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On functions of bounded variation on convex domains in Hilbert spaces

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Dedicated to Matthias Hieber on the occasion of his 60th-Birthday

Abstract. We study functions of bounded variation (and sets of finite perimeter) on a convex open set $\Omega \subseteq X$, X being an infinite-dimensional separable real Hilbert space. We relate the total variation of such functions, defined through an integration by parts formula, to the short-time behaviour of the semigroup associated with a perturbation of the Ornstein–Uhlenbeck operator.

1. Introduction

In this paper we study some properties of functions of bounded variation (BV functions, for short) defined on an open convex subset of a real separable Hilbert space, endowed with a weighted Gaussian measure.

In finite dimension the theory of BV functions is widely developed (see e.g. [3] and the references therein), whereas in the infinite-dimensional setting the analysis is still at the initial stage and many basic properties are unexplored. Besides the interest on its own, the study of BV functions in infinite-dimensional spaces is motivated by problems arising in calculus of variations, stochastic analysis and it is connected with the applications in information technology (see, for example, [19,22–24,26]).

BV functions for Gaussian measures in separable Banach spaces were introduced in [17] using Dirichlet forms. Inspired by the results in finite dimension, which connect the theory of functions of bounded variation to that of semigroups of bounded operators, the authors of [18] have proved an elegant characterisation of BV functions in terms of the short-time behaviour of the Ornstein–Uhlenbeck semigroup. More precisely, in a separable Banach space X, if γ is a centred and nondegenerate

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Gaussian measure on X and u belongs to the Orlicz space $L(\log L)^{1/2}(X,\gamma)$, then $u \in BV(X,\gamma)$ if, and only if,

$$\liminf_{t\to 0^+} \int_X |D_H S(t)u|_H \mathrm{d}\gamma < +\infty,$$

where D_H is the gradient operator along the Cameron–Martin space H (see Sect. 2) and $S(t)$ is the classical Ornstein–Uhlenbeck semigroup defined via the Mehler formula (see (1)). The latter is the analogous, in the Gaussian setting, of the heat semigroup used by De Giorgi in [11] to provide the original definition of BV functions in the Euclidean case. An analytic approach based on geometric measure theory is proposed in [4] to prove, as in the finite-dimensional case, the equivalence of different definitions of $BV(X,\gamma)$ functions also, as in [18], in terms of the Ornstein–Uhlenbeck semigroup $S(t)$ near $t = 0$. Similar De Giorgi-type characterisations of BV functions have been obtained for weighted Gaussian measures and more recently for general Fomin differentiable measures in Hilbert spaces, see [12] and the reference therein.

Beside the difficulty of considering general measures, another difficulty of different nature comes from the consideration of functions defined in domains rather than in the whole space. These difficulties come from the lack of factorisation of the underlying measure (that is lost even for Gaussian measures in domains) and the unavailability of decomposition of the domain through the classical method of local charts. Therefore, the easiest interesting case seems to be that of *convex* domains that are possible to deal with through *global* penalisation techniques. This is the approach we followed in [6] (see also [21]), and in this paper we take advantage of the results proved there. We start from a weighted Gaussian measure $\nu := e^{-U}\gamma$ in a Hilbert space X, where $U : X \to \mathbb{R}$ is convex and sufficiently regular, and consider an open convex domain $\Omega \subseteq X$. After introducing the Cameron–Martin space H and the Malliavin gradient D_H along it, we define the form $(u,v) \mapsto \int_\Omega \langle D_H u, D_H v\rangle_H \mathrm{d}\nu$ on the appropriate Sobolev spaces. The perturbed Ornstein–Uhlenbeck operator L_Ω is then defined in the usual variational way, and it is the generator of an analytic, strongly continuous and contraction semigroup $T_\Omega(t)$ in $L^p(\Omega,\nu)$, for $1 < p < \infty$.

For the latter, differently from the Ornstein–Uhlenbeck semigroup in the whole space, no explicit integral representation which allows for direct computations is known. In this direction, in [21] the authors consider the restrictions to an open convex set $\Omega \subseteq X$ of $BV(X,\gamma)$ functions and they characterise the finiteness of their total variation in Ω in terms of the Neumann Ornstein–Uhlenbeck semigroup defined in Ω.

Following the ideas in [2], we define the $BV(\Omega,\nu)$ space through an integration by parts formula against suitable Lipschitz functions. Then, we show that the functions u of bounded variation in Ω with respect to ν can be characterised by the finiteness of the limit of $\|D_H T_\Omega(t)u\|_{L^1(\Omega,\nu;H)}$ as $t \to 0^+$. The proof of this result relies on a commutation formula between the semigroup $T_\Omega(t)$ and the gradient operator along H (see Proposition 5). This result was already known in the case of the whole space (see [12]). Here, by means of the crucial pointwise gradient estimate (7) and suitable

penalisations Φ_ε of U outside Ω based on the *distance function from Ω along H* (here is a first point where the convexity of Ω comes into the play) and the penalisation $\nu_\varepsilon := e^{-\Phi_\varepsilon}\gamma$ of the measure ν, see Sect. 2.1, we are able to let ε to 0^+ and to come back to Ω.

Finally, we provide a necessary condition in order that a set E is of finite perimeter in Ω with respect to ν (i.e. $\chi_E \in BV(\Omega, \nu)$). This condition is given in terms of the short-time behaviour of the *Ornstein–Uhlenbeck content* $\|T_\Omega(t)\chi_E - \chi_E\|_{L^1(\Omega,\nu)}$ as $t \to 0^+$. Further, a sufficient condition in terms of a related quantity is also shown. This circle of ideas goes back to [20], which originated several researches. Among these, the only infinite-dimensional result, proved for BV functions in space endowed with a Gaussian measure, is in [5].

2. Hypotheses and preliminaries

Let H_1 and H_2 be two real Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_{H_1}$ and $\langle\cdot,\cdot\rangle_{H_2}$, respectively. We denote by $\mathcal{B}(H_1)$ the σ-algebra of Borel subsets of H_1 and by $C_b^k(H_1; H_2), k \in \mathbb{N}\cup\{\infty\}$ the set of k-times Fréchet differentiable functions from H_1 to H_2 with bounded derivatives up to order k ($C_b^k(H_1)$ if $H_2 = \mathbb{R}$). For $\Phi \in C_b^1(H_1; H_2)$ we denote by $D\Phi(x)$ the derivative of Φ at $x \in H_1$: if $f \in C_b^1(H_1)$, for every $x \in H_1$ there exists a unique $k \in H_1$ such that $Df(x)(h) = \langle h, k\rangle_{H_1}$, $h \in H_1$ and we set $Df(x) := k$. Let X be a separable Hilbert space, with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. Let $B \in \mathcal{L}(X)$ (the set of bounded linear operators from X to itself). We say that B is *non-negative* if $\langle Bx, x\rangle \geq 0$ for every $x \in X$ and *positive* if $\langle Bx, x\rangle > 0$ for every $x \in X \setminus \{0\}$. We recall that a non-negative and self-adjoint operator $B \in \mathcal{L}(X)$ is a trace class operator whenever $\mathrm{Tr}(B) := \sum_{n=1}^\infty \langle Be_n, e_n\rangle < \infty$ for some (and hence, every) orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of X.

Let γ be a nondegenerate Gaussian measure on X with mean zero and covariance operator $Q_\infty := -QA^{-1}$, where the operators Q and A satisfy the following assumptions.

Hypotheses 1. (i) *$Q \in \mathcal{L}(X)$ is a self-adjoint and non-negative operator with* $\mathrm{Ker}\, Q = \{0\}$*;*

(ii) *$A : D(A) \subseteq X \to X$ is a self-adjoint operator satisfying $\langle Ax, x\rangle \leq -\omega|x|^2$ for every $x \in D(A)$ and some positive ω;*

(iii) *$Qe^{tA} = e^{tA}Q$ for any $t \geq 0$;*

(iv) $\mathrm{Tr}(-QA^{-1}) < \infty$.

Under Hypotheses 1(i)–(iii), the measure γ is well defined and the Ornstein–Uhlenbeck semigroup defined via the Mehler formula

$$(S(t)f)(x) := \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\mathrm{d}\gamma(y), \qquad x \in X,\ f \in L^1(X,\gamma) \tag{1}$$

is symmetric in $L^2(X,\gamma)$. We fix an orthonormal basis $(v_k)_{k\in\mathbb{N}}$ of X such that

$$Q_\infty v_k = \lambda_k v_k, \qquad k \in \mathbb{N}, \tag{2}$$

where $(\lambda_k)_{k\in\mathbb{N}}$ is the decreasing sequence of eigenvalues of Q_∞. Under Hypothesis 1(iv), the Cameron–Martin space $(H, |\cdot|_H)$

$$H := Q_\infty^{1/2}(X) = \Big\{x \in X \Big| \sum_{k=1}^{\infty} \lambda_k^{-1} \langle x, v_k\rangle^2 < \infty\Big\},$$

where $|\cdot|_H$ is induced by the inner product $\langle h, k\rangle_H := \langle Q_\infty^{-1/2}h, Q_\infty^{-1/2}k\rangle$, is a Hilbert space compactly and densely embedded in X (see [8] and [13] for further details). The sequence $(e_k)_{k\in\mathbb{N}}$, where $e_k = \sqrt{\lambda_k}v_k$ for any $k \in \mathbb{N}$, is an orthonormal basis of H. By Hypotheses 1, the operator $-Q_\infty^{-1} : D(Q_\infty^{-1}) \subseteq X \to X$ ($-Q_\infty^{-1} : D(Q_\infty^{-1}) \subseteq H \to H$, respectively) is the generator of a contractive and strongly continuous semigroup $e^{-tQ_\infty^{-1}}$ on X (on H, respectively), see [14, Proposition p. 84]). If Y is a Banach space with norm $\|\cdot\|_Y$, a function $F : X \to Y$ is said to be H-Lipschitz continuous if there exists a positive constant C such that

$$\|F(x + h) - F(x)\|_Y \le C|h|_H, \tag{3}$$

for every $h \in H$ and γ-a.e. $x \in X$. We denote by $[F]_{H\text{-Lip}}$ the best constant C in (3). For more information see [8, Sections 4.5 and 5.11]. We denote by $\mathcal{H}_2$ the space of the Hilbert–Schmidt operators in H that is the space of the bounded linear operators $B : H \to H$ such that $\|B\|_{\mathcal{H}_2}^2 := \sum_{i=1}^{\infty} |Bg_i|_H^2$ is finite, where $\{g_n \,|\, n \in \mathbb{N}\}$ is any orthonormal basis of H. We say that $f : X \to \mathbb{R}$ is H-differentiable at $x_0 \in X$ if there exists $\ell \in H$ such that

$$f(x_0 + h) = f(x_0) + \langle \ell, h\rangle_H + o(|h|_H), \qquad \text{as } |h|_H \to 0.$$

In such a case we set $D_H f(x_0) := \ell$ and $D_i f(x_0) := \langle D_H f(x_0), e_i\rangle_H$ for any $i \in \mathbb{N}$. The derivative $D_H f(x_0)$ is called the *Malliavin derivative* of f at x_0. In a similar way we say that f is twice H-differentiable at x_0 if f is H-differentiable near x_0 and there exists $\mathcal{B} \in \mathcal{H}_2$ such that

$$f(x_0 + h) = f(x_0) + \langle D_H f(x_0), h\rangle_H + \frac{1}{2}\langle \mathcal{B}h, h\rangle_H + o(|h|_H^2), \qquad \text{as } |h|_H \to 0.$$

In such a case we set $D_H^2 f(x_0) := \mathcal{B}$ and $D_{ij} f(x_0) := \langle D_H^2 f(x_0)e_j, e_i\rangle_H$ for any $i, j \in \mathbb{N}$. If f is twice H-differentiable at x_0, then $D_{ij} f(x_0) = D_{ji} f(x_0)$ for every $i, j \in \mathbb{N}$. Notice that if $f : X \to \mathbb{R}$ is once or twice Fréchet differentiable at x_0, then it is once or twice H-differentiable at x_0 and it holds $D_H f(x_0) = Q_\infty Df(x_0)$, and $D_H^2 f(x_0) = Q_\infty D^2 f(x_0) Q_\infty$, where the equality must be understood as holding in H. For any $k \in \mathbb{N}\cup\{\infty\}$, we denote by $\mathcal{F}C_b^k(X)$, the space of cylindrical C_b^k functions, i.e. the set of functions $f : X \to \mathbb{R}$ such that $f(x) = \varphi(\langle x, h_1\rangle, \dots, \langle x, h_N\rangle)$ for some $\varphi \in C_b^k(\mathbb{R}^N), h_1, \dots, h_N \in H$ and $N \in \mathbb{N}$. By $\mathcal{F}C_b^k(X, H)$ we denote H-valued cylindrical C_b^k functions with finite rank. The Sobolev spaces in the sense of Malliavin $D^{1,p}(X, \gamma)$ and $D^{2,p}(X, \gamma)$ with $p \in [1, \infty)$ are defined as the completions of the *smooth cylindrical functions* $\mathcal{F}C_b^\infty(X)$ in the norms

$$\|f\|_{D^{1,p}(X,\gamma)} := \Big(\|f\|_{L^p(X,\gamma)}^p + \int_X |D_H f|_H^p \mathrm{d}\gamma\Big)^{\frac{1}{p}};$$

$$\|f\|_{D^{2,p}(X,\gamma)} := \left(\|f\|^p_{D^{1,p}(X,\gamma)} + \int_X \|D^2_H f\|^p_{\mathcal{H}_2} \mathrm{d}\gamma\right)^{\frac{1}{p}}.$$

This is equivalent to considering the domain of the closure of the gradient operator, defined on smooth cylindrical functions, in $L^p(X,\gamma)$ (see [8, Section 5.2]). Let $U : X \to \mathbb{R}$ satisfy the following assumptions.

Hypotheses 2. *U is a convex function which belongs to $C^2(X) \cap D^{1,q}(X,\gamma)$ for all $q \in [1,\infty)$ with H-Lipschitz gradient.*

The convexity of the function U guarantees that U is bounded from below by a linear function, and therefore, it decreases at most linearly and by Fernique theorem (see [8, Theorem 2.8.5]) e^{-U} belongs to $L^1(X,\gamma)$. Then, we can consider the finite log-concave measure

$$\nu := e^{-U}\gamma.$$

It is obvious that γ and ν are equivalent measures, hence saying that a statement holds γ-a.e. is the same as saying that it holds ν-a.e. Moreover as $U \in \cap_{q\geq 1} D^{1,q}(X,\gamma)$, the operator $D_H : \mathcal{FC}^1_b(X) \to L^p(X,\nu;H)$ is closable in $L^p(X,\nu)$, $p \in (1,\infty)$ and the space $D^{1,p}(X,\nu)$, $p > 1$ can be defined as the domain of its closure (still denoted by D_H). In a similar way we may define $D^{2,p}(X,\nu)$, $p \in (1,\infty)$ (for more details see [1,9,16]). The Gaussian integration by parts formula $\int_X D_i f \mathrm{d}\gamma = \frac{1}{\sqrt{\lambda_i}} \int_X \langle x, v_i\rangle f \mathrm{d}\gamma$, which holds true for any $f \in \mathcal{FC}^1_b(X)$ and $i \in \mathbb{N}$, yields

$$\int_X \psi D_i\varphi \mathrm{d}\nu + \int_X \varphi D_i\psi \mathrm{d}\nu = \int_X \varphi\psi D_i U \mathrm{d}\nu + \frac{1}{\sqrt{\lambda_i}} \int_X \langle x, v_i\rangle \varphi\psi \mathrm{d}\nu, \qquad i \in \mathbb{N},$$

for any $\varphi \in D^{1,p}(X,\nu)$ $(p > 1)$ and $\psi \in \mathcal{FC}^1_b(X)$.

In what follows Ω denotes an open subset of X. In this case, the spaces $D^{1,p}(\Omega,\nu)$ and $D^{2,p}(\Omega,\nu)$, $p \in (1,\infty)$, can be defined in a similar way as in the whole space, thanks to the following result (see [6, Proposition 1.4]).

Proposition 1. *Assume that Hypotheses 1 and 2 are satisfied. Let $p \in (1,\infty)$ and let Ω be an open subset of X. The operators $D_H : \mathcal{FC}^\infty_b(\Omega) \to L^p(\Omega,\nu;H)$ and*

$$(D_H, D^2_H) : \mathcal{FC}^\infty_b(\Omega) \times \mathcal{FC}^\infty_b(\Omega) \to L^p(\Omega,\nu;H) \times L^p(\Omega,\nu;\mathcal{H}_2)$$

are closable in $L^p(\Omega,\nu)$ and $L^p(\Omega,\nu) \times L^p(\Omega,\nu)$, respectively. Here $\mathcal{FC}^\infty_b(\Omega)$ is the space of the restrictions to Ω of functions in $\mathcal{FC}^\infty_b(X)$.

The spaces $D^{1,p}(\Omega,\nu;H)$, $p \in (1,\infty)$, are defined in a similar way, replacing smooth cylindrical functions with H-valued smooth cylindrical functions with finite rank. We recall that if $F \in D^{1,p}(\Omega,\nu;H)$, then $D_H F(x)$ belongs to $\mathcal{H}_2$ for a.e. $x \in \Omega$. We denote by p' the conjugate exponent to $p \in (1,\infty)$.

2.1. Perturbed Ornstein–Uhlenbeck semigroup on convex domains

In order to consider the initial boundary value problems defined in Ω we define the *distance function along* H

$$d_\Omega(x) := \begin{cases} \inf\{|h|_H \mid h \in H \cap (\Omega - x)\}, & H \cap (\Omega - x) \neq \emptyset; \\ \infty, & H \cap (\Omega - x) = \emptyset, \end{cases}$$

for $x \in X$, and we recall some useful regularity results, (see, for instance, [8, Theorems 2.8.5 and 5.11.2] and [10, Section 3]).

Proposition 2. *If $\Omega \subseteq X$ is an open convex set, then d_Ω^2 is H-differentiable and its Malliavin derivative is H-Lipschitz with H-Lipschitz constant less than or equal to 2, i.e.*

$$|D_H d_\Omega^2(x+h) - D_H d_\Omega^2(x)|_H \leq 2|h|_H,$$

for any $h \in H$ and for ν-a.e $x \in X$. Moreover, $D_H^2 d_\Omega^2$ exists ν-a.e. in X and d_Ω^2 belongs to $D^{2,p}(X,\nu)$ for every $p \in [1,\infty)$.

We require some further regularity on d_Ω^2.

Hypotheses 3. *Let Ω be an open convex subset of X such that $\nu(\partial\Omega) = 0$ and $D_H^2 d_\Omega^2$ is H-continuous γ-a.e. in X, i.e. for γ-a.e. $x \in X$ we have*

$$\lim_{|h|_H \to 0} D_H^2 d_\Omega^2(x+h) = D_H^2 d_\Omega^2(x).$$

Remark 1. As stated in [6, Remark 1.7] there is a rather large class of subsets of X satisfying Hypothesis 3. For instance, if $\partial\Omega$ is (locally) a C^2-embedding in X of an open subset of a hyperplane in X and $\nu(\partial\Omega) = 0$, then Hypothesis 3 is satisfied. Easy examples are open balls and open ellipsoids of X, open hyperplanes of X and every set of the form $\Omega = \{x \in X \mid G(x) < 0\}$, where $G : X \to \mathbb{R}$ is a C^2-convex function such that $D_H G$ is nonzero at every point of $\partial\Omega$.

We consider the semigroup $T_\Omega(t)$ on $L^2(\Omega,\nu)$ and its generator L_Ω:

$$D(L_\Omega) = \Big\{ u \in D^{1,2}(\Omega,\nu) \,\Big|\, \exists v \in L^2(\Omega,\nu) \text{ such that } \int_\Omega \langle D_H u, D_H \varphi\rangle_H \mathrm{d}\nu = -\int_\Omega v\varphi \,\mathrm{d}\nu \ \forall \varphi \in \mathcal{F}C_b^\infty(\Omega) \Big\} \tag{4}$$

with $L_\Omega u := v$ if $u \in D(L_\Omega)$. We recall (see [6, Section 2]) an approximation procedure of $T_\Omega(t)f$, when $f \in L^2(\Omega,\nu)$, through $\mathcal{F}C_b^3(X)$ functions that relies on a reduction to a finite (say n-) dimensional space and on a ε-penalisation argument. Accordingly, the approximation depends on two parameters n and ε. More precisely, we consider the function $\Phi_\varepsilon : X \to \mathbb{R}$ defined by

$$\Phi_\varepsilon(x) := U(x) + \frac{1}{2\varepsilon} d_\Omega^2(x), \qquad x \in X,\ \varepsilon > 0,$$

and the measure ν_ε given by $e^{-\Phi_\varepsilon}\gamma$. Next, we consider the operator L_ε on the whole X defined as

$$D(L_\varepsilon) = \Big\{ u \in D^{1,2}(X,\nu_\varepsilon) \,\Big|\, \exists\, v \in L^2(X,\nu_\varepsilon) \text{ such that } \int_X \langle D_H u, D_H\varphi\rangle_H \mathrm{d}\nu_\varepsilon = -\int_X v\varphi \,\mathrm{d}\nu_\varepsilon \text{ for every } \varphi \in \mathcal{FC}_b^\infty(X)\Big\}, \tag{5}$$

with $L_\varepsilon u := v$ if $u \in D(L_\varepsilon)$, and the semigroup $T_\varepsilon(t)$ generated by L_ε in $L^2(X,\nu_\varepsilon)$. We point out that L_ε acts on smooth cylindrical functions φ as follows

$$\begin{aligned} L_\varepsilon\varphi &= \mathrm{Tr}(D_H^2\varphi) - \sum_{i=1}^\infty \lambda_i^{-1}\langle x, e_i\rangle D_i\varphi - \langle D_H\Phi_\varepsilon, D_H\varphi\rangle_H \\ &= \mathrm{Tr}(D_H^2\varphi) - \sum_{i=1}^\infty \lambda_i^{-1}\langle x, e_i\rangle D_i\varphi - \Big\langle D_H U + \frac{1}{2\varepsilon} D_H d_\Omega^2, D_H\varphi\Big\rangle_H. \end{aligned}$$

Now we recall a useful approximation result whose proof can be found in [6, Theorem 2.8].

Theorem 1. *Under Hypotheses 1, 2 and 3 the following statements hold true.*

(i) *For any $\varepsilon > 0$ and $f \in L^2(X,\nu_\varepsilon)$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^2(X,\nu_\varepsilon)$ converging to f in $L^2(X,\nu_\varepsilon)$ such that $T_\varepsilon(t)f_n$ is in $\mathcal{FC}_b^3(X)$ and*

$$\lim_{n\to\infty} \|T_\varepsilon(t)f_n - T_\varepsilon(t)f\|_{D^{1,2}(X,\nu_\varepsilon)} = 0, \qquad t > 0.$$

In addition, if $f \in D^{1,2}(X,\nu_\varepsilon)$ then the sequence (f_n) can be chosen in a way that $D_H f_n$ converges to $D_H f$ in $L^1(X,\nu_\varepsilon;H)$, as $n\to\infty$.

(i) *For any $f \in L^2(\Omega,\nu)$ there exists an infinitesimal sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $T_{\varepsilon_n}(t)\widetilde{f}$ weakly converges to $T_\Omega(t)f$ in $D^{1,2}(\Omega,\nu)$, where $\widetilde{f}$ is any L^2-extension of f to X.*

We collect some properties of $T_\Omega(t)$, see [6, Proposition 1.10, Theorems 3.1 & 3.3].

Proposition 3. *If Hypotheses 1, 2 and 3 hold true, then*

(i) *the semigroup $T_\Omega(t)$ generated in $L^2(\Omega,\nu)$ can be extended to a positivity preserving contraction semigroup in $L^p(\Omega,\nu)$ for every $1 \le p \le \infty$ and $t \ge 0$, still denoted by $T_\Omega(t)$. It is strongly continuous in $L^p(\Omega,\nu)$ for any $p \in [1,\infty)$ and consistent;*

(ii) *for any $p \in [1,\infty)$, $f \in L^p(\Omega,\nu)$ and $g \in L^\infty(\Omega,\nu)$ it holds*

$$\int_\Omega f T_\Omega(t) g \,\mathrm{d}\nu = \int_\Omega g T_\Omega(t) f \,\mathrm{d}\nu, \qquad t > 0; \tag{6}$$

(iii) *for any $p \in (1,\infty)$, $f \in L^p(\Omega,\nu)$ and $t > 0$ there is $K_p > 0$ such that*

$$|D_H T_\Omega(t) f|_H^p \le K_p t^{-p/2} T_\Omega(t)|f|^p \qquad \nu\text{-a.e. in } \Omega; \tag{7}$$

(iv) *if* $f \in D^{1,p}(\Omega, \nu)$, $t > 0$ *and* $p \in [1, \infty)$ *it holds*

$$|D_H T_\Omega(t) f|^p \le e^{-p\lambda_1^{-1} t} T_\Omega(t) |D_H f|_H^p \qquad \nu\text{-a.e. in } \Omega. \tag{8}$$

We point out that the results in Proposition 3 continue to hold if we replace Ω, ν and $T_\Omega(t)$ by X, ν_ε and $T_\varepsilon(t)$, respectively.

2.2. BV functions in Hilbert spaces: definitions and some known facts

We introduce BV functions in the Wiener space setting. Let Y be a separable Hilbert space with norm $|\cdot|_Y$. We recall that in a separable space X the σ-algebra $\mathcal{B}(X)$ is generated by the family of the cylindrical sets (see e.g. [25]). Denote by $\mathcal{M}(\Omega; Y)$ the set of Borel Y-valued measures on Ω. If $Y = \mathbb{R}$ then we write $\mathcal{M}(\Omega)$. The total variation of $\mu \in \mathcal{M}(\Omega; Y)$ is the positive Borel measure

$$|\mu|(B) := \sup \left\{ \sum_{n=1}^{\infty} |\mu(B_n)|_Y \mid \begin{array}{c} B = \bigcup_{n=1}^{\infty} B_n,\ B_n \in \mathcal{B}(\Omega) \\ B_n \cap B_m = \emptyset,\ \text{if } n \neq m, \end{array} \right\}, \quad B \in \mathcal{B}(\Omega).$$

Let $\mathrm{Lip}_c(\Omega; Y)$ be the set of bounded Lipschitz continuous Y-valued functions $g : \Omega \to Y$ such that $\mathrm{dist}(\mathrm{supp}\, g, X \setminus \Omega) > 0$ and define the space $BV(\Omega, \nu)$ as follows.

Definition 1. Let Ω be an open subset of X. We say that a function $f \in L^2(\Omega, \nu)$ is of bounded variation in Ω, and we write $f \in BV(\Omega, \nu)$, if there exists a measure $\mu \in \mathcal{M}(\Omega; H)$ such that

$$\int_\Omega f \partial_h^* g \, \mathrm{d}\nu = -\int_\Omega g \, d\langle \mu, h \rangle_H,$$

for every $g \in \mathrm{Lip}_c(\Omega)$ and $h \in H$, where ∂_h^* denotes, up to the sign, the adjoint in $L^2(\Omega, \nu)$ of the partial derivative along $h \in H$. In this case we set $D_\nu f := \mu$.

As in the finite-dimensional case, one can characterise functions of bounded variation by their total variation.

Definition 2. Let Ω be an open subset of X and $u \in L^2(\Omega, \nu)$. We define the variation of u in Ω by

$$V_\nu(u, \Omega) := \sup \left\{ \int_\Omega u \, \mathrm{div}_\nu \, g \, \mathrm{d}\nu \;\middle|\; \begin{array}{c} F \subseteq H \text{finite-dimensional}, \\ g \in \mathrm{Lip}_c(\Omega; F),\ \|g\|_\infty \le 1. \end{array} \right\}.$$

Here $\mathrm{div}_\nu \, g = \sum_{i=1}^N \partial_{k_i}^* g_i(x)$ if $g(x) = \sum_{i=1}^N g_i(x) k_i$ and $F = \mathrm{span}\{k_1, \dots, k_N\}$ for some $N \in \mathbb{N}$.

When $\Omega = X$, in the two definitions above we can consider $\mathrm{Lip}_b(X)$ and $\mathrm{Lip}_b(X; F)$, respectively, as test functions spaces.

As announced, in [2, Theorem 5.7] it has been proved that $u \in BV(\Omega, \nu)$ if and only if $V_\nu(u, \Omega)$ is finite. Moreover, in this case

$$|D_\nu u|(\Omega) = V_\nu(u, \Omega). \tag{9}$$

Finally we say that a Borel subset E of X is of finite perimeter in Ω with respect to ν, whenever the function χ_E belongs to $BV(\Omega, \nu)$. In this case we denote by $P_\nu(E, \Omega)$ the total variation of χ_E in Ω.

3. A De Giorgi type characterisation

The main result of this section is the De Giorgi type characterisation of $BV(\Omega, \nu)$ functions in Theorem 3, which relies on a "quasi-commutative" formula between the semigroup $T_\Omega(t)$ and the H-gradient operator D_H; here estimate (8) plays a crucial role. This formula is inspired by an analogous formula proved in [12]. We first define the Sobolev spaces $D^{1,2}(X, \nu_\varepsilon; H)$.

Definition 3. We denote by $D^{1,2}(X, \nu_\varepsilon; H)$ the domain of the closure of the operator $D_H : \mathcal{F}C_b^1(X, H) \to L^2(X, \nu_\varepsilon; \mathcal{H}_2)$ in the $L^2(X, \nu_\varepsilon; H)$ norm (see [7, Section 8.1]). D_H is defined as

$$D_H\Phi(x) = \sum_{i=1}^{n}\sum_{j=1}^{k(i)} \frac{\partial \varphi_i}{\partial \xi_j}(\langle x, x_1\rangle, \dots, \langle x, x_{k(i)}\rangle)((Q_\infty^{1/2}x_j) \otimes e_i),$$

where $\{e_i \mid i \in \mathbb{N}\}$ is an orthonormal basis of H and

$$\Phi(x) = \sum_{i=1}^{n} \varphi_i(\langle x, x_1\rangle, \dots, \langle x, x_{k(i)}\rangle)e_i$$

for some $n \in \mathbb{N}$, $k(i) \in \mathbb{N}$, $x_1, \dots, x_{k(i)} \in X$ and $\varphi_i \in C_b^1(\mathbb{R}^{k(i)})$ for every $i = 1, \dots, n$. In an analogous way we define the space $D^{1,2}(\Omega, \nu; H)$.

We first show a vector-valued version of Theorem 1. Let $\mathbf{L}_\varepsilon$ in $L^2(X, \nu_\varepsilon; H)$ be the operator defined via the quadratic form by

$$(F, G) \mapsto \int_X \langle D_H F, D_H G\rangle_{\mathcal{H}_2} \mathrm{d}\nu_\varepsilon \qquad F, G \in D^{1,2}(X, \nu_\varepsilon; H).$$

In the same way we define the operator $\mathbf{L}_\Omega$ in in $L^2(\Omega, \nu; H)$. We recall that by [14, p. 84] ([14, Corollary 3.17 and Proposition 3.23] and [14, Corollary 4.8], respectively), the operators $\mathbf{L}_\varepsilon$ and $\mathbf{L}_\Omega$ generate strongly continuous semigroups $\mathbf{T}_\varepsilon(t)$ and $\mathbf{T}_\Omega(t)$ (contractive and analytic, respectively).

Proposition 4. *The operators $\mathbf{L}_\varepsilon$, $\mathbf{L}_\Omega$ and the semigroups $\mathbf{T}_\varepsilon(t)$ and $\mathbf{T}_\Omega(t)$ act component by component, i.e. if $F \in D(\mathbf{L}_\varepsilon)$ ($D(\mathbf{L}_\Omega)$, respectively), and it is such that $F = \sum_{i=1}^{\infty} f_i e_i$ for some basis $\{e_n \mid n \in \mathbb{N}\}$ of H, then $f_i \in D(L_\varepsilon)$ ($D(L_\Omega)$, respectively) and*

$$\mathbf{L}_\varepsilon F = \sum_{i=1}^{\infty}(L_\varepsilon f_i)e_i, \qquad \mathbf{L}_\Omega F = \sum_{i=1}^{\infty}(L_\Omega f_i)e_i.$$

Moreover, for every $t > 0$, if $F \in L^2(X, \nu_\varepsilon; H)$ ($L^2(\Omega, \nu; H)$, respectively), and it is such that $F = \sum_{i=1}^\infty f_i e_i$ for some basis $\{e_n \,|\, n \in \mathbb{N}\}$ of H and $f_i \in L^2(X, \nu_\varepsilon)$ ($L^2(\Omega, \nu)$, respectively) then

$$\mathbf{T}_\varepsilon(t)F = \sum_{i=1}^\infty (T_\varepsilon(t) f_i)e_i, \qquad \left(\mathbf{T}_\Omega(t)F = \sum_{i=1}^\infty (T_\Omega(t) f_i)e_i, \text{ respectively}\right).$$

The above identities hold ν_ε-a.e. in X (ν_Ω-a.e. in Ω, respectively).

Proof. We only show the results for $\mathbf{L}_\varepsilon$ and $\mathbf{T}_\varepsilon(t)$. Let $F = \sum_{i=1}^\infty f_i e_i \in D(\mathbf{L}_\varepsilon)$ and let $G = g e_j$ for some $j \in \mathbb{N}$ and $g \in D^{1,2}(X, \nu_\varepsilon)$; then

$$\begin{aligned}\int_X \langle D_H g, D_H f_j\rangle_H \mathrm{d}\nu_\varepsilon &= \int_X \langle D_H G, D_H F\rangle_{\mathcal{H}_2} \mathrm{d}\nu_\varepsilon \\ &= -\int_X \langle G, \mathbf{L}_\varepsilon F\rangle_H \mathrm{d}\nu_\varepsilon = -\int_X g(\mathbf{L}_\varepsilon F)_j \mathrm{d}\nu_\varepsilon.\end{aligned}$$

This shows that $f_j \in D(L_\varepsilon)$ (see (5)) and $L_\varepsilon f_j = (\mathbf{L}_\varepsilon F)_j$. Now observe that

$$D_t(\mathbf{T}_\varepsilon(t)F)_j = (\mathbf{L}_\varepsilon \mathbf{T}_\varepsilon(t)F)_j = L_\varepsilon(\mathbf{T}_\varepsilon(t)F)_j, \qquad (\mathbf{T}_\varepsilon(0)F)_j = f_j.$$

Thus, by the uniqueness of the solution of the Cauchy problem associated with $D_t - L_\varepsilon$ in $L^2(X, \nu_\varepsilon)$, it follows that $(\mathbf{T}_\varepsilon(t)F)_j = T_\varepsilon(t) f_j$ for any $t > 0$. The arbitrariness of $j \in \mathbb{N}$ concludes the proof. □

Remark 2. According to the definition of $\mathbf{T}_\varepsilon(t)$ and $\mathbf{T}_\Omega(t)$ it is immediately seen that for every $F \in L^2(X, \nu_\varepsilon; H)$ and $G \in L^2(\Omega, \nu; H)$

$$|\mathbf{T}_\varepsilon(t)F|^2 \le T_\varepsilon(t)|F|^2, \qquad t \ge 0,\ \nu_\varepsilon\text{-a.e. in } X \tag{10}$$

and

$$|\mathbf{T}_\Omega(t)G|^2 \le T_\Omega(t)|G|^2, \qquad t \ge 0,\ \nu\text{-a.e. in } \Omega. \tag{11}$$

Moreover, taking into account that the semigroups $\mathbf{T}_\Omega(t)$ and $\mathbf{T}_\varepsilon(t)$ act component by component, we can obtain a vector-valued version of Theorem 1.

Theorem 2. *Under Hypotheses 1, 2 and 3, the following statements hold true.*

(i) *For any $\varepsilon > 0$ and $F \in L^2(X, \nu_\varepsilon; H)$, there exists a sequence $(F_n)_n \subseteq L^2(X, \nu_\varepsilon; H)$ such that $\mathbf{T}_\varepsilon(t)F_n$ belongs to $\mathcal{F}C_b^3(X; H)$ and*

$$\begin{aligned}&\lim_{n\to\infty} \|F_n - F\|_{L^2(X,\nu_\varepsilon;H)} = 0,\\ &\lim_{n\to\infty} \|\mathbf{T}_\varepsilon(t)F_n - \mathbf{T}_\varepsilon(t)F\|_{D^{1,2}(X,\nu_\varepsilon;H)} = 0, \qquad t > 0.\end{aligned} \tag{12}$$

If, in addition, $F \in D^{1,2}(X, \nu_\varepsilon; H)$ then $D_H F_n$ converges to $D_H F$ in $L^1(X, \nu_\varepsilon; \mathcal{H}_2)$, as $n \to \infty$.

(ii) *For any $F \in L^2(\Omega, \nu; H)$ there exists an infinitesimal sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\mathbf{T}_{\varepsilon_n}(t)\widetilde{F}$ weakly converges to $\mathbf{T}_\Omega(t)F$ in $D^{1,2}(\Omega, \nu; H)$, where $\widetilde{F}$ is any L^2-extension of F to X.*

Proof. (i) Let $F = \sum_{i=1}^\infty f^{(i)} e_i$ where $f^{(i)} \in L^2(X, \nu_\varepsilon), i \in \mathbb{N}$. For every $i \in \mathbb{N}$, by Theorem 1(i), there exists $(f_k^{(i)})_{k\in\mathbb{N}} \subseteq L^2(X, \nu_\varepsilon)$ converging to $f^{(i)}$ in $L^2(X, \nu_\varepsilon)$ such that $T_\varepsilon(t) f_k^{(i)}$ belongs to $\mathcal{F}C_b^3(X)$ and

$$\lim_{k\to\infty} \|T_\varepsilon(t) f_k^{(i)} - T_\varepsilon(t) f^{(i)}\|_{D^{1,2}(X,\nu_\varepsilon)} = 0, \qquad t, \varepsilon > 0. \tag{13}$$

Observe that (12) follows immediately from (13). Now fix $i, n \in \mathbb{N}$ and consider $k_i \in \mathbb{N}$ such that for every $k \geq k_i$ it holds

$$\int_X |f_k^{(i)} - f^{(i)}|^2 \mathrm{d}\nu_\varepsilon < \frac{1}{n2^i}.$$

Consider the vector field $F_n := \sum_{i=1}^n f_{k_i}^{(i)} e_i$. We claim that (F_n) is the sequence we are looking for. Indeed F_n belongs to $L^2(X, \nu_\varepsilon; H)$ for any $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ be such that $\sum_{i=n_0+1}^\infty \|f^{(i)}\|^2_{L^2(X,\nu_\varepsilon)} \leq \eta/2$ and let $n \geq n_0$ such that $1/n < \eta/2$. We have

$$\begin{aligned}\|F_n - F\|^2_{L^2(X,\nu_\varepsilon;H)} &\leq \sum_{i=1}^n \int_X |f_{k_i}^{(i)} - f^{(i)}|^2 \mathrm{d}\nu_\varepsilon + \sum_{i=n+1}^\infty \|f^{(i)}\|^2_{L^2(X,\nu_\varepsilon)} \\ &\leq \frac{1}{n} + \frac{\eta}{2} \leq \eta.\end{aligned}$$

In a similar way we can prove the other statements.

(ii) is an immediate consequence of Proposition 4 and Theorem 1(ii). □

Before going on, recall that usually in the characterisation of functions of bounded variation in terms of the short-time behaviour of suitable semigroups a crucial tool is an appropriate commutation formula between the semigroup and the gradient operator. For instance, for the Wiener space and the Ornstein–Uhlenbeck semigroup the equality $D_H S(t) f = e^{-t}\mathbf{S}(t) D_H f$ holds true for any $t \geq 0$. Let us prove a (quasi) commutation formula between $T_\Omega(t)$ and D_H, under the following additional assumption.

Hypotheses 4. *The map $(d_\Omega)^{-2}\|D_H^2 d_\Omega^2\|_{\mathcal{H}_2}$ belongs to $L^2(X, \nu)$.*

Remark 3. It is not difficult to show that every open ball and every open ellipsoid of X as well as every open hyperplane of X satisfy Hypothesis 4. We show that Hypothesis 4 is satisfied when Ω is the unit ball B_X centred at zero. The other examples can be discussed in a similar fashion. Observe that, by Proposition 2, $\|D_H^2 d_\Omega^2\|_{\mathcal{H}_2} \leq 2$ and $\|D_H^2 d_\Omega^2(x)\|_{\mathcal{H}_2} = 0$ if $x \in B_X$. Moreover, there exists a constant $C > 0$ such that

$$\begin{aligned}d_{B_X}(x) &\geq C \inf\{|h|_X \,|\, h \in H \cap (B_X - x)\} \geq C \inf\{|h|_X \,|\, h \in (B_X - x)\} \\ &= C \inf\{|x - h|_X \,|\, h \in B_X\} = C\mathrm{dist}(x, B_X) = C||x|_X - 1|,\end{aligned}$$

where $\mathrm{dist}(x, B_X)$ is the distance of x from B_X. So

$$\begin{aligned}\int_X d_{B_X}^{-4}\|D_H d_{B_X}^2\|_{\mathcal{H}_2}^2 \mathrm{d}\nu &\le K\int_{X\setminus B_X}\frac{1}{(|x|_X-1)^4}\mathrm{d}\nu(x)\\ &\le K\int_X \frac{1}{(|x|_X-1)^4}e^{-U(x)}\mathrm{d}\gamma(x) \le K\Big(\int_X e^{-p'U}\mathrm{d}\gamma\Big)^{\frac{1}{p'}}\Big(\int_X\frac{1}{(|x|_X-1)^{4p}}\mathrm{d}\gamma(x)\Big)^{\frac{1}{p}}\\ &\le K\Big(\int_X e^{-p'U}\mathrm{d}\gamma\Big)^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^n}\frac{1}{(|\xi|_{\mathbb{R}^n}-1)^{4p}}\mathrm{d}\gamma_n(\xi)\Big)^{\frac{1}{p}},\end{aligned} \tag{14}$$

where K is a positive constant and γ_n denotes the n-dimensional Gaussian measure, image of γ under the projection on $\mathrm{span}\{v_1,\dots,v_n\}$. To conclude, observe that there exists $n\in\mathbb{N}$ such that the right-hand side of (14) is finite.

Proposition 5. *Under Hypotheses 1, 2, 3 and 4, the formula*

$$\begin{aligned}&D_H T_\Omega(t)f-(e^{-tQ_\infty^{-1}}\mathbf{T}_\Omega(t)D_H f)\\ &\quad=-\int_0^t e^{(s-t)Q_\infty^{-1}}\mathbf{T}_\Omega(t-s)(D_H^2 U D_H T_\Omega(s)f)\mathrm{d}s.\end{aligned} \tag{15}$$

holds true ν-a.e. in Ω, for any $f\in \mathrm{Lip}_c(\Omega)$ and $t>0$.

Proof. In order to prove (15) we show that

$$\begin{aligned}\int_\Omega\langle D_H T_\Omega(t)f,G\rangle_H\mathrm{d}\nu&=\int_\Omega\langle e^{-tQ_\infty^{-1}}\mathbf{T}_\Omega(t)D_H f,G\rangle_H\mathrm{d}\nu\\ &-\int_\Omega\int_0^t\langle e^{(s-t)Q_\infty^{-1}}(\mathbf{T}_\Omega(t-s)(D_H^2UD_HT_\Omega(s)f)),G\rangle_H\mathrm{d}s\mathrm{d}\nu,\end{aligned} \tag{16}$$

for any $f\in\mathrm{Lip}_c(\Omega)$, $G\in C_b(\Omega;H)$ and $t>0$. By performing slight changes in [12, Appendix A] we get

$$\begin{aligned}&D_H T_\varepsilon(t)g-(e^{-tQ_\infty^{-1}}\mathbf{T}_\varepsilon(t)D_H g)\\ &\quad=-\int_0^t e^{(s-t)Q_\infty^{-1}}\mathbf{T}_\varepsilon(t-s)(D_H^2\Phi_\varepsilon D_H T_\varepsilon(s)g)\mathrm{d}s\end{aligned} \tag{17}$$

ν_ε-a.e. in X for any $g\in\mathrm{Lip}_b(X)$ and $\varepsilon>0$, where $T_\varepsilon(t)$ is the semigroup introduced in Sect. 2.1. Now, let $f\in\mathrm{Lip}_c(\Omega)$ and $\widetilde{f}$ be the trivial extension to zero of f in the whole space X. Clearly, $\widetilde{f}$ belongs to $\mathrm{Lip}_b(X)$ and (17) holds true with g replaced by $\widetilde{f}$. Consequently, multiplying (17) by the function G and integrating on Ω with respect to ν yield

$$\begin{aligned}\int_\Omega\langle D_H T_\varepsilon(t)\widetilde{f},G\rangle_H\mathrm{d}\nu&=\int_\Omega\langle e^{-tQ_\infty^{-1}}(\mathbf{T}_\varepsilon(t)D_H\widetilde{f}),G\rangle_H\mathrm{d}\nu\\ &-\int_\Omega\int_0^t\langle e^{(s-t)Q_\infty^{-1}}(\mathbf{T}_\varepsilon(t-s)(D_H^2\Phi_\varepsilon D_H T_\varepsilon(s)\widetilde{f})),G\rangle_H\mathrm{d}s\mathrm{d}\nu\\ &=\int_\Omega\langle e^{-tQ_\infty^{-1}}\mathbf{T}_\varepsilon(t)D_H\widetilde{f},G\rangle_H\mathrm{d}\nu\end{aligned}$$

$$-\int_0^t\int_\Omega \langle e^{(s-t)Q_\infty^{-1}}(\mathbf{T}_\varepsilon(t-s)(D_H^2\Phi_\varepsilon D_H T_\varepsilon(s)\widetilde{f})), G\rangle_H \mathrm{d}\nu\mathrm{d}s, \tag{18}$$

where in the last line we used the Fubini–Tonelli theorem.

The proof of (16) is split in two steps.

Step 1. We argue by approximation on the last terms in (18) and (16).

For every $\varepsilon, s > 0$ we fix a Borel measurable version of $D_H T_\Omega(s)f$ and $D_H T_\varepsilon(s)\widetilde{f}$ in $L^2(\Omega,\nu;H)$ and $L^2(X,\nu_\varepsilon;H)$, respectively. Consider the function

$$\Gamma_\varepsilon(s,x) := \begin{cases} D_H T_\Omega(s)f(x), & x\in\Omega;\\ D_H T_\varepsilon(s)\widetilde{f}(x), & x\in X\setminus\Omega. \end{cases}$$

Observe that the map $x\mapsto \Gamma_\varepsilon(s,x)$ is an extension of $D_H T_\Omega(s)f$ to the whole X. Thus, by Theorem 2 there is a sequence $\varepsilon_n\downarrow 0$ such that for every $\eta>0$ the function $\mathbf{T}_{\varepsilon_n}(t-s)(D_H^2 U\Gamma_\eta(s,\cdot))$ weakly converges to $\mathbf{T}_\Omega(t-s)(D_H^2 U D_H T_\Omega(s)f)$ in $D^{1,2}(\Omega,\nu;H)$. Observe that the set

$$\{\mathbf{T}_{\varepsilon_n}(t-s)(D_H^2 U\Gamma_\eta(s,\cdot)) \,|\, n\in\mathbb{N} \text{ and } s,\eta>0\} \tag{19}$$

is bounded in $L^2(\Omega,\nu;H)$. Indeed by the contractivity of $\mathbf{T}_{\varepsilon_n}(t)$ in the space $L^2(X,\nu_{\varepsilon_n};H)$, the fact that $U\equiv\Phi_\varepsilon$ on Ω and estimate (8) we have

$$\begin{aligned}
&\left\|\mathbf{T}_{\varepsilon_n}(t-s)(D_H^2 U\Gamma_\eta(s,\cdot))\right\|_{L^2(\Omega,\nu;H)}\\
&\quad=\left\|\mathbf{T}_{\varepsilon_n}(t-s)(D_H^2 U\Gamma_\eta(s,\cdot))\right\|_{L^2(\Omega,\nu_{\varepsilon_n};H)}\\
&\quad\le\left\|\mathbf{T}_{\varepsilon_n}(t-s)(D_H^2 U\Gamma_\eta(s,\cdot))\right\|_{L^2(X,\nu_{\varepsilon_n};H)}\\
&\quad\le\left\|D_H^2 U\Gamma_\eta(s,\cdot)\right\|_{L^2(X,\nu_{\varepsilon_n};H)}\le [D_H U]_{H\text{-Lip}}\left\|\Gamma_\eta(s,\cdot)\right\|_{L^2(X,\nu_{\varepsilon_n};H)}\\
&\quad\le [D_H U]_{H\text{-Lip}}\left(\|D_H T_\eta(s)\widetilde{f}\|_{L^2(X,\nu_{\varepsilon_n};H)}+\|D_H T_\Omega(s)f\|_{L^2(\Omega,\nu;H)}\right)\\
&\quad\le [D_H U]_{H\text{-Lip}}e^{-2\lambda_1^{-1}s}\left(\|T_\eta(s)|D_H\widetilde{f}|_H\|_{L^2(X,\nu_{\varepsilon_n})}+\|T_\Omega(s)|D_H f|_H\|_{L^2(\Omega,\nu)}\right)\\
&\quad\le 2[D_H U]_{H\text{-Lip}}(\nu(X))^{\frac12}\|D_H f\|_{L^\infty(\Omega,\nu;H)},
\end{aligned}$$

where in the last line we used the contractivity of $T_\eta(t)$ and $T_\Omega(t)$ in L^∞ and the fact that $\nu_{\varepsilon_n}(X)\le\nu(X)$ for any $n\in\mathbb{N}$. So there exists $M>0$ large enough so that the family in (19) is contained in $B(0,M)$, the ball of $L^2(\Omega,\nu;H)$ with centre 0 and radius M.

Recall that every bounded subset of $L^2(\Omega,\nu;H)$ is weakly metrisable (see [15, Proposition 3.106]) and let $\rho: B(0,M)\times B(0,M)\to\mathbb{R}$ be a metric such that the topology generated by ρ and the weak topology in $B(0,M)$ coincide. Now we use a diagonal argument to pass to the limit in (18). Let $n_1\in\mathbb{N}$ be such that for every $n\ge n_1$ it holds

$$\rho\Big(\mathbf{T}_{\varepsilon_n}(t-s)(D_H^2 U\overline{\Gamma}_1(s,\cdot)), \mathbf{T}_\Omega(t-s)(D_H^2 U D_H T_\Omega(s)f)\Big)\le 1,$$

where $\overline{\Gamma}_j(s,x) = \Gamma_{j^{-1}}(s,x)$ for any $s > 0$ and $x \in X$. Now assume that $n_1, \dots, n_k$ are already constructed and consider $n_{k+1} > n_k$ such that for every $n \geq n_{k+1}$

$$\rho\Big(\mathbf{T}_{\varepsilon_n}(t-s)(D_H^2 U \overline{\Gamma}_{n_k}(s,\cdot)), \mathbf{T}_{\Omega}(t-s)(D_H^2 U D_H T_{\Omega}(s) f)\Big) \leq \frac{1}{2^k}.$$

Consider now the sequence $(\mathbf{T}_{\varepsilon_{n_k}}(t-s)(D_H^2 U \overline{\Gamma}_{n_k}(s,\cdot)))_{k\in\mathbb{N}}$ and observe that it weakly converges to $\mathbf{T}_{\Omega}(t-s)(D_H^2 U D_H T_{\Omega}(s) f)$ in $L^2(\Omega, \nu; H)$ as $k \to \infty$.

Step 2. To complete the proof, we replace ε in (18) by a sequence $\varepsilon_m \downarrow 0$ such that step 1 and Theorems 1, 2 apply. Let us show that we can take the limit as $m \to \infty$. Indeed, from Theorem 1 it follows that for any $f \in L^2(\Omega, \nu)$, $T_{\varepsilon_m}(t)\widetilde{f}$ weakly converges (up to a subsequence) to $T_{\Omega}(t) f$ in $D^{2,2}(\Omega, \nu)$ as $m \to \infty$, hence writing $T_m, \mathbf{T}_m, \Phi_m, \Gamma_m$ in place of $T_{\varepsilon_m}, \mathbf{T}_{\varepsilon_m}, \Phi_{\varepsilon_m}, \Gamma_{\varepsilon_m}$ we obtain

$$\lim_{m\to\infty} \int_{\Omega} \langle D_H T_m(t)\widetilde{f}, G\rangle_H \mathrm{d}\nu = \int_{\Omega} \langle D_H T_{\Omega}(t) f, G\rangle_H \mathrm{d}\nu$$

and by the analogous vector-valued result (see Theorem 2, (10) and again (7))

$$\lim_{m\to\infty} \int_{\Omega} \langle e^{-tQ_\infty^{-1}} \mathbf{T}_m(t) D_H \widetilde{f}, G\rangle_H \mathrm{d}\nu = \int_{\Omega} \langle e^{-tQ_\infty^{-1}} \mathbf{T}_{\Omega}(t) D_H f, G\rangle_H \mathrm{d}\nu.$$

To conclude we have to prove that the last term in the right-hand side of (18) converges to the last term in the right-hand side of (16).

$$\begin{aligned}
&\left|\int_{\Omega} \langle \mathbf{T}_m(t-s)(D_H^2 \Phi_m D_H T_m(s)\widetilde{f})\mathrm{d}\nu - \int_{\Omega} \mathbf{T}_{\Omega}(t-s)(D_H^2 U D_H T_{\Omega}(s) f), G\rangle_H \mathrm{d}\nu\right| \\
&\quad \leq \left|\int_{\Omega} \langle \mathbf{T}_m(t-s)(D_H^2 \Phi_m D_H T_m(s)\widetilde{f}) - \mathbf{T}_m(t-s)(D_H^2 \Phi_m \Gamma_m(s,\cdot)), G\rangle_H \mathrm{d}\nu\right| \\
&\qquad + \left|\int_{\Omega} \langle \mathbf{T}_m(t-s)(D_H^2 \Phi_m \Gamma_m(s,\cdot)) - \mathbf{T}_m(t-s)(D_H^2 U \Gamma_m(s,\cdot)), G\rangle_H \mathrm{d}\nu\right| \\
&\qquad + \left|\int_{\Omega} \langle \mathbf{T}_m(t-s)(D_H^2 U \Gamma_m(s,\cdot)) - \mathbf{T}_{\Omega}(t-s)(D_H^2 U D_H T_{\Omega}(s) f), G\rangle_H \mathrm{d}\nu\right| \\
&\quad =: I_1(m) + I_2(m) + I_3(m).
\end{aligned}$$

Let us estimate I_1. Using that $\Phi_m \equiv U$ on Ω for every $m \in \mathbb{N}$, formula (10) and the invariance property of T_m with respect to $\nu_m := \nu_{\varepsilon_m}$ we have

$$\begin{aligned}
I_1(m) &\leq \int_{\Omega} \Big|\mathbf{T}_m(t-s)(D_H^2 \Phi_m D_H T_m(s)\widetilde{f}) \\
&\qquad - \mathbf{T}_m(t-s)(D_H^2 \Phi_m \Gamma_m(s,\cdot))\Big|_H |G|_H \mathrm{d}\nu \\
&\leq (\nu(X))^{\frac{1}{2}} \|G\|_{\infty} \left(\int_{\Omega} \Big|\mathbf{T}_m(t-s)(D_H^2 \Phi_m D_H T_m(s)\widetilde{f} - D_H^2 \Phi_m \Gamma_m(s,\cdot))\Big|_H^2 \mathrm{d}\nu\right)^{\frac{1}{2}} \\
&\leq (\nu(X))^{\frac{1}{2}} \|G\|_{\infty} \left(\int_{\Omega} T_m(t-s)\Big|D_H^2 \Phi_m D_H T_m(s)\widetilde{f} - D_H^2 \Phi_m \Gamma_m(s,\cdot)\Big|_H^2 \mathrm{d}\nu_m\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq (\nu(X))^{\frac{1}{2}}\|G\|_\infty\bigg(\int_X T_m(t-s)\Big|D_H^2\Phi_m D_H T_m(s)\widetilde{f}-D_H^2\Phi_m\Gamma_m(s,\cdot)\Big|_H^2\mathrm{d}\nu_m\bigg)^{\frac{1}{2}}\\
&=(\nu(X))^{\frac{1}{2}}\|G\|_\infty\bigg(\int_X\Big|(D_H^2\Phi_m D_H T_m(s)\widetilde{f})-(D_H^2\Phi_m\Gamma_m(s,\cdot))\Big|_H^2\mathrm{d}\nu_m\bigg)^{\frac{1}{2}}\\
&\leq(\nu(X))^{\frac{1}{2}}\|G\|_\infty\bigg(\int_\Omega\|D_H^2U\|_{\mathcal{H}_2}^2\big|D_H T_m(s)\widetilde{f}-D_H T_\Omega(s)f\big|_H^2\mathrm{d}\nu\bigg)^{\frac{1}{2}}\\
&\leq[D_HU]_{H\text{-Lip}}(\nu(X))^{\frac{1}{2}}\|G\|_\infty\bigg(\int_\Omega\big|D_H T_m(s)\widetilde{f}-D_H T_\Omega(s)f\big|_H^2\mathrm{d}\nu\bigg)^{\frac{1}{2}}. \qquad (20)
\end{aligned}
$$

The right-hand side of (20) converges to zero as $m\to\infty$: indeed, $D_H T_m(s)\widetilde{f}$ converges pointwise ν-almost everywhere in Ω to $D_H T_\Omega(s)f$. Furthermore, by Proposition 3 we have that ν-a.e. in Ω

$$
\begin{aligned}
|D_H T_{\varepsilon_m}(s)\widetilde{f}-D_H T_\Omega(s)f|_H^2&\leq 2\Big(|D_H T_{\varepsilon_m}(s)\widetilde{f}|_H^2+|D_H T_\Omega(s)f|_H^2\Big)\\
&\leq 2e^{-2\lambda_1^{-1}s}\Big(T_{\varepsilon_m}(s)|D_H\widetilde{f}|_H^2+T_\Omega(s)|D_Hf|_H^2\Big)\leq 2\|D_Hf\|_{L^\infty(\Omega,\nu;H)}.
\end{aligned}
$$

So by the dominated convergence theorem we get that $I_1(m)$ vanishes as $m\to\infty$. Now, using similar arguments we can estimate $I_2(m)$ as follows

$$
\begin{aligned}
I_2(m)&\leq(\nu(X))^{\frac{1}{2}}\|G\|_\infty\bigg(\int_X\Big|(D_H^2\Phi_m\Gamma_m(s,\cdot))-(D_H^2U\Gamma_m(s,\cdot))\Big|_H^2\mathrm{d}\nu_m\bigg)^{\frac{1}{2}}\\
&\leq(\nu(X))^{\frac{1}{2}}\|G\|_\infty\bigg(\int_X\|D_H^2\Phi_m-D_H^2U\|_{\mathcal{H}_2}^2|\Gamma_m(s,\cdot)|_H^2\mathrm{d}\nu_m\bigg)^{\frac{1}{2}}\\
&\leq(\nu(X))^{\frac{1}{2}}\|G\|_\infty\bigg(\int_X\|D_H^2\Phi_m-D_H^2U\|_{\mathcal{H}_2}^2|(D_H T_m(s)\widetilde{f})\chi_{X\setminus\Omega}\\
&\quad+(D_H T_\Omega(s)f)\chi_\Omega|_H^2\mathrm{d}\nu_m\bigg)^{\frac{1}{2}}\\
&\leq(2\nu(X))^{\frac{1}{2}}\|G\|_\infty\bigg(\int_X\|D_H^2\Phi_m-D_H^2U\|_{\mathcal{H}_2}^2\big(|D_H T_m(s)\widetilde{f}|_H^2\\
&\quad+|D_H T_\Omega(s)f|_H^2\chi_\Omega\big)\mathrm{d}\nu_m\bigg)^{\frac{1}{2}}\\
&\leq(2\nu(X))^{\frac{1}{2}}\|G\|_\infty\bigg(\int_X\|D_H^2\Phi_m-D_H^2U\|_{\mathcal{H}_2}^2\big(T_m(s)|D_H\widetilde{f}|_H^2\\
&\quad+(T_\Omega(s)|D_Hf|_H^2)\chi_\Omega\big)\mathrm{d}\nu_m\bigg)^{\frac{1}{2}}\\
&\leq 2(\nu(X))^{\frac{1}{2}}\|D_Hf\|_{L^\infty(\Omega,\nu;H)}\|G\|_\infty\bigg(\int_X\|D_H^2\Phi_m-D_H^2U\|_{\mathcal{H}_2}^2\mathrm{d}\nu_m\bigg)^{\frac{1}{2}}\\
&=2(\nu(X))^{\frac{1}{2}}\|D_Hf\|_{L^\infty(\Omega,\nu;H)}\|G\|_\infty\bigg(\int_X\frac{1}{4\varepsilon_m}\|D_H^2d_\Omega^2\|_{\mathcal{H}_2}^2e^{-U-\frac{1}{2\varepsilon_m}d_\Omega^2}\mathrm{d}\gamma\bigg)^{\frac{1}{2}}. \qquad (21)
\end{aligned}
$$

Now observe that the right-hand side of (21) vanishes as $m \to \infty$. Indeed the function $\frac{1}{4\varepsilon_m}\|D_H^2 d_\Omega^2\|_{\mathcal{H}_2}^2 e^{-U-\frac{1}{2\varepsilon_m}d_\Omega^2}$ identically vanishes in Ω and converges pointwise to 0 ν-almost everywhere in $X\backslash\Omega$ as $m \to \infty$. Furthermore, observe that the function $(0,\infty) \ni \varepsilon \mapsto R(\varepsilon) := \frac{1}{4\varepsilon^2}\left|D_H^2 d_\Omega^2\right|^2 e^{-\frac{1}{2\varepsilon}d_\Omega^2}$ attains its maximum in $\varepsilon = d_\Omega^2/4$ where it equals to $4d_\Omega^{-4}\|D_H^2 d_\Omega^2|_{\mathcal{H}_2}^2$. Thus, using Hypothesis 4 and applying the dominated convergence theorem we infer that also $I_2(m)$ converges to zero as m goes to infinity.

Finally $I_3(m)$ converges to zero as m goes to infinity thanks to step 1 and this concludes the proof. □

Corollary 1. *Assume Hypotheses 1, 2, 3 and 4 hold true. For any $t > 0$ and $p > 1$ there exist two operators $\mathbf{S}_1(t) : L^p(\Omega,\nu;H) \to L^1(\Omega,\nu;H)$ and $S_2(t) : L^p(\Omega,\nu) \to L^1(\Omega,\nu;H)$ such that for every continuous and H-differentiable function $\varphi : \Omega \to \mathbb{R}$ with H-Lipschitz gradient*

$$D_H T_\Omega(t)\varphi = \mathbf{S}_1(t) D_H\varphi + S_2(t)\varphi.$$

Moreover, the adjoint operator $(\mathbf{S}_1(t))^*$ maps $\mathrm{Lip}_c(\Omega;H)$ into $L^\infty(\Omega,\nu;H)$ and verifies $\left\|\mathbf{S}_1^*(t)F\right\|_\infty \le C_1(t)\|F\|_\infty$ for any $F \in \mathrm{Lip}_c(\Omega;H)$ with $C_1(t) \to 1$ as $t \to 0$ and the norm $C_2(t) := \|S_2(t)\|_{\mathcal{L}(L^p,L^1)} \to 0$ as $t \to 0$.

Proof. Setting $\mathbf{S}_1(t) := e^{-tQ_\infty^{-1}}\mathbf{T}_\Omega(t)$, [6, Proposition 1.10] yields that $\mathbf{S}_1^*(t) = \mathbf{T}_\Omega(t)e^{-tQ_\infty^{-1}}$ maps $\mathrm{Lip}_c(\Omega;H)$ into $L^\infty(\Omega,\nu;H)$ and

$$\begin{aligned}\left\|\mathbf{S}_1^*(t)F\right\|_\infty &= \|(e^{-tQ_\infty^{-1}}\mathbf{T}_\Omega(t))^*F\|_\infty \\ &= \|\mathbf{T}_\Omega(t)e^{-tQ_\infty^{-1}}F\|_\infty \le |e^{-tQ_\infty^{-1}}|_{\mathcal{L}(H)}\|F\|_\infty. \end{aligned} \tag{22}$$

Moreover, setting $S_2(t) := -\int_0^t e^{(s-t)Q_\infty^{-1}}\mathbf{T}_\Omega(t-s)(D_H^2 U D_H T_\Omega))\mathrm{d}s$, by the contractivity of $\mathbf{T}_\Omega(t)$ in $L^1(\Omega,\nu;H)$, (7), Hypothesis 2, the contractivity of $e^{-tQ_\infty^{-1}}$ in H, estimate (11) and the invariance property of $T_\Omega(t)$, we get

$$\begin{aligned}&\|\mathbf{S}_2(t)\varphi\|_{L^1(\Omega,\nu;H)} \\ &\le \int_0^t\int_\Omega \left|e^{(s-t)Q_\infty^{-1}}\mathbf{T}_\Omega(t-s)(D_H^2 U D_H T_\Omega(s)\varphi)\right|_H \mathrm{d}\nu\mathrm{d}s \\ &\le \int_0^t \left|e^{(s-t)Q_\infty^{-1}}\right|_{\mathcal{L}(H)}\int_\Omega \left|\mathbf{T}_\Omega(t-s)(D_H^2 U D_H T_\Omega(s)\varphi)\right|_H \mathrm{d}\nu\mathrm{d}s \\ &\le \int_0^t\int_\Omega \left|\mathbf{T}_\Omega(t-s)(D_H^2 U D_H T_\Omega(s)\varphi)\right|_H \mathrm{d}\nu\mathrm{d}s \\ &\le \int_0^t\int_\Omega \left|D_H^2 U D_H T_\Omega(s)\varphi\right|_H \mathrm{d}\nu\mathrm{d}s \\ &\le \int_0^t\int_\Omega \|D_H^2 U\|_{\mathcal{H}_2}|D_H T_\Omega(s)\varphi|_H \mathrm{d}\nu\mathrm{d}s \\ &\le \int_0^t \left(\int_\Omega \|D_H^2 U\|_{\mathcal{H}_2}^{p'}\mathrm{d}\nu\right)^{\frac{1}{p'}}\left(\int_\Omega |D_H T_\Omega(s)\varphi|_H^p \mathrm{d}\nu\right)^{\frac{1}{p}}\mathrm{d}s \end{aligned}$$

$$\leq K_p^{\frac{1}{p}} \|D_H^2 U\|_{L^{p'}(X,\nu;\mathcal{H}_2)} \int_0^t s^{-\frac{1}{2}} \left(\int_\Omega T_\Omega(s)|\varphi|^p \mathrm{d}\nu \right)^{\frac{1}{p}} \mathrm{d}s$$

$$\leq K_p^{\frac{1}{p}} \|D_H^2 U\|_{L^{p'}(X,\nu;\mathcal{H}_2)} \|\varphi\|_{L^p(\Omega,\nu)} \int_0^t s^{-\frac{1}{2}} \mathrm{d}s$$

$$= 2K_p^{\frac{1}{p}} \sqrt{t} \|D_H^2 U\|_{L^{p'}(X,\nu;\mathcal{H}_2)} \|\varphi\|_{L^p(\Omega,\nu)} \tag{23}$$

for any $t > 0$. By the assumption on U we deduce that the operator $S_2(t)$ is bounded from $L^p(\Omega, \nu)$ into $L^1(\Omega, \nu; H)$ for any $t > 0$. Finally, estimates (22) and (23) allow us to complete the proof. □

Now, we are able to prove the main result of this section.

Theorem 3. *Assume Hypotheses 1, 2, 3 and 4 hold true and let $u \in L^2(\Omega, \nu)$. The following statements are true:*

(i) *if* $\liminf_{t\to 0^+} \|D_H T_\Omega(t)u\|_{L^1(\Omega,\nu;H)}$ *is finite, then* $u \in BV(\Omega, \nu)$;
(ii) *if* $u \in BV(\Omega, \nu)$, *then* $\limsup_{t\to 0^+} \|D_H T_\Omega(t)u\|_{L^1(\Omega,\nu;H)} \leq |D_\nu u|(\Omega)$.

Hence, $u \in BV(\Omega, \nu)$ iff $\lim_{t\to 0^+} \|D_H T_\Omega(t)u\|_{L^1(\Omega,\nu;H)} < \infty$. *In this case*

$$|D_\nu u|(\Omega) = \lim_{t\to 0^+} \|D_H T_\Omega(t)u\|_{L^1(\Omega,\nu;H)}. \tag{24}$$

Proof. (i) follows from the strong continuity of $T_\Omega(t)$ in $L^1(\Omega, \nu)$, see Proposition 3(i), and the lower semicontinuity of the norm (9), which imply

$$|D_\nu u|(\Omega) \leq \liminf_{t\to 0^+} \int_\Omega |D_H T_\Omega(t)u|_H \mathrm{d}\nu.$$

To prove (ii) we write the L^1-norm of the gradient of $T_\Omega(t)u$ by duality, as

$$\|D_H T_\Omega(t)u\|_{L^1(\Omega,\nu;H)} = \sup \left\{ \int_\Omega \langle D_H T_\Omega(t)u, F\rangle_H \mathrm{d}\nu \;\middle|\; \begin{array}{l} F \in \mathrm{Lip}_c(\Omega; H), \\ \|F\|_\infty \leq 1 \end{array} \right\}.$$

Taking into account that, for any $F \in \mathrm{Lip}_b(\Omega; H)$ we get

$$\begin{aligned} \int_\Omega \langle D_H T_\Omega(t)u, F\rangle_H \mathrm{d}\nu &= \int_\Omega u(D_H T_\Omega(t))^* F \mathrm{d}\nu \\ &\leq \int_\Omega u\Big(\mathbf{S}_1(t) D_H + S_2(t)\Big)^* F \mathrm{d}\nu = \int_\Omega u(D_H^* \mathbf{S}_1(t)^* F + S_2(t)^* F)\mathrm{d}\nu \\ &\leq |D_\nu u|(\Omega)\|\mathbf{S}_1(t)^* F\|_\infty + \|u\|_{L^2(\Omega,\nu)}\|S_2(t)^* F\|_{L^2(\Omega,\nu)} \\ &\leq (|D_\nu u|(\Omega) C_1(t) + C_2(t))\|F\|_\infty \end{aligned}$$

we deduce that

$$\|D_H T_\Omega(t)u\|_{L^1(\Omega,\nu;H)} \leq C_1(t)|D_\nu u|(\Omega) + C_2(t) \tag{25}$$

for any $t > 0$ where C_i $(i = 1, 2)$ are the positive functions in Corollary 1. Thus, taking the limsup as $t \to 0^+$ in (25) we get

$$\limsup_{t\to 0^+} \int_\Omega |D_H T_\Omega(t)u|_H \mathrm{d}\nu \le |D_\nu u|(\Omega)$$

and the proof is complete. □

It follows from Theorem 3 that functions in $BV(\Omega, \nu)$ may be approximated *in variation* by smooth functions. This result was already known in infinite dimension when $\Omega = X$ and $T_\Omega(t)$ is the Ornstein–Uhlenbeck semigroup and in a convex set, see [21], where the approximation is based on finite-dimensional reductions of the semigroup generated by the Neumann Ornstein–Uhlenbeck operator in Ω.

Proposition 6. *Under Hypotheses 1, 2, 3 and 4, for any $f \in BV(\Omega, \nu)$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq D^{1,2}(\Omega, \nu)$ such that*

$$\text{(i)} \lim_{n\to\infty} \|f_n - f\|_{L^2(\Omega,\nu)} = 0 \text{ and (ii)} \lim_{n\to\infty} \int_\Omega |D_H f_n|_H \mathrm{d}\nu = |D_\nu f|(\Omega). \tag{26}$$

If $C \subseteq \Omega$ is closed and $|D_\nu f|(\partial C) = 0$ then $|D_\nu f|(C) = \lim_{n\to\infty} \int_C |D_H f_n|_H \mathrm{d}\nu$.

Proof. Consider the semigroup $T_\Omega(t)$ generated in $L^2(\Omega, \nu)$ by the operator L_Ω defined in (4). It is known that for any $f \in L^2(\Omega, \nu)$ the function $T_\Omega(t)f$ belongs to $D^{1,2}(\Omega, \nu)$ for any $t > 0$ and by the strong continuity of $T_\Omega(t)$, $T_\Omega(t)f$ converges to f in $L^2(\Omega, \nu)$ as $t \to 0^+$. Moreover, Theorem 3 implies that $\|D_H T_\Omega(t)f\|_{L^1(\Omega,\nu;H)}$ converges to $|D_\nu f|(\Omega)$ as $t \to 0^+$. Thus, (26) is proved. To complete the proof let us observe that, by the lower semicontinuity of the total variation, for any open set $A \subseteq \Omega$

$$|D_\nu f|(A) \le \liminf_{n\to\infty} \int_A |D_H f_n|_H \mathrm{d}\nu \tag{27}$$

(see [21, Corollary 2.5]). Analogously we deduce that

$$|D_\nu f|(C) \ge \limsup_{n\to\infty} \int_C |D_H f_n|_H \mathrm{d}\nu \tag{28}$$

for any closed subset $C \subseteq \Omega$. Indeed, by (27) we obtain

$$\begin{aligned} |D_\nu f|(\Omega) - |D_\nu f|(C) &= |D_\nu f|(\Omega\setminus C) \le \liminf_{n\to\infty} \int_{\Omega\setminus C} |D_H f_n|_H \mathrm{d}\nu \\ &= \liminf_{n\to\infty} \left(\int_\Omega |D_H f_n|_H \mathrm{d}\nu - \int_C |D_H f_n|_H \mathrm{d}\nu \right) \\ &= \lim_{n\to\infty} \int_\Omega |D_H f_n|_H \mathrm{d}\nu - \limsup_{n\to\infty} \int_C |D_H f_n|_H \mathrm{d}\nu \end{aligned}$$

whence, using (26)(ii), estimate (28) follows. Now, using estimates (27), (28) and the fact that $|D_\nu f|(\partial C) = 0$ we obtain

$$|D_\nu f|(C) = |D_\nu f|(\mathring{C}) \le \liminf_{n\to\infty} \int_{\mathring{C}} |D_H f_n|_H \mathrm{d}\nu$$
$$\le \limsup_{n\to\infty} \int_{\mathring{C}} |D_H f_n|_H \mathrm{d}\nu \le |D_\nu f|(C), \tag{29}$$

where $\mathring{C}$ denotes the interior of C. Estimate (29) yields the claim. □

We conclude this section showing that estimate (8) and the previous approximation result allow to improve estimate (25) obtaining (30).

Theorem 4. *Under Hypotheses 1, 2, 3 and 4, if $f \in BV(\Omega, \nu)$ then*

$$\int_\Omega |D_H T_\Omega(t) f|_H \mathrm{d}\nu \le e^{-\lambda_1^{-1} t} |D_\nu f|(\Omega), \qquad t > 0, \tag{30}$$

λ_1 being the maximum eigenvalue of the covariance operator Q_∞, see (2). Moreover, for any open set $A \subset \Omega$ with $\overline{A} \subset \Omega$,

$$\lim_{t\to 0^+} \int_A |D_H T_\Omega(t) f|_H \mathrm{d}\nu = |D_\nu f|(A).$$

Proof. Let $f \in BV(\Omega, \nu)$ and let $(f_n)_{n\in\mathbb{N}} \in D^{1,2}(\Omega, \nu)$ be the sequence given by Proposition 6. By the contractivity of $T_\Omega(t)$ we deduce that $T_\Omega(t) f_n$ converges to $T_\Omega(t) f$ in $L^2(\Omega, \nu)$ as $n \to \infty$. This fact, together with the lower semicontinuity of the total variation, (8) and (6) yield

$$\int_\Omega |D_H T_\Omega(t) f|_H \mathrm{d}\nu \le \liminf_{n\to\infty} \int_\Omega |D_H T_\Omega(t) f_n|_H \mathrm{d}\nu$$
$$\le e^{-\lambda_1^{-1} t} \liminf_{n\to\infty} \int_\Omega T_\Omega(t) |D_H f_n|_H \mathrm{d}\nu \le e^{-\lambda_1^{-1} t} \lim_{n\to\infty} \int_\Omega |D_H f_n|_H \mathrm{d}\nu$$
$$= e^{-\lambda_1^{-1} t} |D_\nu f|(\Omega)$$

whence (30) is proved. The last assertion follows immediately from Proposition 6 taking into account that $T_\Omega(t) f$ satisfies (26). □

4. Sets of finite perimeter in Ω

This section is devoted to provide some sufficient and necessary conditions in order that a Borel set $E \subseteq X$ have finite perimeter in Ω. We consider also the case of $BV(\Omega, \nu)$ functions and $\Omega = X$. There are three semigroups involved: beside $T_\Omega(t)$, we consider the Ornstein–Uhlenbeck semigroup $S(t)$ generated in $L^2(X, \gamma)$ by the realisation of the operator

$$L_{OU}\varphi = \mathrm{Tr}(D_H^2 \varphi) - \sum_{i=1}^{\infty} \lambda_i^{-1} \langle x, e_i \rangle D_i \varphi \quad \varphi \in \mathcal{FC}_b^2(X)$$

and the semigroup $T(t)$ generated in $L^2(X,\nu)$ by the realisation of the operator

$$L\varphi = L_{OU}\varphi - \langle D_H U, D_H\varphi\rangle_H, \quad \varphi \in \mathcal{FC}_b^2(X). \tag{31}$$

Recall that $S(t)$ admits a pointwise representation by means of the Mehler formula (1).

Theorem 5. *Assume Hypotheses 1, 2, 3 and 4 hold true and let $E \subseteq X$ be a Borel set such that $P_\nu(E,\Omega) < \infty$. Then*

$$\limsup_{t\to 0^+} \frac{1}{\sqrt{t}} \|T_\Omega(t)\chi_E - \chi_E\|_{L^1(\Omega,\nu)} < \infty. \tag{32}$$

More precisely

$$\|T_\Omega(t)u - u\|_{L^1(\Omega,\nu)} \le 2\sqrt{K_2 t}\,|D_\nu u|(\Omega) \tag{33}$$

for any $u \in BV(\Omega,\nu)$ and $t > 0$ where K_2 is the constant in (7).

Proof. Clearly, once estimate (33) is proved, (32) follows at once choosing $u = \chi_E$. Thus, let us prove (33). To this aim, we consider $g \in L^\infty(\Omega,\nu)$ and assume first that $u \in D^{1,2}(\Omega,\nu)$. By the self-adjointness of the operators $L_\Omega T_\Omega(s)$ for $s \ge 0$ in $L^2(\Omega,\nu)$ we have

$$\begin{aligned}\int_\Omega g(T_\Omega(t)u - u)\mathrm{d}\nu &= \int_\Omega g\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} T_\Omega(s)u\mathrm{d}s\mathrm{d}\nu = \int_0^t\int_\Omega g(L_\Omega T_\Omega(s)u)\mathrm{d}\nu\mathrm{d}s\\ &= \int_0^t\int_\Omega (L_\Omega T_\Omega(s)g)u\mathrm{d}\nu\mathrm{d}s = -\int_0^t\int_\Omega \langle D_H T_\Omega(s)g, D_H u\rangle_H \mathrm{d}\nu\mathrm{d}s.\end{aligned}$$

The Cauchy–Schwarz inequality and (7) yield

$$\begin{aligned}\int_\Omega g(T_\Omega(t)u - u)\mathrm{d}\nu &\le \int_0^t\int_\Omega |D_H T_\Omega(s)g|_H |D_H u|_H \mathrm{d}\nu\mathrm{d}s\\ &= \int_0^t\int_\Omega (|D_H T_\Omega(s)g|_H^2)^{\frac12}|D_H u|_H\mathrm{d}\nu\mathrm{d}s\\ &\le \sqrt{K_2}\int_0^t s^{-\frac12}\int_\Omega (T_\Omega(s)|g|^2)^{\frac12}|D_H u|_H\mathrm{d}\nu\mathrm{d}s.\end{aligned}$$

From the contractivity of $T_\Omega(t)$ in $L^\infty(\Omega,\nu)$, for any $t > 0$ we deduce

$$\begin{aligned}\int_\Omega g(T_\Omega(t)u - u)\mathrm{d}\nu &\le \sqrt{K_2}\|g\|_{L^\infty(\Omega,\nu)}\int_0^t s^{-\frac12}\int_\Omega |D_H u|_H\mathrm{d}\nu\mathrm{d}s\\ &= 2\sqrt{K_2 t}\|g\|_{L^\infty(\Omega,\nu)}\int_\Omega |D_H u|_H\mathrm{d}\nu.\end{aligned} \tag{34}$$

For $u \in BV(\Omega,\nu)$, from Proposition 6 we get a sequence $u_n \in D^{1,2}(\Omega,\nu)$ converging to u in $L^2(\Omega,\nu)$ with $\lim_{n\to\infty}\int_\Omega |Du_n|_H\mathrm{d}\nu = |D_\nu u|(\Omega)$. Thus, putting u_n in place of u in (34) and letting $n \to \infty$ we get

$$\int_\Omega g(T_\Omega(t)u - u)\mathrm{d}\nu \le 2\sqrt{K_2 t}\|g\|_{L^\infty(\Omega,\nu)}|D_\nu u|(\Omega), \qquad g \in L^\infty(\Omega,\nu).$$

Finally, taking the supremum on the $g \in L^\infty(\Omega, \nu)$ with $\|g\|_\infty \le 1$ we obtain

$$\int_\Omega |T_\Omega(t)u - u| \mathrm{d}\nu \le 2\sqrt{K_2 t}\, |D_\nu u|(\Omega)$$

whence (33) follows. □

Remark 4. Note that condition (32) is equivalent to

$$\limsup_{t\to 0^+} \frac{1}{\sqrt{t}} \int_{E^c\cap\Omega} (T_\Omega(t)\chi_E)\mathrm{d}\nu < \infty.$$

Indeed, $|T_\Omega(t)\chi_E - \chi_E| = (\chi_E - T_\Omega(t)\chi_E)\chi_E + (T_\Omega(t)\chi_E - \chi_E)\chi_{E^c}$. The invariance of $T_\Omega(t)$ with respect to ν in Ω yields

$$\begin{aligned}\int_\Omega (\chi_E - T_\Omega(t)\chi_E)\chi_E \mathrm{d}\nu &= \int_\Omega (T_\Omega(t)\chi_E - \chi_E T_\Omega(t)\chi_E)\mathrm{d}\nu \\ &= \int_\Omega \chi_{E^c} T_\Omega(t)\chi_E \mathrm{d}\nu.\end{aligned}$$

Consequently,

$$\int_\Omega |T_\Omega(t)\chi_E - \chi_E| \mathrm{d}\nu = 2\int_{\Omega\cap E^c} T_\Omega(t)\chi_E \mathrm{d}\nu.$$

Now, we prove a quasi-converse of Theorem 5. We start with a preliminary result for bounded functions.

Proposition 7. *Under Hypotheses 1, 2, 3 and 4, let $u \in L^\infty(X, \nu)$ be such that*

$$\liminf_{t\to 0^+} \frac{1}{\sqrt{t}} \int_\Omega \int_X |u(e^{-t}x + \sqrt{1-e^{-2t}}y) - u(x)| \mathrm{d}\gamma(y)\mathrm{d}\nu(x) = C < \infty. \quad (35)$$

Then $u \in BV(\Omega, \nu)$ and $|D_\nu u|(\Omega) \le C\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}\sqrt{\pi}/2$.

Proof. We divide the proof into two steps.
Step 1. Here we prove that for any $v \in C_b^1(X)$, it holds that

$$\begin{aligned}&\lim_{t\to 0^+} \frac{1}{\sqrt{t}} \int_\Omega \int_X |v(e^{-t}x + \sqrt{1-e^{-2t}}y) - v(x)| \mathrm{d}\gamma(y)\mathrm{d}\nu(x) \\ &\quad = \frac{2}{\sqrt{\pi}} \int_\Omega |Dv(x)| \mathrm{d}\nu(x). \end{aligned} \quad (36)$$

To this aim, we observe that

$$\begin{aligned}K_v(t) &:= \int_\Omega \int_X |v(e^{-t}x + \sqrt{1-e^{-2t}}y) - v(x)| \mathrm{d}\gamma(y)\mathrm{d}\nu(x) \\ &= \int_\Omega \int_X \left| \int_0^t \frac{\mathrm{d}}{\mathrm{d}r} v(e^{-r}x + \sqrt{1-e^{-2r}}y)\mathrm{d}r \right| \mathrm{d}\gamma(y)\mathrm{d}\nu(x) \\ &= \int_\Omega \int_X \left| \int_0^t \langle Dv(e^{-r}x + \sqrt{1-e^{-2r}}y), -e^{-r}x + \frac{e^{-2r}}{\sqrt{1-e^{-2r}}}y\rangle \mathrm{d}r \right| \mathrm{d}\gamma(y)\mathrm{d}\nu(x)\end{aligned}$$

$$\leq \int_0^t \frac{e^{-r}}{\sqrt{1-e^{-2r}}} \cdot \\ \cdot \int_\Omega \int_X \left| \left\langle Dv(e^{-r}x + \sqrt{1-e^{-2r}}y), -\sqrt{1-e^{-2r}}x + e^{-r}y \right\rangle \right| \mathrm{d}\gamma(y)\mathrm{d}\nu(x).$$

Now, for r fixed we perform the "Gaussian rotation"

$$(x, y) \mapsto R_r(x, y) := (e^{-r}x + \sqrt{1-e^{-2r}}y, -\sqrt{1-e^{-2r}}x + e^{-r}y) =: (u, w)$$

to get, thanks to the invariance of γ under R_r,

$$K_v(t) \leq \int_0^t \frac{e^{-r}}{\sqrt{1-e^{-2r}}} \int_X \int_X |\langle Dv(u), w\rangle| \cdot \\ \cdot \chi_\Omega(e^{-r}u - \sqrt{1-e^{-2r}}w) e^{-U(e^{-r}u - \sqrt{1-e^{-2r}}w)} \mathrm{d}\gamma(u) d\gamma(w) \\ =: \int_X \int_X f_v(t, u, w) \mathrm{d}\gamma(w) d\gamma(u).$$

We claim that

$$\lim_{t\to 0^+} \frac{1}{\sqrt{t}} \int_X \int_X f_v(t, u, w) \mathrm{d}\gamma(w) d\gamma(u) = C \int_\Omega |Dv| \mathrm{d}\nu.$$

Indeed, by the convexity of U there exist $z \in X$ and $a \in \mathbb{R}$ such that $U(x) \geq \langle x, z\rangle + a$, hence

$$\frac{1}{\sqrt{t}} f_v(t, u, w) \leq \frac{1}{\sqrt{t}} |Dv(u)||w| e^{|\langle z,u\rangle| + |\langle z,w\rangle| + |a|} \int_0^t \frac{1}{\sqrt{2r}} \mathrm{d}r \\ = \sqrt{2}|Dv(u)||w| e^{|\langle z,u\rangle| + |\langle z,w\rangle| + |a|} \in L^1(X \times X, \gamma \otimes \gamma), \qquad t \in (0, 1)$$

and, using De L'Hôpital's rule, for almost every $(u, w) \in \Omega \times X$

$$\lim_{t\to 0^+} \frac{f_v(t, u, w)}{\sqrt{t}} = \sqrt{2}\chi_\Omega(u)\langle Dv(u), w\rangle e^{-U(u)}.$$

So by the dominated convergence theorem we obtain

$$\limsup_{t\to 0^+} \frac{1}{\sqrt{t}} \int_\Omega \int_X |v(e^{-t}x + \sqrt{1-e^{-2t}}y) - v(x)| \mathrm{d}\gamma(y)\mathrm{d}\nu(x) \\ \leq \frac{2}{\sqrt{\pi}} \int_\Omega |Dv(u)| \mathrm{d}\nu(u) \tag{37}$$

where we used that $\int_X |\langle Dv(u), w\rangle| \mathrm{d}\gamma(w) = \sqrt{2/\pi}|Dv(u)|$. Indeed, using the factorisation $\gamma = \gamma_1 \otimes \gamma^\perp$, where γ_1 is the 1-dimensional standard Gaussian measure on $E = \mathrm{span} Dv(u)$, we get

$$\int_X |\langle Dv(u), w\rangle| \mathrm{d}\gamma(w) = 2 \int_{\{w:\langle Dv(u),w\rangle > 0\}} \langle Dv(u), w\rangle \mathrm{d}\gamma(w)$$

$$= 2|Dv(u)| \int_{E^\perp} \int_0^\infty t \mathrm{d}\gamma_1(t) d\gamma(w') = \sqrt{2/\pi}|Dv(u)|.$$

To conclude, consider the family of linear functionals $L_t : C_b(X \times X) \to \mathbb{R}, t \in (0, 1)$

$$L_t\varphi = \frac{1}{\sqrt{t}} \int_\Omega \int_X \varphi(x, y)(v(e^{-t}x + \sqrt{1 - e^{-2t}}y) - v(x))\mathrm{d}\gamma(y)\mathrm{d}\nu(x).$$

By (37) we get $\limsup_{t\to 0^+} \|L_t\| \le 2(\sqrt{\pi})^{-1}\|Dv\|_{L^1(\Omega,\nu)}$ and arguing as above

$$\lim_{t\to 0^+} L_t\varphi = \sqrt{2} \int_\Omega \int_X \varphi(x, y)\langle Du(x), y\rangle \mathrm{d}\gamma(y)\mathrm{d}\nu(x) =: L_0\varphi.$$

So L_t weakly* converges to L_0 as $t \to 0^+$ and, by lower semicontinuity of the norm we get (36):

$$\begin{aligned}\|L_0\| &= \frac{2}{\sqrt{\pi}} \int_\Omega |Dv(x)|\mathrm{d}\nu(x) \le \liminf_{t\to 0^+} \|L_t\| \\ &\le \limsup_{t\to 0^+} \|L_t\| \le \frac{2}{\sqrt{\pi}} \int_\Omega |Dv(x)|\mathrm{d}\nu(x).\end{aligned}$$

Step 2. For $u \in L^\infty(X, \nu)$, let $(u_j)_{j\in\mathbb{N}} \subseteq C_b^1(X)$ be such that $u_j \to u$ in $L^2(X, \nu)$, almost everywhere in X and satisfying (35) (thanks to the dominated convergence theorem). Using (36), (6) and (8) we have

$$\begin{aligned}\lim_{t\to 0^+} \frac{K_{u_j}(t)}{\sqrt{t}} &= \frac{2}{\sqrt{\pi}} \int_\Omega |Du_j|\mathrm{d}\nu = \frac{2}{\sqrt{\pi}\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}} \int_\Omega \|Q_\infty^{1/2}\|_{\mathcal{L}(X)}|Du_j|\mathrm{d}\nu \\ &\ge \frac{2}{\sqrt{\pi}\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}} \int_\Omega |Q_\infty^{1/2}Du_j|\mathrm{d}\nu = \frac{2}{\sqrt{\pi}\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}} \int_\Omega |Q_\infty Du_j|_H\mathrm{d}\nu \\ &= \frac{2}{\sqrt{\pi}\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}} \int_\Omega |D_H u_j|_H\mathrm{d}\nu = \frac{2}{\sqrt{\pi}\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}} \int_\Omega (T_\Omega(\sigma)|D_H u_j|_H)\mathrm{d}\nu \\ &\ge \frac{2}{\sqrt{\pi}\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}} e^{\sigma\lambda_1^{-1}} \int_\Omega |D_H T_\Omega(\sigma)u_j|_H\mathrm{d}\nu \qquad (38)\end{aligned}$$

for any $\sigma \in (0, 1)$. Now, since the left-hand side of (38) is uniformly bounded from above by the constant C, the L^1-norm of $D_H T_\Omega(\sigma)u_j$ is bounded as well by the same constant for every $j \in \mathbb{N}$ and $\sigma \in (0, 1)$, i.e.

$$e^{\frac{\sigma}{\lambda_1}} \int_\Omega |DT_\Omega(\sigma)u_j|\mathrm{d}\nu \le C\frac{\sqrt{\pi}}{2}\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}, \qquad j \in \mathbb{N}, \sigma > 0.$$

Thus, recalling that $D_H T_\Omega(\sigma)u_j$ converges to $D_H T_\Omega(\sigma)u$ in $L^1(\Omega, \nu)$ as $j \to \infty$ (see (7)), letting first $j \to \infty$ and then $\sigma \to 0^+$ and using formula (24) we get that $|D_\nu u|(\Omega) \le C\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}\sqrt{\pi}/2$. □

The following result is a quasi-converse of Theorem 5. In fact, we give a sufficient condition to have $P_\nu(E, \Omega) < \infty$ in terms of the short-time behaviour of $T(t)$ and not of $T_\Omega(t)$, where $T(t)$ is the semigroup generated by the operator L defined in (31) in $L^2(X, \nu)$.

Theorem 6. *Under Hypotheses 1, 2, 3 and 4, if $E \in \mathcal{B}(X)$ and*

$$C := \liminf_{t\to 0^+} \frac{1}{\sqrt{t}} \|T(t)\chi_E - \chi_E\|_{L^1(\Omega,\nu)} < \infty, \tag{39}$$

then $P_\nu(E,\Omega) \le C\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}\sqrt{\pi}/2$.

Proof. Choosing $u = \chi_E$ in (35) and observing that

$$\int_\Omega \left| \int_X f(x,y)\mathrm{d}\gamma(y) \right| \mathrm{d}\nu(x) = \int_\Omega \int_X |f(x,y)|\mathrm{d}\gamma(y)\mathrm{d}\nu(x)$$

for any f with constant sign, from Proposition 7 we deduce that if

$$L := \liminf_{t\to 0^+} \frac{1}{\sqrt{t}} \|S(t)\chi_E - \chi_E\|_{L^1(\Omega,\nu)} < \infty \tag{40}$$

then $P_\nu(E,\Omega) \le L\|Q_\infty^{1/2}\|_{\mathcal{L}(X)}\sqrt{\pi}/2$. Here $S(t)$ is the Ornstein–Uhlenbeck semigroup in (1). To conclude we prove that condition (40) is equivalent to (39). From the variation-of-constants formula we deduce

$$(T(t)g)(x) = (S(t)g)(x) - \int_0^t (S(t-\sigma)\langle D_H U, D_H T(\sigma)g\rangle_H)(x)d\sigma, \tag{41}$$

for every $g \in \mathcal{FC}_b(X)$, ν-a.e. $x \in X$ and any $t \ge 0$. To prove (41) it suffices that the map $\sigma \mapsto S(t-\sigma)\langle D_H U, D_H T(\sigma)g\rangle_H$ belongs to $L^1((0,t))$ for any $t > 0$. To this aim, let us observe that

$$\int_X \int_0^t S(t-\sigma)\langle D_H U, D_H T(\sigma)g\rangle_H d\sigma \mathrm{d}\nu < \infty$$

for any $g \in \mathcal{FC}_b(X)$. Indeed, the Hölder inequality and the contractivity of $S(t)$ in $L^2(X,\gamma)$ allow us to write

$$\begin{aligned}
&\int_X \int_0^t S(t-\sigma)\langle D_H U, D_H T(\sigma)g\rangle_H d\sigma \mathrm{d}\nu \\
&\quad = \int_0^t \int_X S(t-\sigma)\langle D_H U, D_H T(\sigma)g\rangle_H \mathrm{d}\nu d\sigma \\
&\quad \le \int_0^t \|S(t-\sigma)\langle D_H U, D_H T(\sigma)g\rangle_H\|_{L^1(X,\nu)} d\sigma \\
&\quad \le \|e^{-U}\|_{L^2(X,\gamma)} \int_0^t \|S(t-\sigma)\langle D_H U, D_H T(\sigma)g\rangle_H\|_{L^2(X,\gamma)} d\sigma \\
&\quad \le \|e^{-U}\|_{L^2(X,\gamma)} \int_0^t \|\langle D_H U, D_H T(\sigma)g\rangle_H\|_{L^2(X,\gamma)} d\sigma \\
&\quad \le \sqrt{K_2}\|e^{-U}\|_{L^2(X,\gamma)}\|g\|_\infty \|D_H U\|_{L^2(X,\gamma;H)} \int_0^t \sigma^{-1/2}\mathrm{d}s \\
&\quad = 2\sqrt{K_2 t}\|e^{-U}\|_{L^2(X,\gamma)}\|g\|_\infty \|D_H U\|_{L^2(X,\gamma;H)},
\end{aligned} \tag{42}$$

where in the last line we used estimate (7) which holds true even in the case $\Omega = X$ and $T_\Omega(t)$ replaced by $T(t)$. Hence, formula (41) follows.

Now, integrating (41) in Ω with respect to ν yields

$$\begin{aligned} \|S(t)\chi_E - \chi_E\|_{L^1(\Omega,\nu)} - H(t) &\le \|T(t)\chi_E - \chi_E\|_{L^1(\Omega,\nu)} \\ &\le \|S(t)\chi_E - \chi_E\|_{L^1(\Omega,\nu)} + H(t) \end{aligned} \tag{43}$$

for any $t > 0$ with

$$H(t) := \left| \int_X \int_0^t S(t-s)\langle D_H U, D_H T(s)\chi_E\rangle_H \mathrm{d}s\mathrm{d}\nu \right|, \qquad t > 0.$$

Using estimate (42) with $g = \chi_E$ we infer that $\limsup_{t\to 0^+} \frac{H(t)}{\sqrt{t}} < \infty$. This last estimate, together with (43), prove that (40) is equivalent to (39) and the proof is complete. □

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REFERENCES

[1] D. Addona, G. Cappa, S. Ferrari, *On the domain of elliptic operators defined in subsets of Wiener spaces*, Infin. Dimens. Anal. Quantum Probab. Relat. Top., **23** (2020).

[2] D. Addona, G. Menegatti, M. Miranda Jr, *BV functions on open domains: the Wiener case and the Fomin differentiable case*, Commun. Pure Appl. Anal., **19** (2020), 2679–2711.

[3] L. Ambrosio, N. Fusco, D. Pallara, Functions of bounded variation and free discontinuity problems, OUP Oxford, (2000).

[4] L. Ambrosio, S. Maniglia, M. Miranda Jr, D. Pallara, BV functions in abstract Wiener spaces, J. Funct. Anal., **3** (2015), 212–230.

[5] L. Ambrosio, M. Miranda Jr, D. Pallara, Some fine properties of BV functions on Wiener spaces, Anal. Geom. Metr. Spaces, **3** (2015), 212-230.

[6] L. Angiuli, S. Ferrari, D. Pallara, *Gradient estimates for perturbed Ornstein–Uhlenbeck semigroups on infinite-dimensional convex domains*, J. Evol. Equ., **19** (2019), 677–715.

[7] V. I. Bogachev, Differentiable measures and the Malliavin calculus, Mathematical Surveys and Monographs, vol. 164, American Mathematical Society, Providence, RI, 2010.

[8] V. I. Bogachev, Gaussian measures, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998.

[9] G. Cappa, S. Ferrari, *Maximal Sobolev regularity for solutions of elliptic equations in infinite dimensional Banach spaces endowed with a weighted Gaussian measure*, J. Differential Equations, **261** (2016), 7099–7131.

[10] G. Cappa, S. Ferrari, *Maximal Sobolev regularity for solutions of elliptic equations in Banach spaces endowed with a weighted Gaussian measure: the convex subset case*, J. Math. Anal. Appl., **458** (2018), 300–331.

[11] E. De Giorgi, Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni, Ann. Mat. Pura Appl. **36** (1954), 191-213, and also *Ennio De Giorgi: Selected Papers*, (L. Ambrosio, G. Dal Maso, M. Forti, M. Miranda, S. Spagnolo eds.), Springer, 2006, 79-99. English translation, *Ibid.*, 58-78.

[12] G. Da Prato, A. Lunardi, *BV functions in Hilbert spaces*. Math. Ann. to appear.

[13] G. Da Prato, J. Zabczyk, Second order partial differential equations in Hilbert spaces, London Mathematical Society Lecture Note Series, vol. 293, Cambridge University Press, Cambridge, 2002.

[14] K.-J. Engel, R. Nagel, A short course on operator semigroups, Universitext, New York, 2006.

[15] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, Banach space theory, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011.

[16] S. Ferrari, *Sobolev spaces with respect to a weighted Gaussian measure in infinite dimensions*, Infin. Dimens. Anal. Quantum Probab. Relat. Top., **22** (2019).

[17] M. Fukushima, *BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space*, *J. Funct. Anal.* **174** (2000), 227-249.

[18] M. Fukushima, M. Hino, *On the space of BV functions and a related stochastic calculus in infinite dimensions*, J. Funct. Anal., **183** (2001), 245-268.

[19] M. Hino, H. Hucida, *Reflecting Ornstein-Uhlenbeck processes on pinned path spaces*, Proceedings of RIMS Workshop on Stochastic Analysis and Applications, (2008), 111-128.

[20] M. Ledoux, *Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space*, Bull. Sci. Math., **118** (1994), 485-510.

[21] A. Lunardi, M. Miranda Jr, D. Pallara, BV functions on convex domains in Wiener spaces, Potential Anal., **43** (2015), 23–48.

[22] R. O'Donnell, Analysis of Boolean functions, Cambridge University Press, 2014.

[23] M. Röckner, R.C. Zhu, X.C. Zhu, *The stochastic reflection problem on an infnite dimensional convex set and BV functions in a Gelfand triple*, Ann. Probab., **40** (2012), 1759-1794.

[24] D. Trevisan, BV-regularity for the Malliavin derivative of the maximum of the Wiener process, Electron. Commun. Probab., **18** (2013).

[25] N. N. Vakhania, V. I. Tarieladze, S. A. Chobanyan, Probability distributions on Banach spaces, Mathematics and its applications, vol. 14, Springer Netherlands, D. Reidel Publishing Company, Dordrecht, Holland, 1987.

[26] L. Zambotti, *Integration by parts formulae on convex sets of paths and applications to SPDEs with reflection*, Probab. Theory Related Fields, **123** (2002), 579–600.

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