Correction

Correction to: Weighted Energy-Dissipation approach to doubly nonlinear problems on the half line

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In our paper [2], strong solutions of the Cauchy problem for the doubly nonlinear gradient flow posed on the dual space V^* of a uniformly convex Banach space V ,

$$\xi(t) + \eta(t) = 0, \quad \xi(t) = d_V \psi(u'(t)), \quad \eta(t) \in \partial \phi(u(t)) \quad \text{in } V^* \quad \text{for a.e. } t > 0, \quad (1)$$

$$u(0) = u_0, \quad (2)$$

where $d_V \psi : V \rightarrow V^*$ (resp. $\partial \phi : V \rightarrow 2^{V^*}$) denotes the Gâteaux derivative (resp. subdifferential) of a smooth (resp. lower semicontinuous) convex functional defined on V , are recovered as a limit as $\varepsilon \rightarrow 0+$ of minimizers

$$u_\varepsilon \in K(u_0) \\ := \{u \in W^{1,p}(\mathbb{R}_+, e^{-t/\varepsilon} dt; V) : u(0) = u_0 \text{ and } \phi(u) \in L^1(\mathbb{R}_+, e^{-t/\varepsilon} dt)\} \quad (3)$$

of the WED functional

$$I_\varepsilon(u) = \begin{cases} \int_0^\infty e^{-t/\varepsilon} (\varepsilon \psi(u'(t)) + \phi(u(t))) dt & \text{if } u \in K(u_0), \\ +\infty & \text{otherwise.} \end{cases} \quad (4)$$

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In [2], we make the following assumptions with $1 < p < +\infty$:

$$|u|_V^p \leq C(\psi(u) + 1) \quad \forall u \in V, \psi(0) = 0; \quad (5)$$

$$|d_V \psi(u)|_{V^*}^{p'} \leq C(|u|_V^p + 1) \quad \forall u \in V, p' = p/(p-1); \quad (6)$$

$$D(\phi) \subset X; \text{ for each } c \text{ the set } \{u \in X : \phi(u) \leq c\} \text{ is bounded in } X; \quad (7)$$

$$|\eta|_{X^*} \leq \ell(|u|_V + \phi(u)) \quad \forall \eta \in \partial_X \phi_X(u), \quad (8)$$

where X is a reflexive Banach space densely and compactly embedded into V , $\partial_X \phi_X : X \rightarrow 2^{X^*}$ denotes the subdifferential operator of the restriction ϕ_X of ϕ onto X and $\ell(\cdot)$ is a non-decreasing function. The proof in [2] relies on a regularization of I_ε and hinges on the following uniform estimate:

$$\sup_{t \in [0, \infty)} \phi_\lambda(u_{\varepsilon\lambda}(t)) \leq C \quad (9)$$

for minimizers $u_{\varepsilon\lambda}$ of regularized WED functionals $I_{\varepsilon\lambda}$, which is given by (3) and (4) with ϕ replaced by the Moreau-Yosida regularization $\phi_\lambda : V \rightarrow \mathbb{R}$ of ϕ . The proof of this bound in [2] is however flawed, as pointed out by Mr. Thomas Ruf, whose feedback we gratefully acknowledge. On the other hand, for any $T > 0$, one can derive a uniform estimate

$$\int_0^T \phi_\lambda(u_{\varepsilon\lambda}(t)) \, dt \leq C_T \quad (10)$$

for some constant $C_T > 0$ independent of $\varepsilon, \lambda \in (0, 1)$. Indeed, for any $n \in \mathbb{N}$, it holds that

$$\int_{\varepsilon n}^{\varepsilon(n+1)} \phi_\lambda(u_{\varepsilon\lambda}(t)) \, dt \leq \varepsilon e\phi(u_0)$$

(see p. 60 of [2] with the choice $\tau = n$). Let $N_\varepsilon \in \mathbb{N}$ be such that $\varepsilon N_\varepsilon \leq T < \varepsilon(N_\varepsilon + 1)$. It then follows that

$$\int_0^T \phi_\lambda(u_{\varepsilon\lambda}(t)) \, dt \leq \sum_{n=0}^{N_\varepsilon} \int_{\varepsilon n}^{\varepsilon(n+1)} \phi_\lambda(u_{\varepsilon\lambda}(t)) \, dt \leq (N_\varepsilon + 1)\varepsilon e\phi(u_0) \leq (T + 1)e\phi(u_0)$$

for $\varepsilon \in (0, 1)$. Hence, by replacing (7) and (8) by

$$D(\phi) \subset X; |u|_X^m \leq C(\phi(u) + 1) \quad \forall u \in D(\phi); \quad (11)$$

$$|\eta|_{X^*}^{m'} \leq \ell(|u|_V)(\phi(u) + 1) \quad \forall \eta \in \partial_X \phi_X(u) \quad (12)$$

for some $1 < m < +\infty$ and a non-decreasing function ℓ in $\mathbb{R}$, the argument in [2] entails the validity of the following.

Theorem 1. (WED variational approach). *Assume (5), (6), (11), and (12). Then:*

- (i) *The WED functional I_ε admits global minimizers u_ε in $K(u_0)$. Additionally, if either ϕ or ψ is strictly convex, the minimizer is unique.*

(ii) For every minimizer u_ε of I_ε on $K(u_0)$ there exists $\eta_\varepsilon \in \partial_X \phi_X(u_\varepsilon)$ such that, by letting $\xi_\varepsilon = d_V \psi(u'_\varepsilon)$, the triple $(u_\varepsilon, \eta_\varepsilon, \xi_\varepsilon)$ belonging to

$$[W^{1,p}(\mathbb{R}_+, e^{-t/\varepsilon} dt; V) \cap L^m_{\text{loc}}(\mathbb{R}_+; X)] \times L^{m'}_{\text{loc}}(\mathbb{R}_+; X^*) \\ \times L^{p'}(\mathbb{R}_+, e^{-t/\varepsilon} dt; V^*)$$

is a strong solution of the Euler-Lagrange problem

$$-\varepsilon \xi'_\varepsilon(t) + \xi_\varepsilon(t) + \eta_\varepsilon(t) = 0, \quad \eta_\varepsilon(t) \in \partial_X \phi_X(u_\varepsilon(t)) \quad \text{in } X^*, \quad \text{for a.e. } t > 0, \quad (13)$$

$$\xi_\varepsilon(t) = d_V \psi(u'_\varepsilon(t)) \quad \text{in } V^*, \quad \text{for a.e. } t > 0, \quad (14)$$

$$u_\varepsilon(0) = u_0. \quad (15)$$

Moreover, minimizers of I_ε in $K(u_0)$ and strong solutions of the Euler-Lagrange system coincide.

(iii) For any sequence $\varepsilon_k \rightarrow 0$ there exists a subsequence (denoted by the same letter) such that $(u_{\varepsilon_k}, \eta_{\varepsilon_k}, \xi_{\varepsilon_k}) \rightarrow (u, \eta, \xi)$ weakly in

$$[W^{1,p}(0, T; V) \cap L^m(0, T; X)] \times L^{m'}(0, T; X^*) \times L^{p'}(0, T; V^*)$$

for all $T > 0$, where (u, η, ξ) is a strong solution to the doubly nonlinear problem,

$$\xi(t) + \eta(t) = 0, \quad \xi(t) = d_V \psi(u'(t)), \quad \eta(t) \in \partial \phi(u(t)) \quad \text{in } V^* \quad \text{for a.e. } t > 0, \\ u(0) = u_0.$$

Indeed, combining (10) with (11) and (12), we infer that

$$\int_0^T |J_\lambda u_{\varepsilon\lambda}(t)|_X^m dt + \int_0^T |\eta_{\varepsilon\lambda}(t)|_{X^*}^{m'} dt \leq C_T,$$

where $J_\lambda : V \rightarrow D(\partial \phi)$ is the resolvent of $\partial \phi$, uniformly for $\varepsilon, \lambda \in (0, 1)$. Hence, instead of the corresponding weak convergences in $L^\rho(0, T; X)$ and $L^\rho(0, T; X^*)$ for any $\rho \in (1, +\infty)$ in [2], we use only

$$J_\lambda u_{\varepsilon\lambda} \rightarrow u_\varepsilon \quad \text{weakly in } L^m(0, T; X), \quad (16)$$

$$\eta_{\varepsilon\lambda} \rightarrow \eta_\varepsilon \quad \text{weakly in } L^{m'}(0, T; X^*) \quad (17)$$

as $\lambda \rightarrow 0+$ for any $T > 0$. The weak convergences (16)–(17) are sufficient to identify the limits, e.g., of nonlinear terms (see [1], where the bounded-interval case is treated). The causal limit of u_ε as $\varepsilon \rightarrow 0+$ can be also discussed with the aid of (16) and (17).

By replacing (7)–(8) by (11)–(12), the WED variational approach is no longer applicable to discuss the Kirchhoff-type integrodifferential equation (5.5)–(5.7) of [2] in full generality. However, one can treat the case of a homogeneous α , say $\alpha(s) =$

$|s|^{p-2}s$ for $1 < p < +\infty$. Indeed, one can incorporate the nonlocality $\|\nabla u(t)\|_2$ into $\alpha(u_t)$ by making use of the homogeneity of α and a change of the time-variable. To this end, set

$$u(x, t) = v(x, s), \quad s = \int_0^t \|\nabla u(\tau)\|_2^{-1/(p-1)} d\tau.$$

Here, one has to avoid the case $\|\nabla u_0\|_2 = 0$, for which only the trivial solution exists. Then $u_t(x, t) = \|\nabla u(t)\|_2^{-1/(p-1)} v_s(x, s)$ and $\Delta u(x, t) = \Delta v(x, s)$, and hence, the equation is reduced to

$$\alpha(v_s) = \Delta v \quad \text{in } \Omega \times \mathbb{R}_+,$$

which falls within the scope of Theorem 1.

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