Superexponential stabilizability of evolution equations of parabolic type via bilinear control

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Abstract. We study the stabilizability of a class of abstract parabolic equations of the form

$$u'(t) + Au(t) + p(t)Bu(t) = 0, \quad t \geq 0$$

where the control $p(\cdot)$ is a scalar function, A is a self-adjoint operator on a Hilbert space X that satisfies $A \geq -\sigma I$, with $\sigma > 0$, and B is a bounded linear operator on X . Denoting by $\{\lambda_k\}_{k \in \mathbb{N}^*}$ and $\{\varphi_k\}_{k \in \mathbb{N}^*}$ the eigenvalues and the eigenfunctions of A , we show that the above system is locally stabilizable to the eigensolutions $\psi_j = e^{-\lambda_j t} \varphi_j$ with doubly exponential rate of convergence, provided that the associated linearized system is null controllable. Moreover, we give sufficient conditions for the pair $\{A, B\}$ to satisfy such a property, namely a gap condition for A and a rank condition for B in the direction φ_j . We give several applications of our result to different kinds of parabolic equations.

1. Introduction

In the field of control theory of dynamical systems, a huge amount of works is devoted to the study of models in which the control enters as an additive term (boundary or internal locally distributed controls); see, for instance, the books [22, 23] by Lions. On the other hand, these kinds of control systems are not suitable to describe processes that change their physical characteristics in response to the control. This issue is quite common for the so-called *smart materials* and in many biomedical, chemical and nuclear chain reactions. Indeed, under the process of *catalysis* some materials are able to change their principal parameters (see the examples in the monograph [20] by Khapalov for more details.)

To deal with these situations, an important role is played by multiplicative controls, that is, controls that appear in the equations as coefficients.

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Due to a weaker control action, exact controllability results are not to be expected with multiplicative controls. Nevertheless, approximate controllability has been obtained for different types of initial/target conditions. For instance, in [18] Khapalov proved a result of nonnegative approximate controllability for a 1D semilinear parabolic equation. In [19], the same author proved approximate and exact null controllability for a bilinear parabolic system with reaction term satisfying Newton's law. Paper [16], by Floridia, is devoted to the study of global approximate multiplicative controllability for nonlinear degenerate parabolic problems. In [8, 10], results of approximate controllability of a one-dimensional reaction–diffusion equation via multiplicative control and with sign changing data are proved. Finally, we recall that some results of approximate controllability have been also obtained for hyperbolic equations with bilinear control (see, for instance, [13]).

An even more specific and weaker class of controls are bilinear controls which enter the equation as scalar functions of time as, for instance, in the following system:

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t > 0 \\ u(0) = u_0. \end{cases} \quad (1)$$

Here, $A : D(A) \subset X \rightarrow X$ is a linear operator on a Hilbert space $(X, \langle \cdot, \cdot \rangle)$ and B is a bounded linear operator on X .

A fundamental result in control theory for this type of evolution equations is the one due to Ball et al. [3] which establishes that under the assumption that A is the generator of a C^0 -semigroup of bounded linear operators on X , system (1) fails to be controllable. Indeed, denoting by $u(t; p, u_0)$ the unique solution of (1), it is shown in [3] that the attainable set from u_0 , defined by,

$$S(u_0) = \{u(t; p, u_0); t \geq 0, p \in L^r_{\text{loc}}([0, +\infty), \mathbb{R}), r > 1\}$$

has a dense complement.

On the other hand, when B is unbounded, the possibility of proving a positive controllability result remains open. This idea of exploiting the unboundedness of B was used by Beauchard and Laurent in [5] to study the Schrödinger equation:

$$\begin{cases} iu_t(t, x) + u_{xx}(t, x) + p(t)\mu(x)u(t, x) = 0, & (t, x) \in (0, T) \times (0, 1) \\ u(t, 0) = u(t, 1) = 0. \end{cases} \quad (2)$$

For such an equation, the authors proved local exact controllability around the ground state in a stronger topology than the natural one of $X = H^2 \cap H^1_0(0, 1)$, for which the multiplication operator $Bu(t, x) = \mu(x)u(t, x)$ is unbounded. In other terms, the above result could be regarded as a description of the attainable set from the ground state as a certain submanifold of the original Banach space.

Following the same strategy, Beauchard [4] studied the controllability of the wave equation:

$$\begin{cases} u_{tt}(t, x) - u_{xx}(t, x) - p(t)\mu(x)u(t, x) = 0, & (t, x) \in (0, T) \times (0, 1) \\ u_x(t, 0) = u_x(t, 1) = 0. \end{cases}$$

In both papers [4,5], a key point of the analysis is the application of the inverse mapping theorem, which is made possible by the controllability of the associated linearized problem. This is the reason why, for parabolic equations, the above strategy meets an obstruction: The spaces for which one can prove controllability of the linearized equation are not well adapted to the use of the inverse map technique.

It is important to stress the fact that the negative result by Ball, Marsden and Slemrod still allows for exact controllability to a trajectory $\bar{u}(\cdot; \bar{p}, \bar{u}_0)$ by a bilinear control. With respect to controllability to trajectories with additive controls, we mention the result in [15] by Fernández-Cara, Guerrero, Imanuvilov and Puel for the Navier–Stokes equations.

In this paper, we are interested in studying the possibility of steering the solution of (1), with a bilinear control, to any eigensolution of the operator A . In order to explain what an eigensolution is, let X be a separable Hilbert space, $A : D(A) \subset X \rightarrow X$ be a self-adjoint operator with compact resolvent, satisfying $A \geq -\sigma I$, with $\sigma \geq 0$ (see Sect. 2 for more on notation and assumptions) and let $\{\lambda_k\}_{k \in \mathbb{N}^*}$ be the eigenvalues of A , $(\lambda_k \leq \lambda_{k+1}, \forall k \in \mathbb{N}^*)$, with associated eigenfunctions $\{\varphi_k\}_{k \in \mathbb{N}^*}$. For every $j \geq 1$, the function $\psi_j(t) = e^{-\lambda_j t} \varphi_j$ will be called the j th *eigensolution* of A .

Our approach is based on the null controllability in some time $T > 0$ of the linearized system:

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_j = 0, \\ y(0) = y_0, \end{cases} \quad (3)$$

a property that we call j -null controllability of the pair $\{A, B\}$.

Indeed, in Theorem 3.7 we show that, if $\{A, B\}$ is j -null controllable, then the nonlinear system (1) is locally superexponentially stabilizable to the eigensolution ψ_j of A . The proof relies on a “quadratic estimate” for the difference between the solution of (1) and its linearization along ψ_j (Proposition 4.4).

Finally, we investigate the notion of j -null controllability, providing sufficient conditions for such a property. In Theorem 3.8, we prove that $\{A, B\}$ is j -null controllable if the gap condition

$$\sqrt{\lambda_{k+1} - \lambda_1} - \sqrt{\lambda_k - \lambda_1} \geq \alpha, \quad \forall k \in \mathbb{N}^*, \quad (4)$$

holds true for some $\alpha > 0$, and B satisfies

$$\begin{aligned} \text{(i)} \quad & \langle B\varphi_j, \varphi_k \rangle \neq 0, \quad \forall k \in \mathbb{N}^*, \\ \text{(ii)} \quad & \exists \tau > 0 : \sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_j, \varphi_k \rangle|^2} < +\infty. \end{aligned} \quad (5)$$

We recall that condition (4), which is required to use the moment method (see, e.g., [4,5]), is usually satisfied by elliptic operators in one space dimension. As far as (5) is concerned, we observe that (3) fails to be null controllable if (5) (i) is violated (see Remark 3.9). On the other hand, for specific problems like the Schrödinger equation

one can allow for (5) (i) to be violated for a finite number of eigenstates assuming additional conditions (see [6]).

The above stabilizability result can be used to study several classes of parabolic problems, for which checking the validity of the assumptions on A and B is usually straightforward. For instance, we can treat the heat equation with a controlled source term of the form

$$u_t - u_{xx} + p(t)\mu(x)u = 0$$

with Dirichlet or Neumann boundary conditions, as well as operators with variable coefficients

$$u_t - ((1+x)^2 u_x)_x + p(t)\mu(x)u = 0,$$

or even 3D problems with radial data symmetry such as

$$u_t - \Delta u + p(t)\mu(|x|)u = 0.$$

The above examples are restricted to one space dimension because of the gap condition in Theorem 3.10. On the other hand, the control we construct is just a scalar function of time. As such, its influence on the system is necessarily limited. In order to treat problems in higher dimension, one should probably use bilinear controls depending on both time and space (see, e.g., [9, 17]).

The results of this paper have been applied to degenerate parabolic equations in [12]. Moreover, our method can be adapted to obtain exact controllability to eigensolutions (see [1]) under slightly more restrictive assumptions, as well as to treat equations with an unbounded coefficient B such as the Fokker–Planck equations (see [2]).

The outline of the paper is the following. In Sect. 2, we introduce the notation and the preliminary assumptions on the data. Section 3 is devoted to our main results. In Sects. 4 and 5, we show the proofs of Theorems 3.7 and 3.8. Finally, in Sect. 6 we give applications to several examples of parabolic problems.

2. Preliminaries

Let $(X, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space. We denote by $\|\cdot\|$ the associated norm on X and by $B_R(\varphi)$ the open ball of radius $R > 0$, centered in $\varphi \in X$.

Let $A : D(A) \subset X \rightarrow X$ be a densely defined, linear operator with the following properties:

- (a) A is self-adjoint,
- (b) $\exists \sigma > 0 : \langle Ax, x \rangle \geq -\sigma \|x\|^2, \forall x \in D(A),$ (6)
- (c) $\exists \lambda > -\sigma$ such that $(\lambda I + A)^{-1} : X \rightarrow X$ is compact.

We recall that under the above assumptions A is a closed operator and $D(A)$ is itself a Hilbert space with the scalar product

$$(x|y)_{D(A)} = \langle x, y \rangle + \langle Ax, Ay \rangle, \quad \forall x, y \in D(A).$$

Moreover, $-A$ is the infinitesimal generator of a strongly continuous semigroup on X which will be denoted by e^{-tA} .

Let $\mathbb{N}^*$ denote the real numbers greater or equal than 1. In view of the above assumptions, there exists an orthonormal basis $\{\varphi_k\}_{k \in \mathbb{N}^*}$ in X of eigenfunctions of A , that is, $\varphi_k \in D(A)$, $\|\varphi_k\| = 1$ and $A\varphi_k = \lambda_k \varphi_k$, $\forall k \in \mathbb{N}^*$, where $\{\lambda_k\}_{k \in \mathbb{N}^*} \subset \mathbb{R}$ denote the corresponding eigenvalues. We recall that $\lambda_k \geq -\sigma$, $\forall k \in \mathbb{N}^*$, and we suppose—without loss of generality—that $\{\lambda_k\}_{k \in \mathbb{N}^*}$ is ordered so that $-\sigma \leq \lambda_k \leq \lambda_{k+1} \rightarrow \infty$ as $k \rightarrow \infty$.

The associated semigroup has the following representation:

$$e^{-tA}\varphi = \sum_{k=1}^{\infty} \langle \varphi, \varphi_k \rangle e^{-\lambda_k t} \varphi_k, \quad \forall \varphi \in X. \quad (7)$$

Let $T > 0$ and consider the problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases} \quad (8)$$

where $u_0 \in X$ and $f \in L^2(0, T; X)$. We now recall two definitions of solution of problem (8) (see, e.g., [7, p. 129]):

- the function $u \in C([0, T], X)$ defined by

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s) ds \quad (9)$$

is called the *mild solution* of (8),

- u is a *strong solution* of (8) in $L^2(0, T; X)$ if there exists a sequence $\{u_k\} \subseteq H^1(0, T; X) \cap L^2(0, T; D(A))$ such that

$$\begin{aligned} u_k &\rightarrow u, \text{ and } u'_k - Au_k \rightarrow f \text{ in } L^2(0, T; X), \\ u_k(0) &\rightarrow u_0 \text{ in } X, \text{ as } k \rightarrow \infty. \end{aligned}$$

The well-posedness of the Cauchy problem (8) is a classical result (see, for instance, [7, Proposition 3.1, p. 130]).

Theorem 2.1. *Let $u_0 \in X$ and $f \in L^2(0, T; X)$. Under hypothesis (6), problem (8) has a unique strong solution in $L^2(0, T; X)$. Moreover, u belongs to $C([0, T]; X)$ and coincides with the mild solution (9).*

Furthermore, there exists a constant $C_0(T) > 0$ such that

$$\sup_{t \in [0, T]} \|u(t)\| \leq C_0(T) (\|u_0\| + \|f\|_{L^2(0, T; X)}) \quad (10)$$

and $C_0(T)$ is nondecreasing with respect to T .

3. Main result

We consider the bilinear control problem

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t > 0 \\ u(0) = u_0 \end{cases} \quad (11)$$

where u is the state variable and $p \in L^2_{\text{loc}}([0, +\infty))$ is the control function.

We recall that, in general, the exact controllability problem for system (11) has a negative answer when B is a bounded linear operator, as shown by Ball, Marsden and Slemrod in [3].

We are interested in studying the stabilizability of system (11) to a fixed trajectory. Given an initial condition $u_0 \in X$ and a control $p \in L^2_{\text{loc}}([0, +\infty))$, we denote by $u(\cdot; u_0, p) : [0, +\infty) \rightarrow X$ the corresponding solution of (11).

Definition 3.1. Given $\bar{u}_0 \in X$ and $\bar{p} \in L^2_{\text{loc}}([0, +\infty))$, we say that the control system (11) is *locally stabilizable to $\bar{u}(\cdot; \bar{u}_0, \bar{p})$* if there exists $\delta > 0$ such that, for every $u_0 \in B_\delta(\bar{u}_0)$, there exists a control $p \in L^2_{\text{loc}}([0, +\infty))$ such that

$$\lim_{t \rightarrow +\infty} \|u(t; u_0, p) - \bar{u}(t; \bar{u}_0, \bar{p})\| = 0.$$

Definition 3.2. Given $\bar{u}_0 \in X$ and $\bar{p} \in L^2_{\text{loc}}([0, +\infty))$, we say that the control system (11) is *locally exponentially stabilizable to $\bar{u}(\cdot; \bar{u}_0, \bar{p})$* if for any $\rho > 0$, there exists $R(\rho) > 0$ such that, for every $u_0 \in B_{R(\rho)}(\bar{u}_0)$, there exists a control $p \in L^2_{\text{loc}}([0, +\infty))$ such that

$$\|u(t; u_0, p) - \bar{u}(t; \bar{u}_0, \bar{p})\| \leq Me^{-\rho t}, \quad \forall t > 0,$$

where $M > 0$ is a constant that depends only on A and B .

Definition 3.3. Given $\bar{u}_0 \in X$ and $\bar{p} \in L^2_{\text{loc}}([0, +\infty))$, we say that the control system (11) is *locally superexponentially stabilizable to $\bar{u}(\cdot; \bar{u}_0, \bar{p})$* if for any $\rho > 0$ there exists $R(\rho) > 0$ such that, for every $u_0 \in B_{R(\rho)}(\bar{u}_0)$, there exists a control $p \in L^2_{\text{loc}}([0, +\infty))$ such that

$$\|u(t; u_0, p) - \bar{u}(t; \bar{u}_0, \bar{p})\| \leq Me^{-\rho e^{\omega t}}, \quad \forall t > 0,$$

where $M, \omega > 0$ are suitable constants depending only on A and B .

For any $j \in \mathbb{N}^*$, we recall that we set $\psi_j(t) = e^{-\lambda_j t} \varphi_j$ and we call ψ_j the j th *eigensolution* of A . Observe that ψ_j solves (11) with $p = 0$ and $u_0 = \varphi_j$. We shall study the superexponential stabilizability of (11) to ψ_j .

We observe that, at least for strictly accretive operators A , system (11) is trivially locally exponentially stabilizable to any trajectory ψ_j by choosing the control $p = 0$.

The novelty of our work is the construction of a control function p that steers $u(t)$ arbitrary close to $\psi_j(t)$ at superexponential speed of decay.

Let $B : X \rightarrow X$ be a bounded linear operator. From now on, we denote by C_B the norm of B , that is,

$$C_B = \sup_{\varphi \in X, \|\varphi\|=1} \|B\varphi\| \quad (12)$$

and, without loss of generality, we suppose $C_B \geq 1$.

Let $A : D(A) \subset X \rightarrow X$ be a densely defined linear operator such that (6) hold.

Definition 3.4. Let $T > 0$. The pair $\{A, B\}$ is called *j-null controllable in time T* if there exists a constant $N(T) > 0$ such that for every $y_0 \in X$ one can find a control $p \in L^2(0, T)$ satisfying

$$\|p\|_{L^2(0,T)} \leq N(T)\|y_0\|, \quad (13)$$

and for which $y(T) = 0$, where $y(\cdot)$ is the solution of

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_j = 0, & t \in [0, T] \\ y(0) = y_0. \end{cases} \quad (14)$$

$N(T) > 0$ is called the *control cost*.

Definition 3.5. The pair $\{A, B\}$ is called *j-null controllable* if there exists $T_0 > 0$ such that $\{A, B\}$ is *j-null controllable in time T_0* .

Remark 3.6. Observe that if $\{A, B\}$ is *j-null controllable in time T_0* , it is *j-null controllable in any time $T > T_0$* .

We can now state our main result.

Theorem 3.7. Let $\{A, B\}$ be a *j-null controllable pair*. Then, system (11) is *superexponentially stabilizable to ψ_j* , for any $j \geq 1$.

In the following result, we give sufficient conditions for *j-null controllability*.

Theorem 3.8. Let $A : D(A) \subset X \rightarrow X$ be a densely defined linear operator that satisfies (6) and such that there exists a constant $\alpha > 0$ for which its eigenvalues fulfill the gap condition

$$\sqrt{\lambda_{k+1} - \lambda_1} - \sqrt{\lambda_k - \lambda_1} \geq \alpha, \quad \forall k \in \mathbb{N}^*. \quad (15)$$

Let $B : X \rightarrow X$ be a bounded linear operator such that

$$\begin{aligned} \text{(i)} \quad & \langle B\varphi_j, \varphi_k \rangle \neq 0, \quad \forall k \in \mathbb{N}^*, \\ \text{(ii)} \quad & \exists \tau > 0 : \sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_j, \varphi_k \rangle|^2} < +\infty. \end{aligned} \quad (16)$$

Then, the pair $\{A, B\}$ is *j-null controllable*.

Remark 3.9. We observe that assumption (16) (i) is necessary for the null controllability of the linear system (14). Indeed, fixed any $T > 0$ and $y_0 \in X$, suppose that there exists $p \in L^2(0, T)$ such that $y(T) = 0$. Taking the Fourier expansion of the solution of (14), thanks to (7) and (9), the previous identity is equivalent to

$$\sum_{k \in \mathbb{N}^*} \langle y_0, \varphi_k \rangle e^{-\lambda_k T} \varphi_k = \int_0^T p(s) \sum_{k \in \mathbb{N}^*} \langle B\varphi_j, \varphi_k \rangle e^{-\lambda_k(T-s)} \varphi_k ds.$$

Since the family $\{\varphi_k\}_{k \in \mathbb{N}^*}$ forms an orthonormal basis of X , it holds that

$$\langle y_0, \varphi_k \rangle = \int_0^T e^{\lambda_k s} p(s) \langle B\varphi_j, \varphi_k \rangle ds, \quad \forall k \in \mathbb{N}^*.$$

In particular, if (16) (i) is violated, there exists some $\bar{k} \in \mathbb{N}^*$ such that $\langle B\varphi_j, \varphi_{\bar{k}} \rangle = 0$. Hence, in the $\bar{k}$ th direction we have that

$$\langle y_0, \varphi_{\bar{k}} \rangle = \int_0^T e^{\lambda_{\bar{k}} s} p(s) \langle B\varphi_j, \varphi_{\bar{k}} \rangle ds,$$

which yields to $y_0 \in \varphi_{\bar{k}}^\perp$.

From Theorems 3.7 and 3.8, we easily deduce the following result:

Corollary 3.10. *Let $A : D(A) \subset X \rightarrow X$ be a densely defined linear operator that satisfies (6) and (15). Let $B : X \rightarrow X$ be a bounded linear operator such that (16) holds.*

Then, problem (11) is locally superexponentially stabilizable to any eigensolution ψ_j , $j \geq 1$.

4. Proof of Theorem 3.7

The proof of Theorem 3.7 will be built through a series of propositions. The first result is the well-posedness of the problem

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) + f(t) = 0, & t \in [0, T] \\ u(0) = u_0. \end{cases} \quad (17)$$

We introduce the following notation:

$$\begin{aligned} \|f\|_{2,0} &:= \|f\|_{L^2(0,T;X)}, & \forall f \in L^2(0,T;X), \\ \|f\|_{\infty,0} &:= \|f\|_{C([0,T];X)} = \sup_{t \in [0,T]} \|f(t)\|, & \forall f \in C([0,T];X). \end{aligned}$$

4.1. Well-posedness

Proposition 4.1. *Let $T > 0$. If $u_0 \in X$, $p \in L^2(0, T)$ and $f \in L^2(0, T; X)$, then there exists a unique mild solution of (17), i.e., a function $u \in C([0, T]; X)$ such that*

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}[p(s)Bu(s) + f(s)]ds, \quad \forall t \in [0, T]. \quad (18)$$

Moreover, there exists a constant $C_1(T) > 0$ such that

$$\|u\|_{\infty,0} \leq C_1(T)(\|u_0\| + \|f\|_{2,0}). \quad (19)$$

Hereafter, we denote by C a generic positive constant which may differ from line to line even if the symbol remains the same. Constants which play a specific role will be distinguished by an index, i.e., $C_0, C_B, \dots$

The proof of the existence of the mild solution of (17) is given in [3]. For what concerns the bound for the solution u of (17), it turns out that, by applying estimate (10) of Theorem 2.1 to (17), if $C_0(T)C_B\|p\|_{L^2(0,T)} \leq 1/2$, then we have inequality (19) with $C_1 = C_2$ defined by

$$C_2 := 2C_0(T), \quad (20)$$

where $C_0(T)$ is introduced in (10). Otherwise, to obtain (19), we proceed by subdividing the interval $[0, T]$ into smaller subintervals for which $C_0(T)C_B\|p\|_{L^2} \leq 1/2$ in all of them. In this case, the constant C_1 in inequality (19) is defined by

$$C_1 = (1 + m)(2C_0(T/m))^m, \quad (21)$$

where m is the number of subintervals.

To prove Theorem 3.7, we first start by assuming $\{A, B\}$ to be j -null controllable and such that the j th eigenvalue of A is zero, that is $\lambda_j = 0$, and we prove the local superexponential stabilizability of (11) to the trajectory φ_j . Then, we will recover the case $\lambda_j \neq 0$ and the stabilization to any trajectory ψ_j from the previous one.

4.2. Uniform estimate for the solution of the nonlinear system

Consider the system

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\ u(0) = u_0, \end{cases} \quad (22)$$

and the trajectory φ_j that is a solution of (22) when $p = 0$, $u_0 = \varphi_j$ and $\lambda_j = 0$. Setting $v := u - \varphi_j$, we observe that v is the solution of the following Cauchy problem:

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_j = 0, & t \in [0, T] \\ v(0) = v_0 := u_0 - \varphi_j. \end{cases} \quad (23)$$

Remark 4.2. Applying Theorem 2.1, we find that $v \in C([0, T]; X)$ is a mild solution of (23), that is

$$v(t) = e^{-tA}v_0 - \int_0^t p(s)e^{-(t-s)A}B(v(s) + \varphi_j)ds = V_0(t) + V_1(t), \quad (24)$$

where

$$\begin{aligned} V_0(t) &:= e^{-tA}v_0, \\ V_1(t) &:= - \int_0^t p(s)e^{-(t-s)A}B(v(s) + \varphi_j)ds. \end{aligned}$$

Since $p(\cdot)B(v(\cdot) + \varphi_j) \in L^2(0, T; X)$, recalling Theorem 2.1, we have that $V_1 \in H^1(0, T; X) \cap L^2(0, T; D(A))$, while $V_0 \in C^1((0, T]; X) \cap C((0, T]; D(A))$. Therefore, for every $\varepsilon \in (0, T)$, $v \in H^1(\varepsilon, T; X) \cap L^2(\varepsilon, T; D(A))$ and for almost every $t \in [\varepsilon, T]$ the following equality holds:

$$v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_j = 0. \quad (25)$$

Showing the stabilizability of the solution u of (22) to the trajectory φ_j is equivalent to proving the stabilizability to 0 of system (23).

Proposition 4.3. *Let $\{A, B\}$ be j -null controllable in time T_0 and assume $\lambda_j = 0$. Let $p_0 \in L^2(0, T_0)$ be a control satisfying (13) such that $\bar{v}(T_0) = 0$, where $\bar{v}(\cdot)$ is the solution of the following linear system:*

$$\begin{cases} \bar{v}(t)' + A\bar{v}(t) + p_0(t)B\varphi_j = 0, & t \in [0, T_0] \\ \bar{v}(0) = v_0. \end{cases} \quad (26)$$

Then, the solution v of (23) with $p = p_0$ and $T = T_0$ satisfies

$$\sup_{t \in [0, T_0]} \|v(t)\|^2 \leq e^{(2\sigma + C_B)T_0 + 2C_B N(T_0)\sqrt{T_0}\|v_0\|} (1 + C_B N^2(T_0)) \|v_0\|^2, \quad (27)$$

where $C_B = \|B\| \geq 1$, σ is defined in (6) and $N(T_0)$ is the control cost in (13).

Proof. Thanks to Remark 4.2, taking the scalar product of (25) with v , we obtain

$$\langle v'(t), v(t) \rangle + \langle Av(t), v(t) \rangle + p_0(t) \langle Bv(t) + B\varphi_j, v(t) \rangle = 0, \quad \text{for a.e. } t \in [\varepsilon, T_0]. \quad (28)$$

Thus, we get that for a.e. $t \in [\varepsilon, T_0]$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \langle Av(t), v(t) \rangle &\leq C_B \left(|p_0(t)| \|v(t)\|^2 + |p_0(t)| \|\varphi_j\| \|v(t)\| \right) \\ &\leq C_B \left(|p_0(t)| \|v(t)\|^2 + \frac{1}{2} |p_0(t)|^2 + \frac{1}{2} \|v(t)\|^2 \right). \end{aligned} \quad (29)$$

Therefore, since A satisfies (6), we have that

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|^2 \leq \left(\sigma + C_B |p_0(t)| + \frac{C_B}{2} \right) \|v(t)\|^2 + \frac{1}{2} C_B |p_0(t)|^2, \quad \text{for a.e. } t \in [\varepsilon, T_0].$$

We now integrate the last inequality from ε to t to obtain

$$\begin{aligned} & \int_{\varepsilon}^t \frac{d}{ds} \|v(s)\|^2 ds \\ & \leq \int_{\varepsilon}^t (2\sigma + C_B (2|p_0(s)| + 1)) \|v(s)\|^2 ds + C_B \int_0^{T_0} |p_0(s)|^2 ds, \quad \text{for a.e. } t \in [\varepsilon, T_0], \end{aligned}$$

and, by Gronwall's inequality, we conclude that

$$\|v(t)\|^2 \leq \left(\|v(\varepsilon)\|^2 + C_B \int_0^{T_0} |p_0(s)|^2 ds \right) e^{\int_{\varepsilon}^t (2\sigma + C_B (2|p_0(s)| + 1)) ds}, \quad \text{for a.e. } t \in [\varepsilon, T_0].$$

Taking the limit as $\varepsilon \rightarrow 0$, we find

$$\|v(t)\|^2 \leq \left(\|v_0\|^2 + C_B \int_0^{T_0} |p_0(s)|^2 ds \right) e^{\int_0^t (2\sigma + C_B (2|p_0(s)| + 1)) ds}, \quad \text{for a.e. } t \in [0, T_0].$$

Thus, taking the supremum over the interval $[0, T_0]$, the last inequality becomes

$$\sup_{t \in [0, T_0]} \|v(t)\|^2 \leq e^{2C_B \sqrt{T_0} \|p_0\|_{L^2(0, T_0)} + (2\sigma + C_B) T_0} \left(\|v_0\|^2 + C_B \|p_0\|_{L^2(0, T_0)}^2 \right).$$

Finally, recalling estimate (13) for p_0 , we get (27). $\square$

4.3. Quadratic estimate for the solution of the nonlinear system

Let $p_0 \in L^2(0, T_0)$ be as in Proposition 4.3. We introduce the function $w(t) := v(t) - \bar{v}(t)$, where v is the solution of (23) with $p = p_0$ and $T = T_0$ and $\bar{v}$ is the solution of (26). Then, w satisfies the following Cauchy problem:

$$\begin{cases} w'(t) + Aw(t) + p_0(t)Bv(t) = 0, & t \in [0, T_0] \\ w(0) = 0. \end{cases} \quad (30)$$

We define the constant

$$K_0^2 := C_B^2 N^2(T_0) e^{2(2\sigma + C_B)T_0 + 2C_B \sqrt{T_0} (1 + C_B N^2(T_0))}, \quad (31)$$

where we recall that σ , C_B and $N(T_0)$ are defined in (6), (12) and (13) respectively.

Proposition 4.4. *Let $\{A, B\}$ be j -null controllable in time T_0 and assume $\lambda_j = 0$. If*

$$K_0 \|v_0\| \leq 1, \quad (32)$$

then the solution of (30) satisfies

$$\|w(T_0)\| \leq K_0 \|v_0\|^2. \quad (33)$$

Proof. Since $w \in H^1(0, T_0; X) \cap L^2(0, T_0; D(A))$, taking the scalar product of both members of the equation in (30) with $w(t)$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w(t)\|^2 &\leq \sigma \|w(t)\|^2 + |p_0(t)| \|Bv(t)\| \|w(t)\| \\ &\leq \left(\frac{1}{2} + \sigma\right) \|w(t)\|^2 + C_B^2 \frac{1}{2} |p_0(t)|^2 \|v(t)\|^2, \quad \text{for a.e } t \in [0, T_0]. \end{aligned} \quad (34)$$

Then, by Gronwall's inequality, and appealing to (27) and (13), we get

$$\begin{aligned} \sup_{t \in [0, T_0]} \|w(t)\|^2 &\leq C_B^2 e^{(2\sigma+1)T_0} \|p_0\|_{L^2(0, T_0)}^2 \sup_{t \in [0, T_0]} \|v(t)\|^2 \\ &\leq C_B^2 e^{(4\sigma+C_B+1)T_0+2C_B N(T_0)\sqrt{T_0}\|v_0\|} (1 + C_B N^2(T_0)) \|v_0\|^2 \|p_0\|_{L^2(0, T_0)}^2 \\ &\leq C_B^2 N^2(T_0) e^{(4\sigma+C_B+1)T_0+2C_B N(T_0)\sqrt{T_0}\|v_0\|} (1 + C_B N^2(T_0)) \|v_0\|^4. \end{aligned} \quad (35)$$

Thus, from (32), we obtain that $N(T_0)\|v_0\| \leq 1$. Therefore,

$$\sup_{t \in [0, T_0]} \|w(t)\|^2 \leq K_0^2 \|v_0\|^4,$$

which yields to (33). $\square$

4.4. Iterated quadratic estimate and Proof of Theorem 3.7

Recalling that $\bar{v}(T_0) = 0$, we deduce from (33) that

$$\|v(T_0)\| \leq K_0 \|v_0\|^2. \quad (36)$$

Now, we want to obtain an estimate as (36) for the solution v of problem (23) defined in successive time intervals of the form $[nT, (n+1)T]$, with $n \geq 1$.

Proposition 4.5. *Let $\{A, B\}$ be j -null controllable in time T_0 and assume $\lambda_j = 0$. Let $v_0 \in X$ be such that*

$$K_0 \|v_0\| \leq 1, \quad (37)$$

where K_0 is defined in (31).

Then, there exists a control $p_n \in L^2(nT_0, (n+1)T_0)$ such that the corresponding solution v of (23) on the time interval $[nT_0, (n+1)T_0]$ satisfies

$$\|v((n+1)T_0)\| \leq \frac{1}{K_0} (K_0 \|v_0\|)^{2^{n+1}}, \quad \forall n \geq 0. \quad (38)$$

Proof. We proceed by induction on n . For $n = 0$, the result has been proved in Proposition 4.4. We suppose that it is true for any index until $n - 1$, and we prove it for n . Iterating the construction of the solution v of (23) in consecutive time intervals

of the form $[kT_0, (k+1)T_0]$ until $k+1 = n$, we set $v_{nT_0} = v(nT_0)$ and we consider the system

$$\begin{cases} v'(t) + Av(t) + p_n(t)Bv(t) + p_n(t)B\varphi_j = 0, & t \in [nT_0, (n+1)T_0], \\ v(nT_0) = v_{nT_0}. \end{cases} \quad (39)$$

We shift this problem in the time interval $[0, T_0]$ by introducing the variable $s := t - nT_0$ and the functions $\tilde{v}(s) = v(s + nT_0)$, $\tilde{p}_n(s) = p_n(s + nT_0)$. Then, $\tilde{v}$ is the solution of the following Cauchy problem:

$$\begin{cases} \tilde{v}_t(s) + A\tilde{v}(s) + \tilde{p}_n(s)B\tilde{v}(s) + \tilde{p}_n(s)B\varphi_j = 0, & s \in [0, T_0] \\ \tilde{v}(0) = v_{nT_0}. \end{cases} \quad (40)$$

Since $\{A, B\}$ is j -null controllable in time T_0 , there exists a control $\tilde{p}_n \in L^2(0, T_0)$ such that the associated linear system of (40) is null controllable and moreover $\tilde{p}_n$ satisfies

$$\|\tilde{p}_n\|_{L^2(0, T)} \leq N(T_0)\|v_{nT_0}\|.$$

Therefore, coming back to the original time interval $[nT_0, (n+1)T_0]$ and setting $p_n(t) := \tilde{p}_n(t - nT_0)$, for all $t \in [nT_0, (n+1)T_0]$, we find that

$$\|p_n\|_{L^2(nT_0, (n+1)T_0)} = \|\tilde{p}_n\|_{L^2(0, T_0)} \leq N(T_0)\|v_{nT_0}\|. \quad (41)$$

Moreover, since by inductive hypothesis

$$\|v(nT_0)\| \leq \frac{1}{K_0} (K_0\|v_0\|)^{2^n}, \quad (42)$$

we deduce that

$$K_0\|v(nT_0)\| \leq 1. \quad (43)$$

Hence, we can use Proposition 4.4 for problem (39), obtaining

$$\|v((n+1)T_0)\| \leq K_0\|v(nT_0)\|^2 \leq K_0 \left(\frac{1}{K_0} (K_0\|v_0\|)^{2^n} \right)^2 = \frac{1}{K_0} (K_0\|v_0\|)^{2^{n+1}} \quad (44)$$

and this concludes the induction argument and the proof of the proposition. $\square$

The last step that allows us to prove Theorem 3.7 consists in showing the rapid decay of the solution u of our initial problem (11) to the fixed stationary trajectory φ_j .

Proposition 4.6. *Let $\{A, B\}$ be j -null controllable in time T_0 and assume $\lambda_j = 0$.*

Then, there exists a constant $C_{T_0} > 0$ such that for every $\theta \in (0, 1)$ and for every $\|v_0\| \leq \frac{\theta}{K_0}$, with K_0 defined as in (31), there exists a control $p \in L^2_{\text{loc}}(0, +\infty)$ for which the solution u of (11) satisfies

$$\|u(t) - \varphi_j\| \leq \frac{C_{T_0}}{K_0} \theta^{2^{\frac{t}{T_0}-1}}, \quad \forall t \geq 0. \quad (45)$$

Proof. We have supposed that $\|v_0\| \leq \frac{\theta}{K_0}$, with $\theta \in (0, 1)$. Thus, (38) at rank $n - 1$, becomes

$$\|v(nT_0)\| \leq \frac{\theta^{2^n}}{K_0}, \quad \forall n \geq 1. \quad (46)$$

Consider now the time interval $[nT_0, (n+1)T_0]$. From estimate (19) used here for problem (39) in the time interval $[nT_0, (n+1)T_0]$ and from the bound (41) for the control p_n , we deduce that there exists a constant $C_{T_0} > 0$ such that

$$\|v(t)\| \leq C_{T_0} \|v(nT_0)\|, \quad \forall t \in [nT_0, (n+1)T_0]. \quad (47)$$

Therefore, using (38) in (47), we obtain that

$$\|v(t)\| \leq C_{T_0} \|v(nT_0)\| \leq \frac{C_{T_0}}{K_0} \theta^{2^n} = \frac{C_{T_0}}{K_0} \left(\theta^{2^{(n+1)}} \right)^{1/2}, \quad \forall t \in [nT_0, (n+1)T_0]. \quad (48)$$

Since $n \leq \frac{t}{T_0} \leq (n+1)$ and $\theta \in (0, 1)$, it holds that

$$\|v(t)\| \leq \frac{C_{T_0}}{K_0} \left(\theta^{2^{(n+1)}} \right)^{1/2} \leq \frac{C_{T_0}}{K_0} \left(\theta^{2^{\frac{t}{T_0}}} \right)^{1/2} = \frac{C_{T_0}}{K_0} \theta^{2^{\frac{t}{T_0}-1}}, \quad \forall t \geq 0. \quad (49)$$

By definition, $v(t) = u(t) - \varphi_j$. So, we deduce (45). $\square$

We now prove Theorem 3.7 for $\lambda_j = 0$.

Proof of Theorem 3.7 (case $\lambda_j = 0$) Since $\{A, B\}$ is by assumption j -null controllable, there exists $T_0 > 0$ such that $\{A, B\}$ is j -null controllable in time T_0 . By hypothesis, the j th eigenvalue of A is equal to zero; hence, we have to prove the superexponential stabilizability of system (22) to the eigensolution $\psi_j \equiv \varphi_j$.

Let $\rho > 0$ and let $\theta \in (0, 1)$ be the value for which $\theta = e^{-2\rho}$. Then, setting $R_\rho := \frac{e^{-2\rho}}{K_0}$, from Proposition 4.6, if $\|u_0 - \varphi_j\| \leq R_\rho$, then

$$\|u(t) - \varphi_j\| \leq \frac{C_{T_0}}{K_0} e^{-\rho 2^{\frac{t}{T_0}}} = \frac{C_{T_0}}{K_0} e^{-\rho e^{\frac{\log 2}{T_0} t}}, \quad \forall t \geq 0.$$

Hence, by defining

$$M := \frac{C_{T_0}}{K_0}, \quad \omega := \frac{\log 2}{T_0}, \quad (50)$$

we conclude the proof of local superexponential stabilizability of problem (11) to the j th eigensolution, when $\lambda_j = 0$, by means of a control $p \in L^2_{\text{loc}}(0, +\infty)$ defined as

$$p(t) = \sum_{n=0}^{\infty} p_n(t) \chi_{[nT_0, (n+1)T_0]}(t).$$

$\square$

We now deal with the general case $\lambda_j \neq 0$.

Lemma 4.7. *Let A satisfy hypothesis (6). If $\{A, B\}$ is j -null controllable, then $\{A_j, B\}$ is j -null controllable, where $A_j : D(A) \subset X \rightarrow X$ is defined as*

$$A_j := A - \lambda_j I. \quad (51)$$

Proof. It is immediate to check that A_j verifies (6). Let us show that $\{A_j, B\}$ is j -null controllable in the same time $T_0 > 0$ as $\{A, B\}$.

We have to prove that the following problem

$$\begin{cases} y'(t) + A_j y(t) + q(t)B\varphi_j = 0, & t \in [0, T], \\ y(0) = y_0 \in X \end{cases} \quad (52)$$

is null controllable and that the desired estimate (13) on q holds true. Now, defining $Y(t) := e^{-\lambda_j t} y(t)$ and $p(t) := e^{-\lambda_j t} q(t)$, we can rewrite problem (52) as

$$\begin{cases} Y'(t) + AY(t) + p(t)B\varphi_j = 0, & t \in [0, T], \\ Y(0) = y_0. \end{cases}$$

Since $\{A, B\}$ is j -null controllable, there exists a time $T_0 > 0$, a constant $N(T_0) > 0$ and a control $p \in L^2(0, T_0)$ such that $Y(T_0; y_0, p) = 0$ and

$$\|p\|_{L^2(0, T_0)} \leq N(T_0)\|y_0\|.$$

Hence, we deduce that the solution of (52) satisfies $y(T_0; y_0, q) = 0$ with $q(t) = e^{\lambda_j t} p(t)$. Furthermore,

$$\begin{aligned} \|q\|_{L^2(0, T_0)}^2 &= \int_0^{T_0} e^{2\lambda_j s} |e^{-\lambda_j s} q(s)|^2 ds \leq \max \left\{ 1, e^{2\lambda_j T_0} \right\} \|p\|_{L^2(0, T_0)}^2 \\ &\leq N^2(T_0) \max \left\{ 1, e^{2\lambda_j T_0} \right\} \|y_0\|^2. \end{aligned}$$

□

We are ready to complete the proof of Theorem 3.7.

Proof of Theorem 3.7 (case $\lambda_j \neq 0$)

Now, in order to deal with the case $\{A, B\}$ j -null controllable in time T_0 and $\lambda_j \neq 0$, we consider the operator A_j defined in (51).

From Lemma 4.7, we have that $\{A_j, B\}$ is j -null controllable in time T_0 . Moreover, the eigenvalues of A_j are given by

$$\mu_k = \lambda_k - \lambda_j, \quad \forall k \in \mathbb{N}^*,$$

(in particular, $\mu_j = 0$), with the same eigenfunctions than those of A , $\{\varphi_k\}_{k \in \mathbb{N}^*}$.

Observe that $z(t) := e^{\lambda_j t} u(t)$, with u the solution of (11), solves

$$\begin{cases} z'(t) + A_j z(t) + p(t)Bz(t) = 0, & t > 0, \\ z(0) = u_0. \end{cases} \quad (53)$$

So, we can apply the first part of the proof of Theorem 3.7 (for $\mu_j = 0$) to this problem and deduce that for all $\rho > 0$ there exists $R_\rho > 0$ such that, if $\|u_0 - \varphi_j\| \leq R_\rho$, then

$$\|z(t) - \varphi_j\| \leq M e^{-\rho e^{\omega t}}, \quad \forall t \geq 0, \quad (54)$$

where M, ω are positive constants.

Therefore,

$$\begin{aligned} \|u(t) - \psi_j(t)\| &= \|e^{-\lambda_j t} z(t) - e^{-\lambda_j t} \varphi_j\| \\ &= e^{-\lambda_j t} \|z(t) - \varphi_j\| \leq M e^{-(\rho e^{\omega t} + \lambda_j t)}, \quad \forall t \geq 0. \end{aligned}$$

The conclusion follows by possibly increasing ρ^1 . $\square$

5. Proof of Theorem 3.8

Let us recall the notion of *biorthogonal family* and a result we will use to show that $\{A, B\}$ is j -null controllable under the conditions of Theorem 3.8.

Definition 5.1. Let $\{\zeta_j\}$ and $\{\sigma_k\}$ be two sequences in a Hilbert space H . We say that the two families are *biorthogonal* or, that $\{\zeta_j\}$ (resp. $\{\sigma_k\}$) is *biorthogonal* to $\{\sigma_k\}$ (resp. $\{\zeta_j\}$), if

$$\langle \zeta_j, \sigma_k \rangle_H = \delta_{jk}, \quad \forall j, k \geq 0$$

where δ_{jk} is the Kronecker delta.

The notion of biorthogonal family was used by Fattorini and Russell in [14], where they introduced the moment method. Such a technique was developed later by several authors. We recall below the result proved in [11].

Theorem 5.2. Let $T > 0$, let $\{\omega_k\}_{k \in \mathbb{N}}$ be a nonnegative increasing sequence, and set $\zeta_k(t) := e^{\omega_k t}$, $t \in [0, T]$, $k \geq 0$.

If, for some $\alpha > 0$,

$$\sqrt{\omega_{k+1}} - \sqrt{\omega_k} \geq \alpha, \quad \forall k \in \mathbb{N},$$

then there exists a family $\{\sigma_j\}_{j \geq 0}$ which is biorthogonal to $\{\zeta_k\}_{k \geq 0}$ in $L^2(0, T)$, that is,

$$\int_0^T \sigma_j(t) e^{\omega_k t} dt = \delta_{jk}, \quad \forall k, j \in \mathbb{N}.$$

Furthermore, there exist constants $C_\alpha, C_\alpha(T) > 0$ such that

$$\|\sigma_j\|_{L^2(0, T)}^2 \leq C_\alpha^2(T) e^{-2\omega_j T} e^{C_\alpha \sqrt{\omega_j}/\alpha}, \quad \forall j \in \mathbb{N}. \quad (55)$$

¹For $\lambda_j > 0$, take $\rho = \bar{\rho}$ and for $\lambda_j < 0$ take $\rho \geq \bar{\rho} + \frac{|\lambda_j|}{\omega}$. Then, $\forall \bar{\rho} > 0, \exists R_{\bar{\rho}} > 0$ s.t. if $\|u_0 - \varphi_j\| \leq R_{\bar{\rho}}$, we have that $\|u(t) - \psi_j(t)\| \leq M e^{-\rho e^{\omega t}}$.

Proof of Theorem 3.8. In order to prove that $\{A, B\}$ is j -null controllable, we have to show that there exists $T > 0$ such that the following problem

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_j = 0, & t \in [0, T], \\ y(0) = y_0 \in X \end{cases} \quad (56)$$

is null controllable with $p \in L^2(0, T)$ that satisfies (13).

First, we observe that by the hypothesis on A , the eigenfunctions $\{\varphi_k\}_{k \in \mathbb{N}^*}$ form an orthonormal basis of X .

Let $T > \tau$, where τ has been defined in (16). For any $y_0 \in X$ and $p \in L^2(0, T)$, it follows from Proposition 4.1 that there exists a unique mild solution $y \in C^0([0, T], X)$ of (56) that can be represented by the formula

$$y(t) = e^{-tA}y_0 - \int_0^t e^{-(t-s)A} p(s)B\varphi_j ds. \quad (57)$$

We want to find $p \in L^2(0, T)$ such that $y(T) = 0$; thus, the following equality must hold:

$$\sum_{k \in \mathbb{N}^*} \langle y_0, \varphi_k \rangle e^{-\lambda_k T} \varphi_k = \int_0^T p(s) \sum_{k \in \mathbb{N}^*} \langle B\varphi_j, \varphi_k \rangle e^{-\lambda_k(T-s)} \varphi_k ds. \quad (58)$$

Therefore,

$$\langle y_0, \varphi_k \rangle = \int_0^T e^{\lambda_k s} p(s) \langle B\varphi_j, \varphi_k \rangle ds, \quad \forall k \in \mathbb{N}^*. \quad (59)$$

Thus, proving null controllability of the linear system (56) reduces to finding a function $p \in L^2(0, T)$ that satisfies

$$\int_0^T e^{\lambda_k s} p(s) ds = \frac{\langle y_0, \varphi_k \rangle}{\langle B\varphi_j, \varphi_k \rangle}, \quad \forall k \in \mathbb{N}^*. \quad (60)$$

Taking $q(t) := e^{\lambda_1 s} p(s)$ and $\omega_k := \lambda_k - \lambda_1 \geq 0$, the family of Eqs. (60) can be rewritten as

$$\int_0^T e^{\omega_k s} q(s) ds = \frac{\langle y_0, \varphi_k \rangle}{\langle B\varphi_j, \varphi_k \rangle}, \quad \forall k \in \mathbb{N}^*. \quad (61)$$

Thanks to assumption (15), there exists $\alpha > 0$ such that the gap condition $\sqrt{\omega_{k+1}} - \sqrt{\omega_k} \geq \alpha$ holds for all $k \in \mathbb{N}^*$. Then, Theorem 5.2 ensures the existence of a family $\{\sigma_k\}_{k \in \mathbb{N}^*}$ such that $\int_0^T \sigma_k(t) e^{\omega_j t} dt = \delta_{kj}$, $\forall k, j \in \mathbb{N}^*$.

We claim that the series

$$\sum_{k \in \mathbb{N}^*} \frac{\langle y_0, \varphi_k \rangle}{\langle B\varphi_j, \varphi_k \rangle} \sigma_k(t) \quad (62)$$

is convergent in $L^2(0, T)$. Indeed, appealing to estimate (55) for $\{\sigma_k\}_{k \in \mathbb{N}^*}$

$$\begin{aligned} \sum_{k \in \mathbb{N}^*} \left| \frac{\langle y_0, \varphi_k \rangle}{\langle B\varphi_j, \varphi_k \rangle} \right| \|\sigma_k\|_{L^2(0, T)} &\leq \|y_0\| \left(\sum_{k \in \mathbb{N}^*} \frac{\|\sigma_k\|_{L^2(0, T)}^2}{|\langle B\varphi_j, \varphi_k \rangle|^2} \right)^{1/2} \\ &\leq \|y_0\| \left(C_\alpha^2(T) \sum_{k \in \mathbb{N}^*} \frac{e^{-2\omega_k T} e^{C_\alpha \sqrt{\omega_k}/\alpha}}{|\langle B\varphi_j, \varphi_k \rangle|^2} \right)^{1/2}. \end{aligned}$$

We observe that the above right-hand side is finite because hypothesis (16) (ii) guarantees

$$\begin{aligned} \Lambda_T &:= \left(\sum_{k \in \mathbb{N}^*} \frac{e^{-2\omega_k T} e^{C_\alpha \sqrt{\omega_k}/\alpha}}{|\langle B\varphi_j, \varphi_k \rangle|^2} \right)^{1/2} \\ &< C(T, \tau, \alpha) \left(\sum_{k \in \mathbb{N}^*} \frac{e^{-2\omega_k \tau}}{|\langle B\varphi_j, \varphi_k \rangle|^2} \right)^{1/2} < +\infty \end{aligned} \quad (63)$$

because $T > \tau$.

Therefore, the control function $q(t) := \sum_{k \in \mathbb{N}^*} \frac{\langle y_0, \varphi_k \rangle}{\langle B\varphi_j, \varphi_k \rangle} \sigma_k(t)$ belongs to $L^2(0, T)$ and satisfies

$$\|q\|_{L^2(0, T)} \leq C_\alpha(T) \Lambda_T \|y_0\|,$$

thanks to the properties of $\{\sigma_k\}_{k \in \mathbb{N}^*}$.

Finally, returning to p , we deduce that

$$\|p\|_{L^2(0, T)}^2 = \int_0^T e^{-2\lambda_1 s} |q(s)|^2 ds \leq \max \left\{ 1, e^{-2\lambda_1 T} \right\} \|q\|_{L^2(0, T)}^2. \quad (64)$$

The conclusion follows by taking

$$N(T) := \max \left\{ 1, e^{\lambda_1 T} \right\} C_\alpha(T) \Lambda_T. \quad (65)$$

□

6. Applications

In this section, we discuss examples of bilinear control systems to which we can apply Corollary 3.10. The first problems we study are 1D parabolic equations of the form

$$u_t(t, x) - u_{xx}(t, x) + p(t)Bu(t, x) = 0, \quad (t, x) \in [0, T] \times (0, 1)$$

in the state space $X = L^2(0, 1)$, with Dirichlet or Neumann boundary conditions and with B the following multiplication operators:

$$Bu(t, x) = \mu(x)u(t, x),$$

where the functions μ are in some appropriate Sobolev space.

Then, we prove the superexponential stabilizability of the following one-dimensional equation with variable coefficients

$$u_t(t, x) - ((1+x)^2 u_x(t, x))_x + p(t)Bu(t, x) = 0$$

with Dirichlet boundary conditions.

Finally, we apply Corollary 3.10 to the following parabolic equation

$$u_t(t, x) - \Delta u(t, x) + p(t)Bu(t, x) = 0, \quad (t, x) \in [0, T] \times B^3$$

for radial data in the 3D unit ball B^3 .

In each example, we will denote by $\{\lambda_k\}_{k \in \mathbb{N}^*}$ and $\{\varphi_k\}_{k \in \mathbb{N}^*}$, respectively, the eigenvalues and eigenfunctions of the second-order operator associated with the problem under investigation. We will take $(\bar{u}, \bar{p}) = (\psi_j, 0)$ as reference trajectory-control pair, where $\psi_j = e^{-\lambda_j t} \varphi_j$ are the eigensolutions associated with the second-order operator at hand.

Remark 6.1. All the second-order operators considered in the aforementioned examples are accretive, that is, $\langle Ax, x \rangle \geq 0$, for all $x \in D(A)$. Hence, the eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}^*}$ are nonnegative.

In this case, we claim that it is sufficient to verify the following gap condition to have (15):

$$\exists \alpha > 0 : \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha, \quad \forall k \geq 1. \quad (66)$$

Indeed, if (66) holds, we have that

$$\begin{aligned} \sqrt{\lambda_{k+1} - \lambda_1} - \sqrt{\lambda_k - \lambda_1} &= \frac{\lambda_{k+1} - \lambda_k}{\sqrt{\lambda_{k+1} - \lambda_1} + \sqrt{\lambda_k - \lambda_1}} \\ &\geq \frac{\lambda_{k+1} - \lambda_k}{\sqrt{\lambda_{k+1}} + \sqrt{\lambda_k}} = \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha, \end{aligned}$$

for all $k \geq 1$, that is (15).

6.1. Dirichlet boundary conditions

Let $\Omega = (0, 1)$, $X = L^2(\Omega)$ and consider the problem

$$\begin{cases} u_t(t, x) - u_{xx}(t, x) + p(t)\mu(x)u(t, x) = 0 & x \in \Omega, t > 0 \\ u = 0 & x \in \partial\Omega, t > 0 \\ u(0, x) = u_0(x) & x \in \Omega, \end{cases} \quad (67)$$

where $p \in L^2(0, T)$ is the control function, u the state variable, and μ is a function in $H^3(\Omega)$.

We denote by A the operator defined by

$$D(A) = H^2 \cap H_0^1(\Omega), \quad A\varphi = -\frac{d^2\varphi}{dx^2}. \quad (68)$$

A satisfies all the properties in (6): In particular, it is strictly accretive and its eigenvalues and eigenfunctions have the following explicit expressions:

$$\lambda_k = (k\pi)^2, \quad \varphi_k(x) = \sqrt{2} \sin(k\pi x), \quad \forall k \in \mathbb{N}^*.$$

It is straightforward to prove that the eigenvalues fulfill the required gap property. Indeed,

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k+1)\pi - k\pi = \pi, \quad \forall k \in \mathbb{N}^*.$$

So, (66) is satisfied (hence (15) holds).

In order to apply Corollary 3.10 to system (67) and deduce the superexponential stabilizability to the trajectory ψ_j , we need to prove that

- $\langle B\varphi_j, \varphi_k \rangle \neq 0$, for all $k \in \mathbb{N}^*$,
- there exists $\tau > 0$ such that

$$\sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_j, \varphi_k \rangle|^2} < +\infty.$$

Following the strategy in [5, formula (8)] (for the case $j = 1$), we find that

$$\langle \mu\varphi_j, \varphi_k \rangle = \frac{4}{k^3\pi^2} \left[\mu'(1)(-1)^{k+j} - \mu'(0) \right] - \frac{\sqrt{2}}{(k\pi)^3} \int_0^1 (\mu\varphi_j)'''(x) \cos(k\pi x) dx.$$

Thus, if

$$\mu'(1) \pm \mu'(0) \neq 0 \quad \text{and} \quad \langle \mu\varphi_j, \varphi_k \rangle \neq 0, \quad \forall k \in \mathbb{N}^*, \quad (69)$$

we deduce that $\langle \mu\varphi_j, \varphi_k \rangle$ is of order $1/k^3$ as $k \rightarrow \infty$.

Remark 6.2. An example of a function which satisfies (69) is $\mu(x) = x^2$. Indeed, in this case

$$\langle \mu\varphi_j, \varphi_k \rangle = \begin{cases} \frac{4kj(-1)^{k+j}}{(k^2-j^2)^2}, & k \neq j, \\ \frac{2j^2\pi^2-3}{6j^2\pi^2}, & k = j, \end{cases}$$

and so $\langle \mu\varphi_j, \varphi_k \rangle \neq 0$ for all $k \in \mathbb{N}^*$ and furthermore there exists a constant $C_j > 0$, that depends on j , such that

$$|\langle \mu\varphi_j, \varphi_k \rangle| \geq \frac{C_j}{\lambda_k^{3/2}}, \quad \forall k \in \mathbb{N}^*.$$

We conclude that, under assumption (69),

$$\exists C > 0 \text{ such that } |\langle B\varphi_j, \varphi_k \rangle| \geq Ck^{-3} = C\lambda_k^{-3/2}, \quad \forall k \in \mathbb{N}^*, \quad (70)$$

and thanks to the polynomial behavior of the bound

$$\sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_j, \varphi_k \rangle|^2} < \infty,$$

for all $\tau > 0$.

Therefore, all the hypotheses of Corollary 3.10 are satisfied and system (67), with initial condition u_0 in a neighborhood of the eigenstate φ_j , is superexponentially stabilizable to the eigensolution ψ_j , for all $j \geq 1$.

Remark 6.3. Assumption (70) for problem (67) is not too restrictive. In fact, it is possible to prove that the set of functions in $H^3(\Omega)$ for which (70) holds is dense in $H^3(\Omega)$. For a proof of this fact, see Appendix A in [5].

6.2. Neumann boundary conditions

Now, we look at an example with Neumann boundary conditions: Let $\Omega = (0, 1)$ and consider the following bilinear stabilizability problem:

$$\begin{cases} u_t(t, x) - \partial_x^2 u(t, x) + p(t)\mu(x)u(t, x) = 0 & x \in \Omega, t > 0 \\ u_x = 0 & x \in \partial\Omega, t > 0 \\ u(0, x) = u_0(x) & x \in \Omega \end{cases} \quad (71)$$

Let $X = L^2(\Omega)$. When we rewrite (71) in abstract form, the operators A and B are defined by

$$\begin{aligned} D(A) &= \{\varphi \in H^2(0, 1) : \varphi' = 0 \text{ on } \partial\Omega\}, \quad A\varphi = -\frac{d^2\varphi}{dx^2} \\ D(B) &= X, \quad B\varphi = \mu\varphi. \end{aligned}$$

where μ is a real-valued function in $H^2(\Omega)$.

Operator A satisfies the assumptions in (6), and it is possible to compute explicitly its eigenvalues and eigenfunctions:

$$\begin{aligned} \lambda_0 &= 0, & \varphi_0 &= 1 \\ \lambda_k &= (k\pi)^2, & \varphi_k(x) &= \sqrt{2} \cos(k\pi x), \quad \forall k \geq 1. \end{aligned}$$

Since the eigenvalues are the same of those in Example 6.1 for $k \geq 1$, the gap condition is satisfied for all $k \geq 0$.

Integrating by parts, we find that the scalar product $\langle \mu\varphi_j, \varphi_k \rangle$ is equal to

$$\langle \mu\varphi_j, \varphi_k \rangle = \begin{cases} \frac{\sqrt{2}}{(k\pi)^2} (\mu'(1)(-1)^k - \mu'(0)) - \frac{\sqrt{2}}{(k\pi)^2} \int_0^1 \mu''(x) \cos(k\pi x) dx, & j = 0, \\ \frac{2}{(k\pi)^2} (\mu'(1)(-1)^{k+j} - \mu'(0)) - \frac{2}{(k\pi)^2} \int_0^1 (\mu(x)\varphi_j(x))'' \cos(k\pi x) dx, & j \geq 1. \end{cases}$$

Thus, reasoning as Example 6.1, if $\langle B\varphi_j, \varphi_k \rangle \neq 0$, $\forall k \in \mathbb{N}$ and $\mu'(1) \pm \mu'(0) \neq 0$, we have that

$$\exists C > 0 \text{ such that } |\langle B\varphi_j, \varphi_k \rangle| \geq Ck^{-2} = C\lambda_k^{-1}, \quad \forall k \in \mathbb{N}^*, \quad (72)$$

and therefore the series in (16) is finite for all $\tau > 0$.

Remark 6.4. An example of a suitable function μ for problem (71) that satisfies the above hypothesis is $\mu(x) = x^2$, for which

$$\langle \mu \varphi_0, \varphi_k \rangle = \begin{cases} \frac{2\sqrt{2}(-1)^k}{(k\pi)^2}, & k \geq 1, \\ \frac{1}{3}, & k = 0, \end{cases}$$

and for $j \neq 0$

$$\langle \mu \varphi_j, \varphi_k \rangle = \begin{cases} \frac{4(-1)^{k+j}(k^2+j^2)}{(k^2-j^2)^2\pi^2}, & k \neq j, \\ \frac{1}{3} + \frac{1}{2j^2\pi^2}, & k = j. \end{cases}$$

Applying Corollary 3.10, it follows that system (71), with initial condition u_0 in a neighborhood of φ_j , is superexponentially stabilizable to the eigensolution ψ_j , for any $j \in \mathbb{N}$.

6.3. Dirichlet boundary conditions, variable coefficients

In this example, we analyze the superexponential stabilizability of a parabolic equation in divergence form with nonconstant coefficients in the second-order term.

Let $\Omega = (0, 1)$, $X = L^2(\Omega)$ and consider the problem

$$\begin{cases} u_t(t, x) - ((1+x)^2 u_x(t, x))_x + p(t)\mu(x)u(t, x) = 0 & x \in \Omega, t > 0 \\ u(t, 0) = 0, \quad u(t, 1) = 0, & t > 0 \\ u(0, x) = u_0(x) & x \in \Omega \end{cases} \quad (73)$$

where $p \in L^2(0, T)$ is the control and μ is a function in $H^2(\Omega)$ with some properties to be specified later.

We denote by A the operator

$$A : D(A) \subset X \rightarrow X, \quad Au = -((1+x)^2 u_x)_x$$

where $D(A) = H^2 \cap H_0^1(\Omega)$ and it is possible to prove that A satisfies the properties in (6). The eigenvalues and eigenfunctions of A are computed as follows:

$$\lambda_k = \frac{1}{4} + \left(\frac{k\pi}{\ln 2} \right)^2, \quad \varphi_k(x) = \sqrt{\frac{2}{\ln 2}} (1+x)^{-1/2} \sin \left(\frac{k\pi}{\ln 2} \ln(1+x) \right), \quad \forall k \in \mathbb{N}^*.$$

The gap condition (66) holds:

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \frac{\pi}{\ln 2}, \quad \forall k \in \mathbb{N}^*.$$

Hence, from Remark 6.1, hypothesis (15) is verified.

Now, we check the hypotheses on the operator $B\varphi = \mu\varphi$ needed to apply Corollary 3.10. We recall that we want to prove that:

- $\langle B\varphi_j, \varphi_k \rangle \neq 0$, for all $k \in \mathbb{N}^*$,
- there exists $\tau > 0$ such that

$$\sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_j, \varphi_k \rangle|^2} < +\infty. \quad (74)$$

Let us compute the Fourier coefficients of $B\varphi_j$:

$$\begin{aligned} \langle \mu\varphi_j, \varphi_k \rangle &= \sqrt{\frac{2}{\ln 2}} \int_0^1 \mu(x)\varphi_j(x)(1+x)^{-1/2} \sin\left(\frac{k\pi}{\ln 2} \ln(1+x)\right) dx \\ &= \sqrt{\frac{2}{\ln 2}} \frac{\ln 2}{k\pi} \left(-\mu(x)\varphi_j(x)(1+x)^{1/2} \cos\left(\frac{k\pi}{\ln 2} \ln(1+x)\right) \Big|_0^1 \right. \\ &\quad \left. + \int_0^1 (\mu(x)\varphi_j(x)(1+x)^{1/2})' \cos\left(\frac{k\pi}{\ln 2} \ln(1+x)\right) dx \right) \\ &= \sqrt{\frac{2}{\ln 2}} \left(\frac{\ln 2}{k\pi}\right)^2 \left((\mu(x)\varphi_j(x)(1+x)^{1/2})' (1+x) \sin\left(\frac{k\pi}{\ln 2} \ln(1+x)\right) \Big|_0^1 \right. \\ &\quad \left. - \int_0^1 ((\mu(x)\varphi_j(x)(1+x)^{1/2})' (1+x))' \sin\left(\frac{k\pi}{\ln 2} \ln(1+x)\right) dx \right) \\ &= \sqrt{\frac{2}{\ln 2}} \left(\frac{\ln 2}{k\pi}\right)^3 \left(((\mu(x)\varphi_j(x)(1+x)^{1/2})' (1+x))' (1+x) \cos\left(\frac{k\pi}{\ln 2} \ln(1+x)\right) \Big|_0^1 \right. \\ &\quad \left. - \int_0^1 (((\mu(x)\varphi_j(x)(1+x)^{1/2})' (1+x))' (1+x))' \cos\left(\frac{k\pi}{\ln 2} \ln(1+x)\right) dx \right) \\ &= \sqrt{\frac{2}{\ln 2}} \left(\frac{\ln 2}{k\pi}\right)^3 \left(\sqrt{\frac{2}{\ln 2}} \frac{2j\pi}{\ln 2} (2\mu'(1)(-1)^{k+j} - \mu'(0)) \right. \\ &\quad \left. - \int_0^1 (((\mu(x)\varphi_j(x)(1+x)^{1/2})' (1+x))' (1+x))' \cos\left(\frac{k\pi}{\ln 2} \ln(1+x)\right) dx \right) \end{aligned}$$

Observe that, for the same reason of Example 6.1, if $2\mu'(1) \pm \mu'(0) \neq 0$ and $\langle \mu\varphi_j, \varphi_k \rangle \neq 0, \forall k \in \mathbb{N}^*$, then there exists a constant $C > 0$ such that $|\langle B\varphi_j, \varphi_k \rangle|$ is bounded from below by $C\lambda_k^{-3/2}$, for all $k \in \mathbb{N}^*$. Thus, series (74) is finite for all $\tau > 0$.

Remark 6.5. Let us fix $j = 1$. As an example of a function μ that verifies the lower bound $|\langle \mu\varphi_1, \varphi_k \rangle| \geq C\lambda_k^{-3/2}$, one can consider again $\mu(x) = x$: Indeed, it satisfies the sufficient condition $2\mu'(1) \pm \mu'(0) \neq 0$ and the Fourier coefficients of $(B\varphi_1)(x) = x\varphi_1(x)$ are all different from zero:

$$\begin{aligned} \langle \mu\varphi_1, \varphi_k \rangle &= \\ &= \begin{cases} \frac{2(2(-1)^{k+1}-1)}{(k^2-1)^2 \left(1 + \frac{(k+1)^2\pi^2}{(\ln 2)^2}\right) \left(1 + \frac{(k-1)^2\pi^2}{(\ln 2)^2}\right)} \left(4k^3 + k + 1 + 2k(k^2-1)^2 \frac{\pi}{(\ln 2)^2}\right), & k \geq 2 \\ \frac{1}{\ln 2} \left(\frac{(1-\ln 2) \left(\frac{2\pi}{\ln 2}\right)^3 - \frac{2\pi}{\ln 2}}{1 + \left(\frac{2\pi}{\ln 2}\right)^3} \right), & k = 1 \end{cases} \end{aligned}$$

This concludes the verification of the hypotheses of Corollary 3.10 that imply the superexponential stabilizability of (73) with initial condition u_0 in a neighborhood of φ_j to the eigensolution ψ_j , for all $j \in \mathbb{N}^*$.

6.4. 3D ball with radial data

In this example, we consider an evolution equation in the three-dimensional unit ball B^3 for radial data. The bilinear stabilizability problem is the following:

$$\begin{cases} u_t(t, r) - \Delta u(t, r) + p(t)\mu(r)u(t, r) = 0 & r \in [0, 1], t > 0 \\ u(t, 1) = 0, & t > 0 \\ u(0, r) = u_0(r) & r \in [0, 1] \end{cases} \quad (75)$$

where the Laplacian in polar coordinates for radial data has the form

$$\Delta \varphi(r) = \partial_r^2 \varphi(r) + \frac{2}{r} \partial_r \varphi(r).$$

The function μ is a radial function as well in the space $H_r^3(B^3)$, where the spaces $H_r^k(B^3)$ are defined as follows:

$$\begin{aligned} X &:= L_r^2(B^3) = \left\{ \varphi \in L^2(B^3) \mid \exists \psi : \mathbb{R} \rightarrow \mathbb{R}, \varphi(x) = \psi(|x|) \right\} \\ H_r^k(B^3) &:= H^k(B^3) \cap L_r^2(B^3). \end{aligned}$$

The domain of the Dirichlet Laplacian $A := -\Delta$ in X is $D(A) = H_r^2 \cap H_0^1(B^3)$. We observe that A satisfies the hypotheses required to apply Theorem 3.10. We denote by $\{\lambda_k\}_{k \in \mathbb{N}^*}$ and $\{\varphi_k\}_{k \in \mathbb{N}^*}$ the families of eigenvalues and eigenfunctions of A , $A\varphi_k = \lambda_k \varphi_k$, namely

$$\lambda_k = (k\pi)^2, \quad \varphi_k = \frac{\sin(k\pi r)}{\sqrt{2\pi r}}, \quad \forall k \in \mathbb{N}^*. \quad (76)$$

See [21], section 8.14. The family $\{\varphi_k\}_{k \in \mathbb{N}^*}$ forms an orthonormal basis of X .

In order to prove a superexponential stabilizability result to the eigensolutions ψ_j , we need to verify the remaining hypotheses in Corollary 3.10 regarding the gap condition of the eigenvalues of A and the properties of the operator $B : X \mapsto X$, $B\varphi = \mu\varphi$.

Since the Laplacian in the 3D ball for radial data behaves as a one-dimensional operator, the analysis is very similar to the previous cases. Indeed, since the eigenvalues of the operator A are actually the same of the 1D Dirichlet Laplacian, we have

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = \pi, \quad \forall k \in \mathbb{N}^*.$$

In order to compute a suitable lower bound for the Fourier coefficients of $B\varphi_j$, we recall the following property of radial symmetric functions $f = f(r)$: The integral over the unit ball $B^n \subset \mathbb{R}^n$ of $f = f(r)$ reduces to

$$\int_{B^n} f dV = |S^{n-1}| \int_0^1 f(r) r^{n-1} dr \quad (77)$$

where $|S^{n-1}|$ is the measure of the surface of the sphere S^{n-1} .

Therefore,

$$\begin{aligned}
 \langle \mu \varphi_j, \varphi_k \rangle &= \int_{B^3} \frac{1}{2\pi} \mu(r) \frac{\sin(j\pi r)}{r} \frac{\sin(k\pi r)}{r} dV \\
 &= 4\pi \int_0^1 \frac{1}{2\pi} \mu(r) \frac{\sin(j\pi r)}{r} \frac{\sin(k\pi r)}{r} r^2 dr \\
 &= \int_0^1 2\mu(r) \sin(j\pi r) \sin(k\pi r) dr \\
 &= \frac{4j}{k^3 \pi^2} \left(\mu'(1)(-1)^{k+j} - \mu'(0) \right) + \\
 &\quad - \frac{2}{(k\pi)^3} \int_0^1 (\mu(r) \sin(j\pi r))''' \cos(k\pi r) dr.
 \end{aligned} \tag{78}$$

Following the same argument as in Example 6.1, if all the coefficients $\langle \mu \varphi_j, \varphi_k \rangle$ are different from zero and, moreover, $\mu'(1) \pm \mu'(0) \neq 0$, then there exists a constant $C > 0$ such that

$$|\langle \mu \varphi_j, \varphi_k \rangle| \geq C \lambda_k^{-3/2}, \quad \forall k \in \mathbb{N}^*,$$

and thus the series in (74) is finite also in this case, for all $\tau > 0$.

Remark 6.6. An example of a function $\mu \in H_r^3(B^3)$ with the aforementioned properties is $\mu(r) = r^2$. In this case, the Fourier coefficients of $B\varphi_j$ are defined by

$$\langle B\varphi_j, \varphi_k \rangle = \begin{cases} \frac{4(-1)^{k+j}kj}{(k^2-j^2)^2\pi^2}, & k \neq j, \\ \frac{2j^2\pi^2-3}{6j^2\pi^2}, & k = j. \end{cases}$$

Finally, applying Corollary 3.10, we deduce that, fixed $T > 0$, for all $\rho > 0$, there exists $R_\rho > 0$ such that, if the initial condition u_0 satisfies $\|u_0 - \varphi_j\| \leq R_\rho$, then

$$\|u(t) - \psi_j(t)\| \leq M e^{-\rho e^{\omega t}}, \quad \forall t > 0,$$

where $M, \omega > 0$ are suitable constants.

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