J. Evol. Equ. (2022) 22:64

1424-3199/22/030001-37, *published online* July 16, 2022
https://doi.org/10.1007/s00028-022-00818-2

Journal of Evolution Equations

Zero-contact angle solutions to stochastic thin-film equations

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Abstract. We establish existence of nonnegative martingale solutions to stochastic thin-film equations with quadratic mobility for compactly supported initial data under Stratonovich noise. Based on so-called α-entropy estimates, we show that almost surely these solutions are classically differentiable in space almost everywhere in time and that their derivative attains the value zero at the boundary of the solution's support. From a physics perspective, this means that they exhibit a zero-contact angle at the three-phase contact line between liquid, solid, and ambient fluid. These α-entropy estimates are first derived for almost surely strictly positive solutions to a family of stochastic thin-film equations augmented by second-order linear diffusion terms. Using Itô's formula together with stopping time arguments, Jakubowski's modification of the Skorokhod theorem, and martingale identification techniques, the passage to the limit of vanishing regularization terms gives the desired existence result.

1. Introduction

In this paper, we are concerned with existence results of martingale solutions to stochastic thin-film equations of the generic form

$$\mathrm{d}u = -(m(u)u_{xxx})_x\mathrm{d}t + (\sqrt{m(u)} \circ \mathrm{d}W)_x \tag{1.1}$$

subject to periodic boundary conditions. The deterministic version of (1.1) models the solely surface-tension driven evolution of the height u of a thin viscous liquid film—the noise term is to capture effects of thermal fluctuations.

Gess and Gnann have been the first to consider stochastic thin-film equations with Stratonovich noise. In [18], they proved the global-in-time existence of nonnegative martingale solutions for the choice $m(u) = u^2$. To establish this result, they took advantage of the regularizing effect of Stratonovich noise compared to Itô noise. In fact, for the Itô version of (1.1), i.e.,

$$\mathrm{d}u = -(m(u)u_{xxx})_x\mathrm{d}t + (\sqrt{m(u)}\mathrm{d}W)_x\ , \tag{1.2}$$

Mathematics Subject Classification: 60H15, 76A20, 76D08, 35K35, 35K65, 35Q35, 35R35, 35R37
Keywords: Stochastic partial differential equation, Degenerate parabolic equation, Fourth-order equation, Thin fluid film, Thermal fluctuations, Stratonovich noise, Martingale solution, Free-boundary problem, Compactly supported initial data.

no integral estimates are known. In contrast, the Stratonovich version (1.1) permits to derive stochastic versions of the energy estimate

$$\frac{1}{2}\int_{\mathcal{O}} |u_x(\cdot,t)|^2\,\mathrm{d}x + \int_0^t\int_{\mathcal{O}} u^2 u_{xxx}^2\,\mathrm{d}x\mathrm{d}s \le \frac{1}{2}\int_{\mathcal{O}} |(u_0)_x|^2\,\mathrm{d}x \tag{1.3}$$

and of the so-called entropy estimate

$$\int_{\mathcal{O}} G(u(\cdot,t))\mathrm{d}x + \int_0^t\int_{\mathcal{O}} u_{xx}^2\,\mathrm{d}x\mathrm{d}s \le \int_{\mathcal{O}} G(u_0)\mathrm{d}x\,, \tag{1.4}$$

where $G(\cdot)$ is a second primitive of the reciprocal mobility $m^{-1}(s)$. Already from the deterministic setting, it is well-known that weak solutions to the free-boundary problem associated with the thin-film equation are not unique in general, unless additional conditions are imposed at the free boundary, i.e., the boundary of $\mathrm{supp}[u(\cdot,t)>0]$. In a series of papers [19,21–25] short-time uniqueness results were established for classical solutions of thin-film equations exhibiting a zero-contact angle at the free boundary.

In this spirit, it is the aim of the present paper to construct nonnegative martingale solutions $\tilde{u}$ to Eq. (1.1) under the choice $m(u)=u^2$ which are $\tilde{\mathbb{P}}$-almost surely and almost everywhere in time continuously differentiable in space. Hence, those spatial derivatives of $\tilde{u}$ attain the value zero in roots of $\tilde{u}$. The vanishing of these derivatives comes as the consequence of additional regularity results. While the solutions constructed by Gess and Gnann in [18] for compactly supported initial data do not have the regularity stipulated by the entropy estimate (1.4) (note that $\int_{\mathcal{O}} G(u_0)\mathrm{d}x = +\infty$ in this case) and therefore do not necessarily exhibit zero-contact angles, the solutions presented here are more regular. In fact, they satisfy a stochastic version of a variant of (1.4), the so-called α-entropy estimate. This α-entropy estimate provides H^2-regularity of appropriate powers of the solutions $\tilde{u}$ without requiring initial data to be zero only on sets of Lebesgue measure zero. For an overview on α-entropy estimates and other integral estimates for the thin-film equation in the deterministic setting, we refer to [3,5,8,28] and the references therein.

At this point, it is worth mentioning that in the analysis of the qualitative behavior of deterministic thin-film equations, weighted versions of α-entropy estimates become important. They have been used, e.g., to obtain optimal results on the propagation of the free boundary of solutions or on the regularity at the free boundary.

For an overview of corresponding results, we cite [6,13,27] for finite speed of propagation and [9,14,20,29] for the occurrence and scaling of waiting time phenomena. In the stochastic setting, the techniques of [20] have been generalized by [15,26] to provide sufficient criteria for the occurrence of waiting time phenomena and for qualitative results on finite speed of propagation for stochastic p-Laplace and stochastic porous-media equations. For finite speed of propagation for the latter equations, we also mention [2,17] which use different techniques.

Before giving the outline of the present paper, we report on variants of (1.1) which are meaningful for physical and/or for analytical reasons as they may set up auxiliary problems to construct the more regular solutions to be considered in this paper. First, we mention (1.1) with the generic mobility $m(u) = u^n$ where $n > 0$. The exponent n depends on the flow boundary conditions at the liquid-solid interface—a no-slip boundary conditions entails $n = 3$. Recently, Dareiotis, Gess, Gnann, and the first author of this paper established the existence of martingale solutions [10] for (1.1) with $m(u) = u^n$ in the parameter regime $n \in [8/3, 4)$ which covers in particular the no-slip case. Note that Davidovitch et al. [11] who derived (1.1) with Itô- instead of Stratonovich noise via the dissipation-fluctuation theorem conjectured that noise enhances spreading, changing in particular characteristic spreading laws on intermediate time-scales in expectation. Parallel in time, Grün, Mecke, and Rauscher [30] studied the influence of thermal fluctuations on the dewetting of unstable liquid films. Based on lubrication approximation and Fokker-Planck-type arguments, they came up with an equation of the generic form

$$\mathrm{d}u = -(m(u)(u_{xx} - \mathfrak{F}'(u))_x)_x \mathrm{d}t + (\sqrt{m(u)}\mathrm{d}W)_x \,, \tag{1.5}$$

where the effective interface potential $\mathfrak{F}(u)$ models van der Waals-interactions—a typical example is the potential $\mathfrak{F}(u) := \alpha u^{-p} - \beta u^{-q}$ with $p > q > 0, \alpha > 0$, and $\beta \geq 0$. For the case $m(u) = u^2$, the existence of a.s. positive martingale solutions has been established in [16]—the technically much more involved case of two space dimensions has been studied in [38]—for a very recent result in the spirit of [18] which also provides α-entropy estimates, see [39].

The outline of our paper is as follows. In contrast to the Trotter–Kato scheme, where the stochastic and the deterministic parts of the equation are split and which was used in [18], we will follow an approximation ansatz based on positive solutions to

$$\mathrm{d}u^\varepsilon = -(m(u^\varepsilon)(u^\varepsilon_{xx} - \varepsilon F'(u^\varepsilon))_x)_x \mathrm{d}t + (\sqrt{m(u^\varepsilon)} \circ \mathrm{d}W)_x \,, \tag{1.6}$$

where $\varepsilon \in (0, 1)$ and $F(u) := u^{-p}$, $p > 2$. We take advantage of the fact that under natural assumptions on the coloured noise $W(x, t) = \sum_{k\in\mathbb{Z}} \lambda_k g_k(x)\beta_k(t)$, such that

- $g_k(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin(\frac{2\pi k x}{L}) & k > 0, \\ \frac{1}{\sqrt{L}} & k = 0, \\ \sqrt{\frac{2}{L}} \cos(\frac{2\pi k x}{L}) & k < 0, \end{cases}$
- $\beta_k(\cdot)$, $k \in \mathbb{Z}$, are i.i.d. Brownian motions on $\mathbb{R}$,
- $\lambda_k \in \mathbb{R}^+_0$, $k \in \mathbb{Z}$, appropriate damping parameters,

(see Sect. 2 for the precise assumptions), existence of a.s. positive martingale solutions to (1.6) comes as a consequence of the existence result in [16].

After formulating precise assumptions in Sect. 2, especially on initial data and the noise, we present our main results in Sect. 3. The existence of solutions to (1.6) is the topic of Sect. 4. Our strategy is the following. Since we are dealing with Stratonovich noise in (1.6), we may rewrite it in Itô form with the corresponding correction term added. The equation obtained can be written in the form

$$\mathrm{d}u^\varepsilon = -\left((u^\varepsilon)^2\left(u^\varepsilon_{xx} - \Pi'_\varepsilon\left(u^\varepsilon\right)\right)_x\right)_x \mathrm{d}t + \left(u^\varepsilon \mathrm{d}W\right)_x \tag{1.7}$$

with

$$\Pi_\varepsilon(u) := \begin{cases} \varepsilon u^{-p} + \frac{1}{2}\left(\frac{\lambda_0^2}{L} + \sum_{k=1}^\infty \frac{2\lambda_k^2}{L}\right)(u - \log(u)) & \text{if } u > 0 \\ +\infty & \text{if } u \le 0, \end{cases} \tag{1.8}$$

for more details see (2.6), (2.7), and (4.1). Note that (1.7) is of the generic form (1.5). That way, the equation satisfies—with a grain of salt—the assumptions of the existence result, Theorem 3.2, in [16]. For a comparison of the growth condition in [16] and that one of the present paper, we refer to Sect. 4, in particular to Remark 4.1.

Since solutions in [16] were constructed under the assumption of positive initial data, we shift nonnegative initial data with potentially compact support by a suitable power of ε such that we recover the nonnegative initial data in the limit, cf. (H2ε) in Sect. 4. This is sufficient to establish in Theorem 4.3 existence results for a family of approximate $\tilde{\mathbb{P}}$-almost surely strictly positive solutions u^ε.

The key result for the passage to the limit $\varepsilon \to 0$ is a combined α-entropy-energy estimate in the spirit of the classical α-entropy estimates in [3] (see also [5]) and the energy estimates in [4] both translated to the stochastic setting. The derivation of this estimate is the content of the fifth section. We first introduce suitable stopping times and cut-off versions of our approximate solutions, cf. (5.4) and (5.5), which allow to derive a first version of an α-entropy-energy estimate in Theorem 5.2.

Itô's formula, which is the main tool for the proof, is applied to the energy $\int_{\mathcal{O}} u_x^2 \mathrm{d}x$ and the α-entropy $\int_{\mathcal{O}} \frac{1}{\alpha(\alpha+1)} u^{\alpha+1} - \frac{1}{\alpha} u + \frac{1}{\alpha+1} \mathrm{d}x$, see "Appendix B" for the rigorous justification. Here, the advantages of the usage of the Stratonovich integral become apparent again. Critical terms occurring in Itô's formula are controlled by the S-tratonovich correction term—this way guaranteeing the estimate to be ε-independent.

The passage to the limit $\varepsilon \to 0$ is discussed in Sect. 6. We use the aforementioned α-energy-entropy estimate to apply Jakubowski's theorem, cf. [33]. Based on this, we follow standard arguments encountered in the analysis of PDE's for the convergence of the deterministic terms and make use of the ideas introduced in [7,31] to identify the stochastic integral in Lemma 6.16. The effective interface potential vanishes in the limit, cf. Lemma 6.11.

There are two appendices. In "Appendix A," the equivalence of the different formulations of (1.6) is made explicit. More details on the application of Itô's formula are provided in "Appendix B".

Notation: Besides the standard notation of pde theory and stochastic analysis, we use the following. By C, we denote a generic constant. Throughout the paper, we will use ε as an approximation parameter, subsequences will not be renamed, if it causes no confusion. We consider the spatial domain $\mathcal{O} = [0, L]$ and define $\mathcal{O}_T := (0, L) \times (0, T)$ for numbers $L, T > 0$. For a function f on $\mathcal{O}_T$, $[f > 0]$ is the set $\{(x,t) \in \mathcal{O}_T;\ f(x,t) > 0\}$. Subspaces consisting of periodic functions (w.r.t. space) are marked by the subscript 'per' on the corresponding function space. For $\gamma, \sigma \in (0,1]$, we denote by $C^{\gamma,\sigma}(\bar{\mathcal{O}}_T)$ the space of Hölder-continuous functions on $\bar{\mathcal{O}}_T$ with Hölder exponents γ and σ in the spatial and temporal variables, respectively. The minimum of a and b is denoted by $a \wedge b$. We write $\langle X\rangle$ for the quadratic variation process of a stochastic process X and $\langle X, Y\rangle$ for the quadratic covariation process of X and another process Y. Moreover, for two Hilbert spaces U and V, $L_2(U,V)$ is the set of Hilbert–Schmidt operators from U to V. Note that the dual pairing on a Banach space X is denoted by ${}_{X'}\langle x', x\rangle_X$ for $x' \in X'$ and $x \in X$.

2. Preliminaries

Let us fix some basic assumptions. We are dealing with the stochastic thin-film equation with Stratonovich noise

$$\mathrm{d}u = -(m(u)u_{xxx})_x \mathrm{d}t + (\sqrt{m(u)} \circ \mathrm{d}W)_x \tag{2.1}$$

on $\mathcal{O}_T$ subject to periodic boundary conditions and initial data specified below. For the noise we consider a Q-Wiener process defined by the operator

$$Qg_k = \lambda_k^2 g_k \quad \forall k \in \mathbb{Z}. \tag{2.2}$$

Here, the functions g_k form a basis of $L^2(\mathcal{O})$ consisting of eigenfunctions of the Laplacian on $\mathcal{O}$ subject to periodic boundary conditions.

$$g_k(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin(\frac{2\pi kx}{L}) & k > 0 \\ \frac{1}{\sqrt{L}} & k = 0 \\ \sqrt{\frac{2}{L}}\cos(\frac{2\pi kx}{L}) & k < 0 \end{cases} \tag{2.3}$$

The noise is coloured by the growth condition on the numbers $(\lambda_k)_{k\in\mathbb{Z}}$, cf. (H3) below. We can now give precise assumptions for our main result.

(H1) The mobility is given by $m(u) = u^2$.

(H2) Let Λ^0 be a probability measure on $H^1_{\text{per}}(\mathcal{O})$ equipped with the Borel σ-algebra which is supported on the subset of nonnegative functions such that there is a positive constant C with the property that

$$\operatorname{esssup}_{v\in \operatorname{supp}\Lambda^0} \left\{ \int_{\mathcal{O}} \frac{1}{2}|v_x|^2 \,\mathrm{d}x + \left(\int_{\mathcal{O}} v \mathrm{d}x\right)\right\} \le C\,.$$

(H3) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration such that

- W is a Q-Wiener process on Ω adapted to $(\mathcal{F}_t)_{t\geq 0}$ which admits a decomposition of the form $W = \sum_{k\in\mathbb{Z}} \lambda_k g_k \beta_k$ for a sequence of independent standard Brownian motions β_k and nonnegative numbers $(\lambda_k)_{k\in\mathbb{Z}}$ with

$$\lambda_{-k} = \lambda_k \tag{2.4}$$

for all $k \in \mathbb{N}$,
- the noise is coloured in the sense that

$$\sum_{k\in\mathbb{Z}} k^4 \lambda_k^2 < \infty\,, \tag{2.5}$$

- there exists a $\mathcal{F}_0$-measurable random variable u_0 such that $\Lambda^0 = \mathbb{P} \circ u_0^{-1}$.

Based on these hypotheses, we may rewrite (2.1) in two different ways.

$$\mathrm{d}u = -(u^2(u_{xx} - \mathcal{S}'(u))_x)_x \mathrm{d}t + (u\mathrm{d}W)_x \tag{2.6}$$

with $\mathcal{S}(u) := C_{Strat}\,(u - \log u)$ and $C_{Strat} := \frac{1}{2}\left(\frac{\lambda_0^2}{L} + \sum_{k=1}^{\infty} \frac{2\lambda_k^2}{L}\right)$. Note that this is equivalent to

$$\mathrm{d}u = (-(u^2 u_{xxx})_x + C_{Strat} u_{xx})\mathrm{d}t + (u\mathrm{d}W)_x\,. \tag{2.7}$$

For a justification, we refer to "Appendix A".

3. Main results

In this section, we make our results on the existence of zero-contact angle martingale solutions precise. We begin with the existence result.

Theorem 3.1. *Let (H1), (H2), and (H3) be satisfied and let $T > 0$ be given. Then there exist a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ with a complete, right-continuous filtration, an $\tilde{\mathcal{F}}_t$-adapted Q-Wiener process $\tilde{W} = \sum_{k\in\mathbb{Z}} \lambda_k g_k \tilde{\beta}_k$, a continuous $L^2(\mathcal{O})$-valued $\tilde{\mathcal{F}}_t$-adapted process $\tilde{u} \in L^2(\tilde{\Omega}; L^2(0,T; W^{1,3}_{per}(\mathcal{O}))) \cap L^2(\tilde{\Omega}; C^{\tilde{\gamma},\tilde{\sigma}}(\bar{\mathcal{O}}_T))$, $\tilde{\gamma} < 1/2$, $\tilde{\sigma} < 1/8$, and $\tilde{u}_0 \in L^2(\tilde{\Omega}; H^1_{per}(\mathcal{O}))$ such that the following holds:*

1. *$\tilde{u}$ and $\tilde{u}_0$ are $\tilde{\mathbb{P}}$-almost surely nonnegative,*
2. *for $t \in [0,T]$ and all $\phi \in H^3_{per}(\mathcal{O})$*

$$\begin{aligned}\int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}_0)\phi dx = &\int\int_{[\tilde{u}>0]} \tilde{u}_x^3 \phi_x dx ds + 3\int_0^t \int_{\mathcal{O}} \tilde{u}\tilde{u}_x^2 \phi_{xx}\, dx ds \\ &+ \int_0^t \int_{\mathcal{O}} \tilde{u}^2 \tilde{u}_x \phi_{xxx}\, dx ds \\ &- \frac{1}{2}\int_0^t \int_{\mathcal{O}} \sum_{k\in\mathbb{Z}} \lambda_k^2 g_k (g_k \tilde{u})_x \phi_x\, dx ds \\ &- \sum_{k\in\mathbb{Z}} \int_0^t \int_{\mathcal{O}} \lambda_k g_k \tilde{u} \phi_x dx d\tilde{\beta}_k \end{aligned} \tag{3.1}$$

holds $\tilde{\mathbb{P}}$-almost surely and we have $\Lambda^0 = \tilde{\mathbb{P}} \circ \tilde{u}_0^{-1}$,

3. *$\tilde{u}$ satisfies for arbitrary $q \geq 1$ and $\alpha \in (-\frac{1}{3}, 0)$ the estimate*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_{\mathcal{O}}\frac{1}{2}|\tilde{u}_x|^2\,dx\right)^q\right]+\mathbb{E}\left[\left(\int_0^T\int_{\mathcal{O}}((\tilde{u})^{\frac{\alpha+3}{4}})_x^4\,dxds\right)^q\right]$$
$$+\mathbb{E}\left[\left(\int_0^T\int_{\mathcal{O}}((\tilde{u})^{\frac{\alpha+3}{2}})_{xx}^2\,dxds\right)^q\right]\leq C(T,q,\tilde{u}_0). \tag{3.2}$$

As a consequence of estimate (3.2), we obtain the following regularity result.

Corollary 3.2. *Let $\tilde{u}$ be a solution as constructed in Theorem 3.1 and $\alpha \in (-1/3, 0)$. Then, $\tilde{\mathbb{P}}$-almost surely, $\tilde{u}$ exhibits a zero-contact angle in the following sense:*

For almost all $t_0 \in (0, T]$, the classical derivative $\frac{\partial}{\partial x}\tilde{u}(x_0, t_0, \omega)$ exists in points $x_0 \in \mathcal{O}$ such that $\tilde{u}(x_0, t_0, \omega) = 0$, and it attains the value zero.

4. An existence result for positive approximate solutions

As pointed out in the introduction, our existence result for zero-contact angle solutions relies on new integral estimates which are initially derived for strictly positive approximate solutions. In this section, we provide results on existence and on refined regularity and positivity properties of such solutions. More precisely, we study stochastic thin-film equations

$$\mathrm{d}u^\varepsilon = -((u^\varepsilon)^2(u^\varepsilon_{xx} - \Pi'_\varepsilon(u^\varepsilon))_x)_x\mathrm{d}t + \sum_{k\in\mathbb{Z}}\lambda_k(u^\varepsilon g_k)_x\mathrm{d}\beta^\varepsilon_k, \tag{4.1}$$

$\varepsilon \in (0, 1)$, subject to periodic boundary conditions on $\mathcal{O}$. This class of equations differs from equations (2.6) and (2.7) by the choice of the potential $\mathcal{S}$ (or Π_ε, respectively) and by the assumptions on the positivity properties of initial data. Here, $\Pi_\varepsilon(u)$ is given by

$$\Pi_\varepsilon(u) := \begin{cases}\varepsilon u^{-p} + \mathcal{S}(u) & \text{if } u > 0\\ +\infty & \text{if } u \leq 0\end{cases} \tag{4.2}$$

with a positive number $p > 2$. Note that the potential Π_ε satisfies for every $\varepsilon > 0$ the hypothesis

(H4ε) For $\varepsilon > 0$, the effective interface potential Π_ε has continuous second-order derivatives on $\mathbb{R}^+$ and satisfies for some $p > 2$ and $u > 0$

$$c_1 u^{-p-2} - c_2 \leq \Pi''_\varepsilon(u) \leq C_1(u^{-p-2} + 1)\,,$$
$$\Pi_\varepsilon(u) \geq C u^{-p}\,,$$

where c_1, C_1, and C are positive constants depending on $\varepsilon > 0$.

In addition, we require initial data to be bounded from below by ε^θ with $\theta \in (0, 1/p)$. Combined with Hypothesis (H2), this leads to the modification

(H2ε) Let Λ_0 be a probability measure on $H^1_{\text{per}}(\mathcal{O})$ satisfying Hypothesis (H2). Then, Λ^ε is the probability measure on $H^1_{\text{per}}(\mathcal{O})$ defined by $\Lambda^\varepsilon = \Lambda^0 \circ S_\varepsilon^{-1}$, where $S_\varepsilon : H^1_{\text{per}}(\mathcal{O}) \to H^1_{\text{per}}(\mathcal{O})$ is for $\theta \in (0, 1/p)$ given by $S_\varepsilon(u) = u + \varepsilon^\theta$.

Remark 4.1. Note that the growth condition (H2) of [16] differs from $(H4\varepsilon)$ only by the assumption that $\Pi''_\varepsilon(u)$ in [16] was required to satisfy

$$\Pi''_\varepsilon(u) \le C_1 u^{-p-2}$$

instead of

$$\Pi''_\varepsilon(u) \le C_1(u^{-p-2} + 1)$$

in our paper. It is straightforward to show that this does not affect the validity of the a priori estimates and the existence result in [16].

Definition 4.2. Let Λ^ε be a probability measure on $H^1_{\text{per}}(\mathcal{O})$ satisfying (H2ε). We call a triple $((\Omega^\varepsilon, \mathcal{F}^\varepsilon, (\mathcal{F}^\varepsilon_t)_{t\ge0}, \mathbb{P}^\varepsilon), u^\varepsilon, W^\varepsilon)$ a martingale solution to (4.1) with initial data Λ^ε on the time interval $[0, T]$ provided

(i) $(\Omega^\varepsilon, \mathcal{F}^\varepsilon, (\mathcal{F}^\varepsilon_t)_{t\ge0}, \mathbb{P}^\varepsilon)$ is a stochastic basis with a complete right-continuous filtration,

(ii) W^ε satisfies (H3) with respect to $(\Omega^\varepsilon, \mathcal{F}^\varepsilon, (\mathcal{F}^\varepsilon_t)_{t\ge0}, \mathbb{P}^\varepsilon)$,

(iii) The continuous $L^2(\mathcal{O})$-valued process $u^\varepsilon \in L^2(\Omega^\varepsilon; L^2(0, T; H^3_{\text{per}}(\mathcal{O}))) \cap L^2(\Omega^\varepsilon; C^{\gamma,\sigma}(\bar{\mathcal{O}}_T))$ with $\gamma < 1/2$, $\sigma < 1/8$ is $\mathbb{P}^\varepsilon$-almost surely positive and adapted to $(\mathcal{F}^\varepsilon_t)_{t\ge0}$,

(iv) there is a $\mathcal{F}^\varepsilon_0$-measurable $H^1_{\text{per}}(\mathcal{O})$-valued random variable u^ε_0 with $\Lambda^\varepsilon = \mathbb{P}^\varepsilon \circ (u^\varepsilon_0)^{-1}$ and the equation

$$\begin{aligned} \int_{\mathcal{O}} (u^\varepsilon(t) - (u^\varepsilon_0))\phi \mathrm{d}x &= \int_0^t \int_{\mathcal{O}} m(u^\varepsilon)(u^\varepsilon_{xx} - \Pi'_\varepsilon(u^\varepsilon))_x \phi_x \, \mathrm{d}x\mathrm{d}s \\ &\quad - \sum_{k\in\mathbb{Z}} \int_0^t \int_{\mathcal{O}} \lambda_k g_k \sqrt{m(u^\varepsilon)}\phi_x \mathrm{d}x\mathrm{d}\beta^\varepsilon_k \end{aligned} \tag{4.3}$$

holds $\mathbb{P}^\varepsilon$-almost surely for all $t \in [0, T]$ and all $\phi \in H^1_{\text{per}}(\mathcal{O})$.

We have the following theorem.

Theorem 4.3. *Let the assumptions (H1), (H2ε), (H3), and (H4ε) be satisfied and let $T > 0$ be given. Then, for every $\varepsilon \in (0, 1)$ there exists a martingale solution $((\Omega^\varepsilon, \mathcal{F}^\varepsilon, (\mathcal{F}^\varepsilon_t)_{t\ge0}, \mathbb{P}^\varepsilon), u^\varepsilon, W^\varepsilon)$ to (4.1) in the sense of Definition 4.2, satisfying the additional bound*

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]} \left(\int_{\mathcal{O}} \frac{1}{2}\left|u^\varepsilon_x\right|^2 + \Pi_\varepsilon(u^\varepsilon)dx\right)^q\right] + \mathbb{E}\left[\int_0^T \int_{\mathcal{O}} (u^\varepsilon)^2 \left|(u^\varepsilon_{xx} - \Pi'_\varepsilon(u^\varepsilon))_x\right|^2 dxdt\right] \\ &\quad \le C(\varepsilon, u_0, T, q)\,, \end{aligned} \tag{4.4}$$

where $q \geq 1$ can be chosen arbitrarily.

Remark 4.4. Combining the continuity of the $L^2(\mathcal{O})$-valued process u^ε with (4.4), it follows that u^ε in fact is a continuous process attaining values in $H^1_w(\mathcal{O})$ (i.e., $H^1(\mathcal{O})$ equipped with the weak topology).

Proof. Observe that Hypothesis (H4ε) differs only by the positive additive term C_1 in the upper bound on Π''_ε from Hypothesis (H2) in [16]. It is straightforward to show that all the results in [16] are still true if Hypothesis (H2) of [16] is replaced by Hypothesis (H4ε) of this paper. Noting that Hypothesis (H3) of this paper is stronger than Hypothesis (H4) of [16], we may establish the result as a consequence of Theorem 3.2 in [16]. □

For brevity, we will also call the process u^ε in Theorem 4.3 a solution of (4.1). We now give two auxiliary results we will need later on. The first one is a continuous version of Lemma 4.1 in [16] and can be established by similar arguments. It provides lower bounds on the solutions u^ε constructed in Theorem 4.3.

Lemma 4.5. *Let $\varepsilon \in (0, 1)$ and consider for $u \in H^1(\mathcal{O}, \mathbb{R}^+)$ the functional*

$$H_\varepsilon[u] := \frac{1}{2}\int_{\mathcal{O}} |u_x|^2 + \varepsilon(u)^{-p}dx\,. \tag{4.5}$$

Then there is a positive constant C_p independent of ε such that

$$\sup_{x\in\mathcal{O}}(u)^{-1} \leq \left(\fint_{\mathcal{O}} udx\right)^{-1} + C_p\varepsilon^{\frac{1}{2-p}}H_\varepsilon[u]^{\frac{2}{p-2}}. \tag{4.6}$$

If in addition $H_\varepsilon[u] \leq \sigma^{-1}$ for $\sigma \in (0, 1)$, there is a positive constant $\bar{C}_p$ independent of ε such that

$$\min_{x\in\mathcal{O}} u \geq \bar{C}_p\,\varepsilon^{\frac{1}{p-2}}\sigma^{\frac{2}{p-2}}\,. \tag{4.7}$$

In the next lemma, we collect a number of integral estimates which come as natural consequences of the energy-entropy estimate Lemma 4.4 in [16]. In the analysis presented in that paper, they were not needed and hence they were not made explicit there.

For the reader's convenience, we recall the notation used in [16]:

Remark 4.6. The pressure associated with a liquid film of thickness u^ε is given by $p^\varepsilon := -u^\varepsilon_{xx} + \Pi'_\varepsilon(u^\varepsilon)$, and at the level of finite elements, the discrete pressure p^h is—with a grain of salt— defined by

$$\left(p^h, \phi^h\right)_h = \int_{\mathcal{O}} u^h_x\phi^h_x\mathrm{d}x + \left(\Pi'_\varepsilon(u^h), \phi^h\right)_h \qquad \text{for all } \phi_h \subset V_h$$

where $(.,.)_h$ denotes the lumped masses scalar product and V_h is an appropriate space of continuous linear finite elements (see [16] for the details). Finally, for positive parameters α and κ,

$$R(s) := \alpha + E_h\left[u^h(s)\right] + \kappa S_h\left[u^h(s)\right]$$

is a weighted sum of discrete energy E_h and discrete entropy S_h—for the details, we refer to (4.14) in [16].

Lemma 4.7. *Let u^ε be a solution as constructed in Theorem 4.3. Then there is a positive constant $C = C(\varepsilon)$ such that*

$$\mathbb{E}\left[\int_0^T \int_{\mathcal{O}} ((u^\varepsilon)^2 p_x^\varepsilon)^2 dxdt\right] \leq C\,, \tag{4.8}$$

$$\mathbb{E}\left[\int_0^T \int_{\mathcal{O}} \left|u_{xx}^\varepsilon\right|^2 dxdt\right] \leq C\,, \tag{4.9}$$

$$\mathbb{E}\left[\int_0^T \int_{\mathcal{O}} (u^\varepsilon)^{-p-2} (u_x^\varepsilon)^2 dxdt\right] \leq C\,. \tag{4.10}$$

Proof. We recall that u^ε has been constructed as a limit of solutions $(u^\varepsilon)^h$ to finite dimensional auxiliary problems, cf. Lemma 4.2 in [16]. The convergence of these solutions is based on the a-priori estimate in Lemma 4.4 of that paper which will be for us the starting point to derive the estimates above. Adopting the notation of [16], and for the ease of presentation omitting the index ε, we recall the h-independent estimate

$$\begin{aligned}\mathbb{E}\left[\int_0^T \int_{\mathcal{O}} (M_h(u^h) p_x^h)^2 \mathrm{d}x\mathrm{d}s\right] &\leq C\mathbb{E}\left[\int_0^T \sup_{x\in\mathcal{O}}((u^h)^2) \int_{\mathcal{O}} M_h(u^h)(p_x^h)^2 \mathrm{d}x\mathrm{d}s\right] \\ &\leq C\mathbb{E}\left[\int_0^T R(s) \int_{\mathcal{O}} M_h(u^h)(p_x^h)^2 \mathrm{d}x\mathrm{d}s\right] \leq C(\varepsilon, u_0, T)\,,\end{aligned} \tag{4.11}$$

where $R(s)$ denoted in [16] a weighted sum of energy and entropy (cf. (4.14) in [16]). In particular, we have the upper bound $\int_{\mathcal{O}} (u_x^h)^2(\cdot, s)\mathrm{d}x \leq R(s)$ for all $s \in [0, T]$. Mimicking the arguments from Lemma 5.2 in [16], this bound shows that the sequence $(M_h(u^h) p_x^h)_{h\in(0,1)}$ is tight on the path-space $L^2(\mathcal{O} \times [0, T])$ with respect to the weak topology. Consequently, by Jakubowski's theorem, cf. [33], we find a stochastic basis $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}\right)$ and for each $h > 0$ random variables $\tilde{g}^h : \tilde{\Omega} \to L^2(\mathcal{O} \times [0, T])$ as well as $\tilde{g} : \tilde{\Omega} \to L^2(\mathcal{O} \times [0, T])$ such that

$$\tilde{g}^h \stackrel{\mathcal{D}}{=} M_h(u^h) p_x^h \tag{4.12}$$

under the probability measure $\tilde{\mathbb{P}}$. Moreover,

$$\tilde{g}^h \rightharpoonup \tilde{g}$$

in $L^2(\mathcal{O} \times [0, T])$ $\tilde{\mathbb{P}}$-almost surely. The identifications $\tilde{g}^h = M_h(\tilde{u}^h)\tilde{p}_x^h$ and $\tilde{g} = \tilde{u}^2 \tilde{p}_x$ can be achieved as in Lemma 5.6 and Lemma 5.9 in [16] where $\tilde{p}^h$ and $\tilde{p}$ are the pressures related to $\tilde{u}^h$ and $\tilde{u}$, respectively. Thus, we have

$$M_h(\tilde{u}^h)\tilde{p}_x^h \rightharpoonup \tilde{u}^2 \tilde{p}_x\,. \tag{4.13}$$

By the lower semi-continuity of the L^2-Norm w.r.t. weak convergence and Fatou's lemma, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^T\int_{\mathcal{O}}\left(\tilde{u}^2\tilde{p}_x\right)^2\mathrm{d}x\mathrm{d}s\right] &\le \mathbb{E}\left[\liminf_{h\to 0}\int_0^T\sup_{x\in\mathcal{O}}((\tilde{u}^h)^2)\int_{\mathcal{O}}M_h(\tilde{u}^h)(\tilde{p}^h_x)^2\mathrm{d}x\mathrm{d}s\right]\\ &\le \liminf_{h\to 0}\mathbb{E}\left[\int_0^T\sup_{x\in\mathcal{O}}((\tilde{u}^h)^2)\int_{\mathcal{O}}M_h(\tilde{u}^h)(\tilde{p}^h_x)^2\mathrm{d}x\mathrm{d}s\right]\\ &\le C(\varepsilon,u_0,T)<\infty,\end{aligned}\tag{4.14}$$

where we could use (4.11) due to (4.12). The arguments to show (4.9) and (4.10) are similar. □

5. A combined α-entropy-energy estimate

In this section, we prove a new estimate satisfied by solutions to Eq. (4.1) which is independent of $\varepsilon > 0$. This estimate will be the key to pass to the limit $\varepsilon \to 0$ and this way to establish the existence of martingale solutions to Eq. (2.1). Abbreviating $F(u) := u^{-p}$, Eq. (4.1) can equivalently be written

$$\mathrm{d}u^\varepsilon = -((u^\varepsilon)^2(u^\varepsilon_{xx}-\varepsilon F'(u^\varepsilon))_x)_x\mathrm{d}t + C_{\mathrm{Strat}}u^\varepsilon_{xx}\mathrm{d}t + \sum_{k\in\mathbb{Z}}\lambda_k(g_k u^\varepsilon)_x\mathrm{d}\beta^\varepsilon_k\,. \tag{5.1}$$

Theorem 4.3 guarantees the existence of a family $(u^\varepsilon)_{\varepsilon\in(0,1)}$ of solutions to (5.1).

Theorem 5.1. *Let $\varepsilon \in (0,1)$ and u^ε be a solution to (5.1). Then, for $q \ge 1$ and $\alpha \in (-\frac{1}{3}, 0)$ the ε-independent estimate*

$$\begin{aligned}\mathbb{E}\Bigg[&\sup_{t\in[0,T]}\left(\int_{\mathcal{O}}\frac{1}{2}|u^\varepsilon_x|^2+\varepsilon F(u^\varepsilon)dx\right)^q\\ &+\sup_{t\in[0,T]}\left(\int_{\mathcal{O}}\frac{1}{\alpha(\alpha+1)}(u^\varepsilon)^{\alpha+1}-\frac{1}{\alpha}u^\varepsilon+\frac{1}{\alpha+1}dx\right)^q\\ &+\left(\int_0^T\int_{\mathcal{O}}(u^\varepsilon)^2(u^\varepsilon_{xx}-\varepsilon F'(u^\varepsilon))_x^2\,dxds\right)^q\\ &+\left(\int_0^T\int_{\mathcal{O}}((u^\varepsilon)^{\frac{\alpha+3}{4}})_x^4\,dxds\right)^q+\left(\int_0^T\int_{\mathcal{O}}((u^\varepsilon)^{\frac{\alpha+3}{2}})_{xx}^2\,dxds\right)^q\\ &+\left(\int_0^T\int_{\mathcal{O}}(u^\varepsilon)^{\alpha+1}(u^\varepsilon_x)^2\varepsilon F''(u^\varepsilon)\,dxds\right)^q\Bigg]\le C(T,q,u_0).\end{aligned}\tag{5.2}$$

holds.

Our strategy to prove Theorem 5.1 is to combine Itô's formula and a stopping time argument. Let us for $\varepsilon > 0$ consider energies

$$\mathcal{E}_\varepsilon(u) := \frac{1}{2}\int_{\mathcal{O}}|u_x|^2\,\mathrm{d}x + \varepsilon\int_{\mathcal{O}}F(u)\mathrm{d}x \tag{5.3}$$

as well as random times

$$T_\sigma := T \wedge \inf\{t \in [0,T] | \mathcal{E}_\varepsilon(u^\varepsilon) \geq \sigma^{-1}\} \tag{5.4}$$

for positive parameters σ.

To show that T_σ is indeed a stopping time, according to Theorem 23 in [12], page 51 (see also the notes to Chapter 1 in [34]), we need to convince ourselves that $\mathcal{E}_\varepsilon(u^\varepsilon)$ is progressively measurable as $\left(\tilde{\mathcal{F}}_t\right)_{t\geq 0}$ is right-continuous and $\mathcal{F}_t$ is complete for each $t \geq 0$. For this, let us prove the stronger result that $\mathcal{E}_\varepsilon(u^\varepsilon)$ is predictable and therefore progressively measurable, too. For the first term on the right-hand side in (5.3), this is essentially a consequence of the fact, that u^ε has continuous paths as a mapping to $L^2(\mathcal{O})$, combined with an appropriate convolution argument to derive predictability of u^ε_x as a mapping to $L^2(\mathcal{O})$. Finally, predictability of $\varepsilon F(u^\varepsilon)$ follows from the fact that u^ε is nonnegative almost surely and F is Lipschitz on $[\rho, \infty)$ for any $\rho > 0$. Hence, $\varepsilon \int_\mathcal{O} F(u^\varepsilon + \rho)$ is predictable for any $\rho > 0$ fixed, too. Together with the monotone convergence theorem and preservation of predictability in the limit $\rho \to 0$, the result follows.

We further introduce the following cut-off versions of solutions u^ε, where we skip the index ε:

$$u_\sigma(\cdot,t) := \begin{cases} u^\varepsilon(\cdot,t) & t \in [0,T_\sigma] \\ u^\varepsilon(\cdot,T_\sigma) & t \in (T_\sigma, T]\,. \end{cases} \tag{5.5}$$

Moreover, we set $p_\sigma := -(u_\sigma)_{xx} + \varepsilon F'(u_\sigma)$. Note that the definition of p_σ is slightly different from that one of p^h or p^ε in Sect. 4 as we do not include the Stratonovich correction term within the pressure any longer.

Lemma 5.2. *For* $\alpha \in (-\frac{1}{3}, 0)$, $q \geq 1$, *and a constant* $C(T,q,u_0)$ *that is independent of* ε *we have the estimate*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left(\frac{1}{2}\int_\mathcal{O}(u_\sigma)_x^2 + \varepsilon F(u_\sigma)dx\right)^q\right] \\
&\quad + \mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_\mathcal{O}\frac{1}{\alpha(\alpha+1)}u_\sigma^{\alpha+1} - \frac{1}{\alpha}u_\sigma + \frac{1}{\alpha+1}dx\right)^q\right] \\
&\quad + \mathbb{E}\left[\left(\int_0^{t\wedge T_\sigma}\int_\mathcal{O}(u_\sigma)^2(p_\sigma)_x^2 dxds\right)^q\right] + \mathbb{E}\left[\left(\int_0^{t\wedge T_\sigma}\int_\mathcal{O}(u_\sigma)_{xx}^2(u_\sigma)^{\alpha+1}dxds\right)^q\right] \\
&\quad + \mathbb{E}\left[\left(\int_0^{t\wedge T_\sigma}\int_\mathcal{O}\frac{|\alpha(\alpha+1)|}{3}u_\sigma^{\alpha-1}(u_\sigma)_x^4 dxds\right)^q\right] \\
&\quad + \mathbb{E}\left[\left(\int_0^{t\wedge T_\sigma}\int_\mathcal{O}u_\sigma^{\alpha+1}(u_\sigma)_x^2\varepsilon F''(u_\sigma)dxds\right)^q\right] \leq C(T,q,u_0)\,.
\end{aligned} \tag{5.6}$$

The proof of Lemma 5.2 will be given below. Let us first assume it to hold and see how (5.2) can be derived from it. We need the following lemma which may be proven similarly as Lemma 5.5 in [16].

Lemma 5.3. *We have* $\lim_{\sigma\to 0} T_\sigma = T$ $\mathbb{P}^\varepsilon$*-almost surely.*

Proof of Theorem 5.1. From Lemma 4.5, we infer that u^ε is strictly positive on $\mathcal{O}\times[0,T]$ $\mathbb{P}^\varepsilon$-almost surely. In combination with Lemma 5.3, we find that the sets $A_\sigma := \{\omega\in\Omega : u^\varepsilon(\cdot,\omega)\geq\sigma \text{ on } \mathcal{O}\times[0,T]\}$ tend for $\sigma\to 0$ to Ω up to a set of measure zero. Hence, using nonnegativity of the terms on the left-hand side of (5.6) and monotone convergence, Theorem 5.1 is proven. □

The rest of this section is devoted to the proof of Lemma 5.2.

Proof of Lemma 5.2. Consider the operators

$$E_1 : u \mapsto \frac{1}{2}\int_{\mathcal{O}} u^2 \mathrm{d}x\,, \tag{5.7}$$

$$E_2 : u \mapsto \varepsilon\int_{\mathcal{O}} F(\eta(u))\mathrm{d}x\,, \tag{5.8}$$

$$\mathcal{G}_\alpha : u \mapsto \int_{\mathcal{O}} G_\alpha(\eta(u))\mathrm{d}x\,, \tag{5.9}$$

where η is a positive smooth cut-off function corresponding to the lower bound of u^ε provided by Lemma 4.5. For precise information about its properties, we refer to (B.13) and (B.14). Moreover, G_α is a standard α-entropy used for the thin-film equation, i.e.,

$$G_\alpha(u) = \frac{1}{\alpha(\alpha+1)}u^{\alpha+1} - \frac{1}{\alpha}u + \frac{1}{\alpha+1} > 0\,. \tag{5.10}$$

Itô's formula, applied separately for each of these operators, see "Appendix B" for details, yields

$$\begin{aligned}
&\int_{\mathcal{O}} \frac{1}{2}(u_\sigma(t))_x^2 + \varepsilon F(u_\sigma(t))\mathrm{d}x + \mathcal{G}_\alpha(u_\sigma(t))\\
&\quad+\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}(u_\sigma)^2(p_\sigma)_x^2\mathrm{d}x\mathrm{d}s + \int_0^{t\wedge T_\sigma}\int_{\mathcal{O}} C_{Strat}(u_\sigma)_{xx}^2\mathrm{d}x\mathrm{d}s\\
&\quad+\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}} C_{Strat}(u_\sigma)_x^2\varepsilon F''(u_\sigma)\mathrm{d}x\mathrm{d}s\\
&= E_1((u_0^\varepsilon)_x) + E_2(u_0^\varepsilon) + \mathcal{G}_\alpha(u_0^\varepsilon)\\
&\quad+\sum_{k\in\mathbb{Z}}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}(u_\sigma)_x\lambda_k(u_\sigma g_k)_{xx} + \lambda_k\varepsilon F'(u_\sigma)(u_\sigma g_k)_x\mathrm{d}x\mathrm{d}\beta_k\\
&\quad+\frac{1}{2}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\sum_{k\in\mathbb{Z}}\lambda_k^2(u_\sigma g_k)_{xx}^2\mathrm{d}x\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\sum_{k\in\mathbb{Z}}\lambda_k^2\varepsilon F''(u_\sigma)(u_\sigma g_k)_x^2\mathrm{d}x\mathrm{d}s\\
&+\sum_{k\in\mathbb{Z}}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\lambda_k\left(\frac{1}{\alpha}u_\sigma^\alpha-\frac{1}{\alpha}\right)(u_\sigma g_k)_x\mathrm{d}x\mathrm{d}\beta_k\\
&+\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}(-(u_\sigma)^2(p_\sigma)_x-C_{Strat}(u_\sigma)_x)\left(\frac{1}{\alpha}u_\sigma^\alpha-\frac{1}{\alpha}\right)_x\mathrm{d}x\mathrm{d}s\\
&+\frac{1}{2}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\sum_{k\in\mathbb{Z}}\lambda_k^2u_\sigma^{\alpha-1}(u_\sigma g_k)_x^2\mathrm{d}x\mathrm{d}s\\
&:=E_1((u_0^\varepsilon)_x)+E_2(u_0^\varepsilon)+\mathcal{G}_\alpha(u_0^\varepsilon)+R_1+\cdots+R_6\,. \qquad (5.11)
\end{aligned}
$$

Let us derive estimates for the terms on the right-hand side of Eq. (5.11). We will frequently use the relations (A.3)–(A.5) and (2.5). Choosing σ small enough, we may assume that $\eta(u_0^\varepsilon)=u_0^\varepsilon$. Then, since $\Lambda^\varepsilon=\mathbb{P}\circ(u_0+\varepsilon^\theta)^{-1}$, (H2), $\theta\in(0,\frac{1}{p})$, and $F(x)=x^{-p}$, we get

$$
\begin{aligned}
\mathbb{E}^\varepsilon\left[E_1((u_0^\varepsilon)_x)+E_2(u_0^\varepsilon)\right]&=\mathbb{E}^0\left[\frac{1}{2}\int_{\mathcal{O}}(u_0+\varepsilon^\theta)_x^2+\varepsilon(u_0+\varepsilon^\theta)^{-p}\mathrm{d}x\right]\\
&\le\mathbb{E}^0\left[C\int_{\mathcal{O}}(u_0)_x^2+\varepsilon\varepsilon^{-\theta p}\mathrm{d}x\right]\le C(u_0)
\end{aligned}
$$

and for $\alpha\in(-1,0)$

$$
\mathbb{E}^\varepsilon\left[\mathcal{G}_\alpha(u_0^\varepsilon)\right]=\mathbb{E}^0\left[\int_{\mathcal{O}}\frac{1}{\alpha(\alpha+1)}(u_0+\varepsilon^\theta)^{\alpha+1}-\frac{1}{\alpha}(u_0+\varepsilon^\theta)+\frac{1}{\alpha+1}\mathrm{d}x\right]\le C(u_0)\,. \qquad (5.12)
$$

Here, the superscript indicates that the expectation is computed with respect to $\mathbb{P}^\varepsilon$ and $\mathbb{P}$, respectively.

Ad R_2: We have

$$
\begin{aligned}
&\frac{1}{2}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\sum_{k\in\mathbb{Z}}\lambda_k^2(u_\sigma g_k)_{xx}^2\mathrm{d}x\mathrm{d}s\\
&\quad=\frac{1}{2}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\sum_{k=1}^\infty k^4\lambda_k^2\frac{32\pi^4}{L^5}u_\sigma^2+2\sum_{k=1}^\infty k^2\lambda_k^2\frac{8\pi^2}{L^3}(u_\sigma)_x^2+4\sum_{k=1}^\infty k^2\lambda_k^2\frac{8\pi^2}{L^3}(u_\sigma)_x^2\\
&\qquad+\left(\frac{\lambda_0^2}{L}+\sum_{k=1}^\infty\frac{2\lambda_k^2}{L}\right)(u_\sigma)_{xx}^2\mathrm{d}x\mathrm{d}s\\
&\quad:=A+B+C+D\,. \qquad (5.13)
\end{aligned}
$$

By means of Poincaré's inequality, A, B, and C will become terms to be controlled by a Gronwall argument, while D cancels out against an identical term on the left-hand side of (5.11).

Ad R_3:

$$R_3 = \frac{1}{2}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\sum_{k\in\mathbb{Z}}\lambda_k^2\varepsilon F''(u_\sigma)\left((g_k)_x^2(u_\sigma)^2 + 2(g_k)_x(g_k)u_\sigma(u_\sigma)_x + (g_k)^2(u_\sigma)_x^2\right)\mathrm{d}x\mathrm{d}s$$

$$= \frac{1}{2}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\sum_{k=1}^{\infty}k^2\lambda_k^2\frac{8\pi^2}{L^3}\varepsilon p(p+1)F(u_\sigma)\mathrm{d}x\mathrm{d}s$$

$$+\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\varepsilon\, C_{Strat}F''(u_\sigma)(u_\sigma)_x^2\mathrm{d}x\mathrm{d}s\,, \tag{5.14}$$

where the first term will be a Gronwall term, the second one cancels out against the corresponding term on the left-hand side of (5.11).

Let us discuss the contributions of the entropy. Ad R_5:

$$\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}(-(u_\sigma)^2(p_\sigma)_x - C_{Strat}(u_\sigma)_x)\left(\frac{1}{\alpha}u_\sigma^\alpha - \frac{1}{\alpha}\right)_x \mathrm{d}x\mathrm{d}s$$

$$= \int_0^{t\wedge T_\sigma}\int_{\mathcal{O}} -(u_\sigma)_{xx}^2 u_\sigma^{\alpha+1} + \frac{\alpha(\alpha+1)}{3}u_\sigma^{\alpha-1}(u_\sigma)_x^4 - \varepsilon F''(u_\sigma)u_\sigma^{\alpha+1}(u_\sigma)_x^2\mathrm{d}x\mathrm{d}s$$

$$-\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}} C_{\mathrm{Strat}}u_\sigma^{\alpha-1}(u_\sigma)_x^2\mathrm{d}x\mathrm{d}s\,. \tag{5.15}$$

Since $\alpha(\alpha+1) < 0$, the first integral is a good term while the second one will cancel out as the following calculation shows.

Ad R_6:

$$\frac{1}{2}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\sum_{k\in\mathbb{Z}}\lambda_k^2 u_\sigma^{\alpha-1}(u_\sigma g_k)_x^2\mathrm{d}x\mathrm{d}s$$

$$= \frac{1}{2}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\sum_{k=1}^{\infty}\lambda_k^2 k^2\frac{8\pi^2}{L^3}u_\sigma^{\alpha+1}\mathrm{d}x\mathrm{d}s$$

$$+\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}}\frac{1}{2}\left(\frac{\lambda_0^2}{L}+\sum_{k=1}^{\infty}\frac{2\lambda_k^2}{L}\right)u_\sigma^{\alpha-1}(u_\sigma)_x^2\mathrm{d}x\mathrm{d}s\,. \tag{5.16}$$

Here, the first term can be estimated by means of Young's and Poincaré's inequalities to become a Gronwall term while the last one vanishes by cancellation as indicated above.

Collecting all terms, rearranging, and combining the constants, as well as applying q'th powers, $q \geq 1$, suprema, and expectation, we get with $E(v) := E_1(v_x) + E_2(v)$ for arbitrary $t' \in [0, T]$ that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,t'\wedge T_\sigma]} E(u_\sigma(t))^q + \sup_{t\in[0,t'\wedge T_\sigma]} \mathcal{G}_\alpha(u_\sigma(t))^q\right] \\
&\quad + \mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\int_{\mathcal{O}} (u_\sigma)^2(p_\sigma)_x^2 \mathrm{d}x\mathrm{d}s\right)^q\right] + \mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\int_{\mathcal{O}} (u_\sigma)_{xx}^2 (u_\sigma)^{\alpha+1} \mathrm{d}x\mathrm{d}s\right)^q\right] \\
&\quad + \mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\int_{\mathcal{O}} \frac{|\alpha(\alpha+1)|}{3} u_\sigma^{\alpha-1}(u_\sigma)_x^4 \mathrm{d}x\mathrm{d}s\right)^q\right] \\
&\quad + \mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\int_{\mathcal{O}} \varepsilon F''(u_\sigma) u_\sigma^{\alpha+1}(u_\sigma)_x^2 \mathrm{d}x\mathrm{d}s\right)^q\right] \\
&\quad \le C(u_0,q) + C\mathbb{E}\left[\int_0^{t'\wedge T_\sigma} E(u_\sigma)^q \mathrm{d}s\right] + \mathbb{E}\left[\sup_{t\in[0,t'\wedge T_\sigma]} |R_1|^q\right] + \mathbb{E}\left[\sup_{t\in[0,t'\wedge T_\sigma]} |R_4|^q\right].
\end{aligned}
$$

Before applying the Burkholder–Davis–Gundy inequality, we consider for $s \in [0, T]$ the operator $\tau_1(s) : Q^{\frac{1}{2}} L^2(\mathcal{O}) \to \mathbb{R}$ with

$$
\begin{aligned}
\tau_1(s)(v) := \int_{\mathcal{O}} &(u_\sigma(s))_x (u_\sigma(s) \sum_{i\in\mathbb{Z}} (g_i, v)\, g_i)_{xx} \\
&+ \varepsilon F'(u_\sigma(s))(u_\sigma(s) \sum_{i\in\mathbb{Z}} (g_i, v)\, g_i)_x \mathrm{d}x\,. \qquad (5.17)
\end{aligned}
$$

Let us estimate the Hilbert–Schmidt norm of τ_1. For better readability, we will skip the argument s in the integral terms.

$$
\begin{aligned}
&\| \tau_1(s) \|^2_{L_2(Q^{1/2}L^2(\mathcal{O}),\mathbb{R})} = \sum_{k\in\mathbb{Z}} |\tau_1(s)(g_k\lambda_k)|^2 \\
&\quad = \sum_{k\in\mathbb{Z}} \left| \int_{\mathcal{O}} (u_\sigma)_{xx}(u_\sigma)_x \lambda_k g_k + (u_\sigma)_{xx} u_\sigma \lambda_k (g_k)_x + \varepsilon p(p+1) u_\sigma^{-p-1}(u_\sigma)_x \lambda_k g_k \mathrm{d}x \right|^2 \\
&\quad \le C\Bigg(\sum_{k\in\mathbb{Z}} \lambda_k^2 k^2 \left(\int_{\mathcal{O}} \frac{1}{2}(u_\sigma)_x^2 \mathrm{d}x \right)^2 + \sum_{k\in\mathbb{Z}} \lambda_k^2 k^2 \int_{\mathcal{O}} (u_\sigma)_{xx}^2 u_\sigma^2 \mathrm{d}x \\
&\qquad + \sum_{k\in\mathbb{Z}} \lambda_k^2 k^2 (p+1)^2 \left(\varepsilon \int_{\mathcal{O}} F(u_\sigma) \mathrm{d}x \right)^2 \Bigg) \\
&\quad \le \left(C(p) E(u_\sigma)^2 + C \| u_\sigma^{-\alpha+1} \|_\infty \int_{\mathcal{O}} (u_\sigma)_{xx}^2 u_\sigma^{\alpha+1} \mathrm{d}x \right). \qquad (5.18)
\end{aligned}
$$

Hence,

$$\mathbb{E}\left[\sup_{t\in[0,t'\wedge T_\sigma]}|R_1|^q\right] \le C\mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma} E(u_\sigma)^2 + \| u_\sigma^{-\alpha+1}\|_\infty \int_{\mathcal{O}} (u_\sigma)_{xx}^2 u_\sigma^{\alpha+1}\mathrm{d}x\mathrm{d}s\right)^{\frac{q}{2}}\right]$$
$$\le C(q)\mathbb{E}\left[\int_0^{t'\wedge T_\sigma} E(u_\sigma)^q \mathrm{d}s\right] + C(p,\delta,T)\,\tilde{\delta}\,\mathbb{E}\left[\sup_{t\in[0,t'\wedge T_\sigma]}\| u_\sigma\|_{H^1(\mathcal{O})}^{2q}\right]$$
$$+\tilde{\delta}^{-1}C(q,\delta,T,L)$$
$$+C(q)\delta\mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\int_{\mathcal{O}}(u_\sigma)_{xx}^2 u_\sigma^{\alpha+1}\mathrm{d}x\mathrm{d}s\right)^q\right] \tag{5.19}$$

for positive parameters δ and $\tilde{\delta}$ and arbitrary $t' \in [0, T]$.

The first term on the right-hand side of the last estimate is a Gronwall term while the others can be absorbed. We proceed to do the same for the stochastic integral R_4. The corresponding operator $\tau_2(s) : Q^{\frac{1}{2}}L^2(\mathcal{O}) \to \mathbb{R}$ now reads

$$\tau_2(s)(v) := \int_{\mathcal{O}}\left(\frac{1}{\alpha}u_\sigma^\alpha(s) - \frac{1}{\alpha}\right)(u_\sigma(s)\sum_{i\in\mathbb{Z}}(g_i, v)\, g_i)_x\mathrm{d}x$$

for $s \in [0, T]$. It follows

$$\begin{aligned}\| \tau_2(s)\|_{L_2(Q^{1/2}L^2(\mathcal{O}),\mathbb{R})}^2 &= \sum_{k\in\mathbb{Z}}\left|\int_{\mathcal{O}}\left(\frac{1}{\alpha}u_\sigma^\alpha - \frac{1}{\alpha}\right)(u_\sigma\sum_{i\in\mathbb{Z}}(g_k\lambda_k, g_i)\, g_i)_x\mathrm{d}x\right|^2\\
&\le \sum_{k\in\mathbb{Z}}\int_{\mathcal{O}} u_\sigma^{2\alpha}(u_\sigma)_x^2 g_k^2\lambda_k^2\mathrm{d}x\\
&\le \sum_{k\in\mathbb{Z}}\lambda_k^2\| g_k\|_\infty^2\| u_\sigma^{\frac{3\alpha+1}{2}}\|_\infty^2\int_{\mathcal{O}} u_\sigma^{\frac{\alpha-1}{2}}(u_\sigma)_x^2\mathrm{d}x\\
&\le C\| u_\sigma^{3\alpha+1}\|_\infty\int_{\mathcal{O}} u_\sigma^{\frac{\alpha-1}{2}}(u_\sigma)_x^2\mathrm{d}x\,,\end{aligned} \tag{5.20}$$

where we used $\alpha \in (-\frac{1}{3}, 0)$, the boundedness of g_k, and (2.5). Then we get

$$\mathbb{E}\left[\sup_{t\in[0,t'\wedge T_\sigma]}|R_4|^q\right] \le C\mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\| u_\sigma^{3\alpha+1}\|_\infty\int_{\mathcal{O}} u_\sigma^{\frac{\alpha-1}{2}}(u_\sigma)_x^2\mathrm{d}x\mathrm{d}s\right)^{\frac{q}{2}}\right]$$
$$\le C(q,T)\,\tilde{\delta}\,\mathbb{E}\left[\sup_{t\in[0,t'\wedge T_\sigma]}\| u_\sigma\|_{H^1(\mathcal{O})}^{2q}\right] + \tilde{\delta}^{-1}C(q,T,L)^q$$
$$+C(q)\delta\mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\int_{\mathcal{O}} u_\sigma^{\alpha-1}(u_\sigma)_x^4\mathrm{d}x\mathrm{d}s\right)^q\right]$$
$$+\delta^{-q}C(q,T,L)\,. \tag{5.21}$$

For the last estimate, we have once more used Poincaré's inequality. The first and the third term can be absorbed while the others are independent of ε. After absorption, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,t'\wedge T_\sigma]} E(u_\sigma(t))^q\right] + \mathbb{E}\left[\sup_{t\in[0,t'\wedge T_\sigma]} \mathcal{G}_\alpha(u_\sigma(t))^q\right] \\
&\quad + \mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\int_{\mathcal{O}} (u_\sigma)^2(p_\sigma)_x^2 \mathrm{d}x\mathrm{d}s\right)^q\right] + \mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\int_{\mathcal{O}} (u_\sigma)_{xx}^2(u_\sigma)^{\alpha+1} \mathrm{d}x\mathrm{d}s\right)^q\right] \\
&\quad + \mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\int_{\mathcal{O}} \frac{|\alpha(\alpha+1)|}{3} u_\sigma^{\alpha-1}(u_\sigma)_x^4 \mathrm{d}x\mathrm{d}s\right)^q\right] \\
&\quad + \mathbb{E}\left[\left(\int_0^{t'\wedge T_\sigma}\int_{\mathcal{O}} \varepsilon F''(u_\sigma)u_\sigma^{\alpha+1}(u_\sigma)_x^2 \mathrm{d}x\mathrm{d}s\right)^q\right] \\
&\quad \leq C(T,q,u_0) + C\mathbb{E}\left[\int_0^{t'\wedge T_\sigma} E(u_\sigma)^q \mathrm{d}s\right].
\end{aligned}
\tag{5.22}
$$

To control the second term on the right-hand side, we observe that (5.22) entails the estimate

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq t'} E\left(u_\sigma(t\wedge T_\sigma)\right)^q\right] \\
&\quad \leq C(T,q,u_0) + C\mathbb{E}\left[\int_0^{t'\wedge T_\sigma} E(u_\sigma(s))^q ds\right] \\
&\quad \leq C(T,q,u_0) + C\mathbb{E}\left[\int_0^{t'} \sup_{t\leq s} E(u_\sigma(t\wedge T_\sigma))^q ds\right]
\end{aligned}
\tag{5.23}
$$

which gives the result by a combination of a Gronwall and a Fubini argument. □

6. Convergence of approximate solutions

In this section, we pass to the limit $\varepsilon \to 0$ with the approximate solutions u^ε and finally, we prove the main results of the paper, i.e., Theorem 3.1 and Corollary 3.2.

6.1. Application of Jakubowski's theorem

In this subsection, we will apply Jakubowski's theorem, cf. [33].

We define $v^\varepsilon := ((u^\varepsilon)^{\frac{\alpha+3}{4}})_x$, $z^\varepsilon := ((u^\varepsilon)^{\frac{\alpha+3}{2}})_{xx}$, and for $\tilde{\gamma} < \gamma$ and $\tilde{\sigma} < \sigma$

$$\begin{aligned}
\mathcal{X}_u &:= C^{\tilde{\gamma},\tilde{\sigma}}(\bar{\mathcal{O}}_T)\\
\mathcal{X}_{u_x} &:= L^2(0,T;L^2(\mathcal{O}))_{weak}\\
\mathcal{X}_v &:= L^4(0,T;L^4(\mathcal{O}))_{weak}\\
\mathcal{X}_z &:= L^2(0,T;L^2(\mathcal{O}))_{weak}\\
\mathcal{X}_W &:= C([0,T];L^2(\mathcal{O}))\\
\mathcal{X}_{u_0} &:= H^1_{\text{per}}(\mathcal{O}).
\end{aligned}$$

Moreover, $\mathcal{X} := \mathcal{X}_u \times \mathcal{X}_{u_x} \times \mathcal{X}_v \times \mathcal{X}_z \times \mathcal{X}_W \times \mathcal{X}_{u_0}$. Using standard results on tightness, see for instance Lemma 5.2 in [16], we get the following result.

Theorem 6.1. *The laws μ_{u^ε}, $\mu_{u_x^\varepsilon}$, μ_{v^ε}, μ_{z^ε}, μ_{W^ε}, $\mu_{u_0^\varepsilon}$ of the corresponding random variables are tight on the path spaces $\mathcal{X}_u$, $\mathcal{X}_{u_x}$, $\mathcal{X}_v$, $\mathcal{X}_z$, $\mathcal{X}_W$, and $\mathcal{X}_{u_0}$, respectively.*

Now, we apply Jakubowski's theorem [33].

Theorem 6.2. *For subsequences of u^ε, u_x^ε, v^ε, z^ε, u_0^ε, and W^ε, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a sequence of $L^2(\mathcal{O})$-valued stochastic processes $\tilde{W}^\varepsilon$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, sequences of random variables*

$$\begin{aligned}
\tilde{u}^\varepsilon &: \tilde{\Omega} \to C^{\tilde{\gamma},\tilde{\sigma}}(\bar{\mathcal{O}}_T),\\
\tilde{w}^\varepsilon &: \tilde{\Omega} \to L^2(0,T;L^2(\mathcal{O})),\\
\tilde{v}^\varepsilon &: \tilde{\Omega} \to L^4(0,T;L^4(\mathcal{O})),\\
\tilde{z}^\varepsilon &: \tilde{\Omega} \to L^2(0,T;L^2(\mathcal{O})),\\
\tilde{u}_0^\varepsilon &: \tilde{\Omega} \to H^1_{per}(\mathcal{O}),
\end{aligned}$$

random variables

$$\begin{aligned}
&\tilde{u} \in L^2(\tilde{\Omega}; C^{\tilde{\gamma},\tilde{\sigma}}(\bar{\mathcal{O}}_T)),\\
&\tilde{w} \in L^2(\tilde{\Omega}; L^2(0,T;L^2(\mathcal{O}))),\\
&\tilde{v} \in L^4(\tilde{\Omega}; L^4(0,T;L^4(\mathcal{O}))),\\
&\tilde{z} \in L^2(\tilde{\Omega}; L^2(0,T;L^2(\mathcal{O}))),\\
&\tilde{u}_0 \in L^2(\tilde{\Omega}; H^1_{per}(\mathcal{O})),
\end{aligned}$$

as well as a $L^2(\mathcal{O})$-valued process $\tilde{W}$ such that

1. *for all $\varepsilon \in (0,1)$ the law of $(\tilde{u}^\varepsilon, \tilde{w}^\varepsilon, \tilde{v}^\varepsilon, \tilde{z}^\varepsilon, \tilde{W}^\varepsilon, \tilde{u}_0^\varepsilon)$ on $\mathcal{X}$ w.r.t. the measure $\tilde{\mathbb{P}}$ equals the law of $(u^\varepsilon, u_x^\varepsilon, v^\varepsilon, z^\varepsilon, W^\varepsilon, u_0^\varepsilon)$ w.r.t. $\mathbb{P}^\varepsilon$.*
2. *as $\varepsilon \to 0$, the sequence $(\tilde{u}^\varepsilon, \tilde{w}^\varepsilon, \tilde{v}^\varepsilon, \tilde{z}^\varepsilon, \tilde{W}^\varepsilon, \tilde{u}_0^\varepsilon)$ converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde{u}, \tilde{w}, \tilde{v}, \tilde{z}, \tilde{W}, \tilde{u}_0)$ in the topology of $\mathcal{X}$.*

Next, we identify the new sequences on $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

Lemma 6.3. *We have* $\tilde w^\varepsilon = \tilde u^\varepsilon_x$, $\tilde v^\varepsilon = ((\tilde u^\varepsilon)^{\frac{\alpha+3}{4}})_x$, *as well as* $\tilde z^\varepsilon = ((\tilde u^\varepsilon)^{\frac{\alpha+3}{2}})_{xx}$ $\tilde{\mathbb{P}}$*-almost surely.*

Proof. The mapping

$$L^2(0,T;L^2(\mathcal{O})) \to \mathbb{R}, \quad u \mapsto \int_0^T \int_{\mathcal{O}} u \,\mathrm{d}x\mathrm{d}s$$

is Borel-measurable. Hence, for arbitrary $\phi \in C_c^\infty(\mathcal{O}_T)$ we have due to the equality of laws stated in Theorem 6.2

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^T \int_{\mathcal{O}} \tilde w^\varepsilon \phi \,\mathrm{d}x\mathrm{d}s + \int_0^T \int_{\mathcal{O}} \tilde u^\varepsilon \phi_x \,\mathrm{d}x\mathrm{d}s\right|\right] \\
&= \mathbb{E}\left[\left|\int_0^T \int_{\mathcal{O}} u^\varepsilon_x \phi \,\mathrm{d}x\mathrm{d}s - \int_0^T \int_{\mathcal{O}} u^\varepsilon_x \phi \,\mathrm{d}x\mathrm{d}s\right|\right] \\
&= 0.
\end{aligned}$$

The other statements follow by similar reasoning. □

As in [16], we consider the filtrations $(\tilde F_t)_{t\geq 0}$ and $(\tilde F^\varepsilon_t)_{t\geq 0}$ with

$$\tilde F_t := \sigma(\sigma(r_t \tilde u, r_t \tilde W) \cup \{N \in \tilde F : \tilde{\mathbb{P}}(N) = 0\} \cup \sigma(\tilde u_0)) \tag{6.1}$$

and

$$\tilde F^\varepsilon_t := \sigma(\sigma(r_t \tilde u^\varepsilon, r_t \tilde W^\varepsilon) \cup \{N \in \tilde F : \tilde{\mathbb{P}}(N) = 0\} \cup \sigma(\tilde u^\varepsilon_0)) \,. \tag{6.2}$$

Here, r_t is the restriction of a mapping on $[0, T]$ to the time interval $[0, t]$, $t \in [0, T]$. The proof of the next lemma can be found in [16] Lemma 5.7.

Lemma 6.4. *The processes* $\tilde W^\varepsilon$ *and* $\tilde W$ *are Q-Wiener processes which are adapted to the filtrations* $(\tilde F^\varepsilon_t)_{t\geq 0}$ *and* $(\tilde F_t)_{t\geq 0}$*, respectively. We have*

$$\tilde W^\varepsilon(t) = \sum_{k\in\mathbb{Z}} \lambda_k \tilde\beta^\varepsilon_k(t) g_k \tag{6.3}$$

and

$$\tilde W(t) = \sum_{k\in\mathbb{Z}} \lambda_k \tilde\beta_k(t) g_k \tag{6.4}$$

with families $(\tilde\beta^\varepsilon_k)_{k\in\mathbb{Z}}$ *and* $(\tilde\beta_k)_{k\in\mathbb{Z}}$ *of i.i.d. standard Brownian motions w.r.t.* $(\tilde F^\varepsilon_t)_{t\geq 0}$ *and* $(\tilde F_t)_{t\geq 0}$*, respectively.*

For the limits $\tilde w$, $\tilde v$ and $\tilde z$, we get the following identities.

Lemma 6.5. *We have* $\tilde{\mathbb{P}}$*-almost surely* $\tilde w = \tilde u_x$, $\tilde v = (\tilde u^{\frac{\alpha+3}{4}})_x$, *and* $\tilde z = (\tilde u^{\frac{\alpha+3}{2}})_{xx}$.

Proof. Exemplarily we show the first statement. From our convergence results in Theorem 6.2 and by integration by parts, we deduce for all test functions $\phi \in C_c^\infty(\mathcal{O}_T)$

$$\begin{aligned}\int_0^T \int_{\mathcal{O}} \tilde{w}\phi \,\mathrm{d}x\mathrm{d}s \leftarrow \int_0^T \int_{\mathcal{O}} \tilde{w}^\varepsilon \phi \,\mathrm{d}x\mathrm{d}s &= \int_0^T \int_{\mathcal{O}} \tilde{u}_x^\varepsilon \phi \,\mathrm{d}x\mathrm{d}s \\ &= -\int_0^T \int_{\mathcal{O}} \tilde{u}^\varepsilon \phi_x \,\mathrm{d}x\mathrm{d}s \\ &\to -\int_0^T \int_{\mathcal{O}} \tilde{u}\phi_x \,\mathrm{d}x\mathrm{d}s\end{aligned}$$

$\tilde{\mathbb{P}}$-almost surely. This gives the first equality. □

6.2. Convergence results of the deterministic terms

In the next lemmas, we establish convergence of the deterministic terms, corresponding to the weak formulation (3.1).

Lemma 6.6. *The sequence $\tilde{u}^\varepsilon$ admits a subsequence such that for a function ζ on $\tilde{\Omega} \times \mathcal{O} \times (0, T)$*

$$(\tilde{u}_x^\varepsilon)^3 \rightharpoonup \zeta \tag{6.5}$$

weakly in $L^{\frac{4}{3}}(\mathcal{O}_T)$ $\tilde{\mathbb{P}}$-almost surely.

Proof. From Lemmas 6.2, 6.3, and 6.5 in the previous subsection, we know $((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x \rightharpoonup (\tilde{u}^{\frac{\alpha+3}{4}})_x$ $\tilde{\mathbb{P}}$-almost surely in $L^4(0, T; L^4(\mathcal{O}))$. By the identity

$$(\tilde{u}_x^\varepsilon)^3 = \left(\frac{4}{\alpha+3}\right)^3 (\tilde{u}^\varepsilon)^{\frac{3(-\alpha+1)}{4}} ((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x^3 \,, \tag{6.6}$$

the $\tilde{\mathbb{P}}$-almost surely uniform boundedness of $\tilde{u}^\varepsilon$ in $L^\infty(\mathcal{O}_T)$, and the positivity of $-\alpha + 1$ we conclude

$$\int_0^T \int_{\mathcal{O}} ((\tilde{u}_x^\varepsilon)^3)^{\frac{4}{3}} \,\mathrm{d}x\mathrm{d}s = C \int_0^T \int_{\mathcal{O}} (\tilde{u}^\varepsilon)^{-\alpha+1} ((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x^4 \,\mathrm{d}x\mathrm{d}s \le C$$

$\tilde{\mathbb{P}}$-almost surely. The result then follows by the reflexivity of $L^{\frac{4}{3}}(\mathcal{O})$. □

Let S be a set. $L^{p-}(S)$ denotes the space of functions that are contained in every space $L^q(S)$, where $1 \le q < p$.

Lemma 6.7. *We have $\tilde{\mathbb{P}}$-almost surely*

$$((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x \to ((\tilde{u})^{\frac{\alpha+3}{4}})_x \tag{6.7}$$

strongly in $L^{4-}([\tilde{u} > 0])$.

Proof. For arbitrary $p > 1$, we have the strong convergence of $\tilde{u}^\varepsilon$ in $L^p(\mathcal{O}_T)$ $\tilde{\mathbb{P}}$-almost surely, cf. Theorem 6.2. Thus, $(\tilde{u}^\varepsilon)^{\frac{\alpha+3}{2}} \to \tilde{u}^{\frac{\alpha+3}{2}}$ in $L^2(\mathcal{O}_T)$ $\tilde{\mathbb{P}}$-almost surely. By Riesz' theorem

$$\lim_{h\to 0} \| (\tilde{u}^\varepsilon)^{\frac{\alpha+3}{2}}(\cdot, \cdot + h) - (\tilde{u}^\varepsilon)^{\frac{\alpha+3}{2}}(\cdot, \cdot) \|_{L^2((0,T-h);L^1(\mathcal{O}))} = 0 \tag{6.8}$$

follows. Furthermore, (5.2) implies

$$\sup_{\varepsilon\in(0,1)} \| ((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{2}})_{xx} \|_{L^2(\mathcal{O}_T)} < \infty \tag{6.9}$$

$\tilde{\mathbb{P}}$-almost surely. Hölder's inequality as well as the uniform bound of u^ε in $C^{\tilde{\gamma},\tilde{\sigma}}(\bar{\mathcal{O}}_T)$ show

$$\begin{aligned} \int_0^T \int_{\mathcal{O}} ((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{2}})_x^2 \mathrm{d}x\mathrm{d}s &= C \int_0^T \int_{\mathcal{O}} (\tilde{u}^\varepsilon_x)^2 (\tilde{u}^\varepsilon)^{\frac{\alpha-1}{2}} (\tilde{u}^\varepsilon)^{\frac{\alpha+3}{2}} \,\mathrm{d}x\mathrm{d}s \\ &\le C \left(\int_0^T \int_{\mathcal{O}} (\tilde{u}^\varepsilon_x)^4 (\tilde{u}^\varepsilon)^{\alpha-1} \,\mathrm{d}x\mathrm{d}s \right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathcal{O}} (\tilde{u}^\varepsilon)^{\alpha+3} \,\mathrm{d}x\mathrm{d}s \right)^{\frac{1}{2}} < \infty \end{aligned} \tag{6.10}$$

for all $\varepsilon \in (0, 1)$ $\tilde{\mathbb{P}}$-almost surely. Combining (6.9) and (6.10), we conclude

$$\sup_{\varepsilon\in(0,1)} \| ((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{2}}) \|_{H^2(\mathcal{O}_T)} < \infty \tag{6.11}$$

$\tilde{\mathbb{P}}$-almost surely. By Simon's theorem, c.f. [40], using (6.8), (6.11), and the spaces $H^2(\mathcal{O}) \subset H^1(\mathcal{O}) \subset L^1(\mathcal{O})$, we get

$$((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{2}})_x \to (\tilde{u}^{\frac{\alpha+3}{2}})_x \tag{6.12}$$

strongly in $L^2(\mathcal{O}_T)$ $\tilde{\mathbb{P}}$-almost surely. On the set $[\tilde{u} > 0]$, there exists a subsequence with

$$((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x \to (\tilde{u}^{\frac{\alpha+3}{4}})_x \tag{6.13}$$

pointwise almost surely for $\varepsilon \to 0$. The sequence $((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x$ is uniformly bounded in $L^4(\mathcal{O}_T)$, cf. (5.2), which in turn implies uniform integrability in $L^{4-\delta}(\mathcal{O}_T)$, $\delta \in (0, 1)$. The result now follows with Vitali's theorem. □

Corollary 6.8. *We have $\tilde{\mathbb{P}}$-almost surely*

$$(\tilde{u}^\varepsilon_x)^3 \rightharpoonup \tilde{u}_x^3 \tag{6.14}$$

weakly in $L^{\frac{4}{3}}([\tilde{u} > 0])$ and for $\phi \in H^3_{per}(\mathcal{O})$

$$\lim_{\varepsilon\to 0} \int_{[\tilde{u}=0]} (\tilde{u}^\varepsilon_x)^3 \phi_x dx = 0. \tag{6.15}$$

Proof. The identity (6.6), the uniform convergence of $\tilde{u}^\varepsilon$, and Lemma 6.7 show $(\tilde{u}_x^\varepsilon)^3 \to \tilde{u}_x^3$ strongly in $L^{\frac{4}{3}-}([\tilde{u} > 0])$. By the uniqueness of weak limits, we find $\zeta = (\tilde{u}_x^\varepsilon)^3$ on $[\tilde{u} > 0]$ in Lemma 6.6 which is (6.14). The second statement (6.15) follows with Hölder's inequality:

$$\int\int_{[\tilde{u}=0]} \left|(\tilde{u}_x^\varepsilon)^3\phi_x\right| \mathrm{d}x\mathrm{d}s = \int\int_{[\tilde{u}=0]} (\tilde{u}^\varepsilon)^{\frac{3(\alpha-1)}{4}} \left|\tilde{u}_x^\varepsilon\right|^3 (\tilde{u}^\varepsilon)^{\frac{-3(\alpha-1)}{4}} |\phi_x|\, \mathrm{d}x\mathrm{d}s$$
$$\leq \left(\int\int_{[\tilde{u}=0]} (\tilde{u}^\varepsilon)^{\alpha-1}(\tilde{u}_x^\varepsilon)^4 \mathrm{d}x\mathrm{d}s\right)^{3/4} \left(\int\int_{[\tilde{u}=0]} (\tilde{u}^\varepsilon)^{-3\alpha+3}\phi_x^4 \mathrm{d}x\mathrm{d}s\right)^{1/4} \to 0\,.$$

□

The convergence of $\tilde{u}^\varepsilon(\tilde{u}_x^\varepsilon)^2$ can be shown in a similar way.

Lemma 6.9. *For a subsequence of $\tilde{u}^\varepsilon$, we have $\tilde{\mathbb{P}}$-almost surely*

$$\tilde{u}^\varepsilon(\tilde{u}_x^\varepsilon)^2 \rightharpoonup \tilde{u}(\tilde{u}_x)^2 \tag{6.16}$$

weakly in $L^2(\mathcal{O}_T)$.

Proof. By means of the identity

$$\tilde{u}^\varepsilon(\tilde{u}_x^\varepsilon)^2 = \left(\frac{4}{\alpha+3}\right)^2 (\tilde{u}^\varepsilon)^{\frac{-\alpha+3}{2}}((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x^2\,, \tag{6.17}$$

we conclude as in Lemma 6.6 that $\tilde{u}^\varepsilon(\tilde{u}_x^\varepsilon)^2$ is uniformly bounded in $L^2(\mathcal{O}_T)$ and thus admits a subsequence such that $\tilde{u}^\varepsilon(\tilde{u}_x^\varepsilon)^2 \rightharpoonup \gamma$ $\tilde{\mathbb{P}}$-almost surely in $L^2(\mathcal{O}_T)$. Using $-\alpha > 0$, the strong convergence (6.12) in Lemma 6.7, as well as

$$\tilde{u}^\varepsilon(\tilde{u}_x^\varepsilon)^2 = \left(\frac{2}{\alpha+3}\right)^2 ((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{2}})_x^2(\tilde{u}^\varepsilon)^{-\alpha}, \tag{6.18}$$

we find a subsequence of $\tilde{u}^\varepsilon(\tilde{u}_x^\varepsilon)^2$ that converges pointwise to $\tilde{u}(\tilde{u}_x)^2$ $\tilde{\mathbb{P}}$-almost surely. The uniform bound of (6.18) and Vitali's theorem then give

$$\tilde{u}^\varepsilon(\tilde{u}_x^\varepsilon)^2 \to \tilde{u}(\tilde{u}_x)^2$$

strongly in $L^{2-}(\mathcal{O}_T)$ $\tilde{\mathbb{P}}$-almost surely. The same arguments as in Corollary 6.8 show $\gamma = \tilde{u}(\tilde{u}_x)^2$. □

The next result follows immediately from the strong convergence of $(\tilde{u}^\varepsilon)^2$ and the weak convergence of $\tilde{u}_x^\varepsilon$ in $L^2(\mathcal{O}_T)$, respectively.

Lemma 6.10. *For all $\phi \in H^3_{per}(\mathcal{O}_T)$*

$$\int_0^T \int_{\mathcal{O}} (\tilde{u}^\varepsilon)^2\tilde{u}_x^\varepsilon\phi_{xxx}\, dxds \to \int_0^T \int_{\mathcal{O}} (\tilde{u})^2\tilde{u}_x\phi_{xxx}\, dxds \tag{6.19}$$

holds $\tilde{\mathbb{P}}$-almost surely.

Finally, we show that the term which contains the effective interface potential vanishes in the limit.

Lemma 6.11. *For $\varepsilon \to 0$ and all $\phi \in H^3_{per}(\mathcal{O}_T)$, we have*

$$\mathbb{E}\left[\left|\int_0^T \int_{\mathcal{O}} \varepsilon (\tilde{u}^\varepsilon)^2 \tilde{u}^\varepsilon_x F''(\tilde{u}^\varepsilon)\phi_x \, dx ds\right|\right] \to 0. \tag{6.20}$$

Proof. Using the weighted version of Young's inequality, we find for $\eta > 0$ and $p \in (2, \infty)$ with $F(x) = x^{-p}$

$$\begin{aligned}\left|\int_0^T \int_{\mathcal{O}} \varepsilon (\tilde{u}^\varepsilon)^2 \tilde{u}^\varepsilon_x (\tilde{u}^\varepsilon)^{-p-2}\phi_x \, \mathrm{d}x\mathrm{d}s\right| &= \left|\frac{1}{p-1}\int_0^T \int_{\mathcal{O}} \varepsilon (\tilde{u}^\varepsilon)^{\frac{-p}{2}} (\tilde{u}^\varepsilon)^{\frac{-p+2}{2}} \phi_{xx} \, \mathrm{d}x\mathrm{d}s\right| \\ &\le \frac{C}{4}\int_0^T \int_{\mathcal{O}} \varepsilon^2 \varepsilon^{\eta-1} (\tilde{u}^\varepsilon)^{-p}\phi_{xx}^2 \, \mathrm{d}x\mathrm{d}s \\ &\quad + C\int_0^T \int_{\mathcal{O}} \varepsilon^{1-\eta} (\tilde{u}^\varepsilon)^{-p+2} \, \mathrm{d}x\mathrm{d}s \\ &:= \mathrm{I} + \mathrm{II}\,.\end{aligned}$$

Due to the boundedness of ϕ_{xx} and (5.2), we have for I

$$C\varepsilon^{\eta+1}\mathbb{E}\left[\int_0^T \int_{\mathcal{O}} F(\tilde{u}^\varepsilon)\phi_{xx}^2 \, \mathrm{d}x\mathrm{d}s\right] \le \varepsilon^{\eta+1} C\,\mathbb{E}\left[\sup_{t\in[0,T]} \int_{\mathcal{O}} F(\tilde{u}^\varepsilon(t))\mathrm{d}x\right] \le \varepsilon^{\eta+1} C \to 0$$

$\tilde{\mathbb{P}}$-almost surely. For II, we argue with $\delta > 0$ as follows:

$$\begin{aligned}&\int_0^T \int_{\mathcal{O}} \varepsilon^{1-\eta} (\tilde{u}^\varepsilon)^{-p} (\tilde{u}^\varepsilon)^2 \, \mathrm{d}x\mathrm{d}s \\ &= \int \int_{[\tilde{u}<\varepsilon^{\frac{\eta}{2}+\delta}]} \varepsilon^{1-\eta} (\tilde{u}^\varepsilon)^{-p} (\tilde{u}^\varepsilon)^2 \mathrm{d}x\mathrm{d}s + \int \int_{[\tilde{u}\ge\varepsilon^{\frac{\eta}{2}+\delta}]} \varepsilon^{1-\eta} (\tilde{u}^\varepsilon)^{-p+2} \mathrm{d}x\mathrm{d}s \\ &\le \varepsilon^{1-\eta+\eta+2\delta} \int_0^T \int_{\mathcal{O}} F(\tilde{u}^\varepsilon)\, \mathrm{d}x\mathrm{d}s + \int \int_{[\tilde{u}\ge\varepsilon^{\frac{\eta}{2}+\delta}]} \varepsilon^{1-\eta+(-p+2)(\frac{\eta}{2}+\delta)} \mathrm{d}x\mathrm{d}s\,.\end{aligned}$$

Hence, using (5.2) once more, we have for η and δ chosen appropriately

$$\mathbb{E}\,[\mathrm{II}] \le \varepsilon^{1+2\delta} C + \varepsilon^{1-\eta-\frac{p\eta}{2}+\eta+\delta(-p+2)} C \to 0$$

for $\varepsilon \to 0$. □

6.3. Identification of the stochastic integral

For $\phi \in H^3_{\text{per}}(\mathcal{O})$ arbitrary, but fixed, we consider the processes $\mathcal{M}_{\varepsilon,\phi} : \Omega \times [0,T] \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{M}_{\varepsilon,\phi}(t) := &\int_{\mathcal{O}} (u^\varepsilon(t) - u_0^\varepsilon)\phi \mathrm{d}x - \int_0^t \int_{\mathcal{O}} (u_x^\varepsilon)^3 \phi_x \,\mathrm{d}x\mathrm{d}s - 3\int_0^t \int_{\mathcal{O}} u^\varepsilon (u_x^\varepsilon)^2 \phi_{xx} \,\mathrm{d}x\mathrm{d}s \\ &- \int_0^t \int_{\mathcal{O}} (u^\varepsilon)^2 u_x^\varepsilon \phi_{xxx} \,\mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathcal{O}} (u^\varepsilon)^2 u_x^\varepsilon \varepsilon F''(u^\varepsilon)\phi_x \,\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_{\mathcal{O}} \frac{1}{2} \sum_{k\in\mathbb{Z}} \lambda_k^2 g_k (g_k u^\varepsilon)_x \phi_x \,\mathrm{d}x\mathrm{d}s\,. \end{aligned} \tag{6.21}$$

Note that the right-hand side of (6.21) coincides with the deterministic terms in (4.3) for the choice $\phi \in H^3_{per}(\mathcal{O})$ which follows easily by integration by parts. In particular, the last term in (6.21) is identical with $C_{Strat} \int_0^t \int_{\mathcal{O}} u_x^\varepsilon \phi_x \mathrm{d}x\mathrm{d}s$, cf. (2.6) and (A.6). On the other hand, we have

$$\mathcal{M}_{\varepsilon,\phi}(t) = \sum_{k\in\mathbb{Z}} \int_0^t \int_{\mathcal{O}} \lambda_k (g_k u^\varepsilon)_x \phi \mathrm{d}x\mathrm{d}\beta_k \tag{6.22}$$

for $t \in [0,T]$, i.e., $\mathcal{M}_{\varepsilon,\phi}$ is a continuous, square integrable $\mathcal{F}_t^\varepsilon$-martingale. We will need the following results which have been shown in [16], Lemmas 5.10 and 5.12:

$$\langle \mathcal{M}_{\varepsilon,\phi} \rangle = \int_0^\cdot \sum_{k\in\mathbb{Z}} \lambda_k^2 \left(\int_{\mathcal{O}} (u^\varepsilon g_k)_x \phi \mathrm{d}x \right)^2 \mathrm{d}s, \tag{6.23}$$

$$\langle \mathcal{M}_{\varepsilon,\phi} \rangle \le C \| \phi \|^2_{H^1_{\text{per}}} \int_0^T \| u^\varepsilon \|^2_{L^2(\mathcal{O})} \mathrm{d}s, \tag{6.24}$$

and for $k \in \mathbb{N}$

$$\langle \mathcal{M}_{\varepsilon,\phi}, \beta_k^\varepsilon \rangle = \lambda_k \int_0^\cdot \int_{\mathcal{O}} (u^\varepsilon g_k)_x \phi \mathrm{d}x\mathrm{d}s. \tag{6.25}$$

With these results at hand, we can establish

Corollary 6.12. *Let $k \in \mathbb{N}$ and $\varepsilon \in (0,1)$. The processes*

$$\mathcal{M}^2_{\varepsilon,\phi} - \int_0^\cdot \sum_{k\in\mathbb{Z}} \lambda_k^2 \left(\int_{\mathcal{O}} (u^\varepsilon g_k)_x \phi dx \right)^2 ds \tag{6.26}$$

and

$$\mathcal{M}_{\varepsilon,\phi} \beta_k^\varepsilon - \lambda_k \int_0^\cdot \int_{\mathcal{O}} (u^\varepsilon g_k)_x \phi dx ds \tag{6.27}$$

are continuous $\mathcal{F}_t^\varepsilon$-martingales.

By the equality of laws stated in Theorem 6.2, we also get the analog statements for

$$\begin{aligned}\tilde{\mathcal{M}}_{\varepsilon,\phi}(t) &:= \int_{\mathcal{O}} (\tilde{u}^\varepsilon(t) - \tilde{u}_0^\varepsilon)\phi \mathrm{d}x - \int_0^t \int_{\mathcal{O}} (\tilde{u}_x^\varepsilon)^3 \phi_x \,\mathrm{d}x\mathrm{d}s \\ &\quad - 3\int_0^t \int_{\mathcal{O}} \tilde{u}^\varepsilon (\tilde{u}_x^\varepsilon)^2 \phi_{xx} \,\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathcal{O}} (\tilde{u}^\varepsilon)^2 \tilde{u}_x^\varepsilon \phi_{xxx} \,\mathrm{d}x\mathrm{d}s \\ &\quad + \int_0^t \int_{\mathcal{O}} (\tilde{u}^\varepsilon)^2 \tilde{u}_x^\varepsilon \varepsilon F''(u^\varepsilon)\phi_x \,\mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathcal{O}} \frac{1}{2} \sum_{k\in\mathbb{Z}} \lambda_k^2 g_k (g_k \tilde{u}^\varepsilon)_x \phi_x \,\mathrm{d}x\mathrm{d}s\,.\end{aligned}$$

Lemma 6.13. *For $k \in \mathbb{N}$ and $\varepsilon \in (0,1)$*

$$\tilde{\mathcal{M}}_{\varepsilon,\phi} \tag{6.28}$$

$$\tilde{\mathcal{M}}_{\varepsilon,\phi}^2 - \int_0^\cdot \sum_{k\in\mathbb{Z}} \lambda_k^2 \left(\int_{\mathcal{O}} (\tilde{u}^\varepsilon g_k)_x \phi dx \right)^2 ds \tag{6.29}$$

$$\tilde{\mathcal{M}}_{\varepsilon,\phi} \tilde{\beta}_k^\varepsilon - \lambda_k \int_0^\cdot \int_{\mathcal{O}} (\tilde{u}^\varepsilon g_k)_x \phi dx ds \tag{6.30}$$

are continuous $\tilde{\mathcal{F}}_t^\varepsilon$-martingales. Moreover, on $[0,T]$ we have

$$\langle \tilde{\mathcal{M}}_{\varepsilon,\phi} \rangle_t = \int_0^t \sum_{k\in\mathbb{Z}} \lambda_k^2 \left(\int_{\mathcal{O}} (\tilde{u}^\varepsilon g_k)_x \phi dx \right)^2 ds \tag{6.31}$$

$$\langle \tilde{\mathcal{M}}_{\varepsilon,\phi}, \tilde{\beta}_k^\varepsilon \rangle_t = \lambda_k \int_0^t \int_{\mathcal{O}} (\tilde{u}^\varepsilon g_k)_x \phi \, dx ds. \tag{6.32}$$

The next step is to show that the martingale property is preserved in the limit. We show that for $\phi \in H^3_{\mathrm{per}}(\mathcal{O})$

$$\begin{aligned}\tilde{\mathcal{M}}_{0,\phi}(t) :=& \int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}_0)\phi \mathrm{d}x - \iint_{[\tilde{u}>0]} (\tilde{u}_x)^3 \phi_x \mathrm{d}x\mathrm{d}s - 3\int_0^t \int_{\mathcal{O}} \tilde{u}(\tilde{u}_x)^2 \phi_{xx} \,\mathrm{d}x\mathrm{d}s \\ & - \int_0^t \int_{\mathcal{O}} (\tilde{u})^2 \tilde{u}_x \phi_{xxx} \,\mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathcal{O}} \frac{1}{2} \sum_{k\in\mathbb{Z}} \lambda_k^2 g_k (g_k \tilde{u})_x \phi_x \,\mathrm{d}x\mathrm{d}s\end{aligned} \tag{6.33}$$

has the martingale property.

Lemma 6.14. *For $s, t \in [0,T]$ with $s \le t$ and for all continuous functions $\Psi \colon C^{\tilde{\gamma},\tilde{\sigma}}([0,s] \times \bar{\mathcal{O}}) \times C([0,s]; L^2(\mathcal{O})) \to [0,1]$, we have*

$$\mathbb{E}\left[\Psi(r_s \tilde{u}, r_s \tilde{W}) \left(\tilde{\mathcal{M}}_{0,\phi}(t) - \tilde{\mathcal{M}}_{0,\phi}(s) \right) \right] = 0. \tag{6.34}$$

Proof. We treat the terms inside the expectation in (6.34) one by one. The continuity of Ψ as well as the convergence of $\tilde{u}^\varepsilon$ to $\tilde{u}$ and of $\tilde{W}^\varepsilon$ to $\tilde{W}$ in $C(0,T; L^2(\mathcal{O}))$ show

$$\lim_{\varepsilon\to 0} \Psi(r_s \tilde{u}^\varepsilon, r_s \tilde{W}^\varepsilon) = \Psi(r_s \tilde{u}, r_s \tilde{W}) \tag{6.35}$$

$\tilde{\mathbb{P}}$-almost surely on [0, 1]. To see the convergence of the expected values, we aim to utilize Vitali's theorem; therefore, since Ψ is bounded, it suffices to show uniform boundedness of moments of the integral-terms in (6.34) and use the convergence results already established.

By the strong convergence of $\tilde{u}^\varepsilon$ in $C^{\tilde{\gamma},\tilde{\sigma}}(\bar{\mathcal{O}}_T)$, cf. Theorem 6.2, we have $\tilde{\mathbb{P}}$-almost surely

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}}(\tilde{u}^\varepsilon(t)-\tilde{u}^\varepsilon(s))\phi\mathrm{d}x=\int_{\mathcal{O}}(\tilde{u}(t)-\tilde{u}(s))\phi\mathrm{d}x\,. \tag{6.36}$$

The α-entropy-energy estimate (5.2) gives $\tilde{u}^\varepsilon\in L^{2q}(\tilde{\Omega};L^\infty(\mathcal{O}_T))$ for an arbitrary $q>1$, and thus, the uniform boundedness of a q-th absolute moment.
Weak convergence of $(\tilde{u}^\varepsilon_x)^3$ $\tilde{\mathbb{P}}$-almost surely has been established in Corollary 6.8. Using Hölder's inequality and (5.2), we have for $q>1$

$$\begin{aligned}&\mathbb{E}\left[\left|\int_s^t\int_{\mathcal{O}}(\tilde{u}^\varepsilon_x)^3\phi_x\,\mathrm{d}x\mathrm{d}s\right|^q\right]\\&\quad\le C\,\mathbb{E}\left[\left|\int_s^t\int_{\mathcal{O}}((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x^4\,\mathrm{d}x\mathrm{d}s\right|^q\right]^{\frac34}\mathbb{E}\left[\left|\int_s^t\int_{\mathcal{O}}(\tilde{u}^\varepsilon)^{-3(\alpha-1)}\phi_x^4\,\mathrm{d}x\mathrm{d}s\right|^q\right]^{\frac14}\le C.\end{aligned}$$

The identity

$$(\tilde{u}^\varepsilon_x)^2\tilde{u}^\varepsilon=\left(\frac{4}{\alpha+3}\right)^2((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x^2(\tilde{u}^\varepsilon)^{\frac{-\alpha+3}{2}}$$

yields

$$\begin{aligned}&\mathbb{E}\left[\left|\int_s^t\int_{\mathcal{O}}(\tilde{u}^\varepsilon_x)^2\tilde{u}^\varepsilon\phi_{xx}\,\mathrm{d}x\mathrm{d}s\right|^q\right]\\&\quad\le C\,\mathbb{E}\left[\left|\int_s^t\int_{\mathcal{O}}((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x^4\,\mathrm{d}x\mathrm{d}s\right|^q\right]^{\frac12}\mathbb{E}\left[\left|\int_s^t\int_{\mathcal{O}}(\tilde{u}^\varepsilon)^{-\alpha+3}\phi_{xx}^2\,\mathrm{d}x\mathrm{d}s\right|^q\right]^{\frac12}\le C\end{aligned}$$

which we combine with Lemma 6.9.
By means of Cauchy–Schwarz' inequality

$$\left|\int_s^t\int_{\mathcal{O}}(\tilde{u}^\varepsilon)^2\tilde{u}^\varepsilon_x\phi_{xxx}\,\mathrm{d}x\mathrm{d}s\right|\le C\left(\sup_{\bar{\mathcal{O}}_T}\tilde{u}^\varepsilon\right)^2\left(\sup_{t\in[0,T]}\int_{\mathcal{O}}\left|\tilde{u}^\varepsilon_x\right|^2\mathrm{d}x\right)^{\frac12}\left(\int_{\mathcal{O}}\phi_{xxx}^2\mathrm{d}x\right)^{\frac12}$$

holds, which implies the boundedness of higher moments. Lemma 6.10 states the needed convergence $\tilde{\mathbb{P}}$-almost surely.
In Lemma 6.11, we have seen

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\Psi(r_s\tilde{u}^\varepsilon,r_s\tilde{W}^\varepsilon)\left(\int_0^T\int_{\mathcal{O}}\varepsilon(\tilde{u}^\varepsilon)^2u^\varepsilon_xF''(\tilde{u}^\varepsilon)\phi_x\,\mathrm{d}x\mathrm{d}s\right)\right]=0\,. \tag{6.37}$$

For the Stratonovich correction term, we have due to the convergence of $\tilde{u}^\varepsilon$ $\tilde{\mathbb{P}}$-almost surely and the weak convergence of $\tilde{u}_x^\varepsilon$ on $L^2(0,T;L^2(\mathcal{O}))$, cf. Theorem 6.2,

$$\int_s^t \int_{\mathcal{O}} \frac{1}{2} \sum_{k\in\mathbb{Z}} \lambda_k^2 g_k (g_k \tilde{u}^\varepsilon)_x \phi_x \,\mathrm{d}x\mathrm{d}s \to \int_s^t \int_{\mathcal{O}} \frac{1}{2} \sum_{k\in\mathbb{Z}} \lambda_k^2 g_k (g_k \tilde{u})_x \phi_x \,\mathrm{d}x\mathrm{d}s$$

$\tilde{\mathbb{P}}$-almost surely. Furthermore, by boundedness of the g_k and (2.5),

$$\left| \int_s^t \int_{\mathcal{O}} \frac{1}{2} \sum_{k\in\mathbb{Z}} \lambda_k^2 g_k (g_k \tilde{u}^\varepsilon)_x \phi_x \,\mathrm{d}x\mathrm{d}s \right| \le C \left| \sup_{\bar{\mathcal{O}}_T} \tilde{u}^\varepsilon \right| .$$

From Lemma 6.13, we know that $(\tilde{\mathcal{M}}_{\varepsilon,\phi})_{\varepsilon\in(0,1)}$ are martingales. Thus, the convergence results above yield

$$\mathbb{E}\left[\Psi(r_s\tilde{u}, r_s\tilde{W}) \left(\tilde{\mathcal{M}}_{0,\phi}(t) - \tilde{\mathcal{M}}_{0,\phi}(s) \right) \right] = 0.$$

□

Dynkin's lemma in combination with Lemma 6.14 implies the martingale property, cf. for example [32].

Corollary 6.15. *$\tilde{\mathcal{M}}_{0,\phi}$ is a continuous $\tilde{\mathcal{F}}_t$-martingale.*

By similar arguments as before, cf. also [16] Lemmas 5.14 and 5.15, we can show that for $0 \le s \le t \le T$ and Ψ as in Lemma 6.14

$$\mathbb{E}\left[\Psi(r_s\tilde{u}, r_s\tilde{W}) \left(\tilde{\mathcal{M}}_{0,\phi}^2(t) - \tilde{\mathcal{M}}_{0,\phi}^2(s) - \int_s^t \sum_{k\in\mathbb{Z}} \lambda_k^2 \left(\int_{\mathcal{O}} (\tilde{u} g_k)_x \phi \mathrm{d}x \right)^2 \mathrm{d}s \right) \right] = 0 \tag{6.38}$$

and

$$\mathbb{E}\left[\Psi(r_s\tilde{u}, r_s\tilde{W}) \left((\tilde{\mathcal{M}}_{0,\phi}\tilde{\beta}_k)(t) - (\tilde{\mathcal{M}}_{0,\phi}\tilde{\beta}_k)(s) - \lambda_k \int_s^t \int_{\mathcal{O}} (\tilde{u} g_k)_x \phi \,\mathrm{d}x\mathrm{d}s \right) \right] = 0 \tag{6.39}$$

holds. Following the argumentation of Lemma 5.16 in [16], the identification of the stochastic term is achieved.

Lemma 6.16. *It holds $\tilde{\mathbb{P}}$-almost surely*

$$\tilde{\mathcal{M}}_{0,\phi} = \sum_{k\in\mathbb{Z}} \int_0^{\cdot} \int_{\mathcal{O}} \lambda_k (\tilde{u} g_k)_x \phi dx d\tilde{\beta}_k \,. \tag{6.40}$$

6.4. Proof of the main results

Finally, we provide the proofs for the existence of zero-contact angle martingale solutions.

Proof of Theorem 3.1. From Theorem 6.2, the existence of the stochastic basis, the Q-Wiener process $\tilde{W}$, as well as sequences $(\tilde{u}^\varepsilon)_{\varepsilon\in(0,1)}$, $(\tilde{u}_0^\varepsilon)_{\varepsilon\in(0,1)}$, and random variables $\tilde{u}$ and $\tilde{u}_0$ follows. Moreover, for every $\varepsilon \in (0,1)$ u^ε and $\tilde{u}^\varepsilon$ as well as u_0^ε and $\tilde{u}_0^\varepsilon$ have the same laws, respectively, and for $\varepsilon \to 0$, $\tilde{u}^\varepsilon \to \tilde{u}$ in $C^{\tilde{\gamma},\tilde{\sigma}}(\bar{\mathcal{O}}_T)$ and $\tilde{u}_0^\varepsilon \to \tilde{u}_0$ in $H^1_{\mathrm{per}}(\mathcal{O})$ holds $\tilde{\mathbb{P}}$-almost surely for a subsequence. Owing to the uniform convergence $\tilde{u}^\varepsilon \to \tilde{u}$, we see in particular that $\tilde{u}$ is nonnegative $\tilde{\mathbb{P}}$-almost surely. The weak formulation (3.1) is satisfied due to Lemma 6.16. Since $u_0 + \varepsilon^\theta \to u_0$ pointwise, by Corollary 13.19 in [36] we get

$$\mathbb{P}^\varepsilon \circ (u_0^\varepsilon)^{-1} = \mathbb{P} \circ (u_0 + \varepsilon^\theta)^{-1} \to \mathbb{P} \circ u_0^{-1} = \Lambda^0$$

weakly for $\varepsilon \to 0$. Likewise, from the pointwise convergence $\tilde{u}_0^\varepsilon \to \tilde{u}_0$ $\tilde{\mathbb{P}}$-almost surely, the weak convergence of the laws follows, i.e.,

$$\tilde{\mathbb{P}} \circ (\tilde{u}_0^\varepsilon)^{-1} \to \tilde{\mathbb{P}} \circ \tilde{u}_0^{-1}.$$

From Theorem 6.2, we know that

$$\mathbb{P}^\varepsilon \circ (u_0^\varepsilon)^{-1} = \tilde{\mathbb{P}} \circ (\tilde{u}_0^\varepsilon)^{-1}$$

for all $\varepsilon \in (0,1)$. Thus, by the uniqueness of limits w.r.t. weak convergence on Polish spaces, we get

$$\Lambda^0 = \tilde{\mathbb{P}} \circ \tilde{u}_0^{-1}.$$

Additionally, $\tilde{u}_0$ is $\tilde{\mathbb{P}}$-almost surely nonnegative.

At last, estimate (3.2) follows from Fatou's lemma and (5.2). We have

$$\begin{aligned}\mathbb{E}\left[\liminf_{\varepsilon\to 0} \sup_{t\in[0,T]} \left(\int_{\mathcal{O}} \frac{1}{2} \left|\tilde{u}_x^\varepsilon\right|^2 \mathrm{d}x\right)^q\right] &\le \liminf_{\varepsilon\to 0} \mathbb{E}\left[\sup_{t\in[0,T]} \left(\int_{\mathcal{O}} \frac{1}{2} \left|\tilde{u}_x^\varepsilon\right|^2 \mathrm{d}x\right)^q\right] \\ &\le C(T, q, \tilde{u}_0).\end{aligned}$$

Hence, by the lower semi-continuity of the $L^\infty(0,T;H^1(\mathcal{O}))$ norm w.r.t. the convergence in the distributional sense and Remark 8.3 from [1], we deduce

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left(\frac{1}{2} \int_{\mathcal{O}} |\tilde{u}_x|^2 \mathrm{d}x\right)^q\right] \le C(T, q, \tilde{u}_0).$$

Again, with Fatou's lemma and the lower semi-continuity of the $L^p(\mathcal{O}_T)$-norm w.r.t. to weak convergence of $((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{4}})_x$ and $((\tilde{u}^\varepsilon)^{\frac{\alpha+3}{2}})_{xx}$, we conclude

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathcal{O}} ((\tilde{u})^{\frac{\alpha+3}{4}})_x^4 \,\mathrm{d}x\mathrm{d}s\right)^q\right] \le C(T, q, \tilde{u}_0)$$

and

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathcal{O}} ((\tilde{u})^{\frac{\alpha+3}{2}})_{xx}^2 \,\mathrm{d}x\mathrm{d}s\right)^q\right] \le C(T, q, \tilde{u}_0),$$

where we used the α-entropy-energy estimate respectively. Thus, we have shown the estimate (3.2) and completed the proof of Theorem 3.1. □

Finally, we show the zero-contact-angle property at touch-down points of the solution.

Proof of Corollary 3.2. From the a priori estimate (3.2) combined with (H2), we infer that $\tilde{u}^{\frac{\alpha+3}{4}}(\cdot, \cdot, \omega)$ is element of $L^4((0, T); W^{1,4}(\mathcal{O}))$ $\tilde{\mathbb{P}}$-almost surely. Using the nonnegativity of $\tilde{u}$, the assumption on α, and the fact that $\tilde{u}^{\frac{\alpha+3}{4}}(\cdot, t, \omega) \in W^{1,4}(\mathcal{O})$ for almost all $t \in [0, T]$, the claim follows from the estimate

$$0 \le \tilde{u}(x, t_0, \omega) \le C(\omega, t_0)|x - x_0|^{\frac{3}{\alpha+3}} \qquad \text{for } x \in \mathcal{O}$$

which is a consequence of Sobolev's embedding theorem. □

Concluding remarks. In this paper which is partially based on the master thesis of the second author [35], we have presented a rather elementary proof for the existence of zero-contact angle solutions to the stochastic thin-film Eq. (1.1) for a quadratic mobility $m(\cdot)$ in one space dimension. The strategy has been to derive new regularity results first for approximate solutions which differ from (1.1) by a potential that enhances spreading and that this way entails strict positivity almost surely. We expect that this method can be slightly modified to establish corresponding results in the spatially two-dimensional case, too, this time starting from the existence result in [38]. Moreover, the new integral estimate (3.2) may serve as a starting point to establish results on the qualitative behavior of solutions—like finite speed of propagation or (non)-occurrence of waiting time phenomena. It remains, however, an open problem to which extent this approach may be applied to more general mobilities $m(\cdot)$.

Acknowledgements

L.K. has been supported by the Research-Training-Group 2339 "Interfaces, Complex Structures, and Singular Limits" of German Research Foundation (DFG), Project Number 321821685. G.G. gratefully acknowledges in addition the support of DFG through the project "Free boundary propagation and noise: analysis and numerics of stochastic degenerate parabolic equations," Project Number 397495103. Both authors acknowledge a most valuable comment by one of the anonymous referees.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data availability statement Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

A. Stratonovich correction

We will briefly discuss how to derive Eq. (2.6) from Eq. (2.1). We skip the index ε in this section. The Stratonovich correction term with respect to

$$\sum_{k\in\mathbb{Z}} (\lambda_k g_k u)_x \circ \mathrm{d}\beta_k(t) \tag{A.1}$$

reads

$$\mathcal{C}_S = \frac{1}{2} \sum_{k\in\mathbb{Z}} \lambda_k^2 (g_k (u g_k)_x)_x \,, \tag{A.2}$$

see, e.g., [18]. By the identities

$$g_k g_k' = \frac{2\pi k}{L} g_k g_{-k} \tag{A.3}$$

$$g_{-k} g_{-k}' = -\frac{2\pi k}{L} g_{-k} g_k \tag{A.4}$$

$$g_k^2 + g_{-k}^2 = \frac{2}{L}, \tag{A.5}$$

a straightforward computation shows

$$\mathcal{C}_S = \left(\frac{\lambda_0^2}{L} + \sum_{k=1}^{\infty} \frac{2\lambda_k^2}{L} \right) u_{xx} = C_{\mathrm{Strat}} u_{xx} \,. \tag{A.6}$$

The stochastic thin-film equation with Stratonovich noise can then be written as

$$\mathrm{d}u = (-(u^2 u_{xxx})_x + C_{Strat} u_{xx})\mathrm{d}t + (u\mathrm{d}W)_x \tag{A.7}$$

or equivalently as

$$\mathrm{d}u = -(u^2(u_{xx} - C_{\mathrm{Strat}}(1-u^{-1}))_x)_x \mathrm{d}t + (u\mathrm{d}W)_x \tag{A.8}$$

$$= -(u^2(u_{xx} - \mathcal{S}'(u))_x)_x \mathrm{d}t + (u\mathrm{d}W)_x \tag{A.9}$$

with $\mathcal{S}(u) = C_{\mathrm{Strat}}(u - \log u)$.

B. Itô's formula

In what follows, we show that all the assumptions of the Itô formula in Theorem 3.1 in [37] are satisfied in our setting. Let us first derive weak formulations as in (3.1) of [37] in the spaces $V_1 := H^2_{per}(\mathcal{O})$, $V_2 := H^1_{per}(\mathcal{O})$, and $H := L^2(\mathcal{O})$. By Lemma 4.5, we see that for $t \in [0, T_\sigma]$

$$u^\varepsilon(\cdot, t) \geq \bar{c}_\varepsilon \sigma^{\frac{2}{p-2}}, \tag{B.1}$$

where $\bar{c}_\varepsilon := \bar{C}_p \varepsilon^{\frac{1}{p-2}}$. For the functions u_σ as introduced in (5.5), we get for all $\phi \in H^1_{per}(\mathcal{O})$

$$\begin{aligned}\int_{\mathcal{O}} u_\sigma(t)\phi \mathrm{d}x = \int_{\mathcal{O}} u_0^\varepsilon \phi \mathrm{d}x + \int_0^{t\wedge T_\sigma} \int_{\mathcal{O}} (-(u_\sigma)^2 (p_\sigma)_x - C_{Strat}(u_\sigma)_x)\phi_x \mathrm{d}x\mathrm{d}s \\ - \sum_{k\in\mathbb{Z}} \int_0^{t\wedge T_\sigma} \int_{\mathcal{O}} u_\sigma g_k \phi_x \mathrm{d}x\mathrm{d}\beta_k\end{aligned} \tag{B.2}$$

on $[0, T]$.

By estimate (4.8), the function $-(u_\sigma)^2(p_\sigma)_x - C_{Strat}(u_\sigma)_x$ defines a mapping

$$\xi \in L^2(\Omega \times [0, T]; (H^2_{per}(\mathcal{O}))') \tag{B.3}$$

by

$$v \mapsto \mathbb{E}\left[\int_0^T \int_{\mathcal{O}} (-(u_\sigma)^2(p_\sigma)_x - C_{\mathrm{Strat}}(u_\sigma)_x)_{xx} v \mathrm{d}x\mathrm{d}s\right]. \tag{B.4}$$

Hence, by Riesz' representation theorem there is $\xi^* \in L^2(\Omega \times [0, T]; H^2_{per}(\mathcal{O}))$ such that

$$\int_\Omega \int_0^T {}_{(H^2_{per}(\mathcal{O}))'}\langle \xi, \phi\rangle_{H^2_{per}(\mathcal{O})} \mathrm{d}s\mathrm{d}\mathbb{P}^\varepsilon = \int_\Omega \int_0^T \left(\xi^*, \phi\right)_{H^2(\mathcal{O})} \mathrm{d}s\mathrm{d}\mathbb{P}^\varepsilon \tag{B.5}$$

holds for all $\phi \in L^2(\Omega \times [0, T]; H^2_{per}(\mathcal{O}))$. Similarly, we introduce f^* w.r.t. $H^1_{per}(\mathcal{O})$, i.e., f^* solves

$$\int_\Omega \int_0^T {}_{(H^1_{per}(\mathcal{O}))'}\langle f, \phi\rangle_{H^1_{per}(\mathcal{O})} \mathrm{d}s\mathrm{d}\mathbb{P}^\varepsilon = \int_\Omega \int_0^T \left(f^*, \phi\right)_{H^1(\mathcal{O})} \mathrm{d}s\mathrm{d}\mathbb{P}^\varepsilon \tag{B.6}$$

for every $\phi \in L^2(\Omega \times [0, T]; H^1_{per}(\mathcal{O}))$, where $f \in L^2(\Omega \times [0, T]; (H^1_{per}(\mathcal{O}))')$ is defined via

$$v \mapsto \mathbb{E}\left[\int_0^T \int_{\mathcal{O}} (-(u_\sigma)^2(p_\sigma)_x - C_{\mathrm{Strat}}(u_\sigma)_x) v_x \mathrm{d}x\mathrm{d}s\right]. \tag{B.7}$$

Riesz' representation theorem also shows

$$\mathbb{E}\left[\int_0^T \left(\| u_\sigma \|^2_{H^1(\mathcal{O})} + \| f^* \|^2_{H^1(\mathcal{O})}\right) \mathrm{d}s\right] \leq C(\varepsilon, T), \tag{B.8}$$

due to (4.8). Owing to the regularity of solutions, cf. Definition 4.2 iii),

$$\mathbb{E}\left[\int_0^T \left(\| (u_\sigma)_x \|^2_{H^2(\mathcal{O})} + \| \xi^* \|^2_{H^2(\mathcal{O})}\right) \mathrm{d}s\right] \le C(\varepsilon, T) \tag{B.9}$$

holds as well.

Let us set $\sigma_s^k := (\lambda_k u^\varepsilon g_k)_x$. By standard convolution arguments, we find the processes ξ^*, f^*, σ_s^k, and $(\sigma_s^k)_x$ to be predictable. Thus, we may rewrite (B.2) by means of (B.5) and obtain

$$\begin{aligned}((u_\sigma)_x(t), \phi)_{L^2(\mathcal{O})} &= \left((u_0^\varepsilon)_x, \phi\right)_{L^2(\mathcal{O})} + \int_0^{t\wedge T_\sigma} \left(-\xi^*, \phi\right)_{H^2(\mathcal{O})} \mathrm{d}s \\ &\quad + \sum_{k\in\mathbb{Z}} \int_0^{t\wedge T_\sigma} \left((\sigma_s^k)_x, \phi\right)_{L^2(\mathcal{O})} \mathrm{d}\beta_k\,, \end{aligned} \tag{B.10}$$

where we have multiplied with ϕ_x for $\phi \in H^2_{\text{per}}(\mathcal{O})$ and integrated by parts. On the other hand, with (B.6) we get

$$\begin{aligned}(u_\sigma(t), \phi)_{L^2(\mathcal{O})} &= \left(u_0^\varepsilon, \phi\right)_{L^2(\mathcal{O})} + \int_0^{t\wedge T_\sigma} \left(f^*, \phi\right)_{H^1(\mathcal{O})} \mathrm{d}s \\ &\quad + \sum_{k\in\mathbb{Z}} \int_0^{t\wedge T_\sigma} \left(\sigma_s^k, \phi\right)_{L^2(\mathcal{O})} \mathrm{d}\beta_k \end{aligned} \tag{B.11}$$

for all $\phi \in H^1_{\text{per}}(\mathcal{O})$. Both formulations (B.10) and (B.11) hold for all $t \in [0, T]$. Concerning the assumption

$$\sum_{k\in\mathbb{Z}} \mathbb{E}\left[\int_0^T \| \sigma_s^k \|^2_H \mathrm{d}t\right] < \infty, \tag{B.12}$$

we get with (A.3)–(A.5) and the assumptions on the data (2.5)

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \int_{\mathcal{O}} \lambda_k^2 \left|(u^\varepsilon g_k)_x\right|^2 \mathrm{d}x &= \frac{\lambda_0^2}{L} \int_{\mathcal{O}} (u_x^\varepsilon)^2 \mathrm{d}x + \sum_{k=1}^{\infty} \lambda_k^2 \frac{2}{L} \int_{\mathcal{O}} (u_x^\varepsilon)^2 \mathrm{d}x \\ &\quad + \sum_{k=1}^{\infty} \lambda_k^2 k^2 \frac{8\pi^2}{L^3} \int_{\mathcal{O}} (u^\varepsilon)^2 \mathrm{d}x, \end{aligned}$$

which implies (B.12) due to (4.4). The proof for $(\sigma_s^k)_x$ uses similar arguments and will be omitted.

For convenience, we will state the operators we work with once more. To guarantee well-posedness on H and continuity of their Fréchet derivatives on $H \times H$, we use a cut-off function $\eta \in C^\infty(\mathbb{R})$ such that for an appropriate $\delta > 0$

$$\eta(x) = \begin{cases} |x| & \text{for} \quad |x| \ge \bar{c}_\varepsilon \sigma^{\frac{2}{p-2}} \\ \in \mathbb{R}^+ & \text{for} \quad |x| \in (\bar{c}_\varepsilon \sigma^{\frac{2}{p-2}} - \delta, \bar{c}_\varepsilon \sigma^{\frac{2}{p-2}}) \\ \bar{c}_\varepsilon \sigma^{\frac{2}{p-2}} - \delta & \text{for} \quad |x| \le \bar{c}_\varepsilon \sigma^{\frac{2}{p-2}} - \delta \end{cases} \tag{B.13}$$

and

$$\left|\eta^{(s)}(x)\right| \leq C(s)\delta^{-s},\ s \in (1,2)\,. \tag{B.14}$$

Note that (B.1) implies $\eta(u^\varepsilon) = u^\varepsilon$, $\eta'(u^\varepsilon) = 1$, and $\eta''(u^\varepsilon) = 0$ on $[0, T_\sigma]$. We define

$$E_1 : u \mapsto \frac{1}{2}\int_{\mathcal{O}} u^2 \mathrm{d}x\,, \tag{B.15}$$

$$E_2 : u \mapsto \varepsilon \int_{\mathcal{O}} F(\eta(u))\mathrm{d}x\,, \tag{B.16}$$

$$\mathcal{G}_\alpha : u \mapsto \int_{\mathcal{O}} G_\alpha(\eta(u))\mathrm{d}x\,, \tag{B.17}$$

where G_α is defined by

$$G_\alpha(u) = \frac{1}{\alpha(\alpha+1)}u^{\alpha+1} - \frac{1}{\alpha}u + \frac{1}{\alpha+1} > 0\,. \tag{B.18}$$

For E_1, E_2, and $\mathcal{G}_\alpha$, we compute the Fréchet derivatives

$$DE_1(u)[v] = \int_{\mathcal{O}} uv\mathrm{d}x \qquad D^2E_1(u)[v,w] = \int_{\mathcal{O}} vw\mathrm{d}x \tag{B.19}$$

$$DE_2(u)[v] = \varepsilon\int_{\mathcal{O}} F'(\eta(u))\eta'(u)v\mathrm{d}x \tag{B.20}$$

$$D^2E_2(u)[v,w] = \varepsilon\int_{\mathcal{O}} F''(\eta(u))(\eta'(u))^2vw + F'(\eta(u))\eta''(u)vw\mathrm{d}x \tag{B.21}$$

$$D\mathcal{G}_\alpha(u)[v] = \int_{\mathcal{O}} G'_\alpha(\eta(u))\eta'(u)v\mathrm{d}x = \int_{\mathcal{O}} \left(\frac{1}{\alpha}\eta(u)^\alpha - \frac{1}{\alpha}\right)\eta'(u)v\mathrm{d}x \tag{B.22}$$

$$D^2\mathcal{G}_\alpha(u)[v,w] = \int_{\mathcal{O}} G''_\alpha(\eta(u))(\eta'(u))^2vw + G'_\alpha(\eta(u))\eta''(u)vw\mathrm{d}x\,. \tag{B.23}$$

Due to the cutoff η, the assumptions i) to iv) of Theorem 3.1 in [37] are readily checked for the space H and its dense subsets V_1 (in the case of E_1) and V_2 (in the case of E_2 and $\mathcal{G}_\alpha$, respectively). Hence, we may choose the operators E_1, E_2 and $\mathcal{G}_\alpha$ to use Itô's formula w.r.t. to the weak formulation (B.10) in the first case and w.r.t. (B.11) in the other two cases. We end up with the following equations which hold for $t \in [0, T]$.

$$\begin{aligned}
\frac{1}{2}\int_{\mathcal{O}} (u_\sigma(t))_x^2\mathrm{d}x &= \frac{1}{2}\int_{\mathcal{O}} (u_0^\varepsilon)_x^2\mathrm{d}x + \sum_{k\in\mathbb{Z}}\int_0^{t\wedge T_\sigma}\int_{\mathcal{O}} (u_\sigma)_x\lambda_k(u_\sigma g_k)_{xx}\mathrm{d}x\mathrm{d}\beta_k \\
&\quad + \int_0^{t\wedge T_\sigma}\int_{\mathcal{O}} -(-(u_\sigma)^2(p_\sigma)_x - C_{Strat}(u_\sigma)_x)(u_\sigma)_{xxx}\mathrm{d}x\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^{t\wedge T_\sigma}\sum_{k\in\mathbb{Z}}\lambda_k^2\int_{\mathcal{O}} (u_\sigma g_k)_{xx}^2\mathrm{d}x\mathrm{d}s
\end{aligned} \tag{B.24}$$

and

$$\varepsilon \int_{\mathcal{O}} F(u_\sigma(t))\mathrm{d}x = \varepsilon \int_{\mathcal{O}} F(\eta(u_0^\varepsilon)))\mathrm{d}x + \sum_{k\in\mathbb{Z}} \int_0^{t\wedge T_\sigma} \int_{\mathcal{O}} \varepsilon F'(u_\sigma)\lambda_k(u_\sigma g_k)_x \mathrm{d}x\mathrm{d}\beta_k$$
$$+ \int_0^{t\wedge T_\sigma} \int_{\mathcal{O}} (-(u_\sigma)^2(p_\sigma)_x - C_{Strat}(u_\sigma)_x)(\varepsilon F'(u_\sigma))_x \mathrm{d}x\mathrm{d}s$$
$$+ \frac{1}{2}\int_0^{t\wedge T_\sigma} \sum_{k\in\mathbb{Z}} \lambda_k^2 \int_{\mathcal{O}} \varepsilon F''(u_\sigma)(u_\sigma g_k)_x^2 \mathrm{d}x\mathrm{d}s\,. \tag{B.25}$$

The entropy (B.17) applied to (B.11) then gives

$$\mathcal{G}_\alpha(u_\sigma(t)) = \mathcal{G}_\alpha(u_0^\varepsilon) + \sum_{k\in\mathbb{Z}} \int_0^{t\wedge T_\sigma} \int_{\mathcal{O}} \left(\frac{1}{\alpha}(u_\sigma)^\alpha - \frac{1}{\alpha}\right) \lambda_k(u_\sigma g_k)_x \mathrm{d}x\mathrm{d}\beta_k$$
$$+ \int_0^{t\wedge T_\sigma} \int_{\mathcal{O}} (-(u_\sigma)^2(p_\sigma)_x - C_{Strat}(u_\sigma)_x) \left(\frac{1}{\alpha}u_\sigma^\alpha - \frac{1}{\alpha}\right)_x \mathrm{d}x\mathrm{d}s$$
$$+ \frac{1}{2}\int_0^{t\wedge T_\sigma} \sum_{k\in\mathbb{Z}} \lambda_k^2 \int_{\mathcal{O}} u_\sigma^{\alpha-1}(u_\sigma g_k)_x^2 \mathrm{d}x\mathrm{d}s\,. \tag{B.26}$$

Thus, combining (B.24), (B.25), and (B.26) and adding all terms with a good sign to the left-hand side, we obtain Eq. (5.11).

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Accepted: 5 June 2022