be cast into that form (see [3]). Finally, in the Sect. 5, we show that if the source term is not small, then the corresponding solutions blow up in finite time. It turns out, however, that for the proof of the blow-up result we need that $(1 + \Omega(t, x)) \geq 0$, which seems natural if the initial data are positive. However, for that being true, it is not sufficient to only assume the initial data to be positive; additional conditions are needed that also result in a more elaborated proof. That has been taken care of in the last section.

2. Mathematical preliminaries

2.1. Sobolev spaces on the torus $\mathbb{T}^3$

We consider the solutions on the torus $\mathbb{T}^3$ using Sobolev spaces H^m where m is a nonnegative integer (see, e.g., [28, Chap 3.1], [23, Chap. 5.10]). It is natural to represent functions on the torus by Fourier series and their norms by Fourier coefficients. For a function f, its Fourier series is given by

$$f(x) = \sum_{k\in\mathbb{Z}^3} \widehat{f_k} e^{ix\cdot k}, \tag{2.1}$$

where

$$\widehat{f_k} = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} f(x) e^{-ix\cdot k} \mathrm{d}x,$$

$x \cdot k = x_1 k_1 + x_2 k_2 + x_3 k_3$, $x \in \mathbb{T}^3$, and $k \in \mathbb{Z}^3$.

The H^m norm is given by

$$\|f\|^2_{H^m(\mathbb{T}^3)} = \|f\|^2_{H^m} = |\widehat{f_0}|^2 + \sum_{k\in\mathbb{Z}^3} |k|^{2m} |\widehat{f_k}|^2. \tag{2.2}$$

The homogeneous Sobolev spaces $\dot{H}^m$ are defined by the semi–norm

$$\|f\|^2_{\dot{H}^m(\mathbb{T}^3)} = \|f\|^2_{\dot{H}^m} = \sum_{k\in\mathbb{Z}^3} |k|^{2m} |\widehat{f_k}|^2. \tag{2.3}$$

We decompose the Sobolev space H^m into two orthogonal components

$$H^m = \mathbb{R} \oplus \dot{H}^m. \tag{2.4}$$

A function $f \in H^m$ belongs to $\dot{H}^m$ if and only if it has a zero mean, that is,

$$\frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} f(x) \mathrm{d}x = 0.$$

By Parseval's identity,

$$\|f\|^2_{L^2(\mathbb{T}^3)} = \sum_{k\in\mathbb{Z}^3} |\widehat{f}_k|^2$$

and since $\widehat{(\partial^\alpha f)}_k = k^\alpha \widehat{f}_k$, the following equivalent holds

$$\|f\|^2_{H^m(\mathbb{T}^3)} \simeq \|f\|^2_{L^2(\mathbb{T}^3)} + \sum_{|\alpha|=m} \|\partial^\alpha f\|^2_{L^2(\mathbb{T}^3)}.$$

We also introduce an inner-product, for two vector valid real functions U and V, we set

$$\langle U, V\rangle_m = \widehat{U}_0 \cdot \widehat{V}_0 + \sum_{k\in\mathbb{Z}^3} |k|^{2m} \left(\widehat{U}_k \cdot \overline{\widehat{V}_k}\right). \tag{2.5}$$

The following proposition which is a certain version of Wirtinger's inequality [10] is a simple consequence of the representation of the homogeneous norm (2.3).

Proposition 1 (Estimate for the gradient). *Let* $\partial_x u = (\partial_1 u, \partial_2 u, \partial_3 u)^\intercal$ *and* $u \in \dot{H}^{m+1}$ *and* $m \geq 0$ *be an integer. Then the following holds*

$$\|u\|_{\dot{H}^{m+1}} = \|\partial_x u\|_{\dot{H}^m}.$$

Proof. Since $|\widehat{(\partial_x u)}_k|^2 = |k|^2\, |\widehat{u}_k|^2$, we obtain by the representation (2.4) of the norm that

$$\|\partial_x u\|^2_{\dot{H}^m} = \sum_{0\neq k\in\mathbb{Z}^3} |k|^{2m} |\widehat{(\partial_x u)}_k|^2 = \sum_{0\neq k\in\mathbb{Z}^3} |k|^{2m}|k|^2|\widehat{u}_k|^2 = \|u\|^2_{\dot{H}^{m+1}}$$

which proves the proposition. □

2.2. Calculus in the Sobolev spaces on the torus $\mathbb{T}^3$

We recall that the known properties in Sobolev spaces, defined over $\mathbb{R}^n$ such as multiplication, embedding and Moser type estimates, hold also for Sobolev spaces defined over the torus $\mathbb{T}^n$, see, e.g., [29, Chap. 13].

Proposition 2 (A nonlinear estimate). *Let* $m > \frac{3}{2}$ *and* $a \in H^m$*, then there is a universal constant* $C(A)$*, depending just on the constants of multiplications and embedding, such that*

$$\left\|a(1+u)^3\right\|_{H^m}, \left\|a(1+u)^3\right\|_{L^\infty} \leq C(A)\|a\|_{H^m} \tag{2.6}$$

for all $u \in H^m$ *with* $\|u\|_{H^m} \leq A$.

Proof. By the multiplication property ([29, Proposition 3.7, Chap. 13]), there is a constant C such that

$$\begin{aligned}\left\|a(1+u)^3\right\|_{H^m} &\leq C\,\|a\|_{H^m}\left\|(1+u)^3\right\|_{H^m} \leq C\,\|a\|_{H^m}\,\|(1+u)\|^3_{H^m}\\ &\leq C\,\|a\|_{H^m}\,(1+\|u\|_{H^m})^3 \leq C\,\|a\|_{H^m}\,(1+A)^3.\end{aligned}$$

Using the embedding $\|u\|_{L^\infty} \leq C\|u\|_{H^m}$, we see that (2.6) holds. □

2.3. Estimate of symmetric hyperbolic system

We shall also need the following property of solution to semi-linear symmetric hyperbolic systems. Consider a symmetric hyperbolic system

$$\partial_t U = \sum_{j=1}^{3} A^j(t,x)\partial_j U + F(t,x,U), \tag{2.7}$$

where the matrices $A^j(t,x)$ are symmetric and $F(t,x,U)$ is a smooth function of U. The next proposition provides a uniform modulus of continuity for the difference $U(t,\cdot) - U_0(\cdot)$ in the H^m norm.

Proposition 3 (Modulus of continuity). *Let $m > \frac{5}{2}$, $A^j \in L^\infty([0,T]; H^m)$ for some positive T and $F(t,x,0) \in L^\infty([0,T]; H^m)$. Assume that $U(t) \in C([0,T]; H^m) \cap C^1([0,T]); H^{m-1})$ is the solution to system (2.7) with initial data $U_0 \in H^m$, then there is a constant $C(\|U_0\|_{H^m})$ such that*

$$\|U(t) - U_0\|_{H^{m-1}} \le C(\|U_0\|_{H^m}) t^{\frac{1}{m}} \quad \textit{for } 0 < t < T.$$

Remark 3. We know from the existence theory for quasilinear symmetric hyperbolic system that the solution U belongs to a certain ball around U_0 in the H^m space (see, e.g., [18,21]). So we may assume that $\|U(t)\|_{H^m} \le \|U_0\|_{H^m} + R$ holds for some positive R and $t \in [0,T]$. The same phenomena appears also for quasi-linear wave equations (see, e.g., [15, Theorem 6.4.11]).

Proof. Let $t < T$, then

$$U(t,x) - U_0(x) = \int_0^t \partial_t U(\tau,x) \mathrm{d}\tau.$$

By the Cauchy Schwarz inequality, it follows that

$$\left| U(t,x) - U_0(x) \right|^2 \le t \int_0^t |\partial_t U(\tau,x)|^2 \mathrm{d}\tau.$$

Hence, we conclude that

$$\|U(t,x) - U_0(x)\|_{L^2}^2 \le t \int_0^t \int |\partial_t U(\tau,x)|^2 \mathrm{d}x \mathrm{d}\tau \le t^2 \|\partial_t U\|_{L^\infty([0,T];L^2)}^2. \tag{2.8}$$

Since U satisfies system (2.7), we obtain

$$\begin{aligned} \|\partial_t U(t)\|_{L^2} \le \|\partial_t U(t)\|_{H^{m-1}} &\le \sum_{j=1}^{3} \|A^j(t,x)\partial_j U(t)\|_{H^{m-1}} + \|F(t,x,U(t))\|_{H^{m-1}} \\ &\le C \sum_{j=1}^{3} \|A_j(t,\cdot)\|_{H^m} \|U(t)\|_{H^m} + C(\|U(t)\|_{L^\infty})\|U(t)\|_{H^m} \\ &\quad + \|F(t,\cdot,0)\|_{H^m}. \end{aligned}$$

Here we used the multiplication property and Moser third estimate, see, e.g., [21, Theorem 6.4.1], [28, Proposition 3.9, Chap. 13]. Thus, it follows from the remark that

$$\sup_{[0,T]} \|\partial_t U(t)\|_{L^2} \leq C(\|U_0\|_{H^m}).$$

We now apply the intermediate estimate $\|u\|_{H^r} \leq \|u\|_{H^m}^{m-\frac{r}{m}} \|u\|_{L^2}^{\frac{r}{m}}$ for $0 < r < m$ (see, e.g., [2, Prop. 1.52]) and inequality (2.8), then

$$\begin{aligned} \|U(t) - U_0\|_{H^{m-1}} &\leq \|U(t) - U_0\|_{H^m}^{\frac{(m-1)}{m}} \|U(t) - U_0\|_{L^2}^{\frac{1}{m}} \\ &\leq \|U(t) - U_0\|_{\frac{(m-1)}{m}} C_0(\|U_0\|_{H^m}) t^{\frac{1}{m}}. \end{aligned}$$

Since $\|U(t)\|_{H^m} \leq \|U_0\|_{H^m} + R$, that completes the proof. □

2.4. Gronwall inequality

We shall use the following version of Gronwall's inequality (see e. g. [2]).

Lemma 1 (Gronwall's inequality). *Let g be a C^1 function, f, F, and A continuous function in the interval* $[t_0, T]$. *Suppose that for $t \in [t_0, T]$ g obeys*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} g^2(t) \leq A(t) g^2(t) + f(t) g(t).$$

Then for $t \in [t_0, T]$, we have

$$g(t) \leq e^{\int_{t_0}^t A(\tau)\mathrm{d}\tau} g(t_0) + \int_{t_0}^t e^{\int_\tau^t A(s)\mathrm{d}s} f(\tau)\mathrm{d}\tau.$$

3. The Cauchy problem for a semi-linear wave equation using Fourier series

In the following section we shall investigate the Cauchy problem (1.10a)–(1.10b), however, for convenience, we multiply the wave equation by -1 and denote the unknown by u instead of Ω, which results in the following semi-linear wave equation

$$\partial_t^2 u + 2\varkappa \partial_t u - \Delta u = e^{-\varkappa t} a(t,x)(1+u)^3 \tag{3.1a}$$

$$(u(0,x), \partial_t u(0,x)) = (f(x), g(x)). \tag{3.1b}$$

Here $\varkappa$ is a positive constant, while $a(t,x)$ is a smooth function as we discussed in Sect. 1.1.

We are interested in proving the global existence of classical solutions to the Cauchy problem (3.1a)–(3.1b) for small initial data and $a(t,x)$. We also note, that the Cauchy problem (3.1a)–(3.1b) has some similarities with the Cauchy problem of the damped semi-linear wave equation

$$\partial_t^2 u + 2\varkappa\partial_t u - \Delta u = |u|^p, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \tag{3.2a}$$

$$(u(0,x), \partial_t u(0,x)) = (f(x), g(x)). \tag{3.2b}$$

for which it is known that for $2 \leq p \leq 3$ there exist global solutions for small initial data, for further details we refer to [12,30] and the references therein. However, we did not find in the literature any results concerning the Cauchy problem (3.2a)–(3.2b) on the torus. There is however another difference, between these two Cauchy problems (3.2a)–(3.2b) and (3.1a)–(3.1b). In Eq. (3.1a), the function $a(t,x)$ is essential, in the sense that global existence depends on the smallness of this function, while the structure of the nonlinear term is of less importance.

3.1. Local existence

Before we are going to present our results concerning the global existence of classical solutions we shall discuss the question of local existence and uniqueness of the initial value problem (3.1a)–(3.1b). There are well-known local existence and uniqueness theorems for quasilinear wave equations of the form

$$g^{\alpha\beta}(u,u')\partial_\alpha\partial_\beta u = F(u), \tag{3.3}$$

where $g^{\alpha\beta}(u,u')$ has a Lorentzian signature and $u' = \partial_\alpha u$, $\alpha = 0,1,2,3$, see for example [15, Theorem 6.4.11], [24, Theorem 4.1] and [26, Theorem 5.1]. These references treat the initial value problems in the Sobolev space $H^m(\mathbb{R}^3)$ and under the condition $F(0) = 0$. We consider solutions of Eq. (3.1a) that belong to Sobolev spaces on the torus $\mathbb{T}^3$, $H^m(\mathbb{T}^3)$, and we observe that the right-hand side of (3.3) does not satisfy the condition $F(0) = 0$. Nevertheless, the above existence results can be applied to the Cauchy problem (3.1a)–(3.1b) because of the following reasons.

1. The energy estimates are an indispensable tool for proving local existence for the linearized equation. The energy estimates rely on the formula for integration by parts $\int u\partial_{x_j} v \mathrm{d}x = -\int \partial_{x_j} u v \mathrm{d}x$, which holds for periodic functions and rapidly decreasing functions in $\mathbb{R}^n$. That is why the energy estimates in the above references hold in the Sobolev spaces $H^m(\mathbb{T}^3)$ as well.
2. Moser type inequality, the second important tool, states that $\|F(u) - F(0)\|_{H^m} \leq C\|u\|_{H^m}$ for a sufficiently smooth function F. This nonlinear estimate is valid both for $u \in H^m(\mathbb{R}^3)$ and for $u \in H^m(\mathbb{T}^3)$. In the case the equations are considered on the $\mathbb{R}^3$, the requirement $F(0) = 0$ is needed since the constant function does not belong to Sobolev space $H^m(\mathbb{R}^3)$. However, the situation is different on the torus. Here obviously the constant function belongs to the space $H^m(\mathbb{T}^3)$.

So we conclude that with some minor modifications of [15, Theorem 6.4.11], the following result on local existence and uniqueness.

Theorem 1 (Local existence). *Let $m > \frac{5}{2}$, $a(t,\cdot) \in L^\infty([0,\infty); H^m(\mathbb{T}^3)$, $f \in H^{m+1}(\mathbb{T}^3)$ and $g \in H^m(\mathbb{T}^3)$, then there exists a positive T and a unique solution u to the Cauchy problem (3.1a)–(3.1b) such that*

$$u \in L^\infty([0,T]; H^{m+1}(\mathbb{T}^3) \cap C^{0,1}([0,T]; H^m(\mathbb{T}^3)),$$

where $C^{0,1}$ is a Lipschitz continuous function.

3.2. Global existence

Once the existence and uniqueness of local solutions have been established (by Theorem 1), we turn now to the question of whether global solutions to the Cauchy problem (3.1a)–(3.1b) exist. Again, the energy estimates are the main tool for treating this problem. Those energy estimates are different from the one that has been used for local existence. Our method consists in expanding the solution into Fourier series, which allows us to solve the corresponding ordinary differential equations for the Fourier's coefficients, and use the norm (2.2) to derive the desired estimates.

Our main result in this section is the following theorem.

Theorem 2 (Global existence of classical solutions for small data). *Let $m > \frac{5}{2}$, $f \in H^{m+1}$, $g \in H^m$, and $a \in C([0,\infty); H^m)$. Then there is a suitable constant ε such that if the following holds*

$$\|f\|_{H^{m+1}}, \|g\|_{H^m}, \sup_{[0,\infty)} \|a(t,\cdot)\|_{H^m} < \epsilon,$$

then the Cauchy problem (3.1a)–(3.1b) has a unique global solution of the form

$$u \in C([0,\infty); H^{m+1}).$$

Moreover, there exists a positive constant C_1 such that

$$\|u(t,\cdot)\|_{H^{m+1}} \le C_1. \tag{3.4}$$

3.3. Proof of Theorem 2

The main points and ideas of the proof of Theorem 2 can be described as follows:

1. Obtain an energy estimate for the linearized equation.
2. Use the Banach fixed pointed theorem for the linearized equation.

We start with the energy estimates for the linearized system of Eq. (3.1a). For any function $v \in H^m$ we set

$$F(t,x) \stackrel{\text{def}}{=} a(t,x)\,(1+v)^3$$

and consider the following linear initial value problem

$$\partial_t^2 u + 2\varkappa\partial_t u - \Delta u = e^{-\varkappa t} F(t,x) \tag{3.5a}$$

$$(u(0,x), \partial_t u(0,x)) = (f(x)), g(x)). \tag{3.5b}$$

We now consider the Fourier coefficients

$$\widehat{u}_k(t) = \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} u(t,x)\mathrm{d}x, \quad k \in \mathbb{Z}^3,$$

and the coefficients of the other data of the Cauchy problem (3.5a)–(3.5b) as well. We obtain, by this procedure, for each $k \in \mathbb{Z}^3$ an ordinary differential equation

$$\widehat{u}_k''(t) + 2\varkappa\widehat{u}_k'(t) + |k|^2\widehat{u}_k(t) = e^{-\varkappa t}\widehat{F}_k(t) \tag{3.6a}$$

$$\widehat{u}_k(0) = \widehat{f}_k, \quad \widehat{u}_k'(0) = \widehat{g}_k. \tag{3.6b}$$

We can solve (3.6a)–(3.6b) explicitly, however, since the structure of the solutions depends on $\varkappa$, and in order to work with similar formulas for all $k \neq 0$, we restrict $\varkappa$ to the interval $(0, 1)$. We present the energy estimates in the following proposition

Proposition 4 (Energy estimate for the linearized wave equation). *Let $m \geq 0$ and $0 < \varkappa < 1$, and assume $F \in C([0, \infty); H^m)$, $f \in H^{m+1}$, and $g \in H^m$. Then there exists a unique solution $u \in C([0, \infty); H^{m+1})$ to Eq.* (3.5a) *with initial data* (3.5b)*, and moreover, it obeys*

$$\begin{aligned}
\|u(t,\cdot)\|^2_{H^{m+1}} \leq 2e^{-2\varkappa t}\Big\{&(1+2\varkappa^2)(1+t^2)\|f\|^2_{\dot{H}^{m+1}} + 2(1+t^2)\|g\|^2_{\dot{H}^m} \\
&+t(1+t^2)\int_0^t \|F(\tau,\cdot)\|^2_{\dot{H}^m}\,d\tau\Big\} \\
&+ \widehat{f}_0^2 + \widehat{g}_0^2\left(\frac{1-e^{-2\varkappa t}}{2\varkappa}\right)^2 + \frac{1}{4}\sup_{\tau\in[0,t]}|\widehat{F}_0(\tau)|^2\left(\frac{1-e^{-\varkappa t}}{\varkappa}\right)^4.
\end{aligned} \tag{3.7}$$

Proof. For each $k \neq 0$ the solution of the initial value problem of the ordinary differential equations (3.6a)–(3.6b) is then given by

$$\begin{aligned}
\widehat{u}_k(t) &= e^{-\varkappa t}\left\{\widehat{f}_k\cos\left(\sqrt{|k|^2-\varkappa^2}\,t\right) + \frac{\widehat{g}_k + \varkappa\widehat{f}_k}{\sqrt{|k|^2-\varkappa^2}}\sin\left(\sqrt{|k|^2-\varkappa^2}\,t\right)\right\} \\
&\quad + \frac{1}{\sqrt{|k|^2-\varkappa^2}}\int_0^t e^{-\varkappa(t-\tau)}\sin\left(\sqrt{|k|^2-\varkappa^2}\,(t-\tau)\right)e^{-\varkappa\tau}\widehat{F}_k(\tau)\mathrm{d}\tau \\
&= e^{-\varkappa t}\left\{\widehat{f}_k\cos\left(\sqrt{|k|^2-\varkappa^2}\,t\right) + \frac{\widehat{g}_k + \varkappa\widehat{f}_k}{\sqrt{|k|^2-\varkappa^2}}\sin\left(\sqrt{|k|^2-\varkappa^2}t\right)\right. \\
&\quad \left. + \frac{1}{\sqrt{|k|^2-\varkappa^2}}\int_0^t \sin\left(\sqrt{|k|^2-\varkappa^2}(t-\tau)\right)\widehat{F}_k(\tau)\mathrm{d}\tau\right\},
\end{aligned} \tag{3.8}$$

and for $k = 0$,

$$\widehat{u}_0(t) = \widehat{f}_0 + \widehat{g}_0\left(\frac{1-e^{-2\varkappa t}}{2\varkappa}\right) + \frac{1}{2\varkappa}\int_0^t\left(1-e^{-2\varkappa(t-\tau)}\right)e^{-\varkappa\tau}\widehat{F}_0(\tau)\mathrm{d}\tau. \tag{3.9}$$

We shall now estimate $\|u\|^2_{H^{m+1}}$ by the formula (2.2). For $k \neq 0$, we conclude from equality (3.8) and the trivial inequality $(a+b+c)^2 \leq 2(a^2+b^2+c^2)$ that

$$\begin{aligned}
|k|^{2(m+1)}|\widehat{u}_k(t)|^2 \leq 2e^{-2\varkappa t}|k|^{2(m+1)} \Bigg\{ & |\widehat{f}_k|^2 \left(\cos\left(\sqrt{|k|^2-\varkappa^2}\, t\right)\right)^2 \\
& + \left|\widehat{g}_k + \varkappa \widehat{f}_k\right|^2 \left(\frac{\sin\left(\sqrt{|k|^2-\varkappa^2}\, t\right)}{\sqrt{|k|^2-\varkappa^2}}\right)^2 \\
& + \left(\int_0^t \frac{\sin\left(\sqrt{|k|^2-\varkappa^2}\,(t-\tau)\right)}{\sqrt{|k|^2-\varkappa^2}} \widehat{F}_k(\tau)\mathrm{d}\tau\right)^2 \Bigg\} \\
& = I_k + II_k + III_k.
\end{aligned}$$

The first one is easy to estimate, and we obtain that

$$I_k \leq 2e^{-2\varkappa t}|k|^{2(m+1)}|\widehat{f}_k|^2. \tag{3.10}$$

For the second and third term, we use the inequality $\sqrt{1+\xi^2}|\sin(\xi t)| \leq \xi\sqrt{1+t^2}$, with $\xi = \sqrt{|k|^2-\varkappa^2}$, that implies

$$\frac{\sin\left(\sqrt{|k|^2-\varkappa^2}\, t\right)}{\sqrt{|k|^2-\varkappa^2}} \leq \sqrt{\frac{1+t^2}{1+\xi^2}} = \sqrt{\frac{1+t^2}{|k|^2+1-\varkappa^2}} \leq \frac{\sqrt{1+t^2}}{|k|}.$$

Hence,

$$\begin{aligned}
II_k &\leq 2e^{-2\varkappa t}|k|^{2m}|\widehat{g}_k + \varkappa\widehat{f}_k|^2(1+t^2) \\
&\leq 2e^{-2\varkappa t}|k|^{2m}\left(2|\widehat{g}_k|^2 + 2\varkappa^2|\widehat{f}_k|^2\right)(1+t^2)
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
III_k &\leq 2e^{-2\varkappa t}|k|^{2m} t\int_0^t (1+(t-\tau)^2|\widehat{F}_k(\tau)|^2\mathrm{d}\tau \\
&\leq 2e^{-2\varkappa t}|k|^{2m}|t(1+t^2)\int_0^t |\widehat{F}_k(\tau)|^2\mathrm{d}\tau.
\end{aligned} \tag{3.12}$$

We now turn to the zero's term (3.9), and we start with the integral term of (3.9),

$$\left|\frac{1}{2\varkappa}\int_0^t \left(1-e^{-2\varkappa(t-\tau)}\right)e^{-\varkappa\tau}\widehat{F}_0(\tau)\mathrm{d}\tau\right| \leq \frac{1}{2}\left(\frac{1-e^{-\varkappa t}}{\varkappa}\right)^2 \sup_{[0,t]}|\widehat{F}_0(\tau)| \tag{3.13}$$

This leads to

$$|\widehat{u}_0(t)|^2 \leq 2\left\{\widehat{f}_0^2 + g_0^2\left(\frac{1-e^{-2\varkappa t}}{2\varkappa}\right)^2 + \frac{1}{4}\left(\frac{1-e^{-\varkappa t}}{\varkappa}\right)^4\left(\sup_{[0,t]}|\widehat{F}_0(\tau)|\right)^2\right\}. \tag{3.14}$$

Summing up the inequalities (3.10), (3.11), (3.12) and (3.14) imply that inequality (3.7) holds, and this completes the proof of Proposition 4. □

We turn now to prove the main result of this section, namely, the proof of Theorem 2.

Proof of Theorem 2 by a fixed point argument. Based on the energy estimate (3.7) of the solution to the linear Cauchy problem (3.5a)–(3.5b), we shall show the existence of classical solutions to the Cauchy problem (3.1a)–(3.1b) in the interval $[0,\infty)$ in the Sobolev space H^{m+1} for $m > \frac{5}{2}$, under the assumption that the initial data, as well as $a(t,x)$, are sufficiently small. In order to achieve this, we define a linear operator

$$\mathscr{L} : C([0,\infty); H^{m+1}) \to C([0,\infty); H^{m+1}),$$

as follows. Let $u = \mathscr{L}(v)$ be the solution to the linear equation

$$\partial_t^2 u + 2\varkappa\partial_t u - \Delta u = e^{-\varkappa t}a(t,x)(1+v)^3$$
$$(u(0,x), \partial_t u(0,x)) = (f(x), g(x)).$$

Next, for $R > 0$ we define a bounded set $B_R \subset H^{m+1}$ as follows

$$B_R = \left\{v(t,\cdot) \in C([0,\infty); H^{m+1}) : \sup_{[0,\infty)} \|v(t,\cdot)\|_{H^{m+1}} \leq R,\ v(0,x) = f(x), \partial_t v(0,x) = g(x)\right\}.$$

Obviously, the ball B_R is a closed set in the Banach space $C([0,\infty); H^{m+1})$, and that is why we can apply the Banach fixed point theorem to the operator $\mathscr{L}$, which will enable us to prove the existence of global solutions. In order to apply the Banach fixed point theorem, we need to show:

(a) $\mathscr{L} : B_R \to B_R$, that is, $\mathscr{L}$ maps the ball into itself.
(b) $\mathscr{L} : B_R \to B_R$ is a contraction.

We start with a):

We shall use the energy estimate provided by Proposition 2. So we set

$$M_1 = \max\{2e^{-2\varkappa t}\left((1+2\varkappa^2(1+t^2)\right) : t \geq 0\};$$
$$M_2 = \max\{4e^{-2\varkappa t}\left(1+t^2\right) : t \geq 0\};$$
$$M_3 = \max\{e^{-2\varkappa t}\left(t^2(1+t^2\right)) : t \geq 0\}.$$

Using standard calculus in $H^m(\mathbb{T}^3)$, there is a constant $C(R)$ such that

$$\|(1+v)^3\|_{L^\infty} \leq C_e \|(1+v)^3\|_{H^m} \leq C(R)$$

for any $v \in B_R$, here C_e is the constant of the embedding $L^\infty \hookrightarrow H^m$. We first estimate the integral term and $\widehat{F}_0$ of the right-hand side of (3.7). So

$$\int_0^t \|F(\tau,\cdot)\|^2_{\dot{H}^m} \mathrm{d}\tau \leq t \sup_{[0,t]} \|a(\tau,\cdot)(1+v(\tau,\cdot)^3\|^2_{\dot{H}^m} \leq t C_m^2 \sup_{[0,\infty)} \|a(\tau,\cdot)\|^2_{\dot{H}^m} C^2(R),$$

where C_m is the constant of the multiplication in the Sobolev space H^m. Now,

$$\widehat{F}_0(\tau) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} a(\tau,x)(1+v(\tau,x))^3 \mathrm{d}x, \tag{3.16}$$

By Jensen's inequality (see e. g. [19, Ch. 2]),

$$\begin{aligned} |\widehat{F}_0(\tau)|^2 &= \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} a^2(\tau,x)(1+v(\tau,x))^6 \mathrm{d}x \leq \|(1+v(\tau,\cdot))^3\|^2_{L^\infty} \|a(\tau,\cdot)\|^2_{L^2} \\ &\leq C^2(R)\|a(\tau,\cdot)\|^2_{H^m}. \end{aligned}$$

Now, by Proposition 4, inequality (3.7), $u = \mathscr{L}(v)$ satisfies the inequality

$$\begin{gathered} \|u(t,\cdot)\|^2_{H^{m+1}} \leq M_1 \|f\|^2_{\dot{H}^{m+1}} + M_2 \|g\|^2_{\dot{H}^m} + M_3 C_m^2 C^2(R) \|a(\tau,\cdot)\|^2_{H^m} \\ \widehat{f}_0^2 + \frac{\widehat{g}_0^2}{4\varkappa^2} + \frac{1}{4\varkappa^2} C^2(R)\|a(\tau,\cdot)\|^2_{H^m} \leq R^2, \end{gathered}$$

if

$$\|f\|^2_{H^{m+1}} \leq \frac{R^2}{4\max\{M_1, 1\}}, \tag{3.17}$$

$$\|g\|^2_{H^m} \leq \frac{R^2}{4\max\{M_2, \frac{1}{4\varkappa^2}\}} \tag{3.18}$$

and

$$\sup_{[0,\infty)} \|a(t,\cdot)\|^2_{H^m} \leq \frac{R^2}{2C_m^2 C^2(R)} \frac{1}{4\max\{M_3, \frac{1}{4\varkappa^4}\}}. \tag{3.19}$$

Thus, $\mathscr{L}$ maps the ball into itself provided that (3.17), (3.18) and (3.19) hold.

b) Contraction: Let $w = \mathscr{L}(v_1) - \mathscr{L}(v_2)$, then w satisfies

$$\begin{aligned} &\partial_t^2 w + 2\varkappa \partial_t w - \Delta u = e^{-\varkappa t} a(t,x)\left((1+v_1)^3 - (1+v_2)^3\right), \\ &(w(0,x), \partial_t w(0,x)) = (0,0). \end{aligned}$$

By the energy estimate (3.7), we obtain that

$$\begin{aligned} \|w\|^2_{H^{m+1}} \leq{}& 2e^{-2\varkappa t} t^2(1+t^2) \sup_{[0,t]} \|a(\tau,\cdot)\left((1+v_1(\tau,\cdot))^3 - (1+v_2(\tau,\cdot))^3\right)\|^2_{H^m} \\ &+ \frac{1}{4\varkappa^4} \sup_{[0,t]} |\widehat{F}_0(\tau)|^2. \end{aligned}$$

Note that

$$\left((1+v_1)^3-(1+v_2)^3\right)=(v_1-v_2)\left(3+3(v_1+v_2)+(v_1^2+v_1v_2+v_2^2)\right),$$

and that similar to Eq. (3.16) we obtain

$$\begin{aligned}|\widehat{F}_0(\tau)|^2 \le \|a(\tau,\cdot)\|_{H^m}^2 \|(v_1-v_2)(\tau,\cdot)\\ \times\left(3+3(v_1+v_2)+(v_1^2+v_1v_2+v_2^2)\right)(\tau,\cdot)\|_{L^\infty}^2.\end{aligned}$$

So by the embedding $L^\infty \hookrightarrow H^m$, the multiplication property of H^m and the fact that $v_1, v_2 \in B_R$, there exists a constant $K(R)$ such that

$$\|w(t,\cdot)\|_{H^{m+1}}^2 \le \max\{M_3, \frac{1}{4\varkappa^2}\}K^2(R)\sup_{[0,\infty)}\|(v_1-v_2)(t,\cdot)\|_{H^m}^2 \sup_{[0,\infty)}\|a(t,\cdot)\|_{H^m}^2$$

holds. Thus, the operator $\mathscr{L} : B_R \to B_R$ is a contraction provided that

$$\sup_{[0,\infty)}\|a(t,\cdot)\|_{H^m}^2 \le \frac{1}{2}\frac{1}{\max\{M_3, \frac{1}{4\varkappa^2}\}K^2(R)}. \tag{3.20}$$

So let ϵ be the minimum of the upper-bounds (3.17)–(3.20), then the existence of a unique global solution follows from the application of the Banach fixed point theorem. The solution belongs to the ball B_R, and therefore, inequality (3.4) holds. That completes the proof of Theorem 2. □

4. The wave equation as a modified symmetric hyperbolic system

In this section, we investigate the questions of global existence and asymptotic decay of classical solutions to the Cauchy problem (3.1a)–(3.1b) by using the theory of symmetric hyperbolic systems and the corresponding energy estimates. Since the relativistic Euler equations can be written as a symmetric hyperbolic system (see [3]), it will enable us, in the future, to couple the semi-linear equation (3.1a) to the Euler equations (1.4).

It is a well-known fact that wave equations can be cast into symmetric hyperbolic form. It turns out, however, that we need a modification of this standard procedure, which we will outline in the next subsection. With this new system at hand, we are able to prove results similar to those in Sect. 3. There are, however, some important differences between the results in both sections which we have to point out. We do not require that the initial data have to be small, and we can, even, drop the term $e^{-\varkappa t}$ and yet obtain global existence.

However, since we rely, to a certain extent, on properties of the $\dot{H}^m$ spaces and since the right-hand side of the wave equation (1.10a), the term $(1+u)^3 \notin \dot{H}^m$, we will perform a projection on the wave equation that allows us to obtain a system of an ordinary differential equation and a modified wave equation with a right-hand side, that does belong to $\dot{H}^m$.

4.1. The projection of the wave equation

Based on our observations made in Sect. 2.1 about norms of the spaces H^m and $\dot{H}^m$, and in particular the orthogonal decomposition of H^m (2.4), we define the orthogonal projection $P_0 : H^m \to \mathbb{R}$, by

$$\widehat{u}_0 = P_0(u).$$

We denote by u_h the complementary projection, that is,

$$u_h = (Id - P_0)\, u.$$

Since u_h belongs to $\dot{H}^m$, its norm is given by the formula

$$\|u_h\|^2_{H^m} = \sum_{k\neq 0} |k|^{2m}\, |\widehat{u}_k|$$

and obviously, $\langle \widehat{u}_0, u_h\rangle_m = 0$ holds, where the inner product is given by Eq. (2.5).

We apply now the projections P_0 and $\mathrm{Id} - P_0$ to the wave equation (1.10a), that is,

$$P_0 \left\{\partial_t^2 u + 2\varkappa \partial_t u - \Delta u\right\} = P_0 \left\{e^{-\varkappa t} a(t,x)(1+u)^3\right\}$$

and

$$(\mathrm{Id} - P_0) \left\{\partial_t^2 u + 2\varkappa \partial_t u - \Delta u\right\} = (\mathrm{Id} - P_0) \left\{e^{-\varkappa t} a(t,x)(1+u)^3\right\}.$$

Those projections result in the following system

$$\widehat{u}_0'' + 2\varkappa \widehat{u}_0' = e^{-\varkappa t}\widehat{F}_0 \tag{4.1a}$$

$$\partial_t^2 u_h + 2\varkappa \partial_t u_h - \Delta u_h = e^{-\varkappa t}\left(a(t,x)(1+u)^3 - \widehat{F}_0\right), \tag{4.1b}$$

where

$$\widehat{F}_0 = P_0(a(t,x)\,(1+u)^3) = \int_{\mathbb{T}^3} a(t,x)\,(1+u)^3\, \mathrm{d}x.$$

4.2. A semi-linear wave equation written as a symmetric hyperbolic system

The most common way to write the wave equation as a symmetric hyperbolic system is to consider either the vector valued function

$$V = \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \qquad \text{or} \qquad V = \begin{pmatrix} \partial_t u \\ \partial_x u \\ u \end{pmatrix}$$

as an unknown (here $\partial_x u \stackrel{\text{def}}{=} (\partial_1 u, \partial_2 u, \partial_3 u)^\mathsf{T}$). However, in both cases, for a system with damping terms, the energy estimates obtained are not appropriate to show global existence.

We, therefore, introduce a different unknown by setting

$$V \stackrel{\text{def}}{=} \begin{pmatrix} \partial_t u_h + \varkappa u_h \\ \partial_x u_h \end{pmatrix}. \tag{4.2}$$

Then Eq. (4.1b) can be written as a symmetric hyperbolic system as follows

$$\partial_t V = \sum_{k=1}^{3} B^k \partial_k V - \varkappa V + \varkappa^2 \begin{pmatrix} u_h \\ 0 \\ 0 \\ 0 \end{pmatrix} + e^{-\varkappa t} \begin{pmatrix} a(t,x)(1+u)^3 \\ 0 \\ 0 \\ 0 \end{pmatrix} - e^{-\varkappa t} \begin{pmatrix} \widehat{F_0} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.3}$$

where B^k are constant symmetric matrices,

$$B^k = \begin{pmatrix} 0 & \delta_1^k & \delta_2^k & \delta_3^k \\ \delta_1^k & 0 & 0 & 0 \\ \delta_2^k & 0 & 0 & 0 \\ \delta_3^k & 0 & 0 & 0 \end{pmatrix}.$$

4.3. Energy estimates

Definition 1 (*The energy functional*). The energy functional for the unknown V, given by Eq. (4.2) is

$$E(t) \stackrel{\text{def}}{=} \langle V(t), V(t) \rangle_m = \|\partial_t u_h + \varkappa u_h\|_{H^m}^2 + \|\partial_x u_h\|_{H^m}^2. \tag{4.4}$$

Remark 4 (About the definition of the energy). It might look surprising to define the energy as a scalar product in H^m while the vector V as defined in (4.1b) only contains terms that belong to $\dot{H}^m$. We use this notation since the energy estimates contain terms that belong to H^m.

Proceeding in the usual way, suppose V satisfies (4.3), then differentiation of the energy with respect to time results that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \langle \partial_t V(t), V(t)\rangle_m &= \sum_{k=1}^{3} \langle B^k \partial_k V, V\rangle_m - \varkappa \langle V, V\rangle_m + \varkappa^2 \langle u_h, \partial_t u_h + \varkappa u_h\rangle_m \\ &\quad + e^{-\varkappa t}\langle a(t,\cdot)(1+u)^3, \partial_t u_h + \varkappa u_h\rangle_m - e^{-\varkappa t}\langle \widehat{F_0}, \partial_t u_h + \varkappa u_h\rangle_m \\ &= -\varkappa \|V\|_{H^m}^2 + \varkappa^2 \langle u_h, \partial_t u_h + \varkappa u_h\rangle_m + e^{-\varkappa t}\langle a(t,\cdot)(1+u)^3, \partial_t u_h + \varkappa u_h\rangle_m,\end{aligned}$$

since $\dot{H}^m \perp \mathbb{R}$, and since B^k are symmetric and constant, then by integration by parts that $\langle B^k \partial_k V, V\rangle_m = 0$. By the Cauchy Schwarz inequality, we obtain

$$|\langle u_h, \partial_t u_h + \varkappa u_h\rangle_m| \le \|u_h\|_{H^m} \|\partial_t u_h + \varkappa u_h\|_{H^m} \le \|u_h\|_{H^m} \|V\|_{H^m}$$

and

$$|\langle a(1+u)^3, \partial_t u_h + \varkappa u_h\rangle_m| \leq \|a\,(1+u)^3\,\|_{H^m}\|V\|_{H^m},$$

which allows us to conclude, using the definition of the energy (4.4), that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq -\varkappa E(t) + \left\{\varkappa^2\,\|u_h(t)\|_{H^m} + e^{-\varkappa t}\,\left\|a(t,\cdot)((1+u(t))^3\right\|_{H^m}\right\}\sqrt{E(t)}$$

We now apply Gronwall's inequality, Lemma 1, in the interval $[t_0, t]$ and with $A(t) = -\varkappa$, then we obtain

$$\begin{aligned}\sqrt{E(t)} \leq e^{-\varkappa(t-t_0)}\sqrt{E(t_0)} + \varkappa^2\int_{t_0}^{t} e^{-\varkappa(t-s)}\,\|u_h(s)\|_{H^m}\,\mathrm{d}s \\ + \int_{t_0}^{t} e^{-\varkappa(t-s)}e^{-\varkappa s}\,\left\|a(s,\cdot)\,(1+u(s))^3\right\|_{H^m}\mathrm{d}s\end{aligned}. \tag{4.5}$$

Remark 5 (The role of u_h in the a-priori estimates). We observe that the term $1+u = 1+\widehat{u}_0+u_h$ implies that, for a fixed $\widehat{u}_0$ Eq. (4.1b) is not coupled to (4.1a) and it consists only of the unknown u_h. Obviously, u_h is not a solution to the system (4.1a)–(4.1b), but it enables us to obtain important a-priori estimates for the solution.

We are now in a position to apply this energy estimate to show global existence by a bootstrap argument, which is done in the next section.

4.4. Global existence by a bootstrap argument

In this section, we take the initial data in the homogeneous space,

$$\begin{cases} u(x,0) = f(x), & \partial_t u(x,0) = g(x), \\ f \in \dot{H}^{m+1}, & g \in \dot{H}^m. \end{cases} \tag{4.6}$$

This is not essential for the proof, but it makes it somewhat simpler. The following theorem is the main result of this section.

Theorem 3 (Global existence and decay of solutions). *Let* $0 < \varkappa < 1$ *and* $m > \frac{5}{2}$, *let the initial data be as specified by* (4.6), *and* $a \in C([0,\infty); H^m)$. *There exists a suitable constant* ε *such that if*

$$\sup_{[0,\infty)} \|a(t,\cdot)\|_{H^m} < \varepsilon, \tag{4.7}$$

then system (4.1a)–(4.1b)*, or equivalently Eq.* (3.1a)*, with initial data given by* (4.6) *has a unique solution*

$$u \in C([0,\infty); H^{m+1}). \tag{4.8}$$

Moreover

$$\lim_{t\to\infty} \|u(t)\|_{H^{m+1}} \leq \widetilde{\epsilon}, \tag{4.9}$$

where $\widetilde{\epsilon}$ *depends on the smallness condition* (4.7).

Remark 6 (Comparison with Theorem 2). We emphasize that, contrary to Theorem 2, in Sect. 3.2, the initial data are not required to be small.

Remark 7 (About the asymptotic behavior of the metric). Recall that the physical metric has the following form $g_{\alpha\beta} = \phi^2 \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ denotes the Minkowski metric. In Sect. 1.1 we concluded that the background metric has the following form $(e^{\varkappa t})^2 \eta_{\alpha\beta}$. We shall now use the asymptotic estimate (4.9) of the global solutions to compare the asymptotic of the physical metric with the background metric. We remind that $\phi = e^{\varkappa t}(1+u)$, where u is the solution to the Cauchy problem (3.1a)–(3.1b); therefore, we can conclude that the asymptotic behavior of the metric can be described by the following expression

$$(1-\widetilde{\epsilon})^2 \leq \lim_{t\to\infty} \frac{g_{\alpha\beta}(t,x)}{e^{2\varkappa t}\eta_{\alpha\beta}} = \lim_{t\to\infty}(1+u(t,x))^2 \leq (1+\widetilde{\epsilon})^2.$$

The proof of Theorem 3 is based on the following propositions which we present together with their corresponding proofs. We recall that by the existence theorem, Theorem 1, the solution of the system (4.1a)–(4.1b) exists in a certain time interval $[0, T]$.

Proposition 5 (A priori estimates). *Let $0 < \varkappa < 1$, $1 < \beta < \frac{1}{\varkappa}$ and set $\alpha = \|V(0)\|_{H^m}$. Assume the solution $u = \widehat{u}_0 + u_h$ to (4.1a)–(4.1b) with initial data (4.6) exists for $t \in [0, T]$. If $\|a(t,\cdot)\|_{H^m}$ is sufficiently small, then there exists a T^+, $0 < T^+ \leq T$, such that*

$$\sup_{[0,T^+]} \|u(t)\|_{H^m} \leq \alpha\beta \tag{4.10}$$

and

$$E(T^+) \leq E(0). \tag{4.11}$$

Proof. We start with the proof of inequality (4.10). Recall that although we have written $\alpha = \|V(0)\|_{H^m}$, the initial data are in the homogeneous Sobolev, space and therefore by Proposition 1 and (4.4), we obtain

$$\|u(0)\|_{H^{m+1}} = \|f\|_{H^{m+1}} = \|\partial_x f\|_{H^m} \leq \|V(0)\|_{H^m} = \alpha.$$

Hence, since $\beta > 1$, it follows from the existence Theorem 1 and the continuity property of the corresponding solutions, that there exists $0 < T^+ \leq T$ such that

$$\sup_{[0,T^+]} \|u(t)\|_{H^m} \leq \alpha\beta.$$

We now turn to inequality (4.11). For $t \in [0, T^+]$ we observe, using inequality (4.5) that

$$\begin{aligned}\sqrt{E(t)} &\le e^{-\varkappa t}\sqrt{E(0)} + \varkappa^2 \int_0^t e^{-\varkappa(t-s)}\alpha\beta \mathrm{d}s \\ &\quad + \int_0^t e^{-\varkappa(t-s)} e^{-\varkappa s} \left\| a(s,\cdot)\left(1+u(s)\right)^3 \right\|_{H^m} \mathrm{d}s \\ &\le e^{-\varkappa t}\sqrt{E(0)} + \varkappa\left(1-e^{-\varkappa t}\right)\alpha\beta + te^{-\varkappa t}\sup_{[0,t]}\left\| a(s,\cdot)\left(1+u(s)\right)^3\right\|_{H^m}.\end{aligned}$$

A simple algebraic manipulation shows us that, $E(t) \le E(0)$, if

$$\varkappa\left(e^{\varkappa t}-1\right)\alpha\beta + t\sup_{[0,t]}\left\| a(s,\cdot)\left(1+u(s)\right)^3\right\|_{H^m} \le \left(e^{\varkappa t}-1\right)\sqrt{E(0)},$$

or equivalently

$$t\sup_{[0,t]}\left\| a(s,\cdot)\left(1+u(s)\right)^3\right\|_{H^m} \le \left(e^{\varkappa t}-1\right)\alpha\left(\beta-\varkappa\right). \tag{4.12}$$

Since for $s \in [0, T^+]$, we can conclude that $\|u(s)\|_{H^m} \le \alpha\beta \le 2\alpha\beta$, we can apply Proposition 2 with $A = 2\alpha\beta$, that results in

$$\left\| a(s,\cdot)\left(1+u(s)\right)^3\right\|_{H^m} \le C(2\alpha\beta)\|a(s,\cdot)\|_{H^m}. \tag{4.13}$$

We now set

$$\epsilon_0 = \frac{\varkappa\alpha(\beta-\varkappa)}{C(2\alpha\beta)}. \tag{4.14}$$

Since $\beta - \varkappa > 0$, $\epsilon_0 > 0$, we can demand the smallness condition

$$\sup_{[0,\infty)} \|a(t,\cdot)\|_{H^m} \le \epsilon_0. \tag{4.15}$$

We now let $t = T^+$ in inequality (4.12), then by inequality (4.13), condition (4.15), with (4.14), we conclude that

$$\begin{aligned}T^+\sup_{[0,T^+]}\left\|a(s,\cdot)\left(1+u(s)\right)^3\right\|_{H^m} &\le T^+C(2\alpha\beta)\sup_{[0,T^+]}\|a(s,\cdot)\|_{H^m} \le T^+C(2\alpha\beta)\epsilon_0 \\ &= \varkappa T^+\alpha(\beta-\varkappa) \le (e^{\varkappa T^+}-1)\alpha(\beta-\varkappa),\end{aligned} \tag{4.16}$$

holds and consequently (4.16) implies inequality (4.12). This proves (4.11) and completes the proof of the proposition. In the last step, we used the elementary inequality $x \le e^x - 1$. □

Based on Proposition 5 we define

Definition 2 (*Definition of $T^\star$*).

$$T^* = \sup\left\{T : \sup_{[0,T]}\|u(t)\|_{H^m} \le \alpha\beta \text{ and } E(T) \le E(0)\right\}. \tag{4.17}$$

The following proposition plays a central role in proving Theorem 3.

Proposition 6 ($T^\star$ is not finite). *Under the assumptions of Proposition 5, we obtain*

$$T^* = \infty.$$

It is important to note that we need two conditions in the definition of $T^\star$, as Proposition 5 already suggests. The role of these two conditions will become clearer after we finish the proof and we will come back to this point.
Sketch of the proof:

The proof of this proposition is rather long, and as we said, crucial for Theorem 3 and that is why we sketch here its structure. We prove Proposition 6 by a contradiction argument, in other words, we assume that T^* is finite, and then we show that both conditions of (4.17) hold in a larger interval.

The first step of the proof deals with the extension of the solution for $t > T^*$. In the second step, using the inequality (4.11), we show that there exists a $T^\ddagger > T^*$ such that

$$\sup_{[0,T^\ddagger]} \|u(t)\|_{H^m} \leq \alpha\beta. \tag{4.18}$$

With this inequality proven, we are then able, in the third step, to show that

$$E(T^\ddagger) \leq E(0). \tag{4.19}$$

The existence of these inequalities in the interval $[0, T^\ddagger]$ contradicts the definition of T^*. Therefore, we conclude that $T^* = \infty$.

Proof of Proposition 6.
Step 1 We need to extend the solution beyond $T^\star$, so let $\widetilde{u}$ be the solution to Eq. (3.1a) with initial data $\widetilde{u}(T^*, x) = \widetilde{f}(x)$ and $\partial_t\widetilde{u}(T^*, x) = \widetilde{g}(x)$, where $\widetilde{f}(x) = u(T^*, x)$ and $\widetilde{g}(x) = \partial_t u(T^*, x)$. We will show that

$$u(t) \in H^{m+1} \quad \text{for} \quad t \in [0, T^*], \tag{4.20}$$

which implies $\widetilde{f} \in H^{m+1}$, a fact that is needed in order to apply the existence theorem, Theorem 1. To prove (4.20) we apply Proposition 1 to u_h, which allows us to conclude

$$\|u(t)\|^2_{H^{m+1}} = |\widehat{u}_0(t)|^2 + \|u_h(t)\|^2_{H^{m+1}} = |\widehat{u}_0(t)|^2 + \|\partial_x u_h(t)\|^2_{H^m}. \tag{4.21}$$

Hence, since $u_h(t) \in H^m$, we see by Eq. (4.21) that $\partial_x u_h(t) \in H^m$ and we can conclude $u(t) \in H^{m+1}$. Consequently, by the existence theorem, Theorem 1, there exists a $T_1 > T^*$ such that $\widetilde{u}(t)$ exists for $t \in [T^*, T_1]$.
Step 2 We turn now to the proof of inequality (4.18). Since $\|\widetilde{u}(T^*)\|_{H^m} \leq \alpha\beta$, there exits a T_2, $T^* < T_2 \leq T_1$, such that

$$\sup_{[T^\star,T_2]} \|\widetilde{u}(\tau)\|_{H^m} \leq 2\alpha\beta. \tag{4.22}$$

holds. We now set

$$\widetilde{V} = \begin{pmatrix} \partial_t \widetilde{u}_h + \varkappa \widetilde{u}_h \\ \partial_x \widetilde{u}_h \end{pmatrix}, \tag{4.23}$$

then $(\widehat{\widetilde{u}}_0, \widetilde{u}_h)$ solves system (4.1a)–(4.1b) with the initial data

$$\widehat{\widetilde{u}}_0(T^*) = \widehat{\widetilde{f}}_0, \quad \partial_t \widehat{\widetilde{u}}_0(T^*) = \widehat{\widetilde{g}}_0$$
$$\widetilde{u}_h(T^*, x) = \widetilde{f}_h(x), \quad \partial_t \widetilde{u}_h(T^*, x) = \widetilde{g}_h(x).$$

We shall estimate each component of $(\widehat{\widetilde{u}}_0, \widetilde{u}_h)$ separately. We take $0 < \epsilon_1$ such that $1 < \beta - \epsilon_1$ and then we will prove below the following two inequalities:

$$|\widehat{\widetilde{u}}_0(t)| \leq \frac{\alpha\beta\epsilon_1}{2}, \qquad t \in [T^*, T_4], \tag{4.25}$$

and

$$\sup_{[T^*, T_5]} \|\widetilde{u}_h(t)\|_{H^m} \leq \alpha(\beta - \epsilon_1). \tag{4.26}$$

Here $T_4, T_5 \in (T^*, T_2]$. Setting $T^{\ddagger} = \min\{T_4, T_5\}$, and combining (4.25) and (4.26), we obtain

$$\begin{aligned} \sup_{[T^*, T^{\ddagger}]} \|\widetilde{u}(t)\|^2_{H^m} &= \sup_{[T^*, T^{\ddagger}]} |\widehat{\widetilde{u}}_0(t)|^2 + \sup_{[T^*, T^{\ddagger}]} \|\widetilde{u}_h(t)\|^2_{H^m} \\ &\leq \frac{(\alpha\beta\epsilon_1)^2}{4} + (\alpha(\beta - \epsilon_1))^2 \leq (\alpha\beta)^2, \end{aligned}$$

which proves (4.18). The last inequality requires that $\epsilon_1 \leq \frac{8}{\beta+4}$. We start with the estimate of $\widehat{\widetilde{u}}_0(t)$ for $t \in [T^*, T_2]$. Since it satisfies the initial value problem (3.6a)–(3.6b) with $k = 0$, its solution is given by

$$\begin{aligned} \widehat{\widetilde{u}}_0(t) &= \widehat{\widetilde{f}}_0 + \widehat{\widetilde{g}}_0 \left(\frac{1 - e^{-2\varkappa(t - T^*)}}{2\varkappa} \right) \\ &\quad + \frac{1}{2\varkappa} \int_{T^*}^{t} \left(1 - e^{-2\varkappa(t-\tau)}\right) e^{-\varkappa\tau} \widehat{F}_0(\tau) \mathrm{d}\tau, \end{aligned} \tag{4.27}$$

where

$$\widehat{F}_0(\tau) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} a(\tau, x)(1 + \widetilde{u}(\tau, x))^3 \mathrm{d}x.$$

Since $\widehat{\widetilde{f}}_0 = \widehat{u}_0(T^*)$ and the initial data $f, g \in \dot{H}^m$, we conclude by Eq. (3.9) and estimate (3.13) that

$$|\widehat{\widetilde{f}}_0| \leq \frac{1}{2} \left(\frac{1 - e^{-\varkappa T^*}}{\varkappa} \right)^2 \sup_{[0, T^*]} |\widehat{F}_0(t)| \tag{4.28}$$

holds. By Proposition 2, we obtain

$$|\widehat{F}_0(t)| \leq \sup_{[0,T^*]} \|a(t,\cdot)\|_{L^\infty} C(2\alpha\beta). \tag{4.29}$$

Hence, if we require that

$$\sup_{[0,\infty)} \|a(t,\cdot)\|_{L^\infty} \leq \frac{2\alpha\beta\varkappa^2\epsilon_1}{6C(2\alpha\beta)}, \tag{4.30}$$

then we conclude that

$$|\widehat{\widehat{f}}_0| \leq \frac{\alpha\beta\epsilon_1}{6}. \tag{4.31}$$

holds. Before we proceed, we remark, that the smallness condition (4.30) is given in the term of L^∞ norm, but it can easily be formulated in terms of H^m norm by the Sobolev embedding theorem. Next, since $\lim_{t\to T^*}\left(\frac{1-e^{-2\varkappa(t-T^*)}}{2\varkappa}\right) = 0$, there exists a T_3, with $T^* < T_3 \leq T_2$, such that

$$\left|\widehat{g}_0\right| \left(\frac{1-e^{-2\varkappa(t-T^*)}}{2\varkappa}\right) \leq \frac{\alpha\beta\epsilon_1}{6}, \quad t \in [T^*, T_3]. \tag{4.32}$$

For the third term on the left side of Eq. (4.27), we use inequality (4.22), and then by a similar argument used to estimate the term $|\widehat{\widehat{f}}_0|$, we obtain

$$\begin{aligned} &\left|\frac{1}{2\varkappa}\int_{T^*}^t \left(1-e^{-2\varkappa(t-\tau)}\right) e^{-\varkappa\tau}\widehat{F}_0(\tau)\mathrm{d}\tau\right| \\ &\qquad \leq \frac{1}{2\varkappa^2} C(2\alpha\beta) \sup_{[0,\infty)} \|a(t,\cdot)\|_{L^\infty} \leq \frac{\alpha\beta\epsilon_1}{6}, \end{aligned} \tag{4.33}$$

provided that the condition (4.30) is satisfied. Letting $T_4 = \min\{T_2, T_3\}$, we conclude from inequalities (4.31), (4.32), and (4.33) implies that (4.25) holds.

We now turn to prove inequality (4.26): For a fixed $\widehat{\widehat{u}}_0(t)$, the unknown $\widetilde{V}(t)$, as defined in Eq. (4.23), satisfies the symmetric hyperbolic system (4.3), and that is why we can apply Proposition 3 with initial data $V(T^*, x)$. We first observe by Proposition 1 that

$$\begin{aligned} \|\widetilde{u}_h(t)\|_{H^m} &\leq \|\partial_x \widetilde{u}_h(t)\|_{H^{m-1}} \leq \|\widetilde{V}(t)\|_{H^{m-1}} \\ &\leq \|\widetilde{V}(t) - \widetilde{V}(T^*)\|_{H^{m-1}} + \|\widetilde{V}(T^*)\|_{H^{m-1}} \end{aligned} \tag{4.34}$$

holds. By the definition of T^*, we can conclude that $E(T^*) \leq E(0)$ holds, which then implies the inequality $\|V(T^*,\cdot)\|_{H^m} = \sqrt{E(T^*)} \leq \sqrt{E(0)} = \alpha$. We now can apply Proposition 3 to $\|\widetilde{V}(t) - \widetilde{V}(T^*)\|_{H^{m-1}}$ with C_0 depending on α, combine it with inequality (4.34), and then obtain

$$\|\widetilde{u}_h(t)\|_{H^m} \leq C_0(\alpha)(t-T^*)^{\frac{1}{m}} + \alpha \leq \alpha(\beta - \epsilon_1)$$

provided that $t - T^* \leq \left(\frac{\alpha(\beta-\epsilon_1-1)}{C_0(\alpha)}\right)^m$. Thus, (4.26) holds with $T_5 = T^* + \left(\frac{\alpha(\beta-\epsilon_1-1)}{C_0(\alpha)}\right)^m$.

Step 3 It remains to show that $E(t) \leq E(0)$ for $t \in [T^*, T^\ddagger]$, where $T^\ddagger = \min\{T_4, T_5\}$. We will first establish the inequality

$$\begin{aligned}\sqrt{E(t)} \leq e^{-\varkappa(t-T^*)}\sqrt{E(T^*)} + \varkappa\left(1 - e^{-\varkappa(t-T^*)}\right)\alpha\beta \\ +e^{-\varkappa t}(t-T^*)\sup_{[T^*,t]}\left\|a(\tau,\cdot)\left(1+\widetilde{u}(\tau)\right)^3\right\|_{H^m}.\end{aligned} \tag{4.35}$$

Using the energy estimate (4.5) we observe that

$$\begin{aligned}\sqrt{E(t)} \leq e^{-\varkappa(t-T^*)}\sqrt{E(T^*)} + \varkappa^2\int_{T^*}^t e^{-\varkappa(t-\tau)}\,\|\widetilde{u}_h(\tau)\|_{H^m}\,\mathrm{d}\tau \\ +\int_{T^*}^t e^{-\varkappa(t-\tau)}e^{-\varkappa\tau}\left\|a(\tau,\cdot)\left(1+\widetilde{u}(\tau)\right)^3\right\|_{H^m}\mathrm{d}\tau.\end{aligned} \tag{4.36}$$

Since the inequality $\|\widetilde{u}_h(t)\| \leq \alpha\beta$ holds in the interval $[T^*, T^\ddagger]$, then inequality (4.35) follows by inserting this bound into the energy estimate (4.36). From this inequality, we observe that $\sqrt{E(t)} \leq \sqrt{E(0)}$ holds if

$$\begin{aligned}&\varkappa\left(e^{\varkappa(t-T^*)}-1\right)\alpha\beta + e^{-\varkappa t}e^{\varkappa(t-T^*)}(t-T^*)\sup_{[T^*,t]}\left\|a(\tau,\cdot)\left(1+\widetilde{u}(\tau)\right)^3\right\|_{H^m} \\ &\leq e^{\varkappa(t-T^*)}\sqrt{E(0)} - \sqrt{E(T^*)} = \sqrt{E(0)}\left(e^{\varkappa(t-T^*)}-1\right) + \sqrt{E(0)} - \sqrt{E(T^*)}.\end{aligned}$$

But we already know that $E(T^*) \leq E(0)$ holds, by the definition of T^*. Therefore, it suffices to show that

$$\begin{aligned}&\varkappa\left(e^{\varkappa(t-T^*)}-1\right)\alpha\beta + e^{-\varkappa T^*}(t-T^*)\sup_{[T^*,t]}\left\|a(\tau,\cdot)\left(1+\widetilde{u}(\tau)\right)^3\right\|_{H^m} \\ &\leq \sqrt{E(0)}\left(e^{\varkappa(t-T^*)}-1\right).\end{aligned}$$

Note, that since $\varkappa$ is strictly positive, we conclude inequality $e^{-\varkappa T^*} < 1$, and therefore, we can drop the term $e^{-\varkappa T^*}$. So it is enough to show that

$$(t-T^*)\sup_{[T^*,t]}\left\|a(\tau,\cdot)\left(1+\widetilde{u}(\tau)\right)^3\right\|_{H^m} \leq \left(e^{\varkappa(t-T^*)}-1\right)\alpha(1-\beta\varkappa). \tag{4.37}$$

We now let $t = T^\ddagger$ and we proceed as we did in the proof of Proposition 5. Under the smallness condition on $a(t)$, (4.15) where ϵ_0 is given by (4.14), we conclude that

$$\begin{aligned}&(T^\ddagger - T^*)\sup_{[T^*,T^\ddagger]}\left\|a(\tau,\cdot)\left(1+\widetilde{u}(\tau)\right)^3\right\|_{H^m} \\ &\leq (T^\ddagger - T^*)C(2\alpha\beta)\sup_{[0,\infty)}\|a(\tau,\cdot)\|_{H^m} \\ &\leq \varkappa(T^\ddagger - T^*)\alpha(1-\beta\varkappa) \leq \left(e^{\varkappa(T^\ddagger-T^*)}-1\right)\alpha(1-\beta\varkappa).\end{aligned} \tag{4.38}$$

holds. We observe that the inequalities in expression (4.38) imply inequality (4.37), and thus, we have proved inequality (4.19).

Taking into account the conditions on $a(t)$, namely (4.15) and (4.30), respectively, we conclude that there exists a positive ϵ depending on ϵ_0 and ϵ_1 such that if $\sup_{[0,\infty)} \|a(t,\cdot)\|_{H^m} \leq \epsilon$, the inequalities $\sup_{[0,T^\ddagger]}\{\|u(t)\|_{H^m}\} \leq \alpha\beta$ and $E(T^\ddagger) \leq E(0)$ hold for $t \in [0, T^\ddagger]$. It is important to note that the condition (4.30) also can be formulated in terms of the H^m norm. Therefore, both conditions hold in the larger time interval $[0, T^\ddagger]$. This implies that the assumption that $T^* < \infty$ is false and that completes the proof of Proposition 6. □

Remark 8 (About the definition of $T^\star$). We come back to the question of why we had two conditions in the definition of $T^\star$. One motivation was Proposition 5; however, there is an important difference in the proof of Propositions 5 and 6. While we proved

$$\sup_{[0,T]} \|u(t)\|_{H^m} \leq \alpha\beta$$

in Proposition 5 by a simple continuity argument, we needed condition $E(T^\ddagger) \leq E(0)$ to prove the corresponding inequality in Proposition 6. In other words, both conditions are interconnected appropriately.

We turn now to the proof of Theorem 3.

Proof of Theorem 3. The global existence and the regularity essentially follow from Propositions 5 and 6. The asymptotic behavior (4.9) will be proven by considering $|\widehat{u}_0|$ and $\|u_h\|$ separately.

We start to prove Eq. (4.8). By the existence theorem, Theorem 1, the solution to the initial value problem (3.1a)–(3.1b) exists in a certain time interval $[0, T]$. Consequently, the system (4.1a)–(4.1b) has a solution in the time interval $[0, T]$. Then we can apply Propositions 5 and 6 that provide the existence of a global solution $u \in C([0,\infty); H^m)$. By Proposition 1 we obtain

$$\|u(t)\|^2_{H^{m+1}} = |\widehat{u}_0(t)|^2 + \|u_h(t)\|^2_{H^{m+1}} = |\widehat{u}_0(t)|^2 + \|\partial_x u_h(t)\|^2_{H^m}.$$

Hence, since $V \in C([0,\infty); H^m)$, it follows that $u \in C([0,\infty); H^{m+1})$.

We now turn to the proof of the asymptotic behavior of the global solution as described by Eq. (4.9). The idea is again based on the decomposition $u = \widehat{u}_0 + u_h$ and to show that $\lim_{t\to\infty} \|u_h(t)\|_{H^{m+1}} = 0$. So we set

$$\mu \stackrel{\text{def}}{=} \limsup_{t\to\infty} \|u_h(t)\|_{H^{m+1}}. \tag{4.39}$$

Then for a given $\epsilon > 0$, there exists a t_0 such that $\sup_{[t_0,\infty)} \|u_h(t)\|_{H^{m+1}} \leq \mu + \epsilon$. Using the energy estimate (4.5) for $t > t_0$ and Proposition 2 with $A = \mu + \epsilon$, we obtain

$$\begin{aligned}\sqrt{E(t)} &\le e^{-\varkappa(t-t_0)}\sqrt{E(t_0)} + \varkappa\left(1-e^{-\varkappa(t-t_0)}\right)\sup_{[t_0,t]}\|u_h(\tau)\|_{\dot{H}^m}\\&\quad + e^{-\varkappa t}(t-t_0)\sup_{[t_0,t]}\left\{\|a(\tau,\cdot)(1+u(\tau))^3\|_{H^m}\right\}\\&\le e^{-\varkappa(t-t_0)}\sqrt{E(t_0)} + \varkappa\left(1-e^{-\varkappa(t-t_0)}\right)(\mu+\epsilon)\\&\quad + e^{-\varkappa t}(t-t_0)C(\mu+\epsilon)\sup_{[0,\infty)}\|a(t,\cdot)\|_{H^m}.\end{aligned} \tag{4.40}$$

We conclude from inequality (4.40) that

$$\limsup_{t\to\infty}\sqrt{E(t)} \le \varkappa(\mu+\epsilon)$$

holds. On the other hand, using the fact that $\|u_h(t)\|_{H^{m+1}} = \|\partial_x u_h(t)\|_{H^m} \le \sqrt{E(t)}$ we obtain

$$\mu \le \limsup_{t\to\infty}\sqrt{E(t)} \le \varkappa(\mu+\epsilon). \tag{4.41}$$

We may assume that μ is strictly positive since otherwise there is nothing to be proven. Since $\varkappa$ is strictly smaller than 1, we can choose ϵ to be $\epsilon = (1-\varkappa)\,\mu > 0$. Then, from inequality (4.41), we obtain

$$\mu \le \varkappa(\mu+\epsilon) = \varkappa\mu + \varkappa(1-\varkappa)\,\mu < \varkappa\mu + (1-\varkappa)\,\mu = \mu,$$

which implies that $\mu = 0$ holds. Recalling definition (4.39), we conclude that

$$\lim_{t\to\infty}\|u(t)\|^2_{H^{m+1}} = \lim_{t\to\infty}|\widehat{u}_0(t)|^2 + \lim_{t\to\infty}\|u_h(t)\|^2_{H^{m+1}} = \lim_{t\to\infty}|\widehat{u}_0(t)|^2, \tag{4.42}$$

holds and that is why it remains to estimate the limit of the term $|\widehat{u}_0(t)|$ only. Since we use the following initial data, $f, g \in \dot{H}^m$, we can express $\widehat{u}_0(t)$ explicitly by formula (3.9) and observe that

$$\widehat{u}_0(t) = \frac{1}{2\varkappa}\int_0^t\left(1-e^{-2\varkappa(t-\tau)}\right)e^{-\varkappa\tau}\,\widehat{F}_0(\tau)\mathrm{d}\tau.$$

holds. Using a similar procedure we used to obtain inequalities (4.28), and (4.29) respectively we conclude

$$|\widehat{u}_0(t)| \le \frac{2}{2\varkappa^2}\sup_{[0,\infty)}\|a(t,\cdot)\|_{L^\infty}C(2\alpha\beta).$$

Thus, by the smallness condition (4.30), we obtain

$$|\widehat{u}_0(t)| \le \frac{\alpha\beta\epsilon_1}{3}.$$

and observe that inequality (4.9) holds with $\widetilde{\epsilon} = \frac{\alpha\beta\epsilon_1}{3}$. □

5. Blowup of solutions even for small initial data

In the previous Sects. 3.2 and 4.4, we proved the global existence (and uniqueness) of classical solutions in the Sobolev spaces H^m. It is important to emphasize that the smallness of $a(t,x)$ played an essential role in the proof. That is why we want to drop the smallness assumption of $a(t,x)$ and investigate its consequence. Our main result can be stated as follows.

Theorem 4 (Blowup in finite time). *Let u be the solution to the Cauchy problem (3.1a)–(3.1b) in the interval $[0,T)$, where $0 < T \le \infty$ and assume the following conditions:*

$$\begin{aligned} 0 < a_0 \le a(t,x), &\qquad \forall t \ge 0, \qquad (5.1)\\ 1 + f(x) > 0, \quad \Delta f(x) \ge 0, &\qquad x \in \mathbb{T}^3,\\ \varkappa(1+f(x)) + g(x) \ge 0, &\qquad (5.2)\\ \widehat{g}_0 > 0 &\qquad (5.3)\end{aligned}$$

and

$$\widehat{g}_0^2 - \frac{a_0}{2}(1+\widehat{f}_0)^4 \le 0. \qquad (5.4)$$

Then for sufficiently large a_0, T is finite, and moreover, the following holds:

$$\lim_{t\uparrow T} \|u(t,\cdot)\|_{H^{m+1}} = \infty.$$

We recall that $\widehat{f}_0$ and $\widehat{g}_0$ are the zero Fourier coefficients of f and g respectively. Also, note that condition (5.2) implies that $1 + \widehat{f}_0 > 0$.

Remark 9.

1. Note that for large $a(t,x)$ blow-up occurs in finite time even if the initial data are small.
2. We actually can neglect condition (5.4) since most likely it holds when a_0 is large.
3. We proved the blow-up under the assumptions that $\Delta f(x) \ge 0$ and $\widehat{g}_0 > 0$, where $\partial_t u(0,x) = g(x)$. Our conjecture is that the blow-up holds even without those restrictions.

Proof Sketch: By the local existence theorem, Theorem 1, there exists a regular unique solution u to the Cauchy problem, (3.1a)–(3.1b), namely $u(t,\cdot) \in L^\infty([0,T]; H^{m+1}(\mathbb{T}^3) \cap C^{0,1}([0,T]; H^m(\mathbb{T}^3))$.

We adopt an idea of Yagdjian [32] that was used for a different wave equation, and we set

$$F(t) \overset{\text{def}}{=} \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} u(t,x)\mathrm{d}x = \widehat{u}_0(t) \qquad (5.5)$$

and we shall derive differential inequality (5.9a) for F. For that purpose, we need to apply Jensen's inequality to the right-hand side of Eq. (5.8). In order to do so, we have to ensure that the term $1 + u(t, x)$ is nonnegative in the existence interval $[0, T)$ and we prove it in Sect. 6.

Finally, we use Lemma 2 below, which states that a function that satisfies differential inequality (5.9a) blows up in finite time. Since

$$\|u(t)\|^2_{H^{m+1}} = |F(t)|^2 + \sum_{0 \neq k \in \mathbb{Z}^3} |k|^{2(m+1)} |\widehat{u}_k(t)|^2 \geq |F(t)|^2 \tag{5.6}$$

holds, the blow-up of F implies the blow-up of the solution to the Cauchy problem (3.1a)–(3.1b).

Proof of Theorem 4. We start to derive a differential inequality for F that is defined by Eq. (5.5). First, note that

$$F'(t) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \partial_t u(t, x) \mathrm{d}x = \widehat{u}_0'(t) \quad \text{and} \quad F''(t) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \partial_{tt} u(t, x) \mathrm{d}x.$$

It is a well-known fact that the integral of the Laplacian over the $\mathbb{T}^3$ is zero, or in other words,

$$\int_{\mathbb{T}^3} \Delta u(t, x) \mathrm{d}x = 0. \tag{5.7}$$

One way to prove Eq. (5.7) is to expand u to its Fourier series (2.1), then we observe that the zero coefficient Δu is zero (another possibility to prove Eq. (5.7) is to use Gauss' theorem). So we obtain

$$\begin{aligned} F'' + 2\varkappa F' &= \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} (\partial_{tt} u + 2\varkappa \partial_t u - \Delta u) \, \mathrm{d}x \\ &= e^{-\varkappa t} \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} a(t, x)(1 + u(t, x))^3 \mathrm{d}x. \end{aligned} \tag{5.8}$$

Our idea is to estimate the right-hand side of Eq. (5.8) by Jensen's inequality (see e. g. [19, Ch. 2]), with the convex function s^3, where $s = \sqrt[3]{a(t, x)}(1 + u(t, x))$. The function s^3 is convex, only for $s \geq 0$. Recall that $0 < a(t, x)$ holds by assumption (5.1). The proof of the positivity of $1 + u(t, x)$ is more involved and we refer to Theorem 5 in Sect. 6.

Applying Jensens's inequality as outlined above, we obtain

$$\begin{aligned} &\frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} a(t, x) \, (1 + u(t, x))^3 \, \mathrm{d}x \geq \left(\frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} a^{1/3}(t, x) \, (1 + u(t, x)) \, \mathrm{d}x \right)^3 \\ &\geq a_0 \left(\frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} (1 + u(t, x)) \, \mathrm{d}x \right)^3 = a_0 \, (1 + F(t))^3 \, . \end{aligned}$$

Hence, we have obtained the following initial differential inequality

$$F'' + 2\varkappa F' \geq e^{-\varkappa t} a_0 \, (1 + F)^3 \tag{5.9a}$$

$$F(0) = \widehat{f}_0, \qquad F'(0) = \widehat{g}_0 \tag{5.9b}$$

for $t \in [0, T)$. We now use Lemma 2, stating that F blows up in a finite time interval, then $u(t, x)$ also blows up using Eq. (5.6). $\square$

It remains to state and to prove Lemma 2.

Lemma 2 (Blow-up for the associated differential inequality). *Let F satisfy the differential inequality (5.9a) in the interval* $[0, T)$, *where* $0 < T \leq \infty$ *and with the initial data* (5.9b). *Suppose assumptions* (5.2)–(5.4) *of Theorem* 4 *hold, then for sufficiently large a_0, T is finite, and moreover*

$$\lim_{t \uparrow T} F(t) = \infty.$$

Note, that the differential inequality (5.9a) contains a strong damping term $e^{-\varkappa t}$. So we want to know whether this damping term prevents the blowup in finite time. It turns out, however, that this is not the case.

Proof sketch: The proof consists of three main steps. In the first step, we show that $F'(t)$ is a positive function in the interval of existence $[0, T)$. In the second step, we shall make a variable change in order to transform the differential inequality (5.9a) into an inequality without a first-order term. This together with the positivity of F' will enable us in the third step to integrate the inequalities and to estimate the time of the blow-up.

Proof of Lemma 2.
Step 1 We claim that $F'(t) > 0$ holds in the existence interval $[0, T)$. To see that we set

$$T^* = \sup\{T_1 : F'(t) > 0 \quad \text{for } t \in [0, T_1), 0 \leq T_1 \leq T\}.$$

By the assumptions of Lemma 2, $F'(0) = \widehat{g}_0 > 0$, hence $T^* > 0$. We now assume by contradiction that $T^* < T$, then by the continuity of $F'(t)$, we conclude that $F'(T^*) = 0$. Recall that by assumption (5.2), $1 + F(0) = 1 + \widehat{f}_0 > 0$; hence, $1 + F(T^*) \geq 1 + F(0) > 0$ and consequently

$$F''(T^*) = F''(T^*) + 2\varkappa F'(T^*) \geq e^{-\varkappa T^*} a_0 \left(1 + F(T^*)\right)^3 > 0.$$

This implies that F attains a local minimum at time T^*. But this is impossible since F is an increasing function. Therefore, we conclude that $T^* = T$.
Step 2 We start with a variable change of the form $\tau = \omega(t)$ and define a function $G(\tau)$ such that $F(t) = G(\omega(t))$. Then $F'(t) = \frac{\mathrm{d}G}{\mathrm{d}\tau}(\omega(t))\omega'(t)$ and $F''(t) = \frac{\mathrm{d}^2G}{\mathrm{d}\tau^2}(\omega(t))(\omega'(t))^2 + \frac{\mathrm{d}G}{\mathrm{d}\tau}(\omega(t))\omega''(t)$, which leads to

$$F'' + 2\varkappa F' = \frac{\mathrm{d}^2 G}{\mathrm{d}\tau^2}\left(\omega'\right)^2 + \frac{\mathrm{d}G}{\mathrm{d}\tau}\left(\omega'' + 2\varkappa\omega'\right). \tag{5.10}$$

Now we choose ω such that it satisfies the equation $\omega'' + 2\varkappa\omega' = 0$, and in order to obtain an one-to-one transformation for $t > 0$ we require that $\omega' > 0$. It is straightforward to calculate its general solution

$$\omega(t) = C_1 e^{-2\varkappa t} + C_0,$$

for which we choose $C_1 = -1$ and $C_0 = 2$, and then

$$\tau = \omega(t) = -e^{-2\varkappa t} + 2.$$

Note that ω maps $[0, \infty)$ onto $[1, 2)$ in a one-to-one manner. Taking into account Eq. (5.10), inequality (5.9a) is equivalent to

$$\frac{\mathrm{d}^2 G}{\mathrm{d}\tau^2}\left(\omega'\right)^2 \geq e^{-\varkappa t} a_0 (1+G)^3$$

or

$$\frac{\mathrm{d}^2 G}{\mathrm{d}\tau^2} \geq \frac{e^{-\varkappa t}}{(2\varkappa e^{-2\varkappa t})^2} a_0 (1+G)^3 = \frac{e^{3\varkappa t}}{4\varkappa^2} a_0 (1+G)^3 \geq \frac{a_0}{4\varkappa^2}(1+G)^3.$$

In order to simplify the notation, we set $\frac{\mathrm{d}G}{\mathrm{d}\tau} = G'$ and $G'' = \frac{\mathrm{d}^2 G}{\mathrm{d}\tau^2}$, then G satisfies the initial value inequality

$$G'' \geq \frac{a_0}{4\varkappa^2}(1+G)^3 \tag{5.11a}$$

$$G(1) = \widehat{f}_0, \quad G'(1) = \frac{\widehat{g}_0}{2\varkappa}. \tag{5.11b}$$

We are now in a position to show the blow-up for G at some $1 < \tau_0 < 2$, and consequently, F will blow up at $t_0 = \omega^{-1}(\tau_0)$.

Step 3 We will show that if a_0 is sufficiently large, then there exits $\tau_0 < 2$ such that

$$\lim_{\tau \uparrow \tau_0} G(\tau) = \infty. \tag{5.12}$$

Note that $F' > 0$ holds in the existence interval as it was proven in Step 1, and since $F' = G' \frac{\mathrm{d}\omega}{\mathrm{d}t}$ and $\omega' > 0$, we can conclude that $G' > 0$ holds. Thus, we can multiply inequality (5.11a) by G' and obtain

$$G''G' \geq \frac{a_0}{4\varkappa^2}(1+G)^3 G'.$$

Integrating both sides of inequality (5.11a) from 1 to τ, we conclude that

$$\frac{1}{2}\left(\left(G'(\tau)\right)^2 - \left(G'(1)\right)^2\right) \geq \frac{a_0}{16\varkappa^2}\left((1+G(\tau))^4 - (1+G(1))^4\right).$$

Taking into account the initial values (5.11b), we observe that

$$\begin{aligned}\left(G'(\tau)\right)^2 &\geq \frac{a_0}{8\varkappa^2}(1+G(\tau))^4 + \left(\frac{\widehat{g}_0}{2\varkappa}\right)^2 - \frac{a_0}{8\varkappa^2}\left(1+\widehat{f}_0\right)^4 \\ &= \frac{a_0}{8\varkappa^2}\left\{(1+G(\tau))^4 - \left\{\left(1+\widehat{f}_0\right)^4 - \frac{2\widehat{g}_0^2}{a_0}\right\}\right\}.\end{aligned}$$

The expression $(1+\widehat{f_0})^4 - \frac{2\widehat{g}_0^2}{a_0} \geq 0$ by assumption (5.4), and in order to simplify the calculations we set

$$\lambda^4 \stackrel{\text{def}}{=} (1+\widehat{f_0})^4 - \frac{2\widehat{g}_0^2}{a_0}.$$

Now, since $G'(\tau) > 0$,

$$(1+G(\tau))^4 \geq (1+G(1))^4 = (1+\widehat{f_0})^4 \geq \lambda^4,$$

hence,

$$G'(\tau) \geq \frac{\sqrt{a_0}}{\sqrt{8\varkappa}} \left\{(1+G(\tau))^4 - \lambda^4\right\}^{\frac{1}{2}} \geq \frac{\sqrt{a_0}}{\sqrt{8\varkappa}} \left\{(1+G(\tau))^2 - \lambda^2\right\},$$

or

$$\begin{aligned} &\frac{\sqrt{8}\varkappa}{\sqrt{a_0}} \frac{G'(\tau)}{(1+G(\tau))^2 - \lambda^2} \\ &\quad = \frac{\sqrt{8}\varkappa}{\sqrt{a_0}} \frac{G'(\tau)}{2\lambda} \left(\frac{1}{(1+G(\tau)) - \lambda} - \frac{1}{(1+G(\tau)) + \lambda} \right) \geq 1. \end{aligned} \tag{5.13}$$

Integration of both sides of Eq. (5.13) results in

$$\frac{\sqrt{2}\varkappa}{\lambda\sqrt{a_0}} \left\{ \ln\left(\frac{1+G(\tau)-\lambda}{1+G(\tau)+\lambda}\right) - \ln\left(\frac{1+\widehat{f_0}-\lambda}{1+\widehat{f_0}+\lambda}\right) \right\} \geq \tau - 1,$$

or

$$\ln\left(\frac{1+G(\tau)-\lambda}{1+G(\tau)+\lambda}\right) \geq \frac{\lambda\sqrt{a_0}}{\sqrt{2}\varkappa}(\tau-1) + \ln\left(\frac{1+\widehat{f_0}-\lambda}{1+\widehat{f_0}+\lambda}\right). \tag{5.14}$$

Note that by inequality (5.3) $1+\widehat{f_0}-\lambda > 0$ so inequality (5.14) is well defined. We set

$$\beta \stackrel{\text{def}}{=} \frac{1+\widehat{f_0}-\lambda}{1+\widehat{f_0}+\lambda},$$

then inequality (5.14) is equivalent to

$$\frac{1+G(\tau)-\lambda}{1+G(\tau)+\lambda} \geq e^{\frac{\lambda\sqrt{a_0}}{\sqrt{2}\varkappa}(\tau-1)}\beta,$$

and from this inequality, we obtain that

$$1+G(\tau) \geq \frac{\lambda\left(e^{\frac{\lambda\sqrt{a_0}}{\sqrt{2}\varkappa}(\tau-1)}\beta + 1\right)}{1-\beta e^{\frac{\lambda\sqrt{a_0}}{\sqrt{2}\varkappa}(\tau-1)}}. \tag{5.15}$$

The right-hand side of inequality (5.15) blows up at time τ_0 for which $\ln(1/\beta) = \frac{\lambda\sqrt{a_0}}{\sqrt{2\varkappa}}(\tau_0 - 1)$ or

$$\tau_0 = \frac{\sqrt{2\varkappa}}{\lambda\sqrt{a_0}} \ln\left(\frac{1}{\beta}\right) + 1.$$

Thus, blow-up will occur at time τ_0, however, in order that it will be finite in t, we have to assure that $\tau_0 < 2$, that is,

$$\frac{\sqrt{2\varkappa}}{\lambda\sqrt{a_0}} \ln\left(\frac{1}{\beta}\right) < 1. \tag{5.16}$$

So we now estimate this expression. Recall that

$$\lambda^4 = (1 + \widehat{f_0})^4 - \frac{2\widehat{g}_0^2}{a_0} = (1 + \widehat{f_0})^4 \left(1 - \frac{2\widehat{g}_0^2}{(1 + \widehat{f_0})^4 a_0}\right).$$

We set

$$z = \frac{2\widehat{g}_0^2}{(1 + \widehat{f_0})^4 a_0} \tag{5.17}$$

and we note that z becomes smaller as a_0 grows. Hence,

$$\lambda = (1 + \widehat{f_0})(1 - z)^{\frac{1}{4}} = (1 + \widehat{f_0}) \left(1 - \frac{1}{4} z + o(z)\right),$$

and

$$\frac{1}{\beta} = \frac{(1 + \widehat{f_0}) + \lambda}{(1 + \widehat{f_0}) - \lambda} = \frac{1 + (1 - z)^{\frac{1}{4}}}{1 - (1 - z)^{\frac{1}{4}}} = \frac{2 - \frac{z}{4} + o(z)}{\frac{z}{4} + o(z)} = \frac{8}{z} - 1 + o(z).$$

We now express a_0 by z through condition (5.17), then

$$\frac{\sqrt{2\varkappa}}{\lambda\sqrt{a_0}} \ln\left(\frac{1}{\beta}\right) = \frac{\varkappa(1 + \widehat{f_0})\sqrt{z}}{(1 - z)^{\frac{1}{4}} \widehat{g}_0} \left(\ln\left(\frac{8}{z} - 1 + o(z)\right)\right) \to 0, \quad \text{as } z \to 0.$$

Consequently, inequality (5.16) holds for a_0 sufficiently large and there exists $\tau_0 < 2$ such that Eq. (5.12) is true. $\square$

6. Positivity of the scalar field function

The proof of the blowup of classical solutions presented in Sect. 5 depends on the Jensen inequality. This inequality states that

$$\Phi\left(\int f \,\mathrm{d}\mu\right) \leq \int \Phi(f) \,\mathrm{d}\mu, \tag{6.1}$$

holds, whenever Φ is a convex function and μ is a probabilistic measure. In our case, the function $\Phi(s) = s^3$ is a convex function only if $s \geq 0$. Hence, in order to apply (6.1), we need to show that $1 + u(t, x) \geq 0$. Since the proof of this inequality is a bit lengthy and requires additional tools, we have moved the proof to a separate section.

Another important issue with the positivity of $1+u(t, x) \geq 0$ is the following. The metric of the spacetime is given by $g_{\alpha\beta} = \phi^2 \eta_{\alpha\beta}$, where $\phi(t, x) = e^{\varkappa t}(1 + u(t, x))$. Hence, if $1 + u(t, x) = 0$, then metric vanishes, and the solution, in that case, has no physical meaning.

We recall that u satisfies the initial value problem (3.1a)–(3.1b), and the known existence theorem for semi-linear wave equations, [15, Theorem 6.4.11], assures the existence and uniqueness of C^2-solution in a certain time interval $[0, T)$. Our aim is to show that $1+u(t, x) \geq 0$ provided that $a(t, x) > 0$ and the initial data $1+u(0, x) = 1 + f(x)$ are positive. However, the condition $1 + f(x) > 0$ is not sufficient and further conditions are needed. The question of which additional conditions to impose has also been discussed in other publications in which the solution of the linearized equation is required to be positive, see for example [7] and the celebrated paper by John [16].

In the following, we will present the main result of this section. Since our major interest here is the positivity of $1 + u(t, x)$, we assume the initial data f and g are sufficiently smooth in this section.

Theorem 5 (The positivity of $1 + u(t, x)$). *Assume $u(t, x)$ is a unique C^2 solution of initial value problem (3.1a)–(3.1b) for $t \in [0, T)$, $x \in \mathbb{T}^3$ and for some positive T. Moreover, assume that*

$$a(t, x) > 0 \tag{6.2}$$

$$1 + f(x) > 0, \quad g(x) \geq 0, \quad x \in \mathbb{T}^3 \tag{6.3}$$

$$\Delta f(x) \geq 0, \quad x \in \mathbb{T}^3. \tag{6.4}$$

Then

$$1 + u(t, x) > 0 \quad \textit{for} \;\; (t, x) \in [0, T) \times \mathbb{T}^3.$$

The proof of Theorem 5 is based on the application of Kirchhoff's formula for linear wave equations in $\mathbb{R}^3$ [17, §5] and this is why we have to linearize Eq. (3.1a). We first transform the Eq. (3.1a) with the damping term, to an appropriate form by setting $\phi(t, x) = e^{\varkappa t}(1 + u(t, x))$. We recall that ϕ satisfies the field equation (1.6) with the cosmological constant $\varkappa^2$; thus, the initial value problem (3.1a)–(3.1b) is equivalent to

$$\phi_{tt} - \Delta\phi = e^{-3\varkappa t} a(t, x)\phi^3 + \varkappa^2\phi \tag{6.5a}$$

$$\phi(0, x) = 1 + f(x) \tag{6.5b}$$

$$\phi_t(0, x) = h(x), \tag{6.5c}$$

where

$$h(x) \stackrel{\text{def}}{=} \varkappa(1 + f(x)) + g(x).$$

The linearization of (6.5a), (6.5b) and (6.5c) results in the system

$$v_{tt} - \Delta v = P(t, x) \tag{6.6a}$$

$$v(0, t) = 1 + f(x) \tag{6.6b}$$

$$v_t(0, x) = h(x), \tag{6.6c}$$

where $P(t, x)$ denotes the linearization of the nonlinear right-hand side of (6.5a) and whose precise form it of no importance. By Kirchhoff's formula (see, e.g., [16, §5]) solution of the above system is given by

$$\begin{aligned} v(t, x) =& \frac{t}{4\pi} \int_{|\xi|=1} h(x + t\xi) \mathrm{d}\omega_\xi + \frac{\partial}{\partial t} \left(\frac{t}{4\pi} \int_{|\xi|=1} (1 + f(x + t\xi)) \mathrm{d}\omega_\xi \right) \\ &+ \frac{1}{4\pi} \int_0^t (t - s) \left(\int_{|\xi|=1} P(s, x + (t - s)\xi) \mathrm{d}\omega_\xi \right) \mathrm{d}s, \end{aligned} \tag{6.7}$$

where $\mathrm{d}\omega_\xi$ is the Lebesgue measure of the unit sphere $\mathbb{S}^2$.

Remark 10. In this paper, however, we deal with spatially periodic solutions, while Kirchhoff's formula (6.7) provides solutions in $\mathbb{R}^3$. But if initial data (6.6b) and (6.6c), and the right-hand side of (6.6a) are periodic, then it follows from (6.7) that $v(t, x)$ is also a periodic function of the space variable x.

Proof sketch: Our proof strategy can be described as follows:

1. We first show that the solution to the homogeneous initial value problem (6.6a)–(6.6c) is positive, if the conditions (6.3) and (6.4) are met.
2. We then construct a monotone sequence, $0 < \phi_n \leq \phi_{n+1}$, by an iteration of the linearized equation.
3. We show that whenever $\phi(x, t) > 0$, then

$$\phi_n(t, x) \leq \phi(t, x). \tag{6.8}$$

   We then use (6.8) to show that $\phi(t, x) > 0$ in the entire existence interval.
4. Finally, the positivity of ϕ implies that $1 + u(t, x) = e^{-\varkappa t}\phi(t, x) > 0$.

The iteration scheme

We denote the right-hand side of (6.5a) by $G(\phi)$, that is,

$$G(\phi) = G(\phi, t, x) = e^{-3kt} a(t, x)\phi^3 + k^2\phi.$$

Note that $G(0) = 0$ and

$$\frac{\partial}{\partial \phi} G(\phi) = 3e^{-3kt} a(t, x)\phi^2 + k^2.$$

Hence, if $a(t,x) > 0$, then G is an increasing function of ϕ and non-negative for $\phi \geq 0$. We now let $\phi_0(t,x) = 0$ and set ϕ_{n+1} to be the solution to the linear equation

$$\begin{cases} (\phi_{n+1})_{tt} - \Delta\phi_{n+1} = G(\phi_n) \\ \phi_{n+1}(0,x) = 1 + f(x) \\ (\phi_{n+1})_t(0,x) = h(x) \end{cases} . \tag{6.9}$$

Step 1: The free wave equation. The first step of the iteration consists of showing that the free wave equation

$$\begin{cases} (\phi_1)_{tt} - \Delta\phi_1 = G(0) = 0 \\ \phi_1(0,x) = 1 + f(x) \\ (\phi_1)_t(0,x) = h(x) \end{cases}$$

has a positive solution.

Proposition 7. *Under conditions* (6.3) *and* (6.4) *it follows that* $\phi_1(t,x) > 0$.

Proof. Taking the time derivative in Kirchhoff's formula (6.7), we observe that

$$\phi_1(t,x) = \frac{t}{4\pi}\int_{|\xi|=1} (h(x+t\xi) + \xi\cdot\nabla f(x+t\xi))\, \mathrm{d}\omega_\xi \\ + \frac{1}{4\pi}\int_{|\xi|=1} (1 + f(x+t\xi))\mathrm{d}\omega_\xi .$$

Applying now the Gauss Divergence Theorem to the term $\xi\cdot\nabla f(x+t\xi)$, we obtain that

$$\phi_1(t,x) = \frac{t}{4\pi}\int_{|\xi|=1} (h(x+t\xi))\, \mathrm{d}\omega_\xi + \frac{t^2}{4\pi}\int_{|\xi|\leq 1} \Delta f(x+t\xi)\mathrm{d}\xi \\ + \frac{1}{4\pi}\int_{|\xi|=1} (1 + f(x+t\xi))\mathrm{d}\omega_\xi .$$

Hence, conditions (6.3) and (6.4) imply that $\phi_1(t,x) > 0$. □

Step 2: Monotonicity.

Proposition 8. *Assume conditions* (6.2), (6.3) *and* (6.4) *are satisfied then the sequence* $\{\phi_n\}$ *defined by* (6.9) *is monotone, in other words*

$$\phi_n(t,x) \leq \phi_{n+1}(t,x).$$

holds

Proof. The proof is obviously done by induction. We already have proved that $\phi_1(t,x) > \phi_0(t,x) \equiv 0$ holds. Assume $\phi_{n-1} \leq \phi_n$, then $G(\phi_{n-1}) \leq G(\phi_n)$ since both functions are positive and G is increasing. Hence, by Kirchhoff's formula (6.7), we observe that

$$\begin{aligned}
\phi_{n+1}(t,x) &= \frac{t}{4\pi}\int_{|\xi|=1} h(x+t\xi)\mathrm{d}\omega_\xi + \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{|\xi|=1}(1+f(x+t\xi))\mathrm{d}\omega_\xi\right) \\
&\quad + \frac{1}{4\pi}\int_0^t (t-s)\left(\int_{|\xi|=1} G(\phi_n)(s,x+(t-s)\xi)\mathrm{d}\omega_\xi\right)\mathrm{d}s \\
&\geq \frac{t}{4\pi}\int_{|\xi|=1} h(x+t\xi)\mathrm{d}\omega_\xi + \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{|\xi|=1}(1+f(x+t\xi))\mathrm{d}\omega_\xi\right) \\
&\quad + \frac{1}{4\pi}\int_0^t (t-s)\left(\int_{|\xi|=1} G(\phi_{n-1})(s,x+(t-s)\xi)\mathrm{d}\omega_\xi\right)\mathrm{d}s \\
&= \phi_n(t,x).
\end{aligned}$$

which finishes the proof. □

STEP 3: It remains to show that $\phi(t,x) > 0$ holds.

Proposition 9. *Assume $f(x)$, $h(x)$ and $a(t,x)$ are periodic functions that satisfy the assumptions of Proposition 8. Let ϕ be a C^2 solution to the initial value problem (6.5a)–(6.5c) in the interval $[0,T)$. Then*

$$\phi(t,x) > 0, \quad t \in [0,T) \textit{ and all } x \in \mathbb{T}^3.$$

Proof. We set

$$T^* = \sup\left\{T_1 : \phi(t,x) > 0 \text{ for } t \in [0,T_1) \text{ and } \forall x \in \mathbb{T}^3\right\}. \tag{6.10}$$

Since $1+f(x) > 0$, then by continuity, $0 < T^* \leq T$. The proof consists essentially of two steps. In the first one, we show that $\phi_n(t,x) \leq \phi(t,x)$ for $t \in [0,T^*)$, and in the second one we show that $T^* = T$.

We prove the first step, again, by induction. Proposition 7 implies that $0 = \phi_0(t,x) < \phi(t,x)$ for $t \in [0,T^*)$. Now, assume $\phi_{n-1}(t,x) \leq \phi(t,x)$ for $t \in [0,T^*)$, then in a similar way to the proof of the monotonicity, Proposition 8, we obtain that

$$\begin{aligned}
\phi_n(t,x) &= \frac{t}{4\pi}\int_{|\xi|=1} h(x+t\xi)\mathrm{d}\omega_\xi + \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{|\xi|=1}(1+f(x+t\xi))\mathrm{d}\omega_\xi\right) \\
&\quad + \frac{1}{4\pi}\int_0^t (t-s)\left(\int_{|\xi|=1} G(\phi_{n-1})(s,x+(t-s)\xi)\mathrm{d}\omega_\xi\right)\mathrm{d}s \\
&\leq \frac{t}{4\pi}\int_{|\xi|=1} h(x+t\xi)\mathrm{d}\omega_\xi + \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{|\xi|=1}(1+f(x+t\xi))\mathrm{d}\omega_\xi\right) \\
&\quad + \frac{1}{4\pi}\int_0^t (t-s)\left(\int_{|\xi|=1} G(\phi)(s,x+(t-s)\xi)\mathrm{d}\omega_\xi\right)\mathrm{d}s \\
&= \phi(t,x).
\end{aligned}$$

The last equality follows from the fact that ϕ satisfies the initial value problem (6.5a)–(6.5c).

We turn now to the second step. We claim that $T^* = T$. If not, then $T^* < T$ and we will derive a contradiction. First, we note that there exists a $x_0 \in \mathbb{T}^3$ such that $\phi(T^*, x_0) \leq 0$. Since otherwise, by continuity, there exists $T^{**} > T^*$ such that $\phi(t, x) > 0$ for $t \in [0, T^{**})$ and all $x \in \mathbb{T}^3$, and that obviously contradicts the definition of T^* in (6.10). Now, using monotonicity and the first step, we conclude that

$$0 < \lim_{t \to T^{*-}} \phi_n(t, x_0) \leq \lim_{t \to T^{*-}} \phi(t, x_0) = \phi\left(T^*, x_0\right) \leq 0$$

holds, which is the desired contradiction. □

Step 4 is obvious, and that completes the proof of Theorem 5.

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