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An approximation of solutions to heat equations defined by generalized measure theoretic Laplacians

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Abstract. We consider the heat equation defined by a generalized measure theoretic Laplacian on [0, 1]. This equation describes heat diffusion in a bar such that the mass distribution of the bar is given by a non-atomic Borel probabiliy measure μ, where we do not assume the existence of a strictly positive mass density. We show that weak measure convergence implies convergence of the corresponding generalized Laplacians in the strong resolvent sense. We prove that strong semigroup convergence with respect to the uniform norm follows, which implies uniform convergence of solutions to the corresponding heat equations. This provides, for example, an interpretation for the mathematical model of heat diffusion on a bar with gaps in that the solution to the corresponding heat equation behaves approximately like the heat flow on a bar with sufficiently small mass on these gaps.

1. Introduction

Let μ be a non-atomic Borel probability measure on $[0, 1]$ such that $0, 1 \in \operatorname{supp}(\mu)$, $\mathcal{L}^2([0,1],\mu)$ be the space of measurable functions f such that $\int_a^b f^2 \mathrm{d}\mu < \infty$ and $L^2([0,1],\mu)$ be the corresponding Hilbert space of equivalence classes with inner product $\langle f, g\rangle_\mu := \int_0^1 fg\mathrm{d}\mu$. We define

$$\mathcal{D}_\mu^2 := \Big\{ f \in C^1([0,1]) : \exists \left(f'\right)^\mu \in L^2([0,1],\mu) : \\ f'(x) = f'(0) + \int_0^x \left(f'\right)^\mu (y)\mathrm{d}\mu(y),\ x \in [0,1] \Big\}.$$

The Krein–Feller operator with respect to μ is given as

$$\Delta_\mu : \mathcal{D}_\mu^2 \subseteq L^2([0,1],\mu) \to L^2([0,1],\mu),\ f \mapsto \left(f'\right)^\mu .$$

This definition involves the derivative with respect to μ. If a function f has a representation given by

$$f(x) = \int_0^x \frac{\mathrm{d}}{\mathrm{d}\mu} f(x)\mathrm{d}\mu(x),\ x \in [0,1],$$

then $\frac{\mathrm{d}}{\mathrm{d}\mu} f$ is called the μ-derivative of f. Consequently, in the above definition, $\left(f'\right)^\mu$ is the μ-derivative of f'.

This operator has been widely studied, for example with an emphasis on addressing questions of the spectral asymptotics and further analytical properties [2,3,11–20, 22,23,33,34,36,37], diffusion processes [27,30,31], wave equations [5] and higher-dimensional generalizations [21,35,39].

In order to connect these operators with diffusion equations from a physical point of view, we follow for example [26, Section 1.2] and consider a metallic rod of constant cross-sectional area oriented in the x-direction occupying a region from $x = 0$ to $x = 1$ such that all thermal quantities are constant across a section. We can thus consider the rod as one-dimensional. We investigate the conduction of heat energy on a segment from $x = a$ to $x = b$. Let the temperature at the point $x \in [a, b]$ and time $t \in [0, \infty)$ be denoted by $u(t, x)$ and the total thermal energy in the segment at time t be denoted by $e_{a,b}(t)$. It is well-known that

$$e_{a,b}(t) = \int_a^b u(t, x)\rho(x)\mathrm{d}x,$$

assuming that the rod possesses a mass density $\rho : [0, 1] \to (0, \infty)$. However, if we denote the mass distribution of the rod by μ, we can write

$$e_{a,b}(t) = \int_a^b u(t, x)\mathrm{d}\mu(x).$$

Hence, we can define the total heat energy even if μ has no density. The total heat energy changes only if heat energy flows through the boundaries $x = a$ and $x = b$. We deduce for the rate of change of heat energy

$$\frac{\mathrm{d}}{\mathrm{d}t} e_{a,b}(t) = \phi(t, a) - \phi(t, b), \tag{1}$$

where $\phi(t, x)$ denotes the heat flux density, which gives the rate of thermal energy flowing through x at time t to the right. Assuming sufficient regularity, we can rewrite (1) as

$$\int_a^b \frac{\partial}{\partial t} u(t, x)\mathrm{d}\mu(x) = -\int_a^b \frac{\mathrm{d}}{\mathrm{d}\mu}\phi_t(x)\mathrm{d}\mu(x),$$

where $\phi_t(x) := \phi(t, x)$ and the μ-derivative was defined earlier. With $u_t(x) := u(x, t)$, Fourier's law of heat conduction $\phi = -\frac{\partial u}{\partial x}$ gives

$$\int_a^b \frac{\partial}{\partial t} u(t, x)\mathrm{d}\mu(x) = \int_a^b \frac{\mathrm{d}}{\mathrm{d}\mu}\frac{\mathrm{d}}{\mathrm{d}x} u_t(x)\mathrm{d}\mu(x).$$

Since this is valid for all $a, b \in [0, 1]$, $a < b$, it follows for $t \in [0, \infty)$ and μ-almost all $x \in [0, 1]$

$$\frac{\partial}{\partial t} u(t, x) = \frac{\mathrm{d}}{\mathrm{d}\mu}\frac{\mathrm{d}}{\mathrm{d}x} u_t(x).$$

Applying the definition of the Krein–Feller operator leads to the generalized heat equation

$$\frac{\partial u}{\partial t} = \Delta_\mu u_t,\ t \in [0, \infty) \tag{2}$$

with Dirichlet boundary conditions $u(t, 0) = u(t, 1) = 0$ for all $t \geq 0$ if we assume that the temperature vanishes at the boundaries or with Neumann boundary conditions $\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, 1) = 0$ if the boundaries are assumed to be perfectly insulated. This provides a physical motivation for a mass distribution having full support even if it possesses no Lebesgue density. However, it is still not clear how to interpret the equation if the support of the mass distribution is not the whole interval, in particular for singular measures, such as measures on the Cantor set.

The problem then is to describe heat flow on a rod with massless parts. Krein–Feller operators defined by measures on the classical Cantor set or, more generally, Cantor-like sets with gaps have been extensively studied in recent years (see, e.g., [1,16–19]). In this paper, we give an interpretation of a solution to (2) in the case where μ is not of full support. We approximate the solution by a sequence of solutions to heat equations defined by μ_n for $n \in \mathbb{N}$ such that μ_n is of full support and converges weakly to μ for $n \to \infty$.

To this end, let $b \in \{N, D\}$ represent the boundary condition, where N denotes Neumann and D Dirichlet boundary conditions and we give our basic assumption.

Assumption 1.1. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of non-atomic Borel probability measures on $[0, 1]$ such that $0, 1 \in \operatorname{supp}(\mu_n)$ and $\mu_n \rightharpoonup \mu, n \to \infty$, where $\rightharpoonup$ denotes weak measure convergence.

It is well-known that Δ_μ^b is a non-positive self-adjoint operator (see, e.g., [13]) and thus the generator of a strongly continuous semigroup $\left(T_t^b\right)_{t\geq 0}$ (see, e.g., [24, Lemma 1.3.2]).

If $u_0 \in L^2([0, 1], \mu)$, then the unique mild solution to the initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t) &= \Delta_\mu^b u(t),\ t \in [0, \infty), \\ u(0) &= u_0 \end{aligned} \tag{3}$$

is given by $u(t) = T_t^b u_0$ (see e.g., [10, Proposition II.6.4]).

This motivates the investigation of strong semigroup convergence. However, for different measures, the corresponding semigroups are defined on different spaces. For the special case $\operatorname{supp}(\mu) = \operatorname{supp}(\mu_n) = [0, 1]$ for all $n \in \mathbb{N}$, the results in [6] can be applied to obtain strong semigroup convergence on the space of continuous functions on $[0, 1]$. To formulate a strong semigroup convergence result without that assumption, we restrict the semigroup $\left(T_t^N\right)_{t\geq 0}$ associated to Δ_μ^N on $L^2([0, 1], \mu)$ to the subspace of continuous functions, denoted by $(C[0, 1])_\mu^N$, which is a Banach space with the uniform norm. The semigroup $\left(T_t^D\right)_{t\geq 0}$ is restricted to the Banach space of continuous functions satisfying Dirichlet boundary conditions, denoted by $(C[0,1])_\mu^D$. We show that the restricted semigroup, which we denote by $\left(\bar{T}_t^b\right)_{t\geq 0}$, is, again, a strongly continuous contraction semigroup and the infinitesimal generator is given by

$$\bar{\Delta}_\mu^b f := \Delta_\mu^b f,\ \mathcal{D}\left(\bar{\Delta}_\mu^b\right) := \left\{ f \in \mathcal{D}\left(\Delta_\mu^b\right) : \Delta_\mu^b f \in (C[0,1])_\mu^b \right\}.$$

Moreover, if we assume that $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\mu_n)$, the space $(C[0,1])_\mu^b$ can be continuously embedded in $(C[0,1])_{\mu_n}^b$, where we denote the embedding by π_n. Due to the Trotter-Kato approximation theorem (see, e.g., [10, Theorem I.6.1]), the strong semigroup convergence is equivalent to strong resolvent convergence and strong resolvent convergence is what we will establish. More precisely, let $f \in (C[0,1])_\mu^b$, $\lambda > 0$ and $n \in \mathbb{N}$. We define $\bar{R}_\lambda^b := \left(\lambda - \bar{\Delta}_\mu^b\right)^{-1}$ and $\bar{R}_{\lambda,n}^b := \left(\lambda - \bar{\Delta}_{\mu_n}^b\right)^{-1}$ and prove

$$\left\| \pi_n \bar{R}_\lambda^b f - \bar{R}_{\lambda,n}^b \pi_n f \right\|_\infty \to 0,\ n \to \infty. \tag{4}$$

The main tool for proving (4) is the generalization of the hyperbolic functions sinh and cosh, defined by generalizing the series

$$\sinh(zx) = \sum_{k=0}^\infty z^{2k+1} \frac{x^{2k+1}}{(2k+1)!},\ \cosh(zx) = \sum_{k=0}^\infty z^{2k} \frac{x^{2k}}{(2k)!}.$$

We replace $\frac{x^k}{k!}$ by generalized monomials defined by a measure μ. This extends the theory of measure theoretic functions, developed for trigonometric functions in [1]. Then, we show that the resolvent density of the operator Δ_μ^b is a product of such generalized hyperbolic functions. This leads to strong resolvent convergence for our sequence by proving convergence of these generalized hyperbolic functions. As a consequence we obtain our main result.

Theorem 1.2. *Let $f \in (C[0,1])_\mu^b$ and μ_n be a sequence of measures satisfying Assumption 1.1. Then, for all $t \geq 0$*

$$\lim_{n\to\infty} \left\| \pi_n \bar{T}_t^b f - \bar{T}_{t,n}^b \pi_n f \right\|_\infty = 0,$$

uniformly on bounded time intervals.

For $f \in (C[0,1])_\mu^b$, it follows in the same way as before, that $\left\{u(t) = \bar{T}_t^b f : t \geq 0\right\}$ is the unique mild solution to the initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t) &= \bar{\Delta}_\mu^b u(t),\ t \in [0,\infty), \\ u(0) &= f \end{aligned} \tag{5}$$

Further, $\left\{u_n(t) = \bar{T}_{t,n}^b f : t \geq 0\right\}$ is the unique mild solution to the initial value problem

$$\begin{aligned} \frac{\partial u_n}{\partial t}(t) &= \bar{\Delta}_{\mu_n}^b u_n(t),\ t \in [0,\infty), \\ u_n(0) &= \pi_n f. \end{aligned}$$

Finally, combining these results and Theorem 1.2 yields

$$\lim_{n\to\infty} \|\pi_n u(t) - u_n(t)\|_\infty = 0,$$

uniformly on bounded time intervals.

We obtain a meaningful interpretation for the diffusion of heat in the case of a mass distribution with gaps in that the heat in a rod with mass distribution μ diffuses approximately like the heat on a rod with mass distribution μ_n for sufficiently large n.

This paper is structured as follows. In the following section, we recall definitions related to Krein–Feller operators. In Sect. 3, we introduce the concept of generalized hyperbolic functions and the connection to resolvent operators. Section 4 is devoted to the restriction of the Krein–Feller operator semigroup to the spaces $(C[0,1])_\mu^b$ for $b \in \{N, D\}$. After these preparations, in Sect. 5 we develop the central convergence results, namely the convergence of the hyperbolic functions and the strong resolvent convergence in Sect. 5.1 and finally, the strong semigroup convergence and convergence of solutions to heat equations in Sect. 5.2. In Sect. 6, we show how to apply the results in three examples. Lastly, in Sect. 7, we discuss some open problems.

2. Preliminaries

First, we recall the definition and some properties of the operator Δ_μ^b, where $b \in \{N, D\}$ and μ is a non-atomic Borel probability measure on $[0,1]$ such that $0, 1 \in \operatorname{supp}(\mu)$. If $[0,1] \setminus \operatorname{supp}(\mu) \neq \emptyset$, then $[0,1] \setminus \operatorname{supp}(\mu)$ is open in $\mathbb{R}$ and can be written as

$$[0,1] \setminus \operatorname{supp}(\mu) = \bigcup_{i\geq 1} (a_i, b_i) \tag{6}$$

with $0 < a_i < b_i < 1$, $a_i, b_i \in \operatorname{supp}(\mu)$ for $i \geq 1$. We define

$$\begin{aligned}\mathcal{D}^1 := \Big\{ &f : [0,1] \to \mathbb{R} : \text{there exists } f' \in L^2\left([0,1], \lambda^1\right) : \\ &f(x) = f(0) + \int_0^x f'(y)\mathrm{d}y,\ x \in [0,1] \Big\}\end{aligned}$$

and $H^1\left([0,1],\lambda^1\right)$ to be the space of all $L^2([0,1],\mu)$-equivalence classes possessing a $\mathcal{D}^1$–representative. If $\mu = \lambda^1$ on $[0,1]$, this definition is equivalent to the definition of the Sobolev space W_2^1.

We observe that $H^1\left([0,1],\lambda^1\right)$ is the domain of the non-negative symmetric bilinear form $\mathcal{E}$ on $L^2([0,1],\mu)$ defined by

$$\mathcal{E}(u, v) = \int_0^1 u'(x)v'(x)\mathrm{d}x,\ u, v \in \mathcal{F} := H^1\left([0,1], \lambda^1\right).$$

It is known (see [14, Theorem 4.1]) that $(\mathcal{E}, \mathcal{F})$ defines a Dirichlet form on $L^2([0,1],\mu)$. Hence, there exists an associated non-positive, self-adjoint operator Δ_μ^N on $L^2([0,1],\mu)$ with $\mathcal{F} = \mathcal{D}\left(\left(-\Delta_\mu^N\right)^{\frac{1}{2}}\right)$ such that

$$\langle -\Delta_\mu^N u, v\rangle_\mu = \mathcal{E}(u,v),\ u \in \mathcal{D}\left(\Delta_\mu^N\right), v \in \mathcal{F}$$

and

$$\mathcal{D}\left(\Delta_\mu^N\right) = \left\{f \in L^2([0,1],\mu) : f \text{ has a representative } \bar{f} \text{ with } \bar{f} \in \mathcal{D}_\mu^2 \text{ and } \bar{f}'(0) = \bar{f}'(1) = 0\right\}.$$

The operator Δ_μ^N is called the Neumann Krein–Feller operator with respect to μ. Furthermore, let $\mathcal{F}_0$ be the space of all $L^2([0,1],\mu)$-equivalence classes having a $\mathcal{D}^1$−representative f such that $f(0) = f(1) = 0$. The bilinear form defined by

$$\mathcal{E}(u,v) = \int_0^1 u'(x)v'(x)\mathrm{d}x,\ u,v \in \mathcal{F}_0,$$

is a Dirichlet form, too (see [14, Theorem 4.1]). Again, there exists an associated non-positive, self-adjoint operator Δ_μ^D on $L^2([0,1],\mu)$ with $\mathcal{F}_0 = \mathcal{D}\left(\left(-\Delta_\mu^D\right)^{\frac{1}{2}}\right)$ such that

$$\langle -\Delta_\mu^D u, v\rangle_\mu = \mathcal{E}(u,v),\ u \in \left(\Delta_\mu^D\right),\ v \in \mathcal{F}_0$$

and

$$\mathcal{D}\left(\Delta_\mu^D\right) = \left\{f \in L^2([0,1],\mu) : f \text{ has a representative } \bar{f} \text{ with } \bar{f} \in \mathcal{D}_\mu^2 \text{ and } \bar{f}(0) = \bar{f}(1) = 0\right\}.$$

Then, Δ_μ^D is called the Dirichlet Krein–Feller operator with respect to μ.

By Freiberg [13, Proposition 6.3, Lemma 6.7, Corollary 6.9], there exists an orthonormal basis $\{\varphi_k^b : k \in \mathbb{N}\}$ of $L^2([0,1],\mu)$ consisting of eigenfunctions of $-\Delta_\mu^b$ and for the related ascending ordered eigenvalues $\{\lambda_k^b : k \in \mathbb{N}\}$ we have $0 \le \lambda_1^b \le \lambda_2^b \le \cdots$, where $\lambda_1^D > 0$. Since $\{\varphi_k^b : k \ge 1\}$ is an orthonormal basis of $L^2([0,1],\mu)$, each $f \in L^2([0,1],\mu)$ can be written as $f = \sum_{k\ge1} f_k^b\varphi_k^b$, where $f_k^b := \left\langle f, \varphi_k^b\right\rangle_\mu,\ k \ge 1$. Along with the self-adjointness, we obtain the following formula, called the spectral representation of Δ_μ^b (see, e.g., [24, Section 1.3]):

$$\begin{aligned} -\Delta_\mu^b f &= \sum_{k\ge1} \lambda_k^b f_k^b \varphi_k^b, \\ \mathcal{D}\left(\Delta_\mu^b\right) &= \left\{f \in L^2([0,1],\mu) : \sum_{k\ge1}\left(\lambda_k^b f_k^b\right)^2 < \infty\right\}. \end{aligned} \tag{7}$$

The spectral representation provides a direct way to introduce the associated semi-group. Define for $f \in L^2([0,1],\mu)$

$$T_t^b f := \sum_{k\ge1} e^{-\lambda_k^b t} f_k^b \varphi_k^b,\ t \ge 0. \tag{8}$$

Then, $\left(T_t^b\right)_{t\geq 0}$ is a strongly continuous semigroup on $L^2([0,1],\mu)$ and its infinitesimal generator is Δ_μ^b (see, e.g., [24, Lemma 1.3.2]).

3. Generalized hyperbolic functions and the resolvent operator

Let $b \in \{N, D\}$ and let μ be defined as before. In this section, we develop a useful representation for the resolvent density of Δ_μ^b.

Let $\lambda > 0$. We consider the initial value problem

$$\begin{cases} \Delta_\mu g = \lambda g, \\ g(0) = 1, \ g'(0) = 0 \end{cases} \tag{9}$$

on $L^2([0,1],\mu)$. The problem (9) possesses a unique solution (see [13, Lemma 5.1]), which we denote by $g_{1,N}^\lambda$. Further, under the initial conditions

$$g(1) = 1, \ g'(1) = 0, \tag{10}$$

$$g(0) = 0, \ g'(0) = 1 \tag{11}$$

and

$$g(1) = 0, \ g'(1) = 1, \tag{12}$$

respectively, the above eigenvalue problems also possess unique solutions (see[13, Remark 5.2]), and we denote them by $g_{2,N}^\lambda$, $g_{1,D}^\lambda$ and $g_{2,D}^\lambda$, respectively. The resolvent density is then given as follows.

Lemma 3.1. [13, Theorem 6.1] Let $\lambda > 0$. The resolvent operator $R_\lambda^b := (\lambda - \Delta_\mu^b)^{-1}$ is well-defined and for all $f \in L^2([0,1],\mu)$ we have

$$R_\lambda^b f(x) = \int_0^1 \rho_\lambda^b(x,y) f(y) \mathrm{d}\mu(y), \ x \in [0,1],$$

where the resolvent densities are given by

$$\rho_\lambda^N(x,y) = \rho_\lambda^N(y,x) := \frac{g_{1,N}^\lambda(x) g_{2,N}^\lambda(y)}{\left(g_{1,N}^\lambda\right)'(1)}, \ x, y \in [0,1], \ x \leq y,$$

$$\rho_\lambda^D(x,y) = \rho_\lambda^D(y,x) := -\frac{g_{1,D}^\lambda(x) g_{2,D}^\lambda(y)}{g_{1,D}^\lambda(1)}, \ x, y \in [0,1], \ x \leq y.$$

It is well-known that if $\mu = \lambda^1$, the solutions to (9) and (11) are given by

$$g_{1.N}^\lambda(x) = \cosh\left(\sqrt{\lambda} x\right) \ \text{ and } g_{1.D}^\lambda(x) = \frac{1}{\sqrt{\lambda}} \sinh\left(\sqrt{\lambda} x\right), \ x \in [0,1],$$

respectively. We generalize the notion of hyperbolic functions by solving (9) and (11) for an arbitrary measure μ with the given initial conditions. To this end, we introduce generalized monomials as in [1].

Definition 3.2. *For $x \in [0, 1]$ we set $p_0(x) = q_0(x) = 1$ and for $k \in \mathbb{N}$*

$$p_k(x) := \begin{cases} \int_0^x p_{k-1}(t)\mathrm{d}\mu(t), & \text{if } k \text{ is odd,} \\ \int_0^x p_{k-1}(t)\mathrm{d}t, & \text{if } k \text{ is even,} \end{cases}$$

$$q_k(x) := \begin{cases} \int_0^x q_{k-1}(t)\mathrm{d}t, & \text{if } k \text{ is odd,} \\ \int_0^x q_{k-1}(t)\mathrm{d}\mu(t), & \text{if } k \text{ is even.} \end{cases}$$

We note that for $x \in [0, 1]$ and $k \geq 0$,

$$p_{2k+1}(x) \leq p_{2k}(x) \leq \frac{x^k}{k!}, \ q_{2k+1}(x) \leq p_{2k}(x) \leq \frac{x^k}{k!} \tag{13}$$

(see [18, Lemma 2.3]).

Definition 3.3. *We define for $x \in [0, 1]$, $z \in \mathbb{R}$*

$$\sinh_z(x) := \sum_{k=0}^{\infty} z^{2k+1} q_{2k+1}(x), \ \cosh_z(x) := \sum_{k=0}^{\infty} z^{2k} p_{2k}(x).$$

By (13) for all $z \in \mathbb{R}$

$$\|\sinh_z\|_\infty \leq z e^{z^2}, \ \|\cosh_z\|_\infty \leq e^{z^2}. \tag{14}$$

Example 3.4. If $\mu = \lambda^1$, we have $q_k(x) = \frac{x^k}{k!}$, $k \geq 0$. It follows that in this case

$$\sinh_z(x) = \sum_{k=0}^{\infty} z^{2k+1} \frac{x^{2k+1}}{(2k+1)!} = \sinh(zx)$$

and analogously $\cosh_z(x) = \cosh(zx)$.

Proposition 3.5. *Let $\lambda > 0$. Then, for $x \in [0, 1]$, we have*

$$g^\lambda_{1,N}(x) = \cosh_{\sqrt{\lambda}}(x), \qquad g^\lambda_{1,D}(x) = \frac{1}{\sqrt{\lambda}} \sinh_{\sqrt{\lambda}}(x),$$

$$g^\lambda_{2,N}(x) = \cosh_{\sqrt{\lambda}}(1-x), \quad g^\lambda_{2,D}(x) = -\frac{1}{\sqrt{\lambda}} \sinh_{\sqrt{\lambda}}(1-x).$$

Proof. The assertion for $g_{1,D}^{\lambda}$ was proven in [18, Lemma 2.3]. The proof for $g_{1,N}^{\lambda}$ works analogously. We verify the assertion for $g_{2,N}^{\lambda}$. Let $x \in [0, 1]$. Then,

$$\begin{aligned}
\cosh_{\sqrt{\lambda}}(1-x) &= \sum_{n=0}^{\infty} \lambda^n p_{2n}(1-x) \\
&= 1 + \sum_{n=1}^{\infty} \lambda^n \int_0^{1-x} \int_0^y p_{2n-2}(t)\mathrm{d}\mu(t)\mathrm{d}y \\
&= 1 + \sum_{n=1}^{\infty} \lambda^n \int_0^{1-x} \int_{1-y}^1 p_{2n-2}(1-t)\mathrm{d}\mu(t)\mathrm{d}y \\
&= 1 - \sum_{n=1}^{\infty} \lambda^n \int_x^1 \int_0^y p_{2n-2}(1-t)\mathrm{d}\mu(t)\mathrm{d}y \\
&= 1 - \sum_{n=0}^{\infty} \lambda^{n+1} \int_x^1 \int_0^y p_{2n}(1-t)\mathrm{d}\mu(t)\mathrm{d}y.
\end{aligned}$$

Due to estimate (13) we can use the dominated convergence theorem and obtain

$$\begin{aligned}
\cosh_{\sqrt{\lambda}}(1-x) &= 1 - \lambda \int_x^1 \int_0^y \sum_{n=0}^{\infty} \lambda^n p_{2n}(1-t)\mathrm{d}\mu(t)\mathrm{d}y \\
&= 1 - \lambda \int_x^1 \int_0^y \cosh_{\sqrt{\lambda}}(1-t)\mathrm{d}\mu(t)\mathrm{d}y.
\end{aligned}$$

We set $f(x) := \cosh_{\sqrt{\lambda}}(1-x),\ x \in [0, 1]$ and get

$$f(x) = 1 - \lambda \int_x^1 \int_0^y f(t)\mathrm{d}\mu(t)\mathrm{d}y,\ x \in [0, 1]$$

and in particular

$$f(0) = 1 - \lambda \int_0^1 \int_0^y f(t)\mathrm{d}\mu(t)\mathrm{d}y.$$

It follows that, for $x \in [0, 1]$,

$$f(x) - f(0) = \lambda \int_0^x \int_0^y f(t)\mathrm{d}\mu(t)\mathrm{d}y.$$

The latter equation can be written as $\Delta_\mu f = \lambda f$. It remains to verify the initial conditions. Obviously, $f(1) = \cosh_{\sqrt{\lambda}}(0) = 1$. Using (13) again, we have

$$f'(1) = -\sum_{n=1}^{\infty} \lambda^n p_{2n-1}(0) = 0.$$

The proof for $g_{2,D}^{\lambda}$ follows using the same ideas.

This leads to the following representation for the resolvent density:

Corollary 3.6. *Let* $\lambda > 0$. *We have for* $x, y \in [0, 1]$, $x \leq y$,

$$\rho_\lambda^N(x, y) = \rho_\lambda^N(y, x) = \left(\cosh'_{\sqrt{\lambda}}(1)\right)^{-1} \cosh_{\sqrt{\lambda}}(x) \cosh_{\sqrt{\lambda}}(1 - y),$$
$$\rho_\lambda^D(x, y) = \rho_\lambda^D(y, x) = \frac{1}{\sqrt{\lambda}} \left(\sinh_{\sqrt{\lambda}}(1)\right)^{-1} \sinh_{\sqrt{\lambda}}(x) \sinh_{\sqrt{\lambda}}(1 - y).$$

4. The restricted semigroup

Let $b \in \{N, D\}$ and let μ be defined as before. It is well-known that Δ_μ^b is the generator of a strongly continuous Markovian semigroup $\left(T_t^b\right)_{t\geq 0}$ of contractions on $L^2([0, 1], \mu)$.

Definition 4.1. *For* $(t, x, y) \in (0, \infty) \times [0, 1] \times [0, 1]$, *we define*

$$p_t^b(x, y) := \sum_{k=1}^{\infty} e^{-\lambda_k^b t} \varphi_k^b(x) \varphi_k^b(y).$$

This is called the heat kernel of Δ_μ^b.

The heat kernel is the integral kernel of the semigroup $\left(T_t^b\right)_{t\geq 0}$. That is, for $t > 0$ and $f \in L^2([0, 1], \mu)$, we can write

$$T_t^b f(x) = \int_0^1 p_t^b(x, y) f(y) \mathrm{d}\mu(y), \ x \in [0, 1].$$

In this section, we restrict these semigroups to appropriate spaces of equivalence classes of continuous functions.

Definition 4.2. *(i) We define* $(C[0, 1])_\mu^N$ *as the set of all* $L^2([0, 1], \mu)$*-equivalence classes possessing a continuous representative, formally*

$$(C[0, 1])_\mu^N := \left\{ f \in L^2([0, 1], \mu) : f \text{ possesses a continuous representative} \right\}.$$

(ii) We further define $(C[0, 1])_\mu^D$ *as the set of all* $L^2([0, 1], \mu)$*-equivalence classes possessing a continuous representative that satisfies Dirichlet boundary conditions, formally*

$$(C[0, 1])_\mu^D := \big\{ f \in L^2([0, 1], \mu) : f \text{ possesses a continuous representative } \bar{f} \text{ such that } \bar{f}(0) = \bar{f}(1) = 0 \big\}.$$

The space $(C[0, 1])_\mu^b$ is a Banach space with the norm $\|f\|_{(C[0,1])_\mu^b} := \left\| f|_{\mathrm{supp}(\mu)} \right\|_\infty$. Note that

$$\|f\|_{(C[0,1])_\mu^b} = \left\|\widetilde{f}\right\|_\infty,$$

where $\widetilde{f}$ is the continuous representative of f that is affine on all intervals in $[0,1] \setminus \operatorname{supp}(\mu)$. To simplify the notation, we henceforth write $\|f\|_\infty$ for $\|f\|_{(C[0,1])^b_\mu}$.

Let $u = \sum_{k\geq 1} u_k^b \varphi_k^b \in L^2([0,1],\mu)$ and let $t > 0$. From $\left(f_k^b\right)_{k\geq 1} \in l^2(\mathbb{N})$ it follows that $\left(\left(\lambda_k^b\right)^n e^{-\lambda_k^b t} f_k^b\right)_{k\geq 1} \in l^2(\mathbb{N})$ and thus by (7) and (8)

$$\left(-\Delta_\mu^b\right)^k T_t^b u = \sum_{k\geq 1} \left(\lambda_k^b\right)^k e^{-\lambda_k^b t} u_k^b \varphi_k^b \in L^2([0,1],\mu) \tag{15}$$

and especially $T_t^b u \in \mathcal{D}\left(\Delta_\mu^b\right)$. Hence, the following inclusion holds:

$$T_t^b\left((C[0,1])_\mu^b\right) \subseteq (C[0,1])_\mu^b.$$

This motivates the definition of the restricted semigroup $\left(\bar{T}_t^b\right)_{t\geq 0} := \left(\left(T_t^b\right)_{|_{(C[0,1])_\mu^b}}\right)_{t\geq 0}$, which is for $t \geq 0$ defined by

$$\bar{T}_t^b : (C[0,1])_\mu^b \to (C[0,1])_\mu^b,\ \bar{T}_t^b f = T_t^b f.$$

When evaluating an element of $(C[0,1])_\mu^b$ pointwise, we always evaluate the representative that is affine on all intervals in $[0,1] \setminus \operatorname{supp}(\mu)$.

The goal of this section is to show that $\left(\bar{T}_t^b\right)_{t\geq 0}$ again defines a strongly continuous contraction semigroup. It is obvious that the semigroup property holds. Note that by the Markov property of $(T_t^b)_{t\geq 0}$ for $g \in (C[0,1])_\mu^N$

$$\left|T_t^b g(x)\right| = \left|\int_0^1 p_t^b(x,y) g(y) \mathrm{d}\mu(y)\right| \leq \|g\|_\infty \left|\int_0^1 p_t^b(x,y) \mathrm{d}\mu(y)\right| \leq \|g\|_\infty,\ x \in [0,1].$$

Hence, $(\bar{T}_t^b)_{t\geq 0}$ is a semigroup of contractions. It remains to prove the strong continuity. To this end, we need some preparations. We write $\mathcal{E}(f,f) := \mathcal{E}(f)$ and $\|f\|_\mu^2 := \int_0^1 f(x)^2 \mathrm{d}\mu(x)$.

Lemma 4.3. *If $f \in \mathcal{F}$, then*

$$\|f\|_\infty \leq \mathcal{E}(f)^{\frac{1}{2}} + \|f\|_\mu.$$

Proof. Let $f \in \mathcal{F}$. Then, by the Cauchy–Schwarz inequality for all $x, y \in [0,1]$

$$|f(x) - f(y)| = \left|\int_x^y f'(z)\mathrm{d}z\right| \leq \left(\int_x^y \left(f'\right)^2(z)\mathrm{d}z\right)^{\frac{1}{2}} |x-y|^{\frac{1}{2}} = \mathcal{E}(f)^{\frac{1}{2}} |x-y|^{\frac{1}{2}}.$$

It follows by the reversed triangle inequality and by $|x-y| \leq 1$

$$|f(x)| \leq |f(y)| + \mathcal{E}(f)^{\frac{1}{2}}.$$

Further, by integrating of y w.r.t. μ,

$$|f(x)| \leq \int_0^1 |f(y)|\mathrm{d}\mu(y) + \mathcal{E}(f)^{\frac{1}{2}}$$

and finally by the Cauchy-Schwarz inequality

$$|f(x)| \leq \|f\|_\mu + \mathcal{E}(f)^{\frac{1}{2}}.$$

Lemma 4.4. *Let $f \in (C[0,1])^b_\mu$. Then, $\lim_{t\to 0} \left\|T^b_t f - f\right\|_\infty = 0$.*

Proof. We follow the proof of [28, Proposition 5.2.6]. Let $f \in \mathcal{F}$. By Lemma 4.3 and [28, Lemma B.2.4],

$$\begin{aligned}
\lim_{t\to 0}\left\|T^b_t f - f\right\|_\infty &\leq \lim_{t\to 0} \mathcal{E}\left(T^b_t f - f\right)^{\frac{1}{2}} + \left\|T^b_t f - f\right\|_\mu \\
&\leq \lim_{t\to 0} 2^{\frac{1}{2}} \left(\mathcal{E}\left(T^b_t f - f\right) + \left\|T^b_t f - f\right\|^2_\mu\right)^{\frac{1}{2}} \\
&= 0.
\end{aligned}$$

By the fact that $\mathcal{F}$ is dense in $(C[0,1])^N_\mu$ and that, for $t \geq 0$, T^N_t is continuous on $(C[0,1])^N_\mu$, we obtain the assertion for $b = N$. To verify the case $b = D$, we prove that $\mathcal{F}_0$ is dense in $(C[0,1])^D_\mu$. Let $f \in (C[0,1])^D_\mu$. Then, by the density of $\mathcal{F}$ in $(C[0,1])^N_\mu$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathcal{F}$ for each $n \in \mathbb{N}$ such that

$$\|f - f_n\|_\infty \to 0,\ n \to \infty. \tag{16}$$

We define for $n \in \mathbb{N}$

$$f_{n,0}(x) := f_n(x) - f_n(0) - x(f_n(1) - f_n(0)),\ x \in [0,1],$$

which is an element of $\mathcal{F}_0$. Further, we have that

$$f_0(x) := f(x) - f(0) - x(f(1) - f(0)) = f(x),\ x \in [0,1],$$

since f satisfies Dirichlet boundary conditions. This along with (16) implies for $n \in \mathbb{N}$

$$\begin{aligned}
&\lim_{n\to\infty} \left\|f_{n,0} - f\right\|_\infty \\
&\quad = \lim_{n\to\infty} \left\|f_{n,0} - f_0\right\|_\infty \\
&\quad \leq \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| + |f_n(0) - f(0)| \\
&\qquad + |x\left(f_n(1) - f_n(0) - (f(1) - f(0))\right)| \\
&\quad = 0.
\end{aligned}$$

The main result of this section now follows immediately.

Corollary 4.5. $\left(\bar{T}^b_t\right)_{t\geq 0}$ *is a strongly continuous contraction semigroup on $(C[0,1])^b_\mu$.*

5. Convergence results

5.1. Strong resolvent convergence

Let μ be defined as before and let F be the distribution function of μ. Further, let $(\mu_n)_{n\in\mathbb{N}}$ satisfy Assumption 1.1 and let F_n be the distribution function of μ_n for $n \in \mathbb{N}$.

First, we give convergence results for the generalized hyperbolic functions introduced in Sect. 3 using results from [19]. Let $p_k,\ q_k,\ k \in \mathbb{N}$ be defined by μ and $p_{k,n}, q_{k,n}, k \in \mathbb{N}$ be defined by μ_n for $n \in \mathbb{N}$.

Lemma 5.1. [19, Lemma 3.1] *For $x \in [0,1]$ and $k, n \in \mathbb{N}$ we have*

$$\begin{aligned}
|q_{2k}(x) - q_{2k,n}(x)| &\le 2\frac{\|F - F_n\|_\infty x^k}{(k-1)!},\\
|p_{2k}(x) - p_{2k,n}(x)| &\le 2\frac{\|F - F_n\|_\infty x^k}{(k-1)!},\\
|q_{2k+1}(x) - q_{2k+1,n}(x)| &\le 2\frac{\|F - F_n\|_\infty x^k}{(k-1)!},\\
|p_{2k+1}(x) - p_{2k+1,n}(x)| &\le 2\frac{\|F - F_n\|_\infty x^k}{(k-1)!}.
\end{aligned}$$

Remark 5.2. Since the distribution function of μ is continuous, weak measure convergence implies uniform convergence of the corresponding distribution functions (see [4, Section 8.1]), which is the condition in [19, Lemma 3.1].

For $z \in \mathbb{R}$ let $\cosh_z,\ \sinh_z$ be defined by μ and $\cosh_{z,n},\ \sinh_{z,n}$ be defined by μ_n for $n \in \mathbb{N}$. We obtain a result for the generalized hyperbolic functions, comparable to that for the trigonometric functions in [19].

Lemma 5.3. *Let $z \in \mathbb{R}$. Then,*

$$\begin{aligned}
\left\|\cosh_z - \cosh_{z,n}\right\|_\infty &\le 2z^2 e^{z^2} \|F - F_n\|_\infty,\\
\left\|\cosh_z' - \cosh_{z,n}'\right\|_\infty &\le \left(z^2 + 2z^4 e^{z^2}\right) \|F - F_n\|_\infty,\\
\left\|\sinh_z - \sinh_{z,n}\right\|_\infty &\le 2z^3 e^{z^2} \|F - F_n\|_\infty.
\end{aligned}$$

Proof. Let $x \in [0, 1]$ and $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\left|\cosh_z(x) - \cosh_{z,n}(x)\right| &\leq \sum_{k=1}^{\infty} |p_{2k}(x) - p_{2k,n}(x)|\, z^{2k} \\
&\leq \sum_{k=1}^{\infty} \frac{2\,\|F - F_n\|_\infty}{(k-1)!}\, z^{2k} \\
&= \sum_{k=0}^{\infty} \frac{2\,\|F - F_n\|_\infty}{k!}\, z^{2k+2} \\
&= 2z^2 e^{z^2}\, \|F - F_n\|_\infty\, .
\end{aligned}$$

Further, note that

$$\cosh_z'(x) = \sum_{k=1}^{\infty} p_{2k-1}(x) z^{2k}$$

and

$$\left|p_1(x) - p_{1,n}(x)\right| = |\mu([0,x]) - \mu_n([0,x])| \leq \|F - F_n\|_\infty\, .$$

With that,

$$\begin{aligned}
\left|\cosh_z'(x) - \cosh_{z,n}'(x)\right| &\leq \sum_{k=1}^{\infty} |p_{2k-1}(x) - p_{2k-1,n}(x)|\, z^{2k} \\
&\leq \left(z^2 + 2\sum_{k=2}^{\infty} \frac{z^{2k}}{(k-2)!}\right) \|F - F_n\|_\infty \\
&\leq \left(z^2 + 2z^4 e^{z^2}\right) \|F - F_n\|_\infty\, .
\end{aligned}$$

Finally,

$$\begin{aligned}
\left|\sinh_z(x) - \sinh_{z,n}(x)\right| &\leq \sum_{k=1}^{\infty} |q_{2k+1}(x) - q_{2k+1,n}(x)|\, z^{2k+1} \\
&\leq \sum_{k=1}^{\infty} \frac{2\,\|F - F_n\|_\infty}{(k-1)!}\, z^{2k+1} \\
&\leq \sum_{k=0}^{\infty} \frac{2\,\|F - F_n\|_\infty}{k!}\, z^{2k+3} \\
&\leq 2z^3 e^{z^2}\, \|F - F_n\|_\infty\, .
\end{aligned}$$

We turn to the main result of this section. For $b \in \{N, D\}$ and $\lambda > 0$, let R_λ^b be defined by μ and $R_{\lambda,n}^b$ be defined by μ_n. We assume $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\mu_n)$ for all $n \in \mathbb{N}$. Then, the mapping

$$\pi_n : (C[0,1])_\mu^b \to (C[0,1])_{\mu_n}^b,\ f \mapsto f \tag{17}$$

defines an embedding, where $f \in (C[0,1])^b_{\mu_n}$ denotes the $L^2([0,1],\mu_n)$-equivalence class of the representative of $f \in (C[0,1])^b_{\mu}$ that is affine on each interval $I \subseteq \operatorname{supp}(\mu_n) \setminus \operatorname{supp}(\mu)$.

Theorem 5.4. *Let $\lambda > 0$. Then, for all $f \in (C[0,1])^b_{\mu}$,*

$$\lim_{n\to\infty} \left\| R^b_{\lambda,n}\pi_n f - \pi_n R^b_{\lambda} f \right\|_\infty = 0.$$

Proof. We simplify the notation in this proof by omitting all embeddings. If we evaluate on $\operatorname{supp}(\mu_n) \setminus \operatorname{supp}(\mu)$, we always evaluate the representative that is affine on each interval $I \subseteq \operatorname{supp}(\mu_n) \setminus \operatorname{supp}(\mu)$. First, we consider the case $b = N$. Let $\lambda > 0$, $n \in \mathbb{N}$, $x, y \in [0,1]$ with $x \le y$. Using the triangle inequality,

$$\begin{aligned} &\left|\rho^N_\lambda(x,y) - \rho^N_{\lambda,n}(x,y)\right| \\ &\le \left| \left(\cosh'_{\sqrt{\lambda}}(1)\right)^{-1} - \left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1} \right| \left|\cosh_{\sqrt{\lambda}}(x)\cosh_{\sqrt{\lambda}}(1-y)\right| \\ &+ \left|\cosh_{\sqrt{\lambda}}(x) - \cosh_{\sqrt{\lambda},n}(x)\right| \left| \left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1} \cosh_{\sqrt{\lambda}}(1-y)\right| \\ &+ \left|\cosh_{\sqrt{\lambda}}(1-y) - \cosh_{\sqrt{\lambda},n}(1-y)\right| \left| \left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1} \cosh_{\sqrt{\lambda},n}(x)\right|. \end{aligned} \tag{18}$$

We have

$$\cosh'_{\sqrt{\lambda}}(1) = \sum_{n=1}^{\infty} \lambda^n p_{2n-1}(1) \ge \lambda p_1(1) = \lambda \tag{19}$$

and similarly $\cosh'_{\sqrt{\lambda},n}(1) \ge \lambda$. Applying this along with Lemma 5.3, we get

$$\begin{aligned} \left| \left(\cosh'_{\sqrt{\lambda}}(1)\right)^{-1} - \left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1} \right| &= \left| \frac{\cosh'_{\sqrt{\lambda},n}(1) - \cosh'_{\sqrt{\lambda}}(1)}{\cosh'_{\sqrt{\lambda}}(1)\cosh'_{\sqrt{\lambda},n}(1)} \right| \\ &\le \frac{\left(\lambda + 2\lambda^2 e^{\lambda}\right)\|F - F_n\|_\infty}{\lambda^2} \end{aligned}$$

and thus with (14)

$$\begin{aligned} &\left| \left(\cosh'_{\sqrt{\lambda}}(1)\right)^{-1} - \left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1} \right| \left|\cosh_{\sqrt{\lambda}}(x)\cosh_{\sqrt{\lambda}}(1-y)\right| \\ &\le \frac{\left(e^{2\lambda} + 2\lambda e^{3\lambda}\right)\|F - F_n\|_\infty}{\lambda}. \end{aligned}$$

For the second term on the right-hand side of inequality (18), we calculate

$$\left|\cosh_{\sqrt{\lambda}}(x) - \cosh_{\sqrt{\lambda},n}(x)\right| \left| \left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1} \cosh_{\sqrt{\lambda}}(1-y)\right| \le 2e^{2\lambda}\|F - F_n\|_\infty.$$

Treating the third term analogously and using the above calculations in (18) yields

$$\begin{aligned}
&\lim_{n\to\infty}\max_{x\in[0,1]}\left|\rho_\lambda^N(x,y)-\rho_{\lambda,n}^N(x,y)\right|\\
&\quad\le\lim_{n\to\infty}\frac{\left(e^{2\lambda}+2\lambda e^{3\lambda}\right)\|F-F_n\|_\infty}{\lambda}+4e^{2\lambda}\|F-F_n\|_\infty\\
&\quad=\lim_{n\to\infty}\left(\frac{1}{\lambda}+2e^{\lambda}+4\right)e^{2\lambda}\|F-F_n\|_\infty\\
&\quad=0.
\end{aligned}$$

Further, by (14) and (19),

$$\begin{aligned}
&\left|\int_0^1\rho_\lambda^N(x,y)f(y)\mathrm{d}\mu(y)-\int_0^1\rho_\lambda^N(x,y)f(y)\mathrm{d}\mu_n(y)\right|\\
&\quad\le\left|\left(\cosh'_{\sqrt{\lambda}}(1)\right)^{-1}\cosh_{\sqrt{\lambda}}(x)\right|\\
&\quad\cdot\left|\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)\mathrm{d}\mu(y)-\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)\mathrm{d}\mu_n(y)\right|\\
&\le\frac{e^{\lambda}}{\lambda}\left|\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)\mathrm{d}\mu(y)-\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)\mathrm{d}\mu_n(y)\right|.
\end{aligned}$$

Due to weak measure convergence,

$$\lim_{n\to\infty}\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)\mathrm{d}\mu_n(y)-\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)\mathrm{d}\mu(y)=0$$

and consequently,

$$\lim_{n\to\infty}\max_{x\in[0,1]}\left|\int_0^1\rho_\lambda^N(x,y)f(y)\mathrm{d}\mu(y)-\int_0^1\rho_\lambda^N(x,y)f(y)\mathrm{d}\mu_n(y)\right|=0.$$

We get the same result for $x\ge y$ and obtain

$$\begin{aligned}
&\lim_{n\to\infty}\max_{x\in[0,1]}\left|R_{\lambda,n}^N f(x)-R_\lambda^N f(x)\right|\\
&\le\lim_{n\to\infty}\max_{x\in[0,1]}\left|\int_0^1\rho_\lambda^N(x,y)f(y)\mathrm{d}\mu(y)-\int_0^1\rho_\lambda^N(x,y)f(y)\mathrm{d}\mu_n(y)\right|\\
&+\lim_{n\to\infty}\max_{x\in[0,1]}\left|\int_0^1\left(\rho_\lambda^N(x,y)-\rho_{\lambda,n}^N(x,y)\right)f(y)\mathrm{d}\mu_n\right|\\
&=0.
\end{aligned}$$

Now, let $b = D$. Again using the triangle inequality, for $n \in \mathbb{N}$, $x, y \in [0, 1]$, $x \leq y$,

$$
\begin{aligned}
&\left|\rho_\lambda^D(x, y) - \rho_{\lambda,n}^D(x, y)\right| \\
&\leq \frac{1}{\sqrt{\lambda}}\Bigg(\left|\left(\sinh_{\sqrt{\lambda}}(1)\right)^{-1} - \left(\sinh_{\sqrt{\lambda},n}(1)\right)^{-1}\right|\left|\sinh_{\sqrt{\lambda}}(x)\sinh_{\sqrt{\lambda}}(1-y)\right| \\
&+ \left|\sinh_{\sqrt{\lambda}}(x) - \sinh_{\sqrt{\lambda},n}(x)\right|\left|\left(\sinh_{\sqrt{\lambda},n}(1)\right)^{-1}\sinh_{\sqrt{\lambda}}(1-y)\right| \\
&+ \left|\sinh_{\sqrt{\lambda}}(1-y) - \sinh_{\sqrt{\lambda},n}(1-y)\right|\left|\left(\sinh'_{\sqrt{\lambda},n}(1)\right)^{-1}\sinh_{\sqrt{\lambda},n}(x)\right|\Bigg).
\end{aligned}
\tag{20}
$$

We have

$$
\sinh_{\sqrt{\lambda}}(1) = \sum_{n=0}^{\infty} \lambda^{n+\frac{1}{2}} q_{2n+1}(1) \geq \sqrt{\lambda} q_1(1) = \sqrt{\lambda}
$$

and thus

$$
\left|\left(\sinh_{\sqrt{\lambda}}(1)\right)^{-1} - \left(\sinh_{\sqrt{\lambda},n}(1)\right)^{-1}\right| \leq 2\sqrt{\lambda}e^{\lambda} \left\|F - F_n\right\|_\infty .
$$

Arguing in the same way as before, we get

$$
\begin{aligned}
\lim_{n\to\infty}\max_{x\in[0,1]}\left|\rho_\lambda^D(x, y) - \rho_{\lambda,n}^D(x, y)\right| &\leq \lim_{n\to\infty}\frac{2}{\sqrt{\lambda}}\sqrt{\lambda}e^{\lambda}\left\|F - F_n\right\|_\infty \lambda e^{2\lambda} \\
&\quad + \lim_{n\to\infty}\frac{4}{\sqrt{\lambda}}\lambda^{\frac{3}{2}}e^{\lambda}\left\|F - F_n\right\|_\infty e^{\lambda} \\
&= \lim_{n\to\infty}\left(2e^{\lambda} + 4\right)\lambda e^{2\lambda}\left\|F - F_n\right\|_\infty \\
&= 0.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\max_{x\in[0,1]}\left|\int_0^1 \rho_\lambda^D(x, y) f(y)\mathrm{d}\mu(y) - \int_0^1 \rho_\lambda^D(x, y) f(y)\mathrm{d}\mu_n(y)\right| \\
&\leq \max_{x\in[0,1]}\left|\left(\sqrt{\lambda}\sinh_{\sqrt{\lambda}}(1)\right)^{-1}\sinh_{\sqrt{\lambda}}(x)\right|\left|\int_0^1 \sinh_{\sqrt{\lambda}}(1-y) f(y)\mathrm{d}\mu(y)\right. \\
&\quad\left. - \int_0^1 \sinh_{\sqrt{\lambda}}(1-y) f(y)\mathrm{d}\mu_n(y)\right| \\
&\leq \left|\left(\sqrt{\lambda}\sinh_{\sqrt{\lambda}}(1)\right)^{-1}\right|\left\|\sinh_{\sqrt{\lambda}}\right\|_\infty\left|\int_0^1 \sinh_{\sqrt{\lambda}}(1-y) f(y)\mathrm{d}\mu(y)\right. \\
&\quad\left. - \int_0^1 \sinh_{\sqrt{\lambda}}(1-y) f(y)\mathrm{d}\mu_n(y)\right|.
\end{aligned}
$$

Due to the weak measure convergence, this goes to zero as n tends to ∞. Deducing the same result for $x \geq y$ and combining the above inequalities,

$$\begin{aligned}
\lim_{n\to\infty} \max_{x\in[0,1]} &\left| R^D_{\lambda,n} f(x) - R^D_\lambda f(x) \right| \\
&\leq \lim_{n\to\infty} \max_{x\in[0,1]} \left| \int_0^1 \rho^D_\lambda(x,y) f(y) \mathrm{d}\mu(y) - \int_0^1 \rho^D_\lambda(x,y) f(y) \mathrm{d}\mu_n(y) \right| \\
&+ \lim_{n\to\infty} \max_{x\in[0,1]} \left| \int_0^1 \left(\rho^D_\lambda(x,y) - \rho^D_{\lambda,n}(x,y) \right) f(y) \mathrm{d}\mu_n \right| \\
&= 0.
\end{aligned}$$

Remark 5.5. We made the assumption $\mathrm{supp}(\mu) \subseteq \mathrm{supp}(\mu_n)$ for all $n \in \mathbb{N}$ only to simplify the proofs. Note that our results can be formulated and proven in a very similar way if this condition is not satisfied. If, for example, $\mathrm{supp}(\mu) \supseteq \mathrm{supp}(\mu_n)$ for all $n \in \mathbb{N}$, define $\pi_n : (C[0,1])^b_\mu \to (C[0,1])^b_{\mu_n}, \ f \mapsto f|_{\mathrm{supp}(\mu_n)}$. This is again a bounded linear transformation between Banach spaces and we can follow the same steps as before.

5.2. Strong semigroup convergence

Let μ be defined as before and let $\lambda > 0$. Analogously to the restricted semigroup, we define the restricted resolvent operator by

$$\begin{aligned}
\bar{R}^N_\lambda &: (C[0,1])^N_\mu \to (C[0,1])^N_\mu, \ \bar{R}^N_\lambda f = R^N_\lambda f, \\
\bar{R}^D_\lambda &: (C[0,1])^D_\mu \to (C[0,1])^D_\mu, \ \bar{R}^D_\lambda f = R^D_\lambda f.
\end{aligned}$$

Further, we define the operators $\bar{\Delta}^N_\mu$ and $\bar{\Delta}^D_\mu$ by

$$\begin{aligned}
\bar{\Delta}^N_\mu f &:= \Delta^N_\mu f, \ \mathcal{D}\left(\bar{\Delta}^N_\mu\right) := \left\{ f \in \mathcal{D}\left(\Delta^N_\mu\right) : \Delta^N_\mu f \in (C[0,1])^N_\mu \right\}, \\
\bar{\Delta}^D_\mu f &:= \Delta^D_\mu f, \ \mathcal{D}\left(\bar{\Delta}^D_\mu\right) := \left\{ f \in \mathcal{D}\left(\Delta^D_\mu\right) : \Delta^D_\mu f \in (C[0,1])^D_\mu \right\},
\end{aligned}$$

which are called the part of the operator Δ^N_μ in $C[0,1])^N_\mu$ and the part of the operator Δ^D_μ in $C[0,1])^D_\mu$, respectively. The following Lemma shows how the restricted semigroup, the restricted resolvent and the part of the operator are connected. For that, let $b \in \{N, D\}$.

Lemma 5.6. *(i) The infinitesimal generator of the strongly continuous contraction semigroup $\left(\bar{T}^b_t\right)_{t\geq 0}$ is $\bar{\Delta}^b_\mu$.*
(ii) $\bar{R}^b_\lambda$ is the resolvent of $\bar{\Delta}^b_\mu$.

Proof. For all $f \in L^2([0,1],\mu)$, we have $\|f\|_\infty \geq \|f\|_\mu$, therefore the inclusion map $i : (C[0,1])^b_\mu \to L^2([0,1],\mu), \ f \mapsto f$ is continuous. Moreover, $\left(\bar{T}^b_t\right)_{t\geq 0}$ defines a strongly continuous contraction semigroup on $(C[0,1])^b_\mu$ and $(C[0,1])^b_\mu$ is $\left(\bar{T}^b_t\right)_{t\geq 0}$-invariant (see Corollary 4.5). We thus can apply [9, II.2.3 Proposition] to verify (i).

We turn to part (ii). Let $\lambda > 0$ and let $\widetilde{R}_\lambda^b$ be the resolvent of $\bar{\Delta}_\mu^b$. By part (i) and [9, I.1.10 Theorem], this operator is well-defined and given by

$$\widetilde{R}_\lambda^b f = \int_0^\infty e^{-\lambda s} \bar{T}_s^b f \,\mathrm{d}s, \;\; f \in (C[0,1])_\mu^b.$$

Further, by definition of $\left(\bar{T}_t^b\right)_{t\geq 0}$ and $\bar{R}_\lambda^b$,

$$\bar{R}_\lambda^b f = R_\lambda^b f = \int_0^\infty e^{-\lambda s} T_s^b f \,\mathrm{d}s = \int_0^\infty e^{-\lambda s} \bar{T}_s^b f \,\mathrm{d}s, \;\; f \in (C[0,1])_\mu^b.$$

It follows $\widetilde{R}_\lambda^b = \bar{R}_\lambda^b$ on $(C[0,1])_\mu^b$.

We are now able to establish strong semigroup convergence. To this end, let $(\mu_n)_{n\in\mathbb{N}}$ satisfy Assumption 1.1 and we assume $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\mu_n)$ for all $n \in \mathbb{N}$. For $b \in \{N, D\}$ let $\left(T_t^b\right)_{t\geq 0}$ be defined by μ, $\left(T_{t,n}^b\right)_{t\geq 0}$ be defined by μ_n and analogously the restricted semigroups $\left(\bar{T}_t^b\right)_{t\geq 0}$ and $\left(\bar{T}_{t,n}^b\right)_{t\geq 0}$ be defined by μ and μ_n, respectively.

Proof of Theorem 1.2. For $n \in \mathbb{N}$, π_n is a bounded linear transformation between Banach spaces. Further, $\left(\bar{T}_t^b\right)_{t\geq 0}$ and $\left(\bar{T}_{t,n}^b\right)_{t\geq 0}$, $n \in \mathbb{N}$ are strongly continuous contraction semigroups on their respective spaces (see Corollary 4.5). Hence, due to the first Trotter-Kato approximation theorem (see [10, Theorem I.6.1, Problem I.8.13] for a suitable version), the assertion is a direct consequence of Theorem 5.4.

Remark 5.7. As a further direct consequence of the Trotter-Kato approximation theorem, we obtain the graph norm convergence. That is, for $f \in \mathcal{D}\left(\bar{\Delta}_\mu^b\right)$ there exists $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathcal{D}\left(\bar{\Delta}_{\mu_n}^b\right)$ such that for $n \in \mathbb{N}$

$$\lim_{n\to\infty} \|\pi_n f - f_n\|_\infty + \left\|\pi_n \bar{\Delta}_\mu^b f - \bar{\Delta}_{\mu_n}^b f_n\right\|_\infty = 0.$$

Strong semigroup convergence can be interpreted as convergence of solutions to heat equations. The connection is given as follows (see [9, Proposition VI.6.2]).

Lemma 5.8. *Let A be the generator of a strongly continuous semigroup $(S_t)_{t\geq 0}$ on a Banach space X. Then, for each $f \in \mathcal{D}(A)$ the abstract heat equation*

$$\begin{aligned} \frac{\partial u}{\partial t}(t) &= Au(t), \;\; t \geq 0 \\ u(0) &= f \end{aligned} \tag{21}$$

has a unique classical solution on X given by

$$u : [0,\infty) \to X, \;\; t \mapsto S_t f,$$

meaning that u is continuously differentiable with respect to X, $u(t) \in \mathcal{D}(A)$ and (21) holds for all $t \geq 0$.

Let $T > 0$ and $f \in \mathcal{D}\left(\bar{\Delta}_\mu^b\right)$. Theorem 1.2 implies that the classical solution to

$$\frac{\partial u_n}{\partial t}(t) = \bar{\Delta}_{\mu_n}^b u_n(t),$$
$$u_n(0) = \pi_n f$$

converges uniformly for $(t, x) \in [0, T] \times [0, 1]$ to the classical solution to

$$\frac{\partial u}{\partial t}(t) = \bar{\Delta}_\mu^b u(t),$$
$$u(0) = f$$

as $n \to \infty$, assuming that $\pi_n f \in \mathcal{D}\left(\bar{\Delta}_{\mu_n}^b\right)$. However, the assumption $f \in \mathcal{D}\left(\bar{\Delta}_\mu^b\right)$ and $\pi_n f \in \mathcal{D}\left(\bar{\Delta}_{\mu_n}^b\right)$ for all $n \in \mathbb{N}$ is very restrictive, as the following example illustrates.

Example 5.9. Let μ be a measure satisfying our conditions such that $\operatorname{supp}(\mu)$ is a λ^1-zero set and assume that $\operatorname{supp}(\mu_n) = [0, 1]$ for all $n \in \mathbb{N}$. Further, let $f \in \mathcal{D}\left(\bar{\Delta}_\mu^b\right)$. Then, on any interval $I \subseteq [0, 1] \setminus \operatorname{supp}(\mu)$, $\pi_n f$ is affine. Now, if we assume that $\pi_n f \in \mathcal{D}\left(\bar{\Delta}_{\mu_n}^b\right)$, then $\bar{\Delta}_{\mu_n}^b f(x) = 0$, $x \in I$ and thus $\bar{\Delta}_{\mu_n}^b f = 0 \in (C[0, 1])_{\mu_n}^b$. If $b = D$, we obtain $\pi_n f = 0 \in (C[0, 1])_{\mu_n}^D$ and thus $f = 0 \in (C[0, 1])_\mu^b$ and if $b = N$, $(\pi_n f)' = 0 \in C[0, 1]$ and thus $f' = 0 \in (C[0, 1])_\mu^N$.

This motivates the concept of a mild solution (see [10, Definition II.6.3]).

Definition 5.10. *Let X be a Banach space, $A : \mathcal{D}(A) \subset X \to X$ and $f \in X$. We call a map $u : [0, \infty) \to X,\ t \mapsto u(t)$ a mild solution to the abstract heat equation*

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t}(t) &= Au(t),\ t \geq 0, \\ u(0) &= f \end{aligned} \tag{22}$$

if $\int_0^t u(s)ds \in \mathcal{D}(A)$ and $u(t) = A\int_0^t u(s)ds + f$ for all $t \geq 0$

Using this solution concept, we can establish the desired convergence for any initial condition in the appropriate space.

Theorem 5.11. *Let $f \in (C[0, 1])_\mu^b$ and let $(\mu_n)_{n\in\mathbb{N}}$ satisfy Assumption 1.1. Further, let $\{u(t) : t \geq 0\}$ be the unique mild solution to*

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t}(t) &= \bar{\Delta}_\mu^b u(t),\ t \geq 0, \\ u(0) &= f \end{aligned} \tag{23}$$

and, for $n \geq 1$, let $\{u_n(t) : t \geq 0\}$ be the unique mild solution to

$$\begin{aligned} \frac{\mathrm{d}u_n}{\mathrm{d}t}(t) &= \bar{\Delta}_{\mu_n}^b u_n(t),\ t \geq 0, \\ u_n(0) &= \pi_n f. \end{aligned} \tag{24}$$

Then,

$$\lim_{n\to\infty} \|\pi_n u(t) - u_n(t)\|_\infty = 0, \tag{25}$$

uniformly on bounded time intervals.

Proof. Since $\bar{\Delta}^b_\mu$ is the generator of a strongly continuous semigroup, it follows by [10, Proposition II.6.4] that $t \mapsto \bar{T}^b_t f$ is the unique mild solution to (23). The same argument show that $\bar{T}^b_{t,n}\pi_n f$ is the unique mild solution to (24) for $n \in \mathbb{N}$. Then, (25) is a direct consequence of Theorem 1.2.

6. Applications

Example 6.1. As a first application, we consider a non-atomic Borel probability measure μ on $[0, 1]$ such that $0, 1 \in \operatorname{supp}(\mu)$ and $\operatorname{supp}(\mu) \neq [0, 1]$. We define for $\varepsilon \in (0, 1)$ the approximating probability measure μ_ε by

$$\mu_\varepsilon := \frac{\mu + \varepsilon\lambda^1}{1+\varepsilon}.$$

It is elementary that μ_ε converges weakly to μ as $\varepsilon \to 0$ and Theorem 5.11 is applicable. Let $b \in \{N, D\}$ and $f \in (C[0, 1])^b_\mu$. Then, the unique solution $\{u_\varepsilon(t) : t \geq 0\}$ to

$$\begin{aligned}\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}t}(t) &= \bar{\Delta}^b_{\mu_\varepsilon} u_\varepsilon(t),\\ u_\varepsilon(0) &= \pi_\varepsilon f,\end{aligned}$$

where $\pi_\varepsilon : (C[0, 1])^b_\mu \to (C[0, 1])^b_{\mu_\varepsilon}$ is an embedding as previously defined (see (17)), converges to the unique solution $\{u(t) : t \geq 0\}$ to

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t}(t) &= \bar{\Delta}^b_\mu u(t),\\ u(0) &= f\end{aligned}$$

with respect to the uniform norm as ε tends to zero.

In the previous example, μ could be chosen to be an absolutely continuous measure, for example $\lambda^1_{|[0,\frac{1}{3}]\cup[\frac{2}{3},1]}$, or to be a singular measure, as a self-similar measure on the Cantor set. Furthermore, it is not required that the approximating measures have full support.

Example 6.2. Let $w_1, w_2 \in (0, 1)$ such that $w_1 + w_2 = 1$ and let μ be the unique invariant Borel probabiliy measure on $[0, 1]$ given by the IFS consisting of $S_1(x) = \frac{x}{3}$ and $S_2(x) = \frac{2}{3} + \frac{x}{3}$, $x \in [0, 1]$ and weights w_1 and w_2, i.e., μ is a so-called Cantor

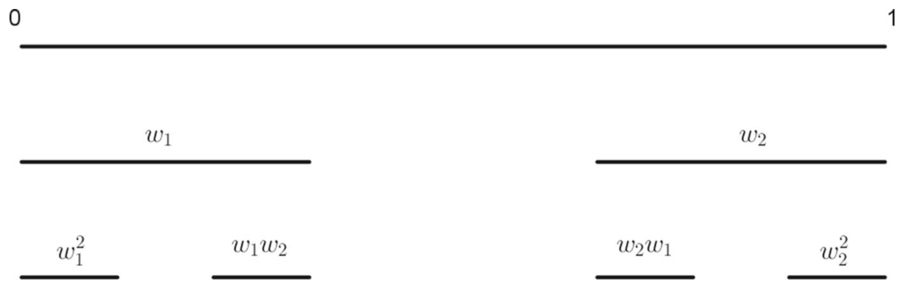

Figure 1. Approximating Cantor measures of levels $n = 0, 1, 2$

measure. Following [19], for $n \in \mathbb{N}$ we define the approximating Cantor measures of level n by

$$\mu_n(B) := 3^n \sum_{x \in \{1,2\}^n} \lambda^1_{|I_x} \prod_{i=1}^{n} \omega_{x_i}, \;\; B \in B([0, 1]),$$

where $I_x := \left(S_{x_1} \circ \cdots \circ S_{x_n}\right)([0, 1]),\;\; x \in \{1, 2\}^n$. The approximating Cantor measures of levels $n = 0, 1, 2$ are illustrated in Fig. 1. We denote the distribution function of μ by F and the distribution function of μ_n by F_n for $n \in \mathbb{N}$. Then, $\|F - F_n\|_\infty \to 0$ (see [19, Proposition 4.2]) as well as $\operatorname{supp}(\mu) \subset \operatorname{supp}(\mu_n)$ for $n \in \mathbb{N}$ and Theorem 5.11 can be applied. Hence, for $f \in (C[0, 1])^b_\mu$, the unique solution $\{u_n(t) : t \geq 0\}$ to

$$\begin{aligned} \frac{\mathrm{d}u_n}{\mathrm{d}t}(t) &= \bar{\Delta}^b_{\mu_n} u_n(t), \\ u_n(0) &= \pi_n f \end{aligned}$$

converges to the unique solution $\{u(t) : t \geq 0\}$ to

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t}(t) &= \bar{\Delta}^b_{\mu} u(t), \\ u(0) &= f \end{aligned}$$

with respect to the uniform norm as n tends to infinity.

Finally, we connect both applications.

Example 6.3. Let $\varepsilon > 0,\;\; n \in \mathbb{N}$ and let μ, μ_n, $\{u(t) : t \geq 0\}$ and $\{u_n(t) : t \geq 0\}$ be defined as in Example 6.2. We define $\mu_{n,\varepsilon}$ by

$$\mu_{n,\varepsilon} := \frac{\mu_n + \varepsilon\lambda^1}{1 + \varepsilon},$$

i.e. analogously to Example 6.1, and $\{u_{n,\varepsilon}(t) : t \geq 0\}$ to be the solution to

$$\begin{aligned} \frac{\mathrm{d}u_{n,\varepsilon}}{\mathrm{d}t}(t) &= \bar{\Delta}^b_{\mu_{n,\varepsilon}} u_{n,\varepsilon}(t), \\ u_{n,\varepsilon}(0) &= \pi_{n,\varepsilon} f, \end{aligned}$$

where $\pi_{n,\varepsilon}$ is an embedding as previously defined. Further, let $t \in [0, \infty)$ and $\delta > 0$. By Example 6.2, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\|u(t) - u_n(t)\|_\infty < \frac{\delta}{2}.$$

By Example 6.1, for each $n \geq n_0$ there exists $\varepsilon_n > 0$ such that for all $\varepsilon < \varepsilon_n$ we have

$$\left\|u_n(t) - u_{n,\varepsilon}(t)\right\|_\infty < \frac{\delta}{2}.$$

Hence, for all $n \geq n_0,\ \varepsilon < \varepsilon_n$ we have

$$\left\|u(t) - u_{n,\varepsilon}(t)\right\|_\infty < \delta.$$

Thus, the heat in a rod with mass distribution given by a Cantor measure diffuses approximately like the heat on a rod possessing a strictly positive mass density which is small off the Cantor set.

7. Directions for further research

Remark 7.1. Consider the heat equation (2) with initial value given by the Delta distribution $\delta_y : g \mapsto g(y)$ for $y \in \text{supp}(\mu)$. Then, the heat kernel

$$p_t(x, y) = \sum_{k \geq 1} e^{-\lambda_k^b t} \varphi_k^b(x) \varphi_k^b(y),\ \ (t, x) \in [0, \infty) \times (0, 1]$$

solves the equation in the distributional sense, where $\left\{\lambda_k^b,\ k \geq 1\right\}$ are the ascending ordered eigenvalues and $\left\{\varphi_k^b,\ k \geq 1\right\}$ the $L_2([0, 1], \mu)$-normed eigenfunctions of Δ_μ^b on $L_2([0, 1], \mu)$. The heat kernel is of particular importance in the context of the associated Markov process (see the remark below) and stochastic partial differential equations (see [7,8]). It is an open question whether weak measure convergence implies pointwise convergence of the corresponding heat kernels for each $(t, x, y) \in (0, \infty) \times [0, 1]^2$.

Remark 7.2. The operator Δ_μ^b on $L_2([0, 1], \mu)$ is the infinitesimal generator of a Markov process, called a quasi-diffusion (see, e.g., [27,30–32]). Convergence of semigroups raises the question of whether the associated Markov processes also converge weakly. If $\mu_n \rightharpoonup \mu$, our results imply that for each $f \in (C[0, 1])_\mu^b$, $t \in [0, \infty)$ and each starting point $x \in [0, 1]$

$$\mathbb{E}\left[f\left(X_n^b(t)\right)\right] = T_{t,n}^b f(x) \to T_t^b f(x) = \mathbb{E}\left[f\left(X^b(t)\right)\right],\ n \to \infty,$$

where X^b is associated to Δ_μ^b and X_n^b is associated to $\Delta_{\mu_n}^b$. We denote by $\mathbb{E}$ the expectation with respect to the law of the appropriate process. This could be extended to a proof of convergence of all finite-dimensional distributions, and tightness would then also be required in order to establish that $X_n^b \to X^b$ weakly in the Skorokhod space of càdlàg functions.

Remark 7.3. Let μ be of full support. Consider the analogue of the wave equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(t) = \Delta_\mu^b u(t), \ t \in [0, \infty)$$

on $L^2([0,1], \mu)$. This hyperbolic equation describes the motion of a vibrating string with mass distribution μ such that, if it is deflected, a tension force drives it back towards its state of equilibrium. If μ were not of full support, the string would have massless parts. It is not clear how to interpret massless parts of a string. We suppose that the motion of such a string behaves approximately like the motion of a string with very little mass on these gaps, analogous to our results about the diffusion of heat.

Assume that $u(0) \in \mathcal{D}\left(\Delta_\mu^b\right)$ and, for reasons of simplicity, that the initial velocity vanishes. Then, there exists a unique solution on $L_2([0,1], \mu)$ given by $u(t) = C(t)u(0)$, $t \geq 0$, where $\{C(t) : t \geq 0\}$ denotes the strongly continuous cosine family of Δ_μ^b (see, e.g., [40]). We have already shown that $\mu_n \rightharpoonup \mu$ implies strong resolvent convergence of the corresponding operators restricted to continuous functions. It is well-known that this implies convergence of the corresponding cosine families $\{C_n(t) : t \geq 0\}$, which implies convergence of the solutions to the corresponding wave equation, provided that there exists $M > 0$ and $w \geq 0$ such that for all $n \geq 1$, $t \geq 0$ $\|C_n(t)\| \leq Me^{w|t|}$ (see [25]). Proving that the restriction of $C(t)$ to $(C[0,1])_\mu^b$ is the cosine family of $\bar{\Delta}_\mu^b$ (and analogously for μ_n) and verifying the above estimate would be a way to establish the desired convergence of solutions to the wave equation.

Remark 7.4. The Krein–Feller operator Δ_μ^b can also be defined with respect to a measure μ with atoms, see, e.g., [29] for a model in this direction. If μ is atomless, the weak convergence implies uniform convergence of the corresponding distribution functions, which we have used to prove convergence of the corresponding generalized monomials (see Remark 5.2). For a measure with atoms, it would be required to find such convergence results without this uniform convergence property. Further, a representation of the resolvent density needs to be developed (see Lemma 3.1 for atomless measures).

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