Smooth measures and capacities associated with nonlocal parabolic operators

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Abstract. We consider a family $\{L_t, t \in [0, T]\}$ of closed operators generated by a family of regular (non-symmetric) Dirichlet forms $\{(B^{(t)}, V), t \in [0, T]\}$ on $L^2(E; m)$. We show that a bounded (signed) measure μ on $(0, T) \times E$ is smooth, i.e. charges no set of zero parabolic capacity associated with $\frac{\partial}{\partial t} + L_t$, if and only if μ is of the form $\mu = f \cdot m_1 + g_1 + \partial_t g_2$ with $f \in L^1((0, T) \times E; dt \otimes m)$, $g_1 \in L^2(0, T; V')$, $g_2 \in L^2(0, T; V)$. We apply this decomposition to the study of the structure of additive functionals in the Revuz correspondence with smooth measures. As a by-product, we also give some existence and uniqueness results for solutions of semilinear equations involving the operator $\frac{\partial}{\partial t} + L_t$ and a functional from the dual $\mathcal{W}'$ of the space $\mathcal{W} = \{u \in L^2(0, T; V) : \partial_t u \in L^2(0, T; V')\}$ on the right-hand side of the equation.

1. Introduction

In the study of parabolic problems of the form

$$-\partial_t u - \Delta_p u = f(\cdot, u) + \mu \quad \text{in } D, \quad u|_{(0,T) \times \partial D} = 0, \quad u(T, \cdot) = \varphi, \quad (1.1)$$

where D is a bounded open set in $\mathbb{R}^d$, Δ_p is the usual p -Laplacian, $p > 1$, and μ is a bounded measure on $D_{0,T} := (0, T) \times D$ charging no set of zero parabolic p -capacity associated with $\frac{\partial}{\partial t} - \Delta_p$ (see below) an important role is played by the result on the decomposition of μ proved by Droniou et al. [6] (see, e.g., [6, 24, 25]; note that in these papers more general than Δ_p operators of the form $A(u) = \operatorname{div} a(t, x, \nabla u)$ are considered). The decomposition proved in [6] says that each such measure μ (we call it diffuse) is of the form

$$\mu = f + \operatorname{div}(G) + \partial_t g \quad (1.2)$$

for some $f \in L^1(D_{0,T})$, $G \in (L^{p'}(D_{0,T}))^d$ and $g \in L^p(0, T; W_0^{1,p}(D) \cap L^2(D))$. Recently, in [16], the converse to this result was proved. The decomposition (1.2) is a counterpart to the decomposition of diffuse measures proved in the stationary case by Boccardo et al. [3]. The decomposition of [3] was extended to the Dirichlet forms setting in [15].

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There has recently been increasing interest in semilinear evolution problems of the form

$$-\frac{\partial u}{\partial t} - L_t u = f(\cdot, u) + \mu, \quad u(T, \cdot) = \varphi, \quad (1.3)$$

involving operators L_t associated with a (possibly nonlocal) Dirichlet form and bounded measure that do not charge the sets of zero parabolic capacity associated with $\partial_t + L_t$ (see [12–14] and the references therein). Motivated by possible applications to problems of the form (1.3), in the present paper we investigate the structure of such measures. We extend the results of [15] to the parabolic setting and at the same time the results of [6] with $p = 2$ to more general parabolic operators. As a by-product, we obtain some results on the existence of solutions to equations of the form (1.3) with $\mu \in \mathcal{W}'_0$ and on the structure of additive functionals associated in the Revuz sense with bounded smooth measures.

Let E be a locally compact separable metric space, $E_{0,T} := (0, T) \times E$ for some $T > 0$, and let m be a Radon measure on E such that $\text{supp}[m] = E$. In the paper we consider smooth measures with respect to parabolic capacities associated with a family $\{L_t, t \in [0, T]\}$ of closed operators generated by a family $\{(B^{(t)}, V), t \in [0, T]\}$ of regular (non-symmetric) Dirichlet forms on $L^2(E; m)$, with common domain V , satisfying some mild regularity assumptions. Our general Dirichlet forms setting allows us to treat both local and nonlocal operators. The results of the paper are new even for local operators. However, in our opinion, the most interesting fact is that we are able to describe the structure of smooth measures (and related additive functionals) for capacities associated with quite large class of parabolic nonlocal operators.

The model example of the family of local operators satisfying our assumptions is the family of divergence form operators

$$L_t = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(t, x) \frac{\partial}{\partial x_j} \right), \quad t \in [0, T], \quad x \in D, \quad (1.4)$$

on $L^2(D; dx)$ with zero Dirichlet boundary conditions. In (1.4), D is a bounded open subset of $\mathbb{R}^d$ and $\{a_{ij}(t, x)\}_{i,j=1,\dots,d}$ is a symmetric uniformly elliptic matrix with bounded measurable elements. In this case the family $\{(B^{(t)}, V), t \in [0, T]\}$ is of the form

$$B^{(t)}(u, v) = \sum_{i,j=1}^d \int_D a_{ij}(t, x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) dx, \quad u, v \in V := H_0^1(D). \quad (1.5)$$

A model example of the family of nonlocal operators satisfying our assumptions is the family consisting of single fractional Laplace operator

$$L_t = -(-\Delta)^{\alpha/2} \quad (1.6)$$

on $L^2(D; dx)$ with zero exterior condition. Here $\alpha \in (0, 2)$ and D is an open subset of $\mathbb{R}^d$. In this case $(B^{(t)}, V) = (B, V)$, $t \in [0, T]$, where

$$B(u, v) = \int_{\mathbb{R}^d} \hat{u}(x) \bar{\hat{v}}(x) |x|^\alpha dx, \quad u, v \in V := \{w \in L^2(D; dx) : \int_{\mathbb{R}^d} |\hat{w}|^2 |x|^\alpha dx < \infty\}$$

and $\hat{u}$ (resp. $\hat{v}$) is the Fourier transform of u (resp. v) (see [8, Example 1.4.1, Theorem 4.4.3]).

In case of problem (1.1), the natural capacity is the p -parabolic capacity cap_p defined for open set $U \subset E_{0,T}$ by

$$\text{cap}_p(U) = \inf\{\|\partial_t u\|_{L^{p'}(0,T;V_p')} + \|u\|_{L^p(0,T;V_p)} : u \geq \mathbf{1}_U \text{ dt} \otimes dx\text{-a.e.}\},$$

where $V_p = W_0^{1,p} \cap L^2(D)$ and V_p' is the dual of V_p (see [6, 28]). To study evolution problems with the operator $\partial_t + L_t$, Pierre [28] introduced the capacity c_2 defined for open $U \subset E_{0,T}$ by

$$c_2(U) = \inf\{\|\partial_t u\|_{L^2(0,T;V')} + \left(\int_0^T B^{(t)}(u(t), u(t)) dt\right)^{1/2} : u \geq \mathbf{1}_U \text{ m}_1\text{-a.e.}\},$$

where $m_1 = dt \otimes m$. In the potential theory, Borel measures on $E_{0,T}$ which do not charge sets of zero capacity c_2 and satisfy some quasi-finiteness condition are called smooth measures. In particular, each bounded Borel measure “absolutely continuous” with respect to c_2 is smooth. Of course, in case of operators of the form (1.4) the classes of diffuse measures and bounded smooth measures coincide.

In our main theorem we extend (1.2) to the Dirichlet form setting. Let $\mathcal{M}_{0,b}(E_{0,T})$ denote the set of all bounded smooth measures on $E_{0,T}$. We show that each $\mu \in \mathcal{M}_{0,b}(E_{0,T})$ admits decomposition of the form

$$\mu = f \cdot m_1 + g_1 + \partial_t g_2 \quad (1.7)$$

with $f \in L^1(E_{0,T}; m_1)$, $g_1 \in \mathcal{V}' = L^2(0, T; V')$, $g_2 \in \mathcal{V} = L^2(0, T; V)$, i.e. for every bounded quasi-continuous $\eta \in \mathcal{W}_0 = \{u \in \mathcal{V} : \partial_t u \in \mathcal{V}', u(0) = 0\}$, we have

$$\int_{E_{0,T}} \eta d\mu = \int_{E_{0,T}} f \eta dt dm + \langle g_1, \eta \rangle - \langle \partial_t \eta, g_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathcal{V}'$ and $\mathcal{V}$. We also show the converse of this theorem. Namely, each $\mu \in \mathcal{M}_b(E_{0,T})$ having decomposition (1.7) is smooth. Note that the converse is an extension of the result proved by Fukushima [7] to time-dependent Dirichlet forms. The proof of the fact that $\mu \in \mathcal{M}_{0,b}(E_{0,T})$ can be written in the form (1.7) is purely analytic. Essential to the proof of the converse part are probabilistic methods.

In applications to (1.1), the analysis of additional properties of the term g appearing in the decomposition (1.2) proved to be important. For instance, crucial to the definition and the existence result of a solution u of (1.1) is the fact that g regularizes u with

respect to time in the sense that $u - g \in W \subset C([0, T], L^p(D))$, where $W = \{u \in L^p(0, T; V_p) : \partial_t u \in L^{p'}(0, T; V_p')\}$. This property together with some other useful properties of g has been proved in [6]. In the present paper, applying the potential theory tools, we show that g_2 from (1.7) enjoys similar properties. We also show some new results on the regularity of g_2 . We show the following useful properties.

- g_2 has an m_1 -version $\tilde{g}_2$ which is quasi-càdlàg (i.e. quasi-right-continuous with left limits; this notion generalizes the notion of quasi-continuity; see Sect. 9) and g_2 is a difference of c_2 -quasi-l.s.c. functions.
- g_2 has an m_1 -version $\tilde{g}_2$ which is c_2 -quasi-bounded, i.e. there exists an increasing sequence $\{F_n\}$ of closed subsets of $E_{0,T}$ such that $c_2(E_{0,T} \setminus F_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\|\tilde{g}_2 \mathbf{1}_{F_n}\|_\infty < \infty$, $n \geq 1$.
- The function $[0, T] \ni t \mapsto g_2(t) \in L^1(E; \rho \cdot m)$ is càdlàg (right-continuous with left limits) for every positive Borel function ρ on E such that $\int_E \rho \, dm < \infty$, and $g_2(T-) := \lim_{t \rightarrow T-} g_2(t) = 0$.
- The measure μ_t defined by $\mu_t(B) = \mu(\{t\} \times B)$ for Borel sets $B \subset E$ is absolutely continuous with respect to m and $\mu_t = (g_2(t) - g_2(t-)) \cdot m$.
- If μ admits decomposition $\mu = f' \cdot m_1 + g'_1 + \partial_t g'_2$ with some $f' \in L^1(E_{0,T}; m_1)$, $g'_1 \in \mathcal{V}'$, $g'_2 \in \mathcal{V}$, then the function $[0, T] \ni t \mapsto (g_2 - g'_2)(t) \in L^1(E; \rho \cdot m)$ belongs to $C([0, T], L^1(E; \rho \cdot m))$ for every positive Borel function ρ on E such that $\int_E \rho \, dm < \infty$.

It is worth pointing out here that the proofs of the above results in the general setting requires us to use quite different methods than those used in [6] for the Leray–Lions-type operators, which are strongly based on the regularization of the measure μ by a convolution operator.

In the proof of our main decomposition theorem, we apply some deep results from the potential theory for evolution operators proved by Pierre [26–28], as well as from the probabilistic potential theory for time dependent or generalized Dirichlet forms developed in the papers by Oshima [20–23], Stannat [30, 31] and Trutnau [32, 33]. In these papers the definitions of the capacity (and hence some quasi-notions) are different. In Sect. 3, which is technical but important for us, we show that all these capacities are in fact equivalent on $E_{0,T}$. This allows us to apply freely the results from the papers mentioned above.

One of the most important ingredient of the proof that each $\mu \in \mathcal{M}_b(E_{0,T})$ having decomposition (1.7) is smooth is an existence result for the Cauchy problem (1.3) with $\mu \in \mathcal{W}'_0$ and f not depending on u . If $\mu \in L^2(0, T; V')$, then the existence of a solution to (1.3) follows from the classical theory of variational inequalities (see [17]). However, if $\mu \in \mathcal{W}'_0$, then the situation is more difficult. To prove the existence of a solution, a decomposition similar to (1.7), but for functionals from $\mathcal{W}'_0$ is needed. In case of (1.1), such a decomposition and an existence result were proved in [6]. In Sect. 5, we prove a similar decomposition result for $\mu \in \mathcal{W}'_0$, and then, in Sect. 6, we deal with the existence of a solution of (1.3) with $\mu \in \mathcal{W}'_0$. Though for our applications we only need the existence result for liner equations, in the paper we show that there

exist a solution for semilinear problem (1.3) under the assumption that $\varphi \in L^2(E; m)$ and $u \mapsto f(\cdot, u)$ is continuous, nonincreasing and satisfies the linear growth condition. We think that this result may be of independent interest. Finally, note here that in the proof that μ given by (1.7) is smooth we use a very recent result from the paper by Beznea and Cîmpean [1] on the characterization of quasimartingale functions.

It is well known that there is a one to one correspondence, called Revuz correspondence, between positive smooth, with respect to the Dirichlet form (B, V) , measures on E and positive continuous additive functionals of the Hunt processes associated with (B, V) (see [8, 19]). In the parabolic case the situation is more subtle. Let $\mathbb{X}$ denote a Hunt process associated with a generalized Dirichlet form $\mathcal{E}$ in the resolvent sense (see Sect. 2) which is generated by the operator $\frac{\partial}{\partial t} + L_t$. In the paper we first show that with each smooth measure μ on $E_{0,T}$ with respect the form $\mathcal{E}$ one can associate uniquely a positive additive functional A^μ of $\mathbb{X}$ which is natural, i.e. has no common discontinuities with the Hunt process associated with $\mathcal{E}$. This is a counterpart to the known result concerning smooth measures on $\mathbb{R} \times E$ (see [21, 23]). Then we analyse more carefully the nature of jumps of A^μ in the case where $\mu \in \mathcal{M}_{0,b}(E_{0,T})$. Roughly speaking, our main result says that for $\mu \in \mathcal{M}_{0,b}(E_{0,T})$ the jumps of A^μ are related to g_2 from decomposition (1.7). We show that g_2 has always a quasi-càdlàg modification $\tilde{g}_2$, i.e. an m_1 -version $\tilde{g}_2$ such that the process $t \mapsto \tilde{g}_2(X_t)$ is right-continuous with left limits, and for every predictable stopping time τ ,

$$\Delta A^\mu_\tau := A^\mu_\tau - A^\mu_{\tau-} = \Delta \tilde{g}_2(X)_\tau.$$

In other words, A^μ has jumps that coincide with the jumps of the process $\tilde{g}_2(X)$ in predictable stopping times. This implies that $\tilde{g}_2$ is quasi-continuous if and only if A^μ is continuous.

2. Preliminaries

In this paper, E is a locally compact separable metric space and m is an everywhere dense Radon measure on E , i.e. m is a positive Borel measure on E , which is finite on compact sets and strictly positive on nonempty open sets.

We set $E^1 = \mathbb{R} \times E$, $m_1 = dt \otimes m$, and for $T > 0$, we set $E_{0,T} = (0, T) \times E$. We denote by $\mathcal{B}(E^1)$ (resp. $\mathcal{B}(E_{0,T})$) the set of all Borel measurable subsets of E^1 (resp. $E_{0,T}$). With the customary abuse of notation, the same symbols are used to denote the sets of real Borel measurable functions on E^1 (resp. $E_{0,T}$). $\mathcal{B}_b(E^1)$ is the set of all real bounded Borel measurable functions on E^1 and $\mathcal{B}_b^+(E^1)$ is the subset of $\mathcal{B}_b(E^1)$ consisting of positive functions. The sets $\mathcal{B}_b(E_{0,T})$, $\mathcal{B}_b^+(E_{0,T})$ are defined analogously.

2.1. Time-dependent Dirichlet forms

Let $H = L^2(E; m)$ and $(\cdot, \cdot)_H$ denote the usual inner product in H . In this paper, we assume that we are given a family $\{(B^{(t)}, V), t \in [0, T]\}$ of regular (non-symmetric)

Dirichlet forms on H with common domain $V \subset H$ (see [19, Chapter I] for the definitions) satisfying the following conditions.

- (a) There is $K \geq 0$ such that $|B_1^{(t)}(\varphi, \psi)| \leq K(B_1^{(t)}(\varphi, \varphi))^{1/2}(B_1^{(t)}(\psi, \psi))^{1/2}$ for all $\varphi, \psi \in V, t \in [0, T]$, where as usual, we set $B_\lambda^{(t)}(\varphi, \psi) = B^{(t)}(\varphi, \psi) + \lambda(\varphi, \psi)_H$ for $\lambda \geq 0$.
- (b) $[0, T] \ni t \mapsto B^{(t)}(\varphi, \psi)$ is measurable for all $\varphi, \psi \in V$.
- (c) There is $c \geq 1$ such that $c^{-1}B^{(0)}(\varphi, \varphi) \leq B^{(t)}(\varphi, \varphi) \leq cB^{(0)}(\varphi, \varphi)$ for all $t \in [0, T]$ and $\varphi \in V$.

To shorten notation, we continue to write B for $B^{(0)}$. By putting $B^{(t)} = B$ for $t \notin [0, T]$, we may and will assume that $B^{(t)}$ is defined and satisfies (c) for all $t \in \mathbb{R}$. We denote by $\tilde{B}^{(t)}$ the symmetric part of $B^{(t)}$, i.e. $\tilde{B}^{(t)}(\varphi, \psi) = \frac{1}{2}(B^{(t)}(\varphi, \psi) + B^{(t)}(\psi, \varphi))$.

Since V is a dense subspace of H and (B, V) is closed, V is a real Hilbert space with respect to $\tilde{B}_1(\cdot, \cdot)$, which is densely and continuously embedded in H . We equip V with the norm $\|\cdot\|_V$ defined by $\|\varphi\|_V^2 = B_1(\varphi, \varphi)$, $\varphi \in V$. We denote by V' the dual space of V , and by $\|\cdot\|_{V'}$ the corresponding norm. For $T > 0$, we set

$$\mathcal{H} = L^2(0, T; H), \quad \mathcal{V} = L^2(0, T; V), \quad \mathcal{V}' = L^2(0, T; V')$$

and

$$\|u\|_{\mathcal{V}}^2 = \int_0^T \|u(t)\|_V^2 dt, \quad \|u\|_{\mathcal{V}'}^2 = \int_0^T \|u(t)\|_{V'}^2 dt.$$

We shall identify H and its dual H' . Then $V \subset H \simeq H' \subset V'$ continuously and densely, and hence $\mathcal{V} \subset \mathcal{H} \simeq \mathcal{H}' \subset \mathcal{V}'$ continuously and densely.

For given $u \in \mathcal{V}$ let $\partial_t u$ denote the derivative in the distribution sense of the function $t \mapsto u(t) \in V$, and let

$$\mathcal{W} = \{u \in \mathcal{V} : \partial_t u \in \mathcal{V}'\}, \quad \|u\|_{\mathcal{W}} = \|u\|_{\mathcal{V}} + \|\partial_t u\|_{\mathcal{V}'}.$$

It is well known that there is a continuous embedding of $\mathcal{W}$ into $C([0, T]; H)$, i.e. for every $u \in \mathcal{W}$ one can find $\bar{u} \in C([0, T]; H)$ such that $u(t) = \bar{u}(t)$ for a.e. $t \in [0, T]$ (with respect to the Lebesgue measure) and

$$\|u\|_{C([0, T]; H)} \leq C\|u\|_{\mathcal{W}} \tag{2.1}$$

for some $C > 0$. In what follows, we adopt the convention that any element of $\mathcal{W}$ is already in $C([0, T]; H)$. With this convention we may define the spaces

$$\mathcal{W}_0 = \{u \in \mathcal{W} : u(0) = 0\}, \quad \mathcal{W}_T = \{u \in \mathcal{W} : u(T) = 0\}.$$

Note that $\mathcal{W}_0, \mathcal{W}_T$ are reflexive spaces as closed linear subspaces of the reflexive space $\mathcal{W}$.

The linear operator ∂_t on $\mathcal{H}$ with domain $\mathcal{W}_T$ will be denoted by Λ . Its adjoint, i.e. the operator $-\partial_t$ with domain $\mathcal{W}_0$, will be denoted by $\hat{\Lambda}$.

We denote by $\mathcal{E}$ the generalized Dirichlet form associated with Λ and the family $\{(B^{(t)}, V), t \in [0, T]\}$, that is

$$\mathcal{E}(u, v) = \begin{cases} \langle -\partial_t u, v \rangle + \mathcal{B}(u, v), & u \in \mathcal{W}_T, v \in \mathcal{V}, \\ \langle \partial_t v, u \rangle + \mathcal{B}(u, v), & u \in \mathcal{V}, v \in \mathcal{W}_0, \end{cases} \quad (2.2)$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$, and

$$\mathcal{B}(u, v) = \int_0^T B^{(t)}(u(t), v(t)) dt. \quad (2.3)$$

Let $D(\mathcal{E}) = (\mathcal{W}_T \times \mathcal{V}) \cup (\mathcal{V} \times \mathcal{W}_0)$. For every $\alpha \geq 0$ we set

$$\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v)_{\mathcal{H}}, \quad (u, v) \in D(\mathcal{E}).$$

The generalized form associated with $\hat{\Lambda}$ and the family $\{(\hat{B}^{(t)}, V), t \in [0, T]\}$, where $\hat{B}^{(t)}(u, v) = B^{(t)}(v, u)$, $u, v \in V$, $t \in [0, T]$, will be denoted by $\hat{\mathcal{E}}$.

In the paper, we denote by $(G_\alpha)_{\alpha>0}$ (resp. $\hat{G}_\alpha)_{\alpha>0}$ the resolvent (resp. coresolvent) associated with the form $\mathcal{E}$, i.e. $(G_\alpha)_{\alpha>0}$, $(\hat{G}_\alpha)_{\alpha>0}$ are strongly continuous resolvents of contractions on $\mathcal{H}$ such that $G_\alpha(\mathcal{H}) \subset \mathcal{W}_T$, $\hat{G}_\alpha(\mathcal{H}) \subset \mathcal{W}_0$ and

$$\mathcal{E}_\alpha(G_\alpha f, g) = (f, g)_{\mathcal{H}} = \mathcal{E}_\alpha(g, \hat{G}_\alpha f), \quad f \in \mathcal{H}, g \in \mathcal{V} \quad (2.4)$$

(for a construction of the resolvents see, e.g., [31, Chapter I]).

Let $\mathcal{W}^1 = \{u \in L^2(\mathbb{R}; V) : \partial_t u \in L^2(\mathbb{R}; V')\}$. We denote by $\mathcal{E}^1$ the time-dependent Dirichlet form associated with $\{(B^{(t)}, V), t \in \mathbb{R}\}$, that is

$$\mathcal{E}^1(u, v) = \begin{cases} \langle -\partial_t u, v \rangle + \mathcal{B}^1(u, v), & u \in \mathcal{W}^1, v \in L^2(\mathbb{R}; V), \\ \langle \partial_t v, u \rangle + \mathcal{B}^1(u, v), & u \in L^2(\mathbb{R}; V), v \in \mathcal{W}^1, \end{cases}$$

where now $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $L^2(\mathbb{R}; V')$ and $L^2(\mathbb{R}; V)$, and

$$\mathcal{B}^1(u, v) = \int_{\mathbb{R}} B^{(t)}(u(t), v(t)) dt.$$

Analogously to the form $\mathcal{E}$ we define $\mathcal{E}_\alpha^1$ for every $\alpha \geq 0$. The resolvent (resp. coresolvent) associated with the form $\mathcal{E}^1$ (see [31, Chapter I]) will be denoted by $(G_\alpha^1)_{\alpha>0}$ (resp. $(\hat{G}_\alpha^1)_{\alpha>0}$).

Note that $\mathcal{E}^1$ can be identified with some generalized Dirichlet forms in the sense considered in [31–33] (see [31, Example I.4.9(iii)]).

Let $\psi \in L^1(E_{0,T}; m_1) \cap \mathcal{B}(E_{0,T})$ be a function such that $0 < \psi \leq 1$. We define the capacity Cap_ψ associated with $\mathcal{E}$ as in [31, Sect. III.2] (see also [33]), that is for an open set $U \subset E_{0,T}$ we set

$$\text{Cap}_\psi(U) = ((G_1 \psi)_U, \psi)_{\mathcal{H}},$$

where $(G_1 \psi)_U$ is the 1-reduced function of $G_1 \psi$, and then for arbitrary $A \subset E_{0,T}$ we set $\text{Cap}_\psi(A) = \inf\{\text{Cap}_\psi(U), U \supset A, U \text{ open}\}$. By [31, Lemma III.2.9] and (2.4),

for any open $U \subset E_{0,T}$, $((G_1\psi)_U, \psi)_{\mathcal{H}} = \mathcal{E}_1((G_1\psi)_U, \hat{G}_1\psi) = \mathcal{E}_1(G_1\psi, (\hat{G}_1\psi)_U) = (\psi, (\hat{G}_1\psi)_U)_{\mathcal{H}}$, so Cap_{ψ} associated with $\mathcal{E}$ coincides with the capacity defined as Cap_{ψ} but for the dual form $\hat{\mathcal{E}}$. In particular, Cap_{ψ} coincides with the capacity considered in [32].

By [31, Proposition III.2.8], Cap_{ψ} is a Choquet capacity. Also note that by [31, Proposition III.2.10], the family of sets $A \subset E_{0,T}$ such that $\text{Cap}_{\psi}(A) = 0$ is the same for all ψ as above.

We will denote by Cap^1 the capacity associated with $\mathcal{E}^1$ and defined in [22] (the definition is also given in [21] and [23, Sect. 6.2]). By [22, Lemma 4.2] (or [23, Lemma 6.2.8]) and [8, Theorem A.1.2], Cap^1 is a Choquet capacity.

Some relations between Cap_{ψ} and Cap^1 , as well as relations between Cap_{ψ} and some other notions of parabolic capacity considered in the literature will be studied in Sect. 3.

We say that a set $A \subset E_{0,T}$ (resp. $A \subset E^1$) is $\mathcal{E}$ -exceptional (resp. $\mathcal{E}^1$ -exceptional) if $\text{Cap}_{\psi}(A) = 0$ (resp. $\text{Cap}^1(A) = 0$), and we say that a property holds $\mathcal{E}$ -quasi-everywhere (resp. $\mathcal{E}^1$ -quasi-everywhere) if the set of those $x \in E_{0,T}$ (resp. $x \in E^1$) for which it does not hold is $\mathcal{E}$ -exceptional (resp. $\mathcal{E}^1$ -exceptional).

Recall that an increasing sequence $\{F_n\}$ of closed subsets of $E_{0,T}$ (resp. E^1) is called an $\mathcal{E}$ -nest (resp. $\mathcal{E}^1$ -nest) if $\text{Cap}_{\psi}(F_n^c) \rightarrow 0$ (resp. $\text{Cap}^1(F_n^c) \rightarrow 0$) as $n \rightarrow \infty$. An increasing sequence $\{F_n\}$ of closed subsets of E^1 is called a generalized $\mathcal{E}^1$ -nest if $\text{Cap}^1(K \setminus F_n) \rightarrow 0$ for every compact $K \subset E_{0,T}$.

A function $u : E_{0,T} \rightarrow \mathbb{R}$ (resp. $u : E^1 \rightarrow \mathbb{R}$) is called $\mathcal{E}$ -quasi-continuous (resp. $\mathcal{E}^1$ -quasi-continuous) if there exists an $\mathcal{E}$ -nest (resp. $\mathcal{E}$ -nest) $\{F_n\}$ such that $u|_{F_n}$ is continuous for every $n \in \mathbb{N}$.

It is known (see [31, Proposition IV.1.8]) that each $u \in \mathcal{W}_0$ has an $\mathcal{E}$ -quasi-continuous m_1 -version. Similarly, each $u \in \mathcal{W}_T$ has an $\mathcal{E}$ -quasi-continuous m_1 -version. We will denote them by $\tilde{u}$.

2.2. Dirichlet forms and Markov processes

Let Δ be adjoint to E^1 as the point at infinity. We adopt the convention that every function f on E^1 (resp. $E_{0,T}$) is extended to $E^1 \cup \{\Delta\}$ (resp. $E_{0,T} \cup \{\Delta\}$) by setting $f(\Delta) = 0$.

Let $\Omega_1 = \{\omega : [0, \infty) \rightarrow E^1 \cup \{\Delta\} : \omega(s) = \Delta, s \geq t \text{ if } \omega(t) = \Delta\}$ and

$$X^1 : \Omega_1 \rightarrow \Omega_1, \quad X_t^1(\omega) = \omega(t).$$

By [20, Theorem 4.2] (see also [22, Theorem 5.1] or [23, Theorem 6.3.1]), there exists a Hunt process $\mathbb{X}^1 = (\Omega_1, (\mathcal{F}_t^1)_{t \geq 0}, (X_t^1)_{t \geq 0}, (P_x^1)_{x \in E^1 \cup \{\Delta\}})$ with life time ζ^1 and cemetery state Δ associated with $\mathcal{E}^1$ in the resolvent sense, i.e. for all $\alpha > 0$ and $f \in \mathcal{B}_b(E^1) \cap L^2(E^1; m_1)$ the resolvent of $\mathbb{X}^1$ defined as

$$R_{\alpha}^1 f(x) = E_x^1 \int_0^{\infty} e^{-\alpha t} f(X_t^1) dt, \quad x \in E^1, \quad f \in \mathcal{B}_b(E^1) \quad (2.5)$$

(E_x^1 stands for the expectation with respect to P_x^1) is an $\mathcal{E}^1$ -quasi-continuous m_1 -version of the resolvent $G_\alpha^1 f$. Let

$$v : \Omega_1 \rightarrow \Omega_1, \quad (v(\omega))(t) = \pi(\omega(0)) + t,$$

where $\pi : E^1 \rightarrow \mathbb{R}$ is the projection on $\mathbb{R}$. Note that under the measure P_x with $x = (s, x^0) \in E^1$, v is the uniform motion to the right, i.e. $v(t) = v(0) + t$, $v(0) = s$. By [20, Theorem 4.2] (or [22, Theorem 5.1]),

$$X_t^1 = (v(t), X_{v(t)}^0), \quad t \geq 0, \quad (2.6)$$

under the measure P_x for every $x \in E^1$.

Recall that $B \in \mathcal{B}(E^1)$ is called $\mathbb{X}^1$ -exceptional if $P_x^1(\sigma_B < \infty) = 0$ for m_1 -a.e. $x \in E^1$, where

$$\sigma_B = \inf\{t > 0 : X_t^1 \in B\}.$$

By the remarks in [22, page 298] (see also [21, Lemma 2.3]),

$$B \text{ is } \mathcal{E}^1\text{-exceptional if and only if } B \text{ is } \mathbb{X}^1\text{-exceptional.} \quad (2.7)$$

Similarly, by [31, Sect. IV.2], there exists a Hunt process $\mathbb{X} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P_x)_{x \in E_{0,T} \cup \{\Delta\}})$ with life time ζ associated with $\mathcal{E}$ in the resolvent sense, i.e. for all $\alpha > 0$ and $f \in \mathcal{B}_b(E_{0,T}) \cap L^2(E_{0,T}; m_1)$,

$$R_\alpha f(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t) dt, \quad x \in E_{0,T}, \quad f \in \mathcal{B}_b(E_{0,T}) \quad (2.8)$$

(E_x stands for the expectation with respect to P_x) is an $\mathcal{E}$ -quasi-continuous m_1 -version of the resolvent $G_\alpha f$. In this case $\Omega = \{\omega : [0, \infty) \rightarrow E_{0,T} \cup \{\Delta\} : \omega(t) = \Delta, s \geq t \text{ if } \omega(s) = \Delta\}$ and

$$X : \Omega \rightarrow \Omega, \quad X_t(\omega) = \omega(t).$$

It is clear that $\Omega \subset \Omega_1$ and $X = X^1$ on Ω .

A set $B \in \mathcal{B}(E_{0,T})$ is called $\mathbb{X}$ -exceptional if $P_x(\sigma_B < \infty) = 0$ for m_1 -a.e. $x \in E_{0,T}$, where σ_B is defined as before but with $\mathbb{X}^1$ replaced by $\mathbb{X}$. By [31, Theorem IV.3.8],

$$B \text{ is } \mathcal{E}\text{-exceptional if and only if } B \text{ is } \mathbb{X}\text{-exceptional.} \quad (2.9)$$

Let $f \in \mathcal{B}(E_{0,T}) \cap L^2(E_{0,T}; m_1)$, $g \in \mathcal{B}(E^1) \cap L^2(E^1; m_1)$ and $v = R_\alpha f$, $u = R_\alpha^1 g$. Since v is an m_1 version of $G_\alpha f$ and u is an m_1 -version of $G_\alpha^1 g$, we have $v \in \mathcal{W}_T$, $u \in \mathcal{W}^1$ and

$$\mathcal{E}_\alpha(v, \eta) = (f, \eta)_{L^2(E_{0,T}; m_1)}, \quad \eta \in \mathcal{W}_0, \quad \mathcal{E}_\alpha^1(u, \eta) = (g, \eta)_{L^2(E^1; m_1)}, \quad \eta \in \mathcal{W}^1. \quad (2.10)$$

Let $g = f$ on $E_{0,T}$ and $g = 0$ on $E^1 \setminus E_{0,T}$. Then, by (2.5) and (2.6), $u|_{E_{0,T}} \in \mathcal{W}_T$. So, in fact $u|_{E_{0,T}}$ is a solution to the first equation in (2.10). By uniqueness of a solution to (2.10), we get $v = u|_{E_{0,T}}$. By this and (2.5) and (2.8), we obtain

$$E_x^1 \int_0^\infty e^{-\alpha t} f(X_t^1) dt = E_x \int_0^\infty e^{-\alpha t} f(X_t) dt \quad (2.11)$$

for every $\alpha \geq 0$ and every $f \in \mathcal{B}^+(E^1)$ such that $\mathbf{1}_{E_{0,T}} f = f$. Let $\Pi : \Omega_1 \rightarrow \Omega$ be defined as

$$\Pi(\omega)(t) = \omega(t), \quad t < T - v(0), \quad \Pi(\omega)(t) = \Delta, \quad t \geq T - v(0).$$

Observe that

$$E_x^1 \int_0^\infty e^{-\alpha t} f(X_t^1) dt = \int_\Omega \int_0^\infty e^{-\alpha t} f(X_t) dt d(P_x^1 \circ \Pi^{-1}).$$

Hence

$$P_x = P_x^1 \circ \Pi^{-1}, \quad x \in E_{0,T}. \quad (2.12)$$

Therefore, for every $x \in E_{0,T}$,

$$\begin{aligned} P_x(\zeta \leq T - v(0)) &= P_x(\omega \in \Omega : \inf\{t \geq 0; X_t(\omega) \notin E_{0,T}\} \leq T - v(0)(\omega)) \\ &= P_x(\omega \in \Omega; \inf\{t \geq 0 : X_t^1(\omega) \notin E_{0,T}\} \leq T - v(0)(\omega)) \\ &= (P_x^1 \circ \Pi^{-1})(\omega \in \Omega : \inf\{t \geq 0; X_t^1(\omega) \notin E_{0,T}\} \leq T - v(0)(\omega)) \\ &= P_x^1(\omega \in \Omega_1 : \inf\{t \geq 0; X_t^1(\Pi(\omega)) \notin E_{0,T}\} \leq T - v(0)(\Pi(\omega))) \\ &= P_x^1(\omega \in \Omega_1 : \inf\{t \geq 0 : (v(t)(\Pi(\omega)), X_{v(t)(\Pi(\omega))}^0(\Pi(\omega))) \notin E_{0,T}\} \\ &\quad \leq T - v(0)(\Pi(\omega))) \\ &\geq P_x^1(\omega \in \Omega_1 : \inf\{t \geq 0 : v(t)(\Pi(\omega)) \notin (0, T)\} \leq T - v(0)(\Pi(\omega))) = 1. \end{aligned}$$

Thus

$$\zeta = \zeta \wedge (T - v(0)) \quad P_x\text{-a.s.} \quad (2.13)$$

In this paper, we denote by $\hat{\mathbb{X}}^1 = (\Omega_1, (\hat{\mathcal{F}}_t^1)_{t \geq 0}, (X_t^1)_{t \geq 0}, (\hat{P}_x^1)_{x \in E^1 \cup \{\Delta\}}, \zeta^1)$ the dual process of $\mathbb{X}^1$, i.e. a Hunt process whose resolvents $\hat{R}_\alpha^1 f$ are $\mathcal{E}^1$ -quasi-continuous m_1 -versions of $\hat{G}_\alpha^1 f$ for any $f \in \mathcal{B}_b(E^1) \cap L^2(E^1; m_1)$. By [22, Theorem 5.1],

$$X_t^1 = (\hat{v}(t), \hat{X}_{\hat{v}(t)}^0), \quad t \geq 0, \quad (2.14)$$

where $\hat{v}$ is the uniform motion to the left, i.e. $\hat{v}(t) = \hat{v}(0) - t$, $\hat{v}(0) = s$ under the measure $\hat{P}_x$ with $x = (s, x^0) \in E^1$.

We denote by $\hat{\mathbb{X}} = (\Omega, (\hat{\mathcal{F}}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\hat{P}_x)_{x \in E_{0,T} \cup \{\Delta\}}, \zeta)$ the dual process of $\mathbb{X}$. It is a Hunt process associated with the dual form $\hat{\mathcal{E}}$. For all $\alpha > 0$ and $f \in \mathcal{B}_b(E_{0,T}) \cap L^2(E_{0,T}; m_1)$ the resolvent of $\hat{\mathbb{X}}$ defined as

$$\hat{R}_\alpha f(x) = \hat{E}_x \int_0^\infty e^{-\alpha t} f(X_t) dt, \quad x \in E_{0,T},$$

where $\hat{E}_x$ denotes the expectation with respect to $\hat{P}_x$, is an $\mathcal{E}$ -quasi-continuous m_1 -version of the coresolvent $\hat{G}_\alpha f$. A modification of the argument used to prove (2.12) and (2.13) shows that

$$\hat{P}_x = \hat{P}_x^1 \circ \hat{\Pi}^{-1}, \quad \zeta = \zeta \wedge \hat{v}(0), \quad \hat{P}_x\text{-a.s.} \quad (2.15)$$

for $x \in E_{0,T}$, where $\hat{\Pi} : \Omega_1 \rightarrow \Omega$ is defined as

$$\hat{\Pi}(\omega)(t) = \omega(t), \quad t < \hat{v}(0), \quad \hat{\Pi}(\omega)(t) = \Delta, \quad t \geq \hat{v}(0).$$

The relation (2.13) [resp. (2.15)] implies that the operator G_α (resp. $\hat{G}_\alpha$) is well defined for $\alpha = 0$, and that $R_0 f$ (resp. $\hat{R} f$) is $\mathcal{E}$ -quasi-continuous m_1 -version of $G_0 f$ (resp. $\hat{G}_0 f$) for every $f \in \mathcal{H}$. Indeed, by (2.13), for every positive $f \in \mathcal{B}(E_{0,T})$,

$$Rf \leq e^{\alpha T} R_\alpha f$$

(we write R instead of R_0). Hence, for every $f \in \mathcal{H} \cap \mathcal{B}(E_{0,T})$, $Rf \in \mathcal{H}$. By the resolvent identity,

$$Rf = R_\alpha f + \alpha R_\alpha Rf.$$

Therefore $Rf \in G_\alpha(\mathcal{H})$, and by (2.4),

$$\mathcal{E}(Rf, g) = (f, g)_{\mathcal{H}}, \quad f \in \mathcal{H}, g \in \mathcal{V}. \quad (2.16)$$

In other words, $G_0 f = Rf$ m_1 -a.e. Of course, Rf is $\mathcal{E}$ -quasi-continuous since $Rf = R_\alpha(f + \alpha Rf)$. A similar argument applies to $\hat{R}_0$ and $\hat{G}_0$.

2.3. Smooth measures

A Borel (signed) measure μ on $E_{0,T}$ is called $\mathcal{E}$ -smooth if it does not charge exceptional sets, i.e. for any Borel set $B \subset E_{0,T}$, if $\text{Cap}_\psi(B) = 0$, then $\mu(B) = 0$, and there exists an $\mathcal{E}$ -nest $\{F_n\}$ of compact subsets of $E_{0,T}$ such that $|\mu|(F_n) < \infty$ for $n \in \mathbb{N}$, where $|\mu|$ denotes the variation of μ . A Borel measure μ on E^1 is called $\mathcal{E}^1$ -smooth if it does not charge $\mathcal{E}^1$ -exceptional sets and there exists a generalized $\mathcal{E}^1$ -nest $\{F_n\}$ such that $|\mu|(F_n) < \infty$ for $n \in \mathbb{N}$.

The set of all $\mathcal{E}$ -smooth (resp. $\mathcal{E}^1$ -smooth) measures will be denoted by $S(E_{0,T})$ (resp. $S^1(E^1)$). $\mathcal{M}_b(E_{0,T})$ (resp. $\mathcal{M}_b^1(E^1)$) denotes the set of all bounded Borel measures on $E_{0,T}$ (resp. E^1), i.e. Borel measures μ such that $|\mu|(E_{0,T}) < \infty$ (resp.

$|\mu|(E^1) < \infty)$. $\mathcal{M}_{0,b}(E_{0,T})$ (resp. $\mathcal{M}_{0,b}^1(E^1)$) denotes the subset of $\mathcal{M}_b(E_{0,T})$ (resp. $\mathcal{M}_b^1(E^1)$) consisting of all $\mathcal{E}$ -smooth (resp. $\mathcal{E}^1$ -smooth) measures.

Let μ be a positive Borel measure on $E_{0,T}$ such that μ does not charge sets of zero capacity. We call it a measure of finite energy integral if there is $C \geq 0$ such that for every positive $\eta \in \mathcal{W}_0$,

$$\int_{E_{0,T}} \tilde{\eta} \, d\mu \leq C \|\eta\|_{\mathcal{W}}. \quad (2.17)$$

If there is $C \geq 0$ such that for every positive $\eta \in \mathcal{W}_T$,

$$\int_{E_T} \tilde{\eta} \, d\mu \leq C \|\eta\|_{\mathcal{W}},$$

we call it a measure of finite co-energy integral. In both cases, $\tilde{\eta}$ denotes an $\mathcal{E}$ -quasi-continuous m_1 -version of η . The set of all positive smooth measures on $E_{0,T}$ of finite energy (resp. co-energy) integral will be denoted by $S_0(E_{0,T})$ (resp. $\hat{S}_0(E_{0,T})$).

LEMMA 2.1. *Let μ be a positive Borel measure on $E_{0,T}$. Then*

(i) $\mu \in S_0(E_{0,T})$ if and only if there exists $u \in \mathcal{V}$ such that

$$\int_{E_{0,T}} \tilde{v} \, d\mu = \mathcal{E}_1(u, v), \quad v \in \mathcal{W}_0. \quad (2.18)$$

(ii) $\mu \in \hat{S}_0(E_{0,T})$ if and only if there exists $u \in \mathcal{V}$ such that

$$\int_{E_{0,T}} \tilde{v} \, d\mu = \mathcal{E}_1(v, u), \quad v \in \mathcal{W}_T. \quad (2.19)$$

Proof. We provide the proof of (i). The proof of (ii) is analogous. If (2.18) is satisfied, then of course $\mu \in S_0(E_{0,T})$. Suppose that μ is of finite energy integral. Then, by [33, Lemma 4.2] applied to the dual form $\hat{\mathcal{E}}$, for every positive $\eta \in \mathcal{W}_0$ there exists $u \in \mathcal{V}$ such that

$$\int_{E_{0,T}} \alpha \widetilde{\hat{G}_\alpha \eta} \, d\mu = \mathcal{E}_1(u, \alpha \hat{G}_\alpha \eta), \quad (2.20)$$

where $\widetilde{\hat{G}_\alpha \eta}$ is a quasi-continuous modification of $\hat{G}_\alpha \eta$. By [30, Proposition 2.7], $\alpha \hat{G}_\alpha \eta \rightarrow \eta$ strongly in $\mathcal{W}_0$. Therefore letting $\alpha \rightarrow \infty$ in (2.20) and applying [31, Corollary III.3.8], Fatou's lemma and [28, Lemma 2], we get (2.17). $\square$

Let $\mu \in S_0(E_{0,T})$. The element $u \in \mathcal{V}$ defined by (2.18) is uniquely determined. We will denote it by $U_1\mu$. Similarly, for $\mu \in \hat{S}_0(E_{0,T})$, the element $u \in \mathcal{V}$ defined by (2.19) is uniquely determined. We will denote it by $\hat{U}_1\mu$. We also set

$$U_\alpha\mu = U_1\mu + (1 - \alpha)G_\alpha U_1\mu, \quad \hat{U}_\alpha\mu = \hat{U}_1\mu + (1 - \alpha)\hat{G}_\alpha \hat{U}_1\mu \quad \alpha \geq 0.$$

It is clear that for any $\alpha \geq 0$,

$$\int_{E_{0,T}} \tilde{v} \, d\mu = \mathcal{E}_\alpha(U_\alpha\mu, v), \quad v \in \mathcal{W}_0, \quad \int_{E_{0,T}} \tilde{v} \, d\mu = \mathcal{E}_\alpha(v, \hat{U}_\alpha\mu), \quad v \in \mathcal{W}_T. \quad (2.21)$$

3. Parabolic capacity

Our basic capacity associated with $\mathcal{E}$ is Cap_ψ . Exceptional sets with respect to Cap_ψ have nice probabilistic interpretation given by (2.9). However, in the literature devoted to partial differential equations, usually some other notions of capacity are used. In this section, we recall some of them and prove that they are all equivalent to Cap_ψ . These results will be needed in the next sections.

LEMMA 3.1. *For any $A \subset E_{0,T}$, $\text{Cap}_\psi(A) = 0$ if and only if $\text{Cap}_{|E_{0,T}}^1(A) = 0$.*

Proof. Since both Cap_ψ and $\text{Cap}_{|E_{0,T}}^1$ are Choquet capacities, it is enough to prove that for any $B \in \mathcal{B}(E_{0,T})$, $\text{Cap}_\psi(B) = 0$ if and only if $\text{Cap}^1(B) = 0$. Suppose that $\text{Cap}^1(B) = 0$. Then, by (2.7), $P_x^1(\sigma_B < \infty) = 0$ for m_1 -a.e. $x \in E^1$. Observe that

$$\{\omega \in \Omega_1; \exists_{t>0} X_t^1(\omega) \in B\} = \Pi^{-1}(\{\omega \in \Omega; \exists_{t>0} X_t(\omega) \in B\}).$$

Hence, by (2.12), $P_x(\sigma_B < \infty) = 0$ for m_1 -a.e. $x \in E_{0,T}$. Consequently, $\text{Cap}_\psi(B) = 0$ by (2.9). Now suppose that $\text{Cap}_\psi(B) = 0$. Then, by [31, Lemma III.2.9] and (2.9), $P_x(\sigma_B < \infty) = \hat{P}_x(\sigma_B < \infty) = 0$ for m_1 -a.e. $x \in E_{0,T}$. From this and (2.6) and (2.12) it follows that

$$P_x^1(\sigma_B < \infty) = 0 \quad \text{for } m_1\text{-a.e. } x \in [0, \infty) \times E. \quad (3.1)$$

Similarly, from (2.14) and (2.15) it follows that

$$\hat{P}_x^1(\sigma_B < \infty) = 0 \quad \text{for } m_1\text{-a.e. } x \in (-\infty, T] \times E. \quad (3.2)$$

By [31, Lemma III.2.9], for every $\eta \in \mathcal{B}_b^+(E^1)$,

$$E_{\eta \cdot m_1}^1 e^{-\sigma_B} \int_{\sigma_B}^{\infty} e^{-t} \eta(X_t^1) dt = \hat{E}_{\eta \cdot m_1}^1 e^{-\sigma_B} \int_{\sigma_B}^{\infty} e^{-t} \eta(X_t^1) dt, \quad (3.3)$$

where $E_{\eta \cdot m_1}^1$ denotes the expectation with respect to the measure $P_{\eta \cdot m_1}^1$ defined as $P_{\eta \cdot m_1}^1(\cdot) = \int_{E^1} P_x(\cdot) \eta(x) m_1(dx)$ and $\hat{E}_{\eta \cdot m_1}^1$ denotes the expectation with respect to $\hat{P}_{\eta \cdot m_1}^1$ defined as $P_{\eta \cdot m_1}^1$ but with P_x replaced by $\hat{P}_x$. Let $\eta = 0$ on $[T, \infty) \times E$ and $\eta > 0$ on $(-\infty, T) \times E$. Then, by (3.2), the right-hand side of (3.3) equals zero, which implies that $P_x^1(\sigma_B < \infty) = 0$ for m_1 -a.e. $x \in (-\infty, T] \times E$. This when combined with (3.1) shows that $P_x^1(\sigma_B < \infty) = 0$ for m_1 -a.e. $x \in E^1$. By (2.7), $\text{Cap}^1(B) = 0$. $\square$

We now recall the capacity considered in Pierre [28] (see also [26, 27]). Let U be a relatively compact open subset of $E_{0,T}$. By [31, Proposition III.1.6], for every $n \geq 1$ there exists a unique solution $e_U^n \in \mathcal{W}_T$ of the following problem

$$\mathcal{E}_1(e_U^n, \eta) = n((e_U^n - \mathbf{1}_U)^-, \eta), \quad \eta \in \mathcal{V}.$$

By [31, Proposition III.1.7], $\{e_U^n\}$ converges strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$. Set $e_U = \lim_{n \rightarrow \infty} e_U^n$. By [31, Proposition III.1.7], for every $\eta \in \mathcal{W}_0$ such that $\eta \geq \mathbf{1}_U$ m_1 -a.e. we have

$$\mathcal{E}_1(e_U, \eta) \geq \mathcal{B}_1(e_U, e_U). \quad (3.4)$$

By Lemma 2.1 (see also [28]), there exists a measure $\mu_U \in S_0(E_{0,T})$ such that $e_U = U_1 \mu_U$. By [27, Proposition II.1], $\text{supp}[\mu_U] \subset \overline{U}$. In [28], the capacity of U is defined by

$$c_0(U) = \mu_U(E_{0,T}).$$

As usual, for an arbitrary $A \subset E_{0,T}$, we define

$$c_0(A) = \inf\{c_0(U), A \subset U, U \text{ open}\}.$$

By [28, Proposition 2], c_0 is a Choquet capacity.

LEMMA 3.2. *For any $A \subset E_{0,T}$, $\text{Cap}_\psi(A) = 0$ if and only if $c_0(A) = 0$.*

Proof. Let U be a relatively compact open subset of $E_{0,T}$. Let $\widetilde{\hat{G}_1\psi}$ be a quasi-continuous modification of $\hat{G}_1\psi$. By the definitions of Cap_ψ and $U_1\mu_U$, and the fact that $\hat{G}_1$ is Markovian, we have

$$\text{Cap}_\psi(U) \leq \int_{E_{0,T}} e_U \psi \, dm_1 = \mathcal{E}_1(e_U, \hat{G}_1\psi) = \int_{E_{0,T}} \widetilde{\hat{G}_1\psi} \, d\mu_U \leq \mu_U(E_{0,T}) = c_0(U).$$

Hence $\text{Cap}_\psi \leq c_0$. Let K be a compact subset of $E_{0,T}$ such that $\text{Cap}_\psi(K) = 0$. Then there exists a nonincreasing sequence U_n of open relatively compact subsets of $E_{0,T}$ such that $\text{Cap}_\psi(U_n) \searrow 0$ and $K \subset U_n$. Let $\eta \in \mathcal{W}_0 \cap C_c(E_{0,T})$ be such that $\eta \geq \mathbf{1}_{U_1}$ (see [27, Lemma II.3]). Then, by (3.4),

$$\mathcal{B}_1(e_{U_n}, e_{U_n}) \leq \mathcal{E}_1(e_{U_n}, \eta) = \int_{E_{0,T}} \tilde{\eta} \, d\mu_{U_n} = c_0(U_n). \quad (3.5)$$

Since $\{U_n\}$ is a nonincreasing, $\{e_{U_n}\}$ is nonincreasing. By [31, Proposition IV.3.4], $e_{U_n} \searrow 0$ m_1 -a.e. By (3.5),

$$\mathcal{B}_1(e_{U_n}, e_{U_n}) \leq c_0(U_1).$$

Therefore there exists a subsequence (still denoted by n) such that $\{e_{U_n}\}$ is weakly convergent in $\mathcal{V}$. Since $e_{U_n} \searrow 0$ m_1 -a.e., in fact $e_{U_n} \rightarrow 0$ weakly in $\mathcal{V}$. Thus $\mathcal{E}_1(e_{U_n}, \eta) \rightarrow 0$, $\eta \in \mathcal{W}_0$. By (3.5),

$$\mathcal{E}_1(e_{U_n}, \eta) \geq c_0(K).$$

Hence $c_0(K) = 0$. Since Cap_ψ and c_0 are Choquet capacities, this implies the desired result. $\square$

Write

$$\mathcal{W}^2 = \{u \in \mathcal{W} : \partial_t u \in L^2(E_{0,T}; m_1)\}, \quad \mathcal{C} = C_c(E_{0,T}) \cap \mathcal{W}^2,$$

where $C_c(E_{0,T})$ is the set of all real continuous functions on $E_{0,T}$ having compact support. For a compact set $K \subset E_{0,T}$, we set

$$\text{CAP}(K) = \inf\{\|u\|_{\mathcal{W}} : u \in \mathcal{C}, u \geq \mathbf{1}_K\}.$$

Next, for an open $U \subset E_{0,T}$, we set $\text{CAP}(U) = \sup\{\text{CAP}(K) : K \subset U, K\text{-compact}\}$, and finally, for arbitrary $A \subset E_{0,T}$, we set $\text{CAP}(A) = \inf\{\text{CAP}(U) : A \subset U, U \text{ open}\}$. We will also consider the capacity c_2 (see [28]) defined as

$$c_2(U) = \inf\{\|u\|_{\mathcal{W}} : u \in \mathcal{W}, u \geq \mathbf{1}_U \text{ } m_1\text{-a.e.}\}$$

for an open set $U \subset E_{0,T}$, and then by $c_2(A) = \inf\{c_2(U) : A \subset U, U \text{ open}\}$ for an arbitrary set $A \subset E_{0,T}$. By [28, Proposition 2], c_2 is a Choquet capacity.

LEMMA 3.3. *For any $A \subset E_{0,T}$, $c_2(A) = 0$ if and only if $\text{Cap}_{\psi}(A) = 0$.*

Proof. Follows from Lemma 3.2 because by [28, Theorem 1], for any $A \subset E_{0,T}$, $c_2(A) = 0$ if and only if $c_0(A) = 0$. $\square$

LEMMA 3.4. *For any $A \subset E_{0,T}$, $\text{CAP}(A) = 0$ if and only if $\text{Cap}_{\psi}(A) = 0$.*

Proof. By virtue of Lemma 3.3, it suffices to prove that CAP is equivalent to c_2 . Let K be a compact subset of $E_{0,T}$ and let $\varepsilon > 0$, $a > 1$. Choose $\eta_{\varepsilon} \in \mathcal{C}$ such that $\|\eta_{\varepsilon}\|_{\mathcal{W}} \leq \text{CAP}(K) + \varepsilon$ and $\eta_{\varepsilon} \geq \mathbf{1}_K$. Since η_{ε} is continuous, there exists an open set V_a such that $a\eta_{\varepsilon} \geq \mathbf{1}_{V_a}$ and $K \subset V_a$. We have

$$c_2(K) \leq c_2(V_a) \leq \|a\eta_{\varepsilon}\|_{\mathcal{W}} \leq a(\text{CAP}(K) + \varepsilon).$$

Letting $\varepsilon \downarrow 0$ and then $a \downarrow 1$ we see that $c_2(K) \leq \text{CAP}(K)$ for every compact $K \subset E_{0,T}$. From this and the fact that c_2 is a Choquet capacity we conclude that $c_2 \leq \text{CAP}$.

Now suppose that $c_2(K) = 0$ for some compact set $K \subset E_{0,T}$. Let $U_n = \{(s, x) \in E_{0,T} : \text{dist}((s, x), K) < n^{-1}\}$. Since $E_{0,T}$ is locally compact and K is compact, we may assume that U_n are relatively compact. Since c_2 is a Choquet capacity, $b_n := c_2(U_n) \searrow 0$. Fix $\varepsilon > 0$ and choose $\eta_n^{\varepsilon} \in \mathcal{W}$ such that $\eta_n^{\varepsilon} \geq \mathbf{1}_{U_n}$ m_1 -a.e. and $\|\eta_n^{\varepsilon}\|_{\mathcal{W}} \leq b_n + \varepsilon$. By putting $\eta_n^{\varepsilon}(t) = \eta_n^{\varepsilon}(T)$ for $t \geq T$ and $\eta_n^{\varepsilon}(t) = \eta_n^{\varepsilon}(0)$ for $t \leq 0$, we may assume that $\eta_n^{\varepsilon} \in \mathcal{W}^1$. Write

$$J_n(\eta_n^{\varepsilon}) = 2n \int_0^{1/(2n)} \eta_n^{\varepsilon}(t+s) j(2ns) ds,$$

where j is a smooth positive function with support in $[0, 1]$ such that $\int_{\mathbb{R}} j(t) dt = 1$. It is clear that $J_n(\eta_n^{\varepsilon}) \in \mathcal{W}^2$, $\|J_n(\eta_n^{\varepsilon})\|_{\mathcal{W}} \leq \|\eta_n^{\varepsilon}\|_{\mathcal{W}}$ and

$$J_n(\eta_n^{\varepsilon}) \geq \mathbf{1}_{U_{n-1}} \text{ } m_1\text{-a.e.} \quad (3.6)$$

By regularity of the forms $B^{(t)}$, for every $a > 0$ there exists $\eta_n^{\varepsilon,a} \in \mathcal{C}$ such that

$$\|J_n(\eta_n^\varepsilon) - \eta_n^{\varepsilon,a}\|_{\mathcal{W}^2} \leq a \quad (3.7)$$

(see [22, Lemma 1.1]). Let $\xi_n \in \mathcal{C}$ be a positive function such that $\xi_n = 1$ on K and $\text{supp}[\xi_n] \subset U_{n-1}$. The existence of ξ_n follows from [22, Lemma 1.1]. We put

$$\tilde{\eta}_n^{\varepsilon,a} = \eta_n^{\varepsilon,a} + (\xi_n - \eta_n^{\varepsilon,a})^+.$$

Observe that $\tilde{\eta}_n^{\varepsilon,a} \in \mathcal{C}$ and $\tilde{\eta}_n^{\varepsilon,a} \geq \mathbf{1}_K$. Let $a_l \downarrow 0$. By (3.7), up to a subsequence, $(\xi_n - \eta_n^{\varepsilon,a_l})^+ \rightarrow (\xi_n - J_n(\eta_n^\varepsilon))^+$ weakly in $\mathcal{W}^2$ as $l \rightarrow \infty$. Hence, up to a subsequence,

$$\frac{1}{k} \sum_{l=1}^k (\xi_n - \eta_n^{\varepsilon,a_l})^+ \rightarrow (\xi_n - J_n(\eta_n^\varepsilon))^+ \quad (3.8)$$

strongly in $\mathcal{W}^2$. Furthermore,

$$\text{CAP}(K) \leq \left\| \frac{1}{k} \sum_{l=1}^k \tilde{\eta}_n^{\varepsilon,a_l} \right\|_{\mathcal{W}} \leq \left\| \frac{1}{k} \sum_{l=1}^k \eta_n^{\varepsilon,a_l} \right\|_{\mathcal{W}} + \left\| \frac{1}{k} \sum_{l=1}^k (\xi_n - \eta_n^{\varepsilon,a_l})^+ \right\|_{\mathcal{W}^2}.$$

By (3.6), $(\xi_n - J_n(\eta_n^\varepsilon))^+ = 0$, so letting $k \rightarrow \infty$ in the above inequality and using (3.7) and (3.8) we get

$$\text{CAP}(K) \leq \|J_n(\eta_n^\varepsilon)\|_{\mathcal{W}} \leq \|\eta_n^\varepsilon\|_{\mathcal{W}} \leq b_n + \varepsilon.$$

Hence $\text{CAP}(K) = 0$. Since c_2 is a Choquet capacity, this implies that $\text{CAP}(A) = 0$ if $c_2(A) = 0$. $\square$

4. Smooth measures and associated additive functionals

The results of this section will be needed in Sects. 8 and 9. Recall that a positive additive functional (AF in abbreviation) A^1 of $\mathbb{X}^1$ is said to be in the Revuz correspondence with a positive $\mu \in S^1(E^1)$ if

$$\lim_{\alpha \rightarrow \infty} \alpha E_{m_1}^1 \int_0^\infty e^{-\alpha t} f(X_t^1) dA_t^1 = \int_{E^1} f(x) \mu(dx), \quad f \in \mathcal{B}_b^+(E^1), \quad (4.1)$$

where $E_{m_1}^1$ denotes the expectation with respect to the measure $P_{m_1}^1$ defined as $P_{m_1}^1(\cdot) = \int_{E^1} P_x(\cdot) m_1(dx)$. Similarly, we say that a positive AF A of $\mathbb{X}$ is in the Revuz correspondence with a positive $\mu \in S(E_{0,T})$ if

$$\lim_{\alpha \rightarrow \infty} \alpha E_{m_1} \int_0^\infty e^{-\alpha t} f(X_t) dA_t = \int_{E_{0,T}} f(x) \mu(dx), \quad f \in \mathcal{B}_b^+(E_{0,T}), \quad (4.2)$$

where E_{m_1} denotes the expectation with respect to $P_{m_1}(\cdot) = \int_{E_{0,T}} P_x(\cdot) m_1(dx)$.

Let $A_t^f = \int_0^t f(X_r) dA_r$. We have

$$\lim_{\alpha \rightarrow \infty} \alpha^2 E_{m_1} \int_0^\infty e^{-\alpha t} A_t^f dt = \lim_{\alpha \rightarrow \infty} \int_0^\infty s e^{-s} \frac{\alpha}{s} E_{m_1} A_{s/\alpha}^f ds. \quad (4.3)$$

Since $t \mapsto E_{m_1} A_t^f$ is subadditive, $(1/t)E_{m_1} A_t^f$ increases as t decreases, and moreover, $\lim_{t \downarrow 0} (1/t)E_{m_1} A_t^f = \sup_{t>0} (1/t)E_{m_1} A_t^f$. Therefore, letting $\alpha \rightarrow \infty$ in (4.3), shows that

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \alpha E_{m_1} \int_0^\infty e^{-\alpha t} dA_t^f &= \lim_{\alpha \rightarrow \infty} \alpha^2 E_{m_1} \int_0^\infty e^{-\alpha t} A_t^f dt \\ &= \sup_{t>0} \frac{1}{t} E_{m_1} A_t^f \cdot \int_0^\infty s e^{-s} ds = \sup_{t>0} \frac{1}{t} E_{m_1} A_t^f. \end{aligned}$$

A similar result holds for A^1 .

It is known (see [12, Sect. 2]) that for every $\mu \in S^1(E^1)$ there exists a unique positive natural AF A^1 of $\mathbb{X}^1$ (i.e. a positive AF of $\mathbb{X}^1$ such that A^1 and $\mathbb{X}^1$ have no common discontinuities) such that A^1 is in the Revuz correspondence with μ . In what follows we denote it by $A^{1,\mu}$. In fact, $A^{1,\mu}$ is a predictable process (see [10, Theorem 5.3]). In the proposition below, we show a similar result for positive smooth measures on $E_{0,T}$.

PROPOSITION 4.1. *Let $\mu \in S(E_{0,T})$ be positive. There exists a unique positive natural AF A^μ of $\mathbb{X}$ in the Revuz correspondence with μ . Moreover, for q.e. $x \in E_{0,T}$,*

$$E_x \int_0^\infty e^{-\alpha t} f(X_t) dA_t^\mu = E_x \int_0^\infty e^{-\alpha t} f(X_t^1) dA_t^{1,\bar{\mu}}, \quad \alpha \geq 0, \quad f \in \mathcal{B}_b^+(E_{0,T}), \quad (4.4)$$

where $\bar{\mu}$ denotes the extension of μ to E^1 such that $\mu(E^1 \setminus E_{0,T}) = 0$ and $A^{1,\bar{\mu}}$ is the positive natural AF of $\mathbb{X}^1$ in the Revuz correspondence with $\bar{\mu}$.

Proof. We first assume that $\mu \in S_0(E_{0,T})$. Since μ is zero outside $E_{0,T}$, by Lemma 3.1 it is clear that $\bar{\mu} \in S_0(E^1)$. By the remark preceding Proposition 4.1, there exists a positive natural AF $A^{1,\bar{\mu}}$ of $\mathbb{X}^1$ in the Revuz correspondence with $\bar{\mu}$. Let $R^1 \bar{\mu}(x) = E_x^1 A^{1,\bar{\mu}}$. By (4.1),

$$\mathbf{1}_{E_{0,T}} R^1 \bar{\mu} = R^1 \bar{\mu} \quad (4.5)$$

for $x \in E_{0,T}$. Hence, by [10, Theorem 9.3] and (2.11) (for the dual process), for every positive $\eta \in \mathcal{B}(E^1) \cap L^2(E^1; m_1)$ such that $\eta = 0$ outside $E_{0,T}$ we have

$$\begin{aligned} \int_{E^1} R^1 \bar{\mu} \eta dm_1 &= \int_{E^1} \hat{R}^1 \eta d\bar{\mu} = \int_{E_{0,T}} \hat{R}^1 \eta d\bar{\mu} \\ &= \int_{E_{0,T}} \hat{R} \eta d\mu = \mathcal{E}(U\mu, \hat{R}\eta) = \int_{E_{0,T}} U\mu \eta dm_1. \end{aligned}$$

Thus $R^1\bar{\mu} = U\mu$ m_1 -a.e. on $E_{0,T}$. By (2.12) and (4.5), for all $\tau \leq \zeta$ and $x \in E_{0,T}$,

$$E_x(R^1\bar{\mu})(X_\tau) = E_x^1(R^1\bar{\mu})(X_\tau^1 \circ \Pi) = E_x^1(R^1\bar{\mu})(X_\tau^1) \leq E_x^1(A_\zeta^{1,\bar{\mu}} - A_\tau^{1,\bar{\mu}}).$$

Hence $R^1\bar{\mu}$ is a natural potential with respect to $\mathbb{X}$. By [2, Theorem IV.4.22], there exists a unique positive natural AF A of $\mathbb{X}$ such that

$$R^1\bar{\mu} = E.A_\zeta \quad \text{q.e. on } E_{0,T}. \quad (4.6)$$

From this and the fact that $R^1\bar{\mu} = U\mu$ m_1 -a.e. on $E_{0,T}$ we get

$$\lim_{t \rightarrow 0} \frac{1}{t} E_{\eta \cdot m_1} A_t = (R^1\bar{\mu}, \frac{1}{t}(\eta - T_t\eta))_{\mathcal{H}} = (U\mu, \frac{1}{t}(\eta - T_t\eta))_{\mathcal{H}} = \mathcal{E}(U\mu, \eta) = \int_{E_{0,T}} \eta \, d\mu$$

for every $\eta \in \mathcal{W}_T \cap C_b(E_{0,T})$. Thus μ is a Revuz measure of A . Let $A^\mu := A$. From (4.6) we easily get (4.4).

To prove the existence of A^μ in the general case, we choose an $\mathcal{E}$ -nest $\{F_n\}$ of compact subsets of $E_{0,T}$ such that $\mu_n = \mathbf{1}_{F_n} \cdot \mu \in S_0(E_{0,T})$. Such a nest exists by [33, Theorem 4.7]. By what has already been proved, for all $\alpha \geq 0$ and $f \in \mathcal{B}_b^+(E_{0,T})$,

$$E_x \int_0^\infty e^{-\alpha t} f(X_t) \, dA_t^{\mu_n} = E_x^1 \int_0^\infty e^{-\alpha t} f(X_t^1) \, dA_t^{1,\bar{\mu}_n} \quad (4.7)$$

for q.e. $x \in E_{0,T}$. Since the additive functional in the Revuz correspondence with a smooth measure is uniquely determined, we have $dA^{\mu_n} \leq dA^{\mu_m}$ for $n \leq m$ and $A_t^{1,\bar{\mu}_n} = \int_0^t \mathbf{1}_{F_n}(X_r) \, dA_r^{1,\bar{\mu}}$, $t \geq 0$, P_x^1 -a.s. We now set

$$A_t^\mu = \sup_{n \geq 1} A_t^{\mu_n}, \quad t \geq 0. \quad (4.8)$$

By (4.7),

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \alpha E_{m_1} \int_0^\infty e^{-\alpha t} f(X_t) \, dA_t^\mu &= \lim_{n \rightarrow \infty} \lim_{\alpha \rightarrow \infty} \alpha E_{m_1}^1 \int_0^\infty e^{-\alpha t} \mathbf{1}_{F_n} f(X_t^1) \, dA_t^{1,\bar{\mu}} \\ &= \int_{\bigcup_{n \geq 1} F_n} f \, d\bar{\mu} = \int_{E_{0,T}} f \, d\mu, \end{aligned}$$

so A^μ defined by (4.8) is in the Revuz correspondence with μ . By [18, Lemma 1, page 182], A^μ is a positive natural AF of $\mathbb{X}$. By (2.12) and [31, Remark IV.3.6], for m_1 -a.e. $x \in E_{0,T}$,

$$\begin{aligned} 0 &= P_x(\lim_{n \rightarrow \infty} \sigma_{E_{0,T} \setminus F_n} < \zeta) \\ &= P_x\left(\omega \in \Omega : \lim_{n \rightarrow \infty} \inf_{t > 0} \{X_t(\omega) \in E_{0,T} \setminus F_n\} < \zeta(\omega)\right) \\ &= P_x^1 \circ \Pi^{-1}\left(\omega \in \Omega : \lim_{n \rightarrow \infty} \inf_{t > 0} \{X_t(\omega) \in E_{0,T} \setminus F_n\} < \zeta(\omega)\right) \\ &= P_x^1\left(\omega \in \Omega^1 : \lim_{n \rightarrow \infty} \inf_{t > 0} \{X_t^1(\Pi(\omega)) \in E_{0,T} \setminus F_n\} < \zeta(\Pi(\omega))\right) \\ &= P_x^1\left(\omega \in \Omega^1 : \lim_{n \rightarrow \infty} \inf_{t > 0} \{X_t^1(\omega) \in E_{0,T} \setminus F_n\} < \zeta(\omega)\right). \end{aligned}$$

Therefore, letting $n \rightarrow \infty$ in (4.7), yields (4.4). $\square$

REMARK 4.2. Since the capacities Cap_ψ associated with $\mathcal{E}$ and the dual form $\hat{\mathcal{E}}$ coincide (see Sect. 2.1), the set of smooth measures $S(E_{0,T})$ associated with $\mathcal{E}$ coincides with the set of smooth measures on $E_{0,T}$ associated with the dual form $\hat{\mathcal{E}}$. Therefore from Proposition 4.1 applied to $\hat{\mathcal{E}}$ and $\hat{\mathbb{X}}$ it follows that for a positive $\mu \in S(E_{0,T})$ there exists a unique predictable AF of $\hat{\mathbb{X}}$ in the Revuz correspondence with μ . We will denote it by $\hat{A}^\mu$.

In what follows, for a positive smooth measure μ on $E_{0,T}$, we put

$$R_\alpha \mu(x) = E_x \int_0^\zeta e^{-\alpha t} dA_t^\mu, \quad \hat{R}_\alpha \mu(x) = \hat{E}_x \int_0^\zeta e^{-\alpha t} d\hat{A}_t^\mu, \quad x \in E_{0,T}.$$

By [10, Theorem 9.3], for every $\mu \in \hat{S}_0(E_{0,T})$ we have

$$\hat{U}_\alpha \mu(x) = \hat{R}_\alpha \mu(x) \quad (4.9)$$

for m_1 -a.e. $x \in E_{0,T}$, and for every $\mu \in S_0(E_{0,T})$,

$$U_\alpha \mu(x) = R_\alpha \mu(x) \quad (4.10)$$

for m_1 -a.e. $x \in E_{0,T}$.

LEMMA 4.3. Let $\alpha \geq 0$. For any positive $\mu, \nu \in S(E_{0,T})$,

$$\int_{E_{0,T}} R_\alpha \nu d\mu = \int_{E_{0,T}} \hat{R}_\alpha \mu d\nu, \quad \int_{E_{0,T}} \hat{R}_\alpha \nu d\mu = \int_{E_{0,T}} R_\alpha \mu d\nu. \quad (4.11)$$

Proof. First we assume that $\mu \in \hat{S}_0(E_{0,T})$ and $\nu = \eta \cdot m_1$ for some $\eta \in \mathcal{B}^+(E_{0,T}) \cap \mathcal{H}$. Then, by (2.21), the fact that $R_\alpha \eta$ is an m_1 -version of $G_\alpha \eta$ and (2.4),

$$\int_{E_{0,T}} R_\alpha \eta d\mu = \mathcal{E}_\alpha(R_\alpha \eta, \hat{U}_\alpha \mu) = \mathcal{E}_\alpha(G_\alpha \eta, \hat{U}_\alpha \mu) = \int_{E_{0,T}} \eta \hat{U}_\alpha \mu dm_1.$$

From this and (4.9) it follows that the first equality in (4.11) is satisfied. Now, suppose that $\mu \in S(E_{0,T})$ and $\nu = \eta \cdot m_1$ for some $\eta \in \mathcal{B}^+(E_{0,T})$. Let $\eta_n = \frac{ng}{1+ng}(\eta \wedge n)$ for some strictly positive $g \in \mathcal{H}$ and choose an $\mathcal{E}$ -nest $\{F_n\}$ of compact sets such that $\mu_n = \mathbf{1}_{F_n} \cdot \mu \in \hat{S}_0(E_{0,T})$. Such a nest exists by [33, Theorem 4.7]. By what has already been proved,

$$\int_{E_{0,T}} R_\alpha \eta_n d\mu_n = \int_{E_{0,T}} \eta_n \hat{R}_\alpha \mu_n dm_1, \quad n \geq 1.$$

Letting $n \rightarrow \infty$ and using [31, Remark IV.3.6] yields the first equality in (4.11). To show it in the general case, we set $\eta_n = n R_n \nu$. By what has already been proved,

$$\int_{E_{0,T}} R_\alpha \eta_n d\mu = \int_{E_{0,T}} n \hat{R}_n \hat{R}_\alpha \mu d\nu, \quad n \geq 1.$$

By a direct calculation, we get $R_\alpha \eta_n \nearrow R_\alpha \nu$ and $n \hat{R}_n \hat{R}_\alpha \mu \nearrow \hat{R}_\alpha \mu$. Therefore letting $n \rightarrow \infty$ in the above equation yields the first equality in (4.11). The proof of the second equality is similar, so we omit it. $\square$

Given a positive smooth measure μ on $E_{0,T}$, we denote by $R_\alpha \circ \mu$ the Borel measure on $E_{0,T}$ defined by the formula

$$\int_{E_{0,T}} \eta d(R_\alpha \circ \mu) = \int_{E_{0,T}} R_\alpha \eta d\mu, \quad \eta \in \mathcal{B}^+(E_{0,T}),$$

and by $\hat{R}_\alpha \circ \mu$ we denote the Borel measure on $E_{0,T}$ defined by

$$\int_{E_{0,T}} \eta d(\hat{R}_\alpha \circ \mu) = \int_{E_{0,T}} \hat{R}_\alpha \eta d\mu, \quad \eta \in \mathcal{B}^+(E_{0,T}).$$

By Lemma 4.3, the measures $R_\alpha \circ \mu$, $\hat{R}_\alpha \circ \mu$ are absolutely continuous with respect to m_1 and

$$R_\alpha \circ \mu = (\hat{R}_\alpha \mu) \cdot m_1, \quad \hat{R}_\alpha \circ \mu = (R_\alpha \mu) \cdot m_1.$$

5. Decomposition of smooth measures

In what follows, we denote by $\langle \langle \cdot, \cdot \rangle \rangle$ the duality pairing between $\mathcal{W}'_0$ and $\mathcal{W}_0$.

In this section, we prove that each $\mu \in \mathcal{M}_b(E_{0,T})$ admits decomposition (1.7). The converse statement will be proved in Sect. 8. We start with a decomposition of elements of the space $\mathcal{W}'_0$. For a proof we defer the reader to [4, Proposition 8.14] or [6, Lemma 2.24].

PROPOSITION 5.1. *Let $g \in \mathcal{W}'_0$. Then there exist $g_1 \in \mathcal{V}'$, $g_2 \in \mathcal{V}$ such that*

$$\|g_1\|_{\mathcal{V}'} \leq \|g\|_{\mathcal{W}'_0}, \quad \|g_2\|_{\mathcal{V}} \leq \|g\|_{\mathcal{W}'_0} \quad (5.1)$$

and for every $v \in \mathcal{W}_0$,

$$\langle \langle g, v \rangle \rangle = \langle g_1, v \rangle - \langle \partial_t v, g_2 \rangle.$$

LEMMA 5.2. *If $v \in \mathcal{W}_0$, then*

$$\alpha \hat{G}_\alpha v \rightarrow v \text{ weakly in } \mathcal{W}_0 \text{ as } \alpha \rightarrow \infty. \quad (5.2)$$

Proof. By [31, Proposition I.III.7], for every $w \in \mathcal{V}$,

$$\alpha G_\alpha w \rightarrow w, \quad \alpha \hat{G}_\alpha w \rightarrow w \quad \text{in } \mathcal{V}. \quad (5.3)$$

By (5.3), for $w \in \mathcal{V}$ we have

$$\begin{aligned} \langle \hat{\Lambda}(\alpha \hat{G}_\alpha v), w \rangle &= \mathcal{B}(w, \alpha \hat{G}_\alpha v) - \mathcal{E}(w, \alpha \hat{G}_\alpha v) \\ &= \mathcal{B}(w, \alpha \hat{G}_\alpha v) - \mathcal{E}(\alpha G_\alpha w, v) \rightarrow \mathcal{B}(w, v) - \mathcal{E}(w, v) = \langle \hat{\Lambda} v, w \rangle. \end{aligned}$$

Since $\mathcal{V}$ is reflexive, $\{\hat{\Lambda}(\alpha \hat{G}_\alpha v)\}$ is weakly convergent in $\mathcal{V}'$ as $\alpha \rightarrow \infty$. From this, (5.3) and Proposition 5.1 we get (5.2). $\square$

REMARK 5.3. If $\mu \in S_0(E_{0,T})$, then $\mu \in \mathcal{W}'_0$. This follows directly from the definition of the space $S_0(E_{0,T})$. Formally, the embedding $i : S_0(E_{0,T}) \rightarrow \mathcal{W}'_0$ can be defined as follows. We first define the mapping $j : \mathcal{V} \rightarrow \mathcal{W}'_0$ as

$$\langle j(u), v \rangle = \mathcal{E}_1(u, v), \quad u \in \mathcal{V}, v \in \mathcal{W}_0, \quad (5.4)$$

and then we define the embedding operator i by $i(\mu) = j(U_1\mu)$. By the definition of $U_1\mu$ we then have

$$\langle i(\mu), \eta \rangle = \int_{E_{0,T}} \tilde{\eta} d\mu, \quad \eta \in \mathcal{W}_0.$$

PROPOSITION 5.4. Let $\mu \in \mathcal{M}_{0,b}(E_{0,T})$. Then there exist $f \in L^1(E_{0,T}; m_1)$ and $g \in \mathcal{W}'_0$ such that

$$\mu = f \cdot m_1 + g, \quad (5.5)$$

i.e. for every bounded $v \in \mathcal{W}_0$ we have

$$\int_{E_{0,T}} \tilde{v} d\mu = \int_{E_{0,T}} f v dm_1 + \langle g, v \rangle. \quad (5.6)$$

Proof. Let $\mu = \mu^+ - \mu^-$ be the Hahn decomposition of μ . Then $\mu^+, \mu^- \in \mathcal{M}_{0,b}(E_{0,T})$. Therefore, without loss of generality, we can assume that μ is positive. By [33, Theorem 4.7], there exists a nest $\{F_n\}$ such that $\mathbf{1}_{F_n} \cdot |\mu| \in S_0(E_{0,T})$ for every $n \in \mathbb{N}$. Set $\mu_n = \mathbf{1}_{F_{n+1} \setminus F_n} \cdot \mu$. Then $|\mu_n| \in S_0(E_{0,T})$ and

$$\sum_{n=1}^{\infty} \mu_n = \mu \quad (5.7)$$

because the set $(\bigcup_{n=1}^{\infty} F_n)^c$ is exceptional. Let $\mu_n^\alpha = \alpha \hat{R}_\alpha \circ \mu_n$. Then for $v \in \mathcal{W}_0$ we have

$$\begin{aligned} \mathcal{E}_1(U_1\mu_n^\alpha, v) &= \int_{E_{0,T}} \tilde{v} d\mu_n^\alpha = \int_{E_{0,T}} \alpha \hat{R}_\alpha v d\mu_n, \\ \mathcal{E}_1(U_1\mu_n, \alpha \hat{R}_\alpha v) &= \int_{E_{0,T}} \alpha \hat{R}_\alpha v d\mu_n. \end{aligned}$$

Hence

$$\mathcal{E}_1(U_1\mu_n^\alpha, v) = \mathcal{E}_1(U_1\mu_n, \alpha \hat{R}_\alpha v),$$

so by Lemma 5.2,

$$\mathcal{E}_1(U_1\mu_n^\alpha, v) \rightarrow \mathcal{E}_1(U_1\mu_n, v). \quad (5.8)$$

Let $j : \mathcal{V} \rightarrow \mathcal{W}'_0$ be defined by (5.4). From (5.8) and the fact that $\mathcal{W}_0$ is a Hilbert space it follows that the sequence $\{j(U_1\mu_n^\alpha)\}$ converges to $j(U_1\mu_n)$ weakly in $\mathcal{W}'_0$ as

$\alpha \rightarrow \infty$. By the Banach–Saks theorem, there is a sequence $\{\alpha_l\}$ such that $\alpha_l \rightarrow \infty$ as $l \rightarrow \infty$ and

$$j(U_1 \hat{f}_k(\mu_n)) \rightarrow j(U_1 \mu_n) \quad \text{in } \mathcal{W}'_0$$

as $k \rightarrow \infty$, where

$$\hat{f}_k(\mu_n) = \frac{1}{k} \sum_{l=1}^k \mu_n^{\alpha_l}.$$

Therefore one can find a subsequence $\{k_n\}$ such that for every $n \in \mathbb{N}$,

$$\|j(U_1(\mu_n - \hat{f}_{k_n}(\mu_n)))\|_{\mathcal{W}'_0} \leq 2^{-n}. \quad (5.9)$$

By (4.11), $\hat{f}_k(\mu_n) = f_k(\mu_n) \cdot m_1$, where

$$f_k(\mu_n) = \frac{1}{k} \sum_{l=1}^k \alpha_l U_{\alpha_l} \mu_n.$$

Set

$$g = \sum_{n=1}^{\infty} j(U_1(\mu_n - (\hat{f}_{k_n}(\mu_n)))), \quad f = \sum_{n=1}^{\infty} f_{k_n}(\mu_n). \quad (5.10)$$

By (5.9), $g \in \mathcal{W}'_0$. Since m_1 is σ -finite, there exists a sequence $\{V_i\}$ of Borel subsets of $E_{0,T}$ such that $\bigcup_{i=1}^{\infty} V_i = E_{0,T}$, $V_i \subset V_{i+1}$ and $m_1(V_i) < \infty$, $i \in \mathbb{N}$. By (4.10) and Lemma 4.3, for every $i \in \mathbb{N}$ we have

$$\int_{E_{0,T}} \mathbf{1}_{V_i} f \, dm_1 = \sum_{n=1}^{\infty} \int_{E_{0,T}} \mathbf{1}_{V_i} f_{k_n}(\mu_n) \, dm_1 = \sum_{n=1}^{\infty} \int_{E_{0,T}} \frac{1}{k_n} \sum_{l=1}^{k_n} \alpha_l \hat{R}_{\alpha_l}(\mathbf{1}_{V_i}) \, d\mu_n. \quad (5.11)$$

Since $\alpha \hat{R}_{\alpha}$ is a Markov operator, it follows that

$$\int_{E_{0,T}} \mathbf{1}_{V_i} f \, dm_1 \leq \sum_{n=1}^{\infty} \|\mu_n\|_{TV} = \|\mu\|_{TV}$$

Hence $\|f\|_{L^1(E_T; m_1)} < \infty$ by the monotone convergence theorem. Since

$$\int_{E_{0,T}} \tilde{v} \, d\mu_n = \mathcal{E}_1(U_1 \mu_n, v)$$

and

$$\int_{E_{0,T}} f_{k_n}(\mu_n) v \, dm_1 = \int_{E_{0,T}} \tilde{v} \, d\hat{f}_{k_n}(\mu_n) = \mathcal{E}_1(U_1(\hat{f}_{k_n}(\mu_n)), v),$$

for every $N \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{n=1}^N \int_{E_{0,T}} \tilde{v} \, d\mu_n &= \sum_{n=1}^N \mathcal{E}_1(U_1(\mu_n - \hat{f}_{k_n}(\mu_n)), v) + \sum_{n=1}^N \int_{E_{0,T}} f_{k_n}(\mu_n) v \, dm_1 \\ &= \left\langle \left\langle \sum_{n=1}^N j(U_1(\mu_n - \hat{f}_{k_n}(\mu_n))), v \right\rangle \right\rangle + \sum_{n=1}^N \int_{E_{0,T}} (f_{k_n}(\mu_n) v) \, dm_1. \end{aligned}$$

Letting $N \rightarrow \infty$ and using (5.7), (5.10) we obtain (5.6), which completes the proof of the proposition. $\square$

The idea of the proof of Proposition 5.4 is similar to that in the case of Leray–Lions-type operators on bounded domains in $\mathbb{R}^d$ (see [6, Theorem 2.27]). In the crucial first step we decompose μ as a sum of measures $\mu_n \in \mathcal{W}'_0$ and then we regularize each measure μ_n . However, contrary to [6], in the Dirichlet forms setting, we define μ_n by using the notion of nests (and we do not use any integral representation of smooth measures; see [6, Theorem 2.23]). Furthermore, in our general setting we cannot regularize μ_n by convolution. Therefore, instead of considering measures of the form $\mu_n * j_{1/\alpha}$ we consider the measures $\alpha \hat{R}_\alpha \circ \mu_n$. Each measure $\alpha \hat{R}_\alpha \circ \mu_n$ has density $\alpha U_\alpha \mu_n$ with respect to the reference measure m_1 and at the same time belongs to $\mathcal{W}'_0$. What is important, as in the case of regularization by convolution, our regularization by means of resolvents is a contraction with respect to the L^1 -norm and with respect to the $\mathcal{W}_0$ -norm.

REMARK 5.5. The decomposition (5.5) also holds for arbitrary $\mu \in S(E_{0,T})$. In this case, in general, $f \notin L^1(E_{0,T}; m_1)$ but f is quasi-integrable, i.e. there exists a nest $\{F_n\}$ of compact subsets of $E_{0,T}$ such that $\mathbf{1}_{F_n} \cdot f \in L^1(E_{0,T}; m_1)$, $n \geq 1$. The proof of the decomposition (5.5) in case $\mu \in S(E_{0,T})$ runs as the proof of Proposition 5.4 with the only difference that in (5.11) we replace V_i by F_i defined in the proof of Proposition 5.4. We then get

$$\int_{E_{0,T}} \mathbf{1}_{F_i} f \, dm_1 \leq \|\mathbf{1}_{F_i} \cdot \mu\|_{TV}, \quad i \geq 1.$$

Combining Proposition 5.1 with Proposition 5.4 we get the following theorem.

THEOREM 5.6. *Let $\mu \in \mathcal{M}_{0,b}$. Then there exist $f \in L^1(E_{0,T}; m_1)$, $g_1 \in \mathcal{V}'$, $g_2 \in \mathcal{V}$ such that*

$$\int_{E_{0,T}} \tilde{v} \, d\mu = \int_{E_{0,T}} f v \, dm_1 + \langle g_1, v \rangle - \langle \partial_t v, g_2 \rangle \quad (5.12)$$

for every bounded $v \in \mathcal{W}_0$.

Note that f, g_1, g_2 in Theorem 5.6 are not uniquely determined. In what follows, any triplet of functions having the same properties as the triplet (f, g_1, g_2) appearing in Theorem 5.6 will be called a decomposition of μ .

REMARK 5.7. From (5.10) it follows that if $\mu \in \mathcal{M}_{0,b}(E_{0,T})$ is positive, then the L^1 part f of the decomposition of μ can be chosen to be positive.

6. Semilinear equations with right-hand side in $\mathcal{W}'_0$

In this section, using the decomposition of elements of $\mathcal{W}'_0$ given in Proposition 5.1, we will show an existence and uniqueness result for the following Cauchy problem

$$-\partial_t u - L_t u = f(\cdot, u) + g, \quad u(T, \cdot) = \varphi. \quad (6.1)$$

In (6.1), $g \in \mathcal{W}'_0$, $f : E_{0,T} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\varphi \in L^2(E; m)$.

We will need the following assumption: there exist $\lambda \in \mathbb{R}$, $M \geq 0$ and a positive $\varrho \in \mathcal{H}$ such for all $x \in E_{0,T}$ and $y, y' \in \mathbb{R}$,

$$|f(x, y)| \leq \varrho(x) + M|y|, \quad (f(x, y) - f(x, y'))(y - y') \leq \lambda|y - y'|^2. \quad (6.2)$$

DEFINITION. We say that $u \in \mathcal{V}$ is a solution of (6.1) if

$$\langle \partial_t \eta, u \rangle + \mathcal{B}(u, \eta) = (\varphi, \eta(T, \cdot))_H + (f(\cdot, u), \eta)_{\mathcal{H}} + \langle \langle g, \eta \rangle \rangle, \quad \eta \in \mathcal{W}_0, \quad (6.3)$$

where $\mathcal{B}$ is defined by (2.3).

In what follows, for a function $\eta \in \mathcal{W}$ and $\varepsilon > 0$, $t \in [0, T]$, we set $\eta^t = \eta \mathbf{1}_{[t, T] \times E}$ and

$$\begin{aligned} \eta_\varepsilon^t(s) &= 0, \quad s \in [0, t], \quad \eta_\varepsilon^t(s) = \varepsilon^{-1} \eta(t + \varepsilon)(s - t), \quad t \in [t, t + \varepsilon], \\ \eta_\varepsilon^t(s) &= \eta(s), \quad s \in [t + \varepsilon, T]. \end{aligned}$$

It is clear that for every $\eta \in \mathcal{W}$, $\eta_\varepsilon^t \rightarrow \eta^t$ strongly in $\mathcal{V}$ as $\varepsilon \searrow 0$.

THEOREM 6.1. Assume that f is a Carathéodory function such that (6.2) is satisfied. Then there exists a unique solution of (6.1).

Proof. Uniqueness. Let $u_1, u_2 \in \mathcal{V}$ be solutions of (6.1). Then, by (6.3),

$$\langle \partial_t \eta, u \rangle + \mathcal{B}(u, \eta) = (f(\cdot, u_1) - f(\cdot, u_2), \eta)_{\mathcal{H}}, \quad \eta \in \mathcal{W}_0, \quad (6.4)$$

where $u = u_1 - u_2$. We see that $u = G(f(\cdot, u_1) - f(\cdot, u_2))$. In particular, $u \in \mathcal{W}_T$. Taking $\eta = u_\varepsilon^t$ in (6.4) and letting $\varepsilon \searrow 0$ we get

$$\begin{aligned} \frac{1}{2} \|u(t)\|_H^2 + \int_t^T B^{(s)}(u(s), u(s)) \, ds &= \int_t^T (f(s, u_1(s)) - f(s, u_2(s)), u(s))_H \, ds \\ &\leq \lambda^+ \int_t^T \|u(s)\|_H^2 \, ds. \end{aligned}$$

Applying Gronwall's lemma shows that $u = 0$.

Existence. Without loss of generality we may and will assume that $\lambda \leq 0$. Let g_1, g_2 be as in (5.1). Define $g^\alpha \in \mathcal{V}'$ by

$$\langle \langle g^\alpha, \eta \rangle \rangle = \langle \langle g, \alpha \hat{G}_\alpha \eta \rangle \rangle, \quad \eta \in \mathcal{V},$$

and to simplify notation, let g_2^α stand for $\alpha \hat{G}_\alpha g_2$. Then

$$\langle \langle g^\alpha, \eta \rangle \rangle = \langle g_1, \alpha \hat{G}_\alpha \eta \rangle - \langle \partial_t \eta, g_2^\alpha \rangle, \quad \eta \in \mathcal{V}. \quad (6.5)$$

By [17, Theorem 6.2], there exists $u^\alpha \in \mathcal{W}$ such that

$$\langle \partial_t \eta, u^\alpha \rangle + \mathcal{B}(u^\alpha, \eta) = (\varphi, \eta(T, \cdot))_H + (f(\cdot, u^\alpha), \eta)_\mathcal{H} + \langle \langle g^\alpha, \eta \rangle \rangle, \quad \eta \in \mathcal{V}.$$

By this and (6.5), for every $\eta \in \mathcal{W}_0$ we have

$$\begin{aligned} & \langle u^\alpha + g_2^\alpha, \partial_t \eta \rangle + \mathcal{B}(u^\alpha + g_2^\alpha, \eta) \\ &= (\varphi, \eta(T, \cdot))_H + (f(\cdot, u^\alpha), \eta)_\mathcal{H} + \langle g_1, \alpha \hat{G}_\alpha \eta \rangle + \mathcal{B}(g_2^\alpha, \eta). \end{aligned} \quad (6.6)$$

Taking $\eta = (u^\alpha + g_2^\alpha)_\varepsilon^t$ as a test function in (6.6) and letting $\varepsilon \searrow 0$ we obtain

$$\begin{aligned} & \frac{1}{2} \|(u^\alpha + g_2^\alpha)(t)\|_H^2 + \int_t^T B^{(s)}((u^\alpha + g_2^\alpha)(s), (u^\alpha + g_2^\alpha)(s)) \, ds \\ &= \frac{1}{2} \|\varphi\|_H^2 + \int_t^T (f(\cdot, u^\alpha(s)), (u^\alpha + g_2^\alpha)(s))_H \, ds + \langle g_1, \alpha \hat{G}_\alpha (u^\alpha + g_2^\alpha)^t \rangle \\ &+ \int_t^T B^{(s)}(g_2^\alpha(s), (u^\alpha + g_2^\alpha)(s)) \, ds. \end{aligned} \quad (6.7)$$

By [31, Proposition I.3.7],

$$\langle g_1, \alpha \hat{G}_\alpha (u^\alpha + g_2^\alpha)^t \rangle \leq C \|g_1\|_{\mathcal{V}'} \|u^\alpha + g_2^\alpha\|_{\mathcal{V}}.$$

By (6.2),

$$\begin{aligned} & \int_t^T (f(\cdot, u^\alpha(s)), (u^\alpha + g_2^\alpha)(s))_H \, ds \\ & \leq M \int_t^T \|(u^\alpha + g_2^\alpha)(s)\|_H^2 \, ds + M \|g_2^\alpha\|_{\mathcal{H}} \|u^\alpha + g_2^\alpha\|_{\mathcal{H}} + \|\varrho\|_{\mathcal{H}} \|u^\alpha + g_2^\alpha\|_{\mathcal{H}}. \end{aligned}$$

Combining the above inequalities with (6.7) we get

$$\begin{aligned} & \frac{1}{2} \|(u^\alpha + g_2^\alpha)(t)\|_H^2 + \int_t^T B^{(s)}((u^\alpha + g_2^\alpha)(s), (u^\alpha + g_2^\alpha)(s)) \, ds \\ & \leq \frac{1}{2} \|\varphi\|_H^2 + M \int_t^T \|(u^\alpha + g_2^\alpha)(s)\|_H^2 \, ds + M \|g_2^\alpha\|_{\mathcal{H}} \|u^\alpha + g_2^\alpha\|_{\mathcal{H}} \\ & + \|\varrho\|_{\mathcal{H}} \|u^\alpha + g_2^\alpha\|_{\mathcal{H}} + C \|g_1\|_{\mathcal{V}'} \|u^\alpha + g_2^\alpha\|_{\mathcal{V}} + \|g_2^\alpha\|_{\mathcal{V}} \|u^\alpha + g_2^\alpha\|_{\mathcal{V}}. \end{aligned}$$

Applying Gronwall's lemma yields

$$\begin{aligned} & \|(u^\alpha + g_2^\alpha)(t)\|_H^2 + \int_0^T B^{(s)}((u^\alpha + g_2^\alpha)(s), (u^\alpha + g_2^\alpha)(s)) \, ds \\ & \leq C(M, T) \left(\|\varphi\|_H^2 + \|g_2^\alpha\|_{\mathcal{H}} \|u^\alpha + g_2^\alpha\|_{\mathcal{H}} + \|\varrho\|_{\mathcal{H}} \|u^\alpha + g_2^\alpha\|_{\mathcal{H}} \right. \\ & \quad \left. + \|g_1\|_{\mathcal{V}'} \|u^\alpha + g_2^\alpha\|_{\mathcal{V}} + \|g_2^\alpha\|_{\mathcal{V}} \|u^\alpha + g_2^\alpha\|_{\mathcal{V}} \right). \end{aligned}$$

Integrating the above inequality with respect to t on $[0, T]$ we get

$$\begin{aligned} \|u^\alpha + g_2^\alpha\|_{\mathcal{V}}^2 &\leq \tilde{C}(M, T) \left(\|\varphi\|_H^2 + \|g_2^\alpha\|_{\mathcal{H}} \|u^\alpha + g_2^\alpha\|_{\mathcal{H}} + \|\varrho\|_{\mathcal{H}} \|u^\alpha + g_2^\alpha\|_{\mathcal{H}} \right. \\ &\quad \left. + \|g_1\|_{\mathcal{V}'} \|u^\alpha + g_2^\alpha\|_{\mathcal{V}} + \|g_2^\alpha\|_{\mathcal{V}} \|u^\alpha + g_2^\alpha\|_{\mathcal{V}} \right). \end{aligned}$$

By Young's inequality,

$$\|u^\alpha + g_2^\alpha\|_{\mathcal{V}}^2 \leq \hat{C}(M, T) \left(\|\varphi\|_H^2 + \|g_2^\alpha\|_{\mathcal{H}}^2 + \|\varrho\|_{\mathcal{H}}^2 + \|g_1\|_{\mathcal{V}'}^2 + \|g_2^\alpha\|_{\mathcal{V}}^2 \right).$$

By [31, Proposition I.3.7] and (5.1),

$$\|u^\alpha + g_2^\alpha\|_{\mathcal{V}}^2 \leq C'(M, T) \left(\|\varphi\|_H^2 + \|\varrho\|_{\mathcal{H}}^2 + \|g\|_{\mathcal{W}_0'}^2 \right). \quad (6.8)$$

Therefore, up to a subsequence, $\{u^\alpha + g_2^\alpha\}$ is weakly convergent in $\mathcal{V}$ to some $v \in \mathcal{V}$. By [31, Proposition I.3.7], $\{g_2^\alpha\}$ strongly converges in $\mathcal{V}$ to g_2 , so

$$u^\alpha \rightarrow u \quad \text{weakly in } \mathcal{V} \quad (6.9)$$

as $\alpha \rightarrow \infty$, where $u = v - g_2$. Observe that by (6.6) and (6.8), the sequence $\{u^\alpha + g_2^\alpha\}$ is bounded in $\mathcal{W}$. In particular, we have $u^\alpha + g_2^\alpha, u + g_2 \in \mathcal{W}$. We now take η_ε^t with $\eta \in \mathcal{W}_0$ as a test function in (6.6) and integrate by parts in the first term on the left hand side of (6.6). Next we pass to the limit with $\varepsilon \searrow 0$ and then pass to the limit with $t \nearrow T$. We then get $(u^\alpha + g_2^\alpha)(T) = \varphi$. Therefore (6.6) can be rewritten as

$$-\langle \partial_t(u^\alpha + g_2^\alpha), \eta \rangle + \mathcal{B}(u^\alpha, \eta) = (f(\cdot, u^\alpha), \eta)_{\mathcal{H}} + \langle g_1, \alpha \hat{G}_\alpha \eta \rangle, \quad \eta \in \mathcal{W}_0. \quad (6.10)$$

It is clear that the above equation also holds true for $\eta \in \mathcal{V}$. Let $\langle \cdot, \cdot \rangle_{V', V}$ denote the duality between V' and V . Let $\eta \in \mathcal{V}$. Taking $(u^\alpha + g_2^\alpha - \eta)^t$ in place of η in (6.10) we get

$$\begin{aligned} & - \int_t^T \langle \partial_s(u^\alpha + g_2^\alpha)(s), (u^\alpha + g_2^\alpha - \eta)(s) \rangle_{V', V} \, ds \\ & + \int_t^T \mathcal{B}^{(s)}(u^\alpha(s), (u^\alpha + g_2^\alpha - \eta)(s)) \, ds \\ & = \int_t^T (f(\cdot, u^\alpha)(s), (u^\alpha + g_2^\alpha - \eta)(s))_H \, ds + \langle g_1, \alpha \hat{G}_\alpha (u^\alpha + g_2^\alpha - \eta)^t \rangle. \end{aligned} \quad (6.11)$$

By [31, Proposition I.3.7],

$$\langle g_1, \alpha \hat{G}_\alpha (u^\alpha + g_2^\alpha - \eta)^t \rangle \rightarrow \langle g_1, (u + g_2 - \eta)^t \rangle \quad (6.12)$$

as $\alpha \rightarrow \infty$. Thanks to the monotonicity of L_t and f , we get by a pseudomonotonicity argument that

$$\begin{aligned} & \int_t^T B^{(s)}(u(s), (u + g_2 - \eta)(s)) \, ds - \int_t^T (f(\cdot, u)(s), (u + g_2 - \eta)(s))_H \, ds \\ & \leq \liminf_{\alpha \rightarrow \infty} \left(\int_t^T B^{(s)}(u^\alpha(s), (u^\alpha + g_2^\alpha - \eta)(s)) \, ds \right. \\ & \quad \left. - \int_t^T (f(\cdot, u^\alpha)(s), (u^\alpha + g_2^\alpha - \eta)(s))_H \, ds \right). \end{aligned} \quad (6.13)$$

Now observe that for $\eta \in \mathcal{W}_0$,

$$\begin{aligned} & - \int_t^T \langle \partial_s(u^\alpha + g_2^\alpha)(s), (u^\alpha + g_2^\alpha - \eta)(s) \rangle_{V', V} \, ds \\ & = -\frac{1}{2} \|\varphi - \eta(T)\|_H^2 + \frac{1}{2} \|(u^\alpha + g_2^\alpha - \eta)(t)\|_H^2 \\ & \quad - \int_t^T \langle \partial_s \eta(s), (u^\alpha + g_2^\alpha - \eta)(s) \rangle_{V', V} \, ds. \end{aligned} \quad (6.14)$$

We know that $\{u^\alpha + g_2^\alpha\}$ is bounded in $\mathcal{W}$ and converges to $u + g_2$ in $\mathcal{V}$ as $\alpha \rightarrow \infty$. Therefore, by the Banach–Saks theorem, there is a sequence $\{\alpha_k\}$ such that $\alpha_k \rightarrow \infty$ as $k \rightarrow \infty$ and the Cesàro means $(1/n) \sum_{k=1}^n (u^{\alpha_k} + g_2^{\alpha_k})$ converge to $u + g_2$ strongly in $\mathcal{W}$. Since $(u^\alpha + g_2^\alpha)(T) = \varphi$ and $\mathcal{W} \subset C([0, T]; H)$, it follows that $(u + g_2)(T) = \varphi$. This when combined with (6.14) and the proved convergences shows that for every bounded positive measurable function ψ on $[0, T]$ we have

$$\begin{aligned} & \int_0^T \psi(t) \int_t^T \langle -\partial_s(u + g_2)(s), (u + g_2 - \eta)(s) \rangle_{V', V} \, ds \, dt \\ & = \int_0^T \psi(t) \left(-\frac{1}{2} \|\varphi - \eta(T)\|_H^2 + \frac{1}{2} \|(u + g_2 - \eta)(t)\|_H^2 \right. \\ & \quad \left. - \int_t^T \langle \partial_s \eta(s), (u + g_2 - \eta)(s) \rangle_{V', V} \, ds \right) dt \\ & \leq \int_0^T \psi(t) \left(-\frac{1}{2} \|\varphi - \eta(T)\|_H^2 \right) dt + \liminf_{\alpha \rightarrow \infty} \int_0^T \psi(t) \left(\frac{1}{2} \|(u^\alpha + g_2^\alpha - \eta)(t)\|_H^2 \right) dt \\ & \quad - \lim_{\alpha \rightarrow \infty} \int_0^T \psi(t) \left(\int_t^T \langle \partial_s \eta(s), (u^\alpha + g_2^\alpha - \eta)(s) \rangle_{V', V} \, ds \right) dt \\ & = \liminf_{\alpha \rightarrow \infty} \int_0^T \psi(t) \int_t^T \langle -\partial_s(u^\alpha + g_2^\alpha)(s), (u^\alpha + g_2^\alpha - \eta)(s) \rangle_{V', V} \, ds \, dt. \end{aligned}$$

This and (6.11)–(6.13) show that

$$\begin{aligned} & \int_0^T \psi(t) \int_t^T \langle -\partial_s(u + g_2)(s), \eta(s) \rangle_{V', V} \, ds \, dt + \int_0^T \psi(t) \int_t^T B^{(s)}(u(s), \eta(s)) \, ds \, dt \\ & \leq \int_0^T \psi(t) \int_t^T (f(\cdot, u)(s), \eta(s))_H \, ds \, dt + \int_0^T \psi(t) \langle g_1, \eta^t \rangle_{V', V} \, dt \end{aligned}$$

for $\eta \in \mathcal{W}_0$. From this we get

$$-\langle \partial_t(u + g_2), \eta \rangle + \mathcal{B}(u, \eta) = (f(\cdot, u), \eta)_{\mathcal{H}} + \langle g_1, \eta \rangle, \quad \eta \in \mathcal{W}_0.$$

Since we know that $(u + g_2)(T) = \varphi$, the above equation implies (6.3). $\square$

In the proposition below, we provide a stochastic representation of the solution of (6.1) with $f = 0$ and $g \in \mathcal{W}'_0 \cap \mathcal{M}_{0,b}(E_{0,T})$. It will be needed in Sect. 9.

PROPOSITION 6.2. *Let $g \in \mathcal{W}'_0$, $\varphi \in H$. Assume that there exist $\mu \in \mathcal{M}_{0,b}(E_{0,T})$ such that*

$$\langle \langle g, \eta \rangle \rangle = \int_{E_{0,T}} \tilde{\eta} \, d\mu$$

for every bounded $\eta \in \mathcal{W}_0$. Let u be a solution of (6.1) with $f = 0$. Then for m_1 -a.e. $x \in E_{0,T}$,

$$u(x) = E_x \left(\varphi(X_T^0) + \int_0^\zeta dA_t^\mu \right).$$

Proof. We adopt the notation from the proof of Theorem 6.1. We first assume that $\varphi = 0$. Let $\mu_\alpha = \alpha \hat{R}_\alpha \circ \mu$. By (4.11) and the definition of a solution of (6.1),

$$\int_{E_{0,T}} \eta R \mu_\alpha \, dm_1 = \int_{E_{0,T}} \hat{R} \eta \, d\mu_\alpha = \int_{E_{0,T}} \alpha R_\alpha(\hat{R} \eta) \, d\mu = \langle \langle g^\alpha, \hat{R} \eta \rangle \rangle = (u^\alpha, \eta)_{\mathcal{H}}$$

for every bounded $\eta \in \mathcal{W}_0$. By (6.9), for every $\eta \in \mathcal{H}$ we have

$$\lim_{\alpha \rightarrow \infty} (u^\alpha, \eta)_{\mathcal{H}} = (u, \eta)_{\mathcal{H}},$$

whereas by (4.11), for every $\eta \in \mathcal{W}_0 \cap C_b(E_{0,T})$,

$$\lim_{\alpha \rightarrow \infty} \int_{E_{0,T}} \eta R \mu_\alpha \, dm_1 = \lim_{\alpha \rightarrow \infty} \int_{E_{0,T}} R \mu \cdot \alpha R_\alpha \eta \, dm_1 = \int_{E_{0,T}} \eta R \mu \, dm_1.$$

Therefore $u = R \mu$ m_1 -a.e. In the general case, we put $w(x) = E_x \varphi(X_T^0)$, $x \in E_{0,T}$. By [12, Theorem 3.7], w is a solution of (6.1) with $f = 0$, $g = 0$. Hence $v = u - w$ is a solution of (6.1) with $f = 0$ and $\varphi = 0$. By what has been already proved, $v = R \mu$ m_1 -a.e. Consequently, $u = w + R \mu$ m_1 -a.e. $\square$

7. Further properties of g_2

We know from Theorem 5.6 that each $\mu \in \mathcal{M}_{0,b}(E_{0,T})$ admits decomposition of the form

$$\mu = f \cdot m_1 + g_1 + \partial_t g_2 \tag{7.1}$$

with $f \in L^1(E_{0,T}; m_1)$, $g_1 \in \mathcal{V}'$ and $g_2 \in \mathcal{V}$. In this section, we prove some further regularity results for g_2 .

In the sequel, for a given Banach space F we denote by $D([0, T]; F)$ the set consisting of all functions $u \in L^1(0, T; F)$ having a version $\tilde{u}$ (i.e. $\int_0^T \|\tilde{u}(t) - u(t)\|_F dt = 0$) such that $[0, T] \ni t \mapsto \tilde{u}(t) \in F$ is càdlàg, i.e. right-continuous with left limits. Of course, $C([0, T]; F) \subset D([0, T]; F)$.

LEMMA 7.1. Assume that $\varphi \in L^1(E; m)$ and ρ is a Borel function on E such that $0 \leq \rho \leq 1$ and $\int_E \rho dm < \infty$.

(i) Let $f \in L^1(E_{0,T})$. Define

$$u(x) = E_x \left(\varphi(X_T^0) + \int_0^\zeta f(X_t) dt \right), \quad x \in E_{0,T}.$$

Then $u \in C([0, T]; L^1(E; \rho \cdot m))$.

(ii) Let $\mu \in \mathcal{M}_{0,b}(E_{0,T})$. Define

$$v(x) = E_x \left(\varphi(X_T^0) + \int_0^\zeta dA_r^\mu \right), \quad x \in E_{0,T}.$$

Then $u \in D([0, T]; L^1(E; \rho \cdot m))$.

Proof. Choose $\{\varphi_n\} \subset H \cap L^1(E; m)$, $\{f_n\} \subset \mathcal{H} \cap L^1(E_{0,T}; m_1)$ and a nest $\{F_n\}$ so that $\|\varphi_n - \varphi\|_{L^1(E;m)} + \|f_n - f\|_{L^1(E_{0,T};m_1)} \rightarrow 0$ as $n \rightarrow \infty$, and $\mu_n := \mathbf{1}_{F_n} \cdot |\mu| \in S_0(E_{0,T})$. By Proposition 6.2, the functions

$$u_n(x) = E_x \left(\varphi_n(X_T^0) + \int_0^\zeta f_n(X_t) dt \right), \quad v_n(x) = E_x \left(\varphi_n(X_T^0) + \int_0^\zeta dA_r^{\mu_n} \right), \\ x \in E_{0,T},$$

are unique solutions of the Cauchy problems

$$-\partial_t u_n - L_t u_n = f_n, \quad u_n(T) = \varphi_n, \quad -\partial_t u_n - L_t u_n = \mu_n, \quad u_n(T) = \varphi_n,$$

respectively. Let $(T_{s,t})_{t>s}$ (resp. $(\hat{T}_{s,t})_{t>s}$) denote the semigroup determined by the form $B^{(s)}$ (resp. dual form $\hat{B}^{(s)}$).

It is clear that $u_n \in \mathcal{W} \subset C([0, T]; H) \subset C([0, T]; L^1(E; \rho \cdot m))$. We have

$$\int_E E_{s,x^0} \int_0^\zeta |(f_n - f)(X_t)| dt m(dx^0) \leq \int_s^T (|T_{s,t}(f_n - f)(t)|, 1)_H dt \\ = \int_s^T (|(f_n - f)(t)|, \hat{T}_{s,t} 1)_H dt \\ \leq \|f_n - f\|_{L^1(E_{0,T}; m_1)}$$

and

$$\int_E E_{s,x^0} |(\varphi_n - \varphi)(X_T^0)| m(dx^0) = (T_{s,T} |\varphi - \varphi_n|, 1)_H \\ = (|\varphi - \varphi_n|, \hat{T}_{s,T} 1)_H \leq \|\varphi - \varphi_n\|_{L^1(E;m)}. \quad (7.2)$$

Hence

$$\sup_{t \in [0, T]} \|u_n(t) - u(t)\|_{L^1(E; \rho \cdot m)} \leq \|\varphi - \varphi_n\|_{L^1(E; m)} + \|f_n - f\|_{L^1(E_{0, T}; m_1)}.$$

Since $u_n \in C([0, T]; L^1(E; \rho \cdot m))$, $n \geq 1$, this proves (i).

By [12, Proposition 3.4, Theorem 3.7], $v_n \in D([0, T]; H) \subset D([0, T]; L^1(E; \rho \cdot m))$. Let $v = \delta_{\{s\}} \otimes m$. Then, by Lemma 4.3

$$\begin{aligned} \int_E \left| E_{s, x^0} \int_0^\zeta dA_t^{\mu} - E_{s, x^0} \int_0^\zeta dA_t^{\mu_n} \right| m(dx^0) &\leq E_v \int_0^\zeta dA_t^{|\mu - \mu_n|} \\ &= \int_{E_{0, T}} \hat{R}v d|\mu_n - \mu|. \end{aligned} \quad (7.3)$$

Since $v \in S(E_{0, T})$, by using Lemma 4.3 with $\alpha = 0$ we get

$$\|\hat{R}v\|_\infty = \sup_{\|\eta\|_{L^1} \leq 1} \int_{E_{0, T}} \eta \hat{R}v dm_1 = \sup_{\|\eta\|_{L^1} \leq 1} \int_{E_{0, T}} R\eta dv = \sup_{\|\eta\|_{L^1} \leq 1} \|(R\eta)(s)\|_{L^1(E; m)}.$$

But $\int_E R\eta(s, x^0) m(dx^0) \leq \int_s^T (T_{s, t} \eta(t), 1)_H dt \leq 1$. Hence $\|\hat{R}v\|_\infty \leq 1$. Therefore from (7.2) and (7.3) it follows that

$$\begin{aligned} \sup_{t \in [0, T]} \|v_n(t) - v(t)\|_{L^1(E; \rho \cdot m)} &\leq \|\varphi - \varphi_n\|_{L^1(E; m)} + \|\mu_n - \mu\|_{TV} \\ &= \|\varphi - \varphi_n\|_{L^1(E; m)} + |\mu|(E_{0, T} \setminus F_n). \end{aligned}$$

Since $v_n \in D([0, T]; L^1(E; \rho \cdot m))$, $n \geq 1$, and the right-hand side of the above inequality converges to zero as $n \rightarrow \infty$, we get (ii). $\square$

The following corollary extends [6, Lemma 2.29].

COROLLARY 7.2. *Let $\mu \in \mathcal{M}_{0, b}(E_{0, T})$ and (f, g_1, g_2) , $(\bar{f}, \bar{g}_1, \bar{g}_2)$ be two decompositions of μ . Then $g_2 - \bar{g}_2 \in C([0, T]; L^1(E; \rho \cdot m))$ for any function ρ on E such that $0 \leq \rho \leq 1$ and $\int_E \rho dm = 1$.*

Proof. For every bounded $\eta \in \mathcal{W}_0$, we have

$$\langle \partial_t \eta, g_2 - \bar{g}_2 \rangle = -\langle g_1 - \bar{g}_1, \eta \rangle - \int_{E_{0, T}} (f - \bar{f}) \eta dm_1.$$

Hence, for every bounded $\eta \in \mathcal{W}_0$,

$$\langle \partial_t \eta, g_2 - \bar{g}_2 \rangle + \mathcal{B}(g_2 - \bar{g}_2, \eta) = -\langle g_1 - \bar{g}_1, \eta \rangle + \langle \chi, \eta \rangle - \int_{E_{0, T}} (f - \bar{f}) \eta dm_1,$$

where χ is an element of $\mathcal{V}'$ such that $\langle \chi, \eta \rangle = \mathcal{B}(g_2 - \bar{g}_2, \eta)$, $\eta \in \mathcal{V}$. Let $u = g_2 - \bar{g}_2$ and $v \in \mathcal{W}_T$ be a solution to the Cauchy problem

$$-\partial_t v - L_t v = -(g_1 - \bar{g}_1) + \chi, \quad v(T) = 0.$$

Then

$$\langle \partial_t \eta, u - v \rangle + \mathcal{B}(u - v, \eta) = - \int_{E_{0,T}} (f - \bar{f}) \eta \, dm_1$$

for all bounded $\eta \in \mathcal{W}_0$. By Proposition 6.2, for m_1 -a.e. $x \in E_{0,T}$ we have

$$(u - v)(x) = E_x \int_0^\zeta (\bar{f} - f)(X_t) \, dt.$$

By Lemma 7.1, $u - v \in C([0, T]; L^1(E; \rho \cdot m))$. Since $v \in \mathcal{W}_T \subset C([0, T]; H) \subset C([0, T]; L^1(E; \rho \cdot m))$, we get the desired result. $\square$

PROPOSITION 7.3. *Let $\mu \in \mathcal{M}_{0,b}(E_{0,T})$ and let (f, g_1, g_2) be a decomposition of μ . Let ρ be a Borel function on E such that $0 \leq \rho \leq 1$ and $\int_E \rho \, dm < \infty$.*

- (i) $g_2 \in D([0, T]; L^1(E; \rho \cdot m))$ and $g_2(T-) = 0$.
 - (ii) For $t \in (0, T)$, let μ_t be the measure defined as $\mu_t(B) = \mu(\{t\} \times B)$, $B \in \mathcal{B}(E)$.
- Then*

$$\mu_t = (g_2(t-) - g_2(t)) \cdot m. \quad (7.4)$$

Proof. Without loss of generality we may assume that $f = 0$ (otherwise, we consider the measure $\nu = \mu - f \cdot m_1$ instead of μ). Then $\mu \in \mathcal{W}'_0$. By Proposition 6.2, the function $u(x) = E_x \int_0^\zeta dA_r^\mu$, $x \in E$, is a solution to the problem

$$-\partial_t u - L_t u = \mu, \quad u(T, \cdot) = 0,$$

so by the definition,

$$\langle \partial_t \eta, u \rangle + \mathcal{B}(u, \eta) = \langle g_1, \eta \rangle - \langle \partial_t \eta, g_2 \rangle, \quad \eta \in \mathcal{W}_0. \quad (7.5)$$

Hence

$$\langle \partial_t \eta, u - g_2 \rangle + \mathcal{B}(u - g_2, \eta) = \chi(\eta), \quad \eta \in \mathcal{W}_0, \quad (7.6)$$

where $\chi(\eta) = -\mathcal{B}(g_2, \eta) + \langle \eta, g_1 \rangle$. From this we conclude that $u - g_2 \in \mathcal{W}_T \subset C([0, T]; H) \subset D([0, T]; L^1(E; \rho \cdot m))$. By Lemma 7.1, $u \in D([0, T]; L^1(E; \rho \cdot m))$. Therefore $g_2 \in D([0, T]; L^1(E; \rho \cdot m))$. For $t \in [0, T)$, we set

$$\begin{aligned} \rho_\varepsilon^t(s) &= 0, \quad s \in [0, t - \varepsilon], \quad \rho_\varepsilon^t(s) = \varepsilon^{-1}(s - t + \varepsilon), \quad s \in [t - \varepsilon, t] \\ \rho_\varepsilon^t(s) &= 1, \quad s \in [t, T]. \end{aligned}$$

Let $\xi \in V \cap C_c(E)$ and $\eta_\varepsilon^t = \rho_\varepsilon^t \xi$. Because $u \in D([0, T]; L^1(E; \rho \cdot m))$ and $u - g_2 \in C([0, T]; H)$, replacing η by η_ε^t in (7.5) and (7.6), and then letting $\varepsilon \downarrow 0$ and $t \uparrow T$ shows that $u(T-) = 0$ and $(u - g_2)(T) = 0$. Hence $g_2(T-) = 0$. Taking η_ε^t as a test function in (5.12) and letting $\varepsilon \downarrow 0$ we obtain

$$\int_{[t,T) \times E} \xi \, d\mu = \langle \mathbf{1}_{[t,T) \times E} g_1, \xi \rangle - \int_E \xi g_2(t-) \, dm \quad (7.7)$$

from which we conclude that (7.4) is satisfied. $\square$

The following proposition is a counterpart to [25, Theorem 1.1].

PROPOSITION 7.4. *Let $\mu \in \mathcal{M}_{0,b}(E_{0,T})$. Then for every $\varepsilon > 0$ there exists a measure $\mu_\varepsilon \in S_0(E_{0,T}) - S_0(E_{0,T})$ such that $\|\mu_\varepsilon - \mu\|_{TV} \leq \varepsilon$ and μ_ε admits decomposition of the form (5.12) with $f = 0$ and $g_2 \in \mathcal{V} \cap L^\infty(E_{0,T}; m_1)$.*

Proof. Without loss of generality we can assume that μ is positive. Let $\{F_n^1\}$ be a nest such that $\mathbf{1}_{F_n^1} \cdot \mu \in S_0(E_{0,T})$. Let $u = R\mu$, $F_n^2 = \{u \leq n\}$ and $F_n = F_n^1 \cap F_n^2$, $\mu_n = \mathbf{1}_{F_n} \cdot \mu$ and $u_n = R\mu_n$. By Itô's formula (see also 9.4),

$$|u_n(x)|^2 \leq 2E_x \int_0^\zeta u_n(X_{t-}) \mathbf{1}_{F_n}(X_{t-}) dA_t^\mu \leq 2nE_x \int_0^\zeta \mathbf{1}_{F_n}(X_{t-}) dA_t^\mu = 2nu_n(x).$$

Hence $u_n(x) \leq 2n$ for q.e. $x \in E_{0,T}$. Consequently, $u_n \in L^\infty(E_{0,T}; m_1)$. By Proposition 6.2, u_n is a solution to the Cauchy problem

$$-\partial_t u_n - L_t u_n = \mu_n, \quad u_n(T) = 0.$$

In other words, for every bounded $v \in \mathcal{W}_0$,

$$\int_{E_{0,T}} \tilde{v} d\mu_n = \langle \chi_n, v \rangle + \langle \partial_t v, u_n \rangle,$$

where χ_n is an element of $\mathcal{V}'$ such that $\langle \chi_n, \eta \rangle = \mathcal{B}(u_n, v)$, $v \in \mathcal{V}$. Thus $\mu_n = \chi_n + \partial_t u_n$, i.e. μ_n admits the decomposition (5.12) with $f = 0$ and $g_2 := u_n \in \mathcal{V} \cap L^\infty(E_{0,T}; m_1)$. Furthermore, $\|\mu_n - \mu\|_{TV} = \mu(E_{0,T} \setminus F_n) \rightarrow 0$ as $n \rightarrow \infty$, which completes the proof. $\square$

It is known (see [25, Example 3.1]) that not every bounded smooth measure can be written in the form (7.1) with bounded g_2 . We are going to show that g_2 is always quasi-bounded with respect to the capacity c_2 . To this end, we first recall the definition of a parabolic potential (see, e.g., [26]).

DEFINITION. A measurable function $u \in \mathcal{V} \cap L^\infty(0, T; H)$ is called a parabolic potential if for every positive $v \in \mathcal{W}_0$,

$$\langle \partial_t v, u \rangle + \mathcal{B}(u, v) \geq 0. \quad (7.8)$$

The set of parabolic potentials will be denoted by $\mathcal{P}^2$. By [27, Proposition I.1], for every $u \in \mathcal{P}^2$ there exists a unique Borel measure μ_u such that for every $\eta \in \mathcal{W}_0 \cap C_b(E_{0,T})$,

$$\langle \partial_t \eta, u \rangle + \mathcal{B}(u, \eta) = \int_{E_{0,T}} \eta d\mu_u.$$

Let $\mu \in S_0$ be positive. Then, by [13, Proposition 3.1], $\mu \in \mathcal{P}^2$ and $\mu_{R\mu} = \mu$. Following [28], for $u \in \mathcal{P}^2$ we set

$$\|u\|_\Lambda^2 = \text{ess sup}_{t \in (0,T)} \|u(t)\|_H^2 + \|u\|_{\mathcal{V}}^2.$$

For an open set $U \subset E_{0,T}$, we define

$$c_1(U) = \inf\{\|\eta\|_\Lambda : \eta \in \mathcal{P}^2, \eta \geq \mathbf{1}_U \text{ } m_1\text{-a.e.}\}.$$

By [28, Theorem 1], there exists $\alpha > 0$ such that

$$c_0 \leq \alpha c_1. \quad (7.9)$$

For $k \geq 0$, we set

$$T_k(s) = \max\{-k, \min\{k, s\}\}, \quad s \in \mathbb{R}.$$

LEMMA 7.5. *Let $\mu \in \mathcal{M}_{0,b}^+$. Then $T_k(R\mu) \in \mathcal{P}^2$, $k \geq 0$, and there exists $c > 0$ such that*

$$\|T_k(R\mu)\|_\Lambda^2 \leq ck\|\mu\|_{TV}.$$

Proof. By [12, Theorem 3.12], $T_k(R\mu) \in \mathcal{V}$. Let $\{F_n\}$ be a Cap_ψ -nest of compact sets such that $\mu_n := \mathbf{1}_{F_n} \cdot \mu \in S_0$. Since $\{F_n\}$ is a Cap_ψ -nest, $R\mu_n \nearrow R\mu$ Cap_ψ -q.e., hence m_1 -a.e. Since $R\mu_n \in \mathcal{P}^2$, by [27, Corollary I.1], $T_k(R\mu_n) \in \mathcal{P}^2$. Hence, by [26, Lemma III-1], there exists $c > 0$ such that

$$\|T_k(R\mu_n)\|_\Lambda^2 \leq ck\|\mu_{T_k(R\mu_n)}\|_{TV}.$$

By [26, Lemma II-5],

$$\|T_k(R\mu_n)\|_\Lambda^2 \leq ck\|\mu_{T_k(R\mu_n)}\|_{TV} \leq ck\|\mu_{R\mu_n}\|_{TV} = ck\|\mu_n\|_{TV} \leq ck\|\mu\|_{TV}. \quad (7.10)$$

By [12, Theorem 3.12], $\sup_{n \geq 1} \|T_k(R\mu_n)\|_{\mathcal{V}} < \infty$. Since $T_k(R\mu_n) \rightarrow T_k(R\mu)$ m_1 -a.e. as $n \rightarrow \infty$, it follows that, up to a subsequence, $T_k(R\mu_n) \rightarrow T_k(R\mu)$ weakly in $\mathcal{V}$. It is clear that $T_k(R\mu)$ satisfies (7.8), which when combined with (7.10) implies that $T_k(R\mu) \in \mathcal{P}^2$ and the desired inequality holds true. $\square$

PROPOSITION 7.6. *Assume that $\mu \in \mathcal{M}_b(E_{0,T})$ is of the form (1.7). Then g_2 has an m_1 -version $\tilde{g}_2$ which is c_0 -quasi-bounded.*

Proof. Set $\nu = \mu - f \cdot m_1$ and

$$v(x) = -E_x \int_0^\zeta dA_t^\nu,$$

where A^ν is a natural AF of $\mathbb{X}$ in the Revuz correspondence with ν . By Proposition 6.2, v is a solution to (6.1) with $f = 0$, $\varphi = 0$ and g replaced by $-v$. Observe that $w = v - g_2$ is a solution to the Cauchy problem

$$-\partial_t w - L_t w = -g_1 - \chi, \quad w(T) = 0,$$

where χ is an element of $\mathcal{V}'$ such that $\langle \chi, \eta \rangle = \mathcal{B}(g_2, \eta)$, $\eta \in \mathcal{V}$. Since $g_1 + \chi \in \mathcal{V}'$, we have $w \in \mathcal{W}_T$. Therefore

$$g_2 = w - E.A_\zeta^v \quad m_1\text{-a.e.} \quad (7.11)$$

for some $w \in \mathcal{W}_T$. By [27, Theorem III.1], w has an m_1 -version $\tilde{w}$ which is c_0 -quasi-continuous, so c_0 -quasi-bounded. Therefore we only need to show that $v_+(x) := E_x A_\zeta^{v^+}$ and $v_-(x) := E_x A_\zeta^{v^-}$ have m_1 -versions which are c_0 -quasi-bounded. We will show this for v_+ . The proof for v_- is analogous. Let $v_+^k = T_k(v_+)$. By Lemma 7.5, $v_+^k \in \mathcal{P}^2$, so by [27, Corollary III.3], there exists an m_1 -version $\tilde{v}_+^k$ of v_+^k which is c_0 -quasi-l.s.c. Hence $\tilde{v}_+ := \sup_{k \geq 1} \tilde{v}_+^k$ is also c_0 -quasi-l.s.c., and of course, it is an m_1 -version of v_+ . By Lemma 7.5 and (7.9),

$$c_0(\tilde{v}_+ > n) = c_0(\tilde{v}_+^{n+1} > n) \leq \alpha n^{-2} \|(v_+^{n+1})\|_\Lambda^2 \leq 2c\alpha n^{-1} \|v^+\|_{TV}. \quad (7.12)$$

Let $\{F_n^1\}$ be a c_0 -nest such that $(\tilde{v}_+)|_{F_n}$ is l.s.c. for each $n \geq 1$. Set $F_n = F_n^1 \cap \{\tilde{v}_+ \leq n\}$. Then F_n is closed. Moreover,

$$c_0(E_{0,T} \setminus F_n) \leq c_0(\tilde{v}_+ > n) + c_0(E_{0,T} \setminus F_n^1) \rightarrow 0$$

as $n \rightarrow \infty$, which proves that $\tilde{v}_+$ is c_0 -quasi-bounded. $\square$

8. Smoothness of measures in $\mathcal{W}'_0$

We already know that each bounded smooth measure admits a decomposition of the form (1.7). The problem whether a bounded measure admitting decomposition (1.7) is smooth is more delicate. In this section, we give positive answer to this question.

In the proof of our result, we will make use of Fukushima's decomposition, which we now recall. For an AF A of $\mathbb{X}$ its energy is defined by

$$e(A) = \frac{1}{2} \lim_{\alpha \rightarrow \infty} \alpha^2 E_{m_1} \int_0^\infty e^{-\alpha t} |A_t|^2 dt$$

whenever the limit exists in $[0, \infty]$. By [31, Proposition IV.1.8], each $u \in \mathcal{W}$ has a quasi-continuous m_1 -modification, which we will denote by $\tilde{u}$. By [32, Theorem 4.5], for any $w \in \mathcal{W}$ there exists a unique martingale AF of $\mathbb{X}$ of finite energy $M^{[u]}$ and a unique continuous AF of $\mathbb{X}$ of zero energy $N^{[u]}$ such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \geq 0, \quad P_x\text{-a.s.} \quad (8.1)$$

for q.e. $x \in E_{0,T}$. The decomposition (8.1) is called Fukushima's decomposition.

PROPOSITION 8.1. *Assume that $u \in \mathcal{W}_T$. Then*

$$e(M^{[u]}) \leq \mathcal{B}(u, u).$$

Proof. Let $A_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0)$, $t \geq 0$. We have

$$\begin{aligned} \alpha^2 E_{m_1} \int_0^\infty e^{-\alpha t} |A_t^{[u]}|^2 dt &= \alpha^2 \int_{E_{0,T}} (R_\alpha u - 2u R_\alpha u + \alpha^{-1} u) dm_1 \\ &= 2\alpha(u - \alpha R_\alpha u, u)_{\mathcal{H}} - \alpha(u^2, 1 - \alpha \hat{R}_\alpha 1)_{\mathcal{H}} \\ &= 2\mathcal{E}(\alpha G_\alpha u, u) - \alpha(u^2, 1 - \alpha \hat{R}_\alpha 1)_{\mathcal{H}}. \end{aligned}$$

By (2.15), $\alpha \hat{R}_\alpha 1(x) = 1 - \hat{E}_x e^{-\alpha \zeta}$, $x \in E_{0,T}$. Hence

$$\alpha(u^2, 1 - \alpha \hat{R}_\alpha 1)_{\mathcal{H}} = \int_{E_{0,T}} \alpha |u|^2 \hat{E}_x e^{-\alpha \zeta} m_1(dx) \geq \int_0^T \alpha |u(s)|_H^2 e^{-\alpha s} ds,$$

which converges to $\|u(0)\|_H^2$ as $\alpha \rightarrow \infty$. By the above calculations,

$$e(A^{[u]}) = \limsup_{\alpha \rightarrow \infty} \left(\mathcal{E}(\alpha G_\alpha u, u) - \frac{1}{2} \alpha(u^2, 1 - \alpha \hat{R}_\alpha 1)_{\mathcal{H}} \right),$$

so by [30, Proposition 2.7],

$$e(A^{[u]}) \leq \mathcal{E}(u, u) - \frac{1}{2} \|u(0)\|_H^2 = \mathcal{B}(u, u).$$

On the other hand, by (8.1), $e(A^{[u]}) = e(M^{[u]})$, which proves the proposition. $\square$

LEMMA 8.2. Assume that A, A^n , $n \geq 1$, are additive functionals of $\mathbb{X}$.

- (i) If $E_{\eta \cdot m_1} \sup_{t \geq 0} |A_t|^p < \infty$ for some $p \geq 1$ and strictly positive $\eta \in \mathcal{B}(E_{0,T})$, then $E_x \sup_{t \geq 0} |A_t|^p < \infty$ for q.e. $x \in E_{0,T}$.
- (ii) If $E_{\eta \cdot m_1} \sup_{t \geq 0} |A_t^n - A_t^m|^p \rightarrow 0$ as $n, m \rightarrow \infty$ for some $p \geq 0$ and strictly positive $\eta \in \mathcal{B}(E_{0,T})$, then there exists a subsequence (still denoted by n) such that $E_x \sup_{t \geq 0} |A_t^n - A_t^m|^p \rightarrow 0$ as $n, m \rightarrow \infty$ for q.e. $x \in E_{0,T}$.

Proof. We will prove (ii). The proof of (i) is analogous. We may assume that

$$E_{\eta \cdot m_1} \sup_{t \geq 0} |A_t^n - A_t^{n+1}|^p \leq 2^{-n}, \quad n \geq 1, \quad (8.2)$$

and $\int_{E_{0,T}} \eta dm_1 = 1$. Let $\tilde{m}_1 = \eta \cdot m_1$ and

$$B = \liminf_{n \rightarrow \infty} \left\{ \sup_{t \geq 0} |A_t^n - A_t^{n+1}|^p > 2^{-2n} \right\}, \quad N = \{x \in E_{0,T} : P_x(B) > 0\}.$$

It is clear that N is a nearly Borel set. We will show that $\text{Cap}_\psi(N) = 0$. Since Cap_ψ is a Choquet capacity, we may and will assume that N is compact. We have

$$P_{\tilde{m}_1}(\sigma_N < \infty) \leq P_{\tilde{m}_1}(X_{\sigma_N} \in N) = P_{\tilde{m}_1}(E_{X_{\sigma_N}} \mathbf{1}_B > 0).$$

By the strong Markov property,

$$P_{\tilde{m}_1}(E_{X_{\sigma_N}} \mathbf{1}_B > 0) = P_{\tilde{m}_1}(E_x(\mathbf{1}_B \circ \theta_{\sigma_N} | \mathcal{F}_{\sigma_N}) > 0).$$

Since A is additive,

$$\begin{aligned} B \circ \theta_{\sigma_N} &= \liminf_{n \rightarrow \infty} \left\{ \sup_{t \geq 0} |A_t^n \circ \theta_{\sigma_N} - A_t^{n+1} \circ \theta_{\sigma_N}|^p > 2^{-n} \right\} \\ &\subset \liminf_{n \rightarrow \infty} \left\{ \sup_{t \geq 0} |A_t^n - A_t^{n+1}|^p > 2^{-n+p} \right\} =: B'. \end{aligned}$$

Hence

$$P_{\tilde{m}_1}(\sigma_N < \infty) \leq \int_{E_{0,T}} P_x(E_x(\mathbf{1}_{B'} | \mathcal{F}_{\sigma_N}) > 0) \tilde{m}_1(dx).$$

But by (8.2) and the Borel–Cantelli lemma, $P_{\tilde{m}_1}(B') = 0$, which implies that $P_x(B') = 0$ for m_1 -a.e. $x \in E_{0,T}$. Hence, for m_1 -a.e. $x \in E_{0,T}$, $P_x(E_x(\mathbf{1}_{B'} | \mathcal{F}_{\sigma_N}) > 0) = 0$, so $P_{\tilde{m}_1}(\sigma_N < \infty) = 0$. Equivalently, $\text{Cap}_\psi(N) = 0$, as claimed. $\square$

LEMMA 8.3. *The set $C = \{\sum_{i=1}^n \xi_i v_i : \xi_i \in C_c^\infty((0, T))$, $v_i \in V \cap C_c(E)\}$ is dense in $\mathcal{W}_0^2$.*

Proof. Follows from [22, Lemma 1.1]. $\square$

THEOREM 8.4. *Let $\mu \in \mathcal{M}_b(E_{0,T})$. If there exist $f \in L^1(E_{0,T}; m_1)$, $g_1 \in \mathcal{V}'$, $g_2 \in \mathcal{V}$ such that (5.12) is satisfied for all $\eta \in \mathcal{W}_0 \cap C_b(E_{0,T})$, then $\mu \in \mathcal{M}_{0,b}(E_{0,T})$.*

Proof. Since the measure $f \cdot m_1$ is smooth, without loss of generality we may assume that $f = 0$. Let Φ be a functional on $\mathcal{W}_0$ defined by the right-hand side of (5.12) (with $f = 0$). It is clear that $\Phi \in \mathcal{W}_0'$. Let $u \in \mathcal{V}$ be a solution to (6.1) with $\varphi = 0$, $f = 0$ and $g = \Phi$.

Step 1. We will show that u is a difference of excessive functions. By (5.12) and the definition of solution to (6.1), for every $\eta \in \mathcal{W}_0 \cap C_b(E_{0,T})$,

$$|\mathcal{E}(u, \eta)| \leq \|\mu\|_{TV} \|\eta\|_\infty. \quad (8.3)$$

Let $\mathcal{C} = \{\sum_{i=1}^n \xi_i v : \xi_i \in H^1(0, T)$, $\xi_i(0) = 0$, $v \in V \cap C_c(E)\}$. Then $\mathcal{C} \subset \mathcal{W}_0 \cap C_b(E_{0,T})$, so (8.3) holds for every $\eta \in \mathcal{C}$. By Lemma 8.3, $\mathcal{C}$ is dense in $\mathcal{W}_0^2$. Let $\eta \in \mathcal{W}_0^2$ be bounded. Write $c = \|\eta\|_\infty$ and choose $\{\eta_n\} \subset \mathcal{C}$ such that $\eta_n \rightarrow \eta$ in $\mathcal{W}_0^2$. Then $T_c(\eta_n) \rightarrow T_c(\eta) = \eta$ in $\mathcal{W}_0^2$, so (8.3) holds for every bounded $\eta \in \mathcal{W}_0^2$. Suppose now that $\eta \in \mathcal{W}_0$ and η is bounded. Then $\alpha \hat{R}_\alpha \eta \in \mathcal{W}_0^2$, $\|\alpha \hat{R}_\alpha \eta\|_\infty \leq \|\eta\|_\infty$ and $\alpha \hat{R}_\alpha \eta \rightarrow \eta$ in $\mathcal{W}_0$ as $\alpha \rightarrow \infty$. Therefore (8.3) holds for every bounded $\eta \in \mathcal{W}_0$. By [1, Proposition 4.5], there exist excessive functions v and w such that $u = v - w$ m_1 -a.e. and for every $\eta \in \mathcal{H} \cap L^\infty(E_{0,T}; m_1)$,

$$\int_{E_{0,T}} (v + w) |\eta| dm_1 < \infty, \quad (8.4)$$

$$\frac{1}{t} \int_{E_{0,T}} |P_t v - v| \eta dm_1 + \frac{1}{t} \int_{E_{0,T}} |P_t w - w| \eta dm_1 \leq c \|\eta\|_\infty. \quad (8.5)$$

Since each excessive function finite m_1 -a.e. is finite q.e., we may assume that $v(x) + w(x) < \infty$, $x \in E_{0,T} \setminus N$, where N is some m_1 -inessential set (see [10, Proposition 6.12]).

Step 2. Let

$$\tilde{u} = v - w, \quad u_n = nR_n\tilde{u}, \quad v_n = (n\tilde{u} - n^2R_n\tilde{u}) \cdot m_1.$$

By [31, Proposition I.3.7], $u_n \rightarrow u$ strongly in $\mathcal{V}$. By Fukushima's decomposition, there exists an m_1 -inessential set N such that for every $x \in E_{0,T} \setminus N$,

$$u_n(X_t) = u_n(X_0) - \int_0^t dA_r^{v_n} + \int_0^t dM_r^{[u_n]}, \quad t \leq \zeta, \quad P_x\text{-a.s.} \quad (8.6)$$

We will show the uniform convergence of $\{M^{[u_n]}\}$ as $n \rightarrow \infty$. Let $\langle M^{[u_n]} \rangle$ denote the sharp bracket of $M^{[u_n]}$ (see, e.g., [8, Sect. A.3]). We have

$$\begin{aligned} e(M^{[u_n]}) &= \frac{1}{2} \lim_{\alpha \rightarrow \infty} \alpha^2 E_{m_1} \int_0^\infty e^{-\alpha t} \langle M^{[u_n]} \rangle_t dt \\ &= \frac{1}{2} \lim_{\alpha \rightarrow \infty} \int_0^\infty s e^{-s} \frac{\alpha}{s} E_{m_1} \langle M^{[u_n]} \rangle_{s/\alpha} ds. \end{aligned} \quad (8.7)$$

Since $t \mapsto E_{m_1} \langle M^{[u_n]} \rangle_t$ is subadditive, $(1/t)E_{m_1} \langle M^{[u_n]} \rangle_t$ increases as t decreases, and $\lim_{t \downarrow 0} (1/t)E_{m_1} \langle M^{[u_n]} \rangle_t = \sup_{t>0} (1/t)E_{m_1} \langle M^{[u_n]} \rangle_t$. Therefore, letting $\alpha \rightarrow \infty$ in (8.7), shows that

$$e(M^{[u_n]}) = \frac{1}{2} \sup_{t>0} \frac{1}{t} E_{m_1} \langle M^{[u_n]} \rangle_t \cdot \int_0^\infty s e^{-s} ds = \frac{1}{2} \sup_{t>0} \frac{1}{t} E_{m_1} \langle M^{[u_n]} \rangle_t.$$

Hence

$$E_{m_1} \langle M^{[u_n]} \rangle_\zeta \leq 2Te(M^{[u_n]}),$$

so applying Doob's inequality we get

$$E_{m_1} \sup_{t \geq 0} |M_t^{[u_n]} - M_t^{[u_m]}|^2 \leq 8Te(M^{[u_n - u_m]})$$

for all $n, m \geq 1$. By Proposition 8.1 and Lemma 8.2, there exists an m_1 -inessential set N such that, up to a subsequence,

$$\lim_{n,m \rightarrow \infty} E_x \sup_{t \geq 0} |M_t^{[u_n]} - M_t^{[u_m]}|^2 = 0, \quad x \in E_{0,T} \setminus N.$$

Step 3. We will show that there exists a natural AF A of $\mathbb{X}$ of finite variation and a martingale AF M of $\mathbb{X}$ such that

$$\tilde{u}(X_t) = \tilde{u}(X_0) - \int_0^t dA_r + \int_0^t dM_r, \quad t \leq \zeta, \quad P_x\text{-a.s.} \quad (8.8)$$

for $x \in E_{0,T} \setminus N$. Set

$$M_t = \liminf_{n \rightarrow \infty} M_t^{[u_n]}, \quad A_t = \liminf_{n \rightarrow \infty} A_t^{v_n}, \quad t \geq 0.$$

By the definition of $\tilde{u}$, for every $x \in E_{0,T}$,

$$u_n(X_t) \rightarrow \tilde{u}(X_t), \quad t \geq 0, \quad P_x\text{-a.s.}$$

Therefore, letting $n \rightarrow \infty$ in (8.6), shows that (8.8) is satisfied. Moreover, M is a square integrable martingale under the measure P_x for $x \in E_{0,T} \setminus N$, and

$$E_x \sup_{t \geq 0} |M_t^{[u_n]} - M_t|^2 \rightarrow 0, \quad A_t = \lim_{n \rightarrow \infty} A_t^{v_n}, \quad t \geq 0, \quad P_x\text{-a.s.} \quad (8.9)$$

for every $x \in E_{0,T} \setminus N$. By [2, Theorem III.5.7], $v(X)$, $w(X)$ are càdlàg supermartingales under the measure P_x for $x \in E_{0,T} \setminus N$. In particular, $\tilde{u}(X)$ and A are càdlàg processes under P_x for $x \in E_{0,T} \setminus N$. By the resolvent identity,

$$\begin{aligned} R|v_n| &= nR|u - nR_n u| \leq nR|v - nR_n v| + nR|w - nR_n w| \\ &= nR(v - nR_n v) + nR(w - nR_n w) = nR_n v + nR_n w \leq v + w. \end{aligned} \quad (8.10)$$

Hence

$$\sup_{n \geq 1} E_x |A^{v_n}|_\zeta \leq v(x) + w(x) < \infty, \quad x \in E_{0,T} \setminus N. \quad (8.11)$$

Let $t_i^k = iT/2^k$, $i = 0, \dots, 2^k$. Since A is càdlàg,

$$|A|_T = \lim_{k \rightarrow \infty} \sum_{i=1}^k |A_{t_i^k} - A_{t_{i-1}^k}| = \sup_{k \geq 1} \sum_{i=1}^k |A_{t_i^k} - A_{t_{i-1}^k}|.$$

Hence

$$\begin{aligned} E_x |A|_T &= E_x \lim_{k \rightarrow \infty} \sum_{i=1}^k |A_{t_i^k} - A_{t_{i-1}^k}| = \lim_{k \rightarrow \infty} E_x \sum_{i=1}^k |A_{t_i^k} - A_{t_{i-1}^k}| \\ &\leq \lim_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} E_x \sum_{i=1}^k |A_{t_i^k}^{v_n} - A_{t_{i-1}^k}^{v_n}| \leq \liminf_{n \rightarrow \infty} E_x \sup_{k \geq 1} \sum_{i=1}^k |A_{t_i^k}^{v_n} - A_{t_{i-1}^k}^{v_n}| \\ &= \liminf_{n \rightarrow \infty} E_x \lim_{k \rightarrow \infty} \sum_{i=1}^k |A_{t_i^k}^{v_n} - A_{t_{i-1}^k}^{v_n}| = \liminf_{n \rightarrow \infty} E_x |A^{v_n}|_T. \end{aligned}$$

From this and (8.11) we get

$$E_x |A|_\zeta < \infty, \quad x \in E_{0,T} \setminus N. \quad (8.12)$$

Let

$$\Lambda = \{\omega \in \Omega : A_t(\omega) = \limsup_{n \rightarrow \infty} A_t^n(\omega), \quad t \geq 0, \quad A_\cdot(\omega) \text{ is càdlàg of finite variation}\}.$$

By what has already been proved, $P_x(\Lambda) = 1$ for $x \in E_{0,T} \setminus N$. Observe that $\theta_t(\Lambda) \subset \Lambda$. Moreover, for all $s, t \geq 0$ and $\omega \in \Lambda$,

$$A_t(\theta_s(\omega)) = \lim_{n \rightarrow \infty} A_t^{v_n}(\theta_s(\omega)) = \lim_{n \rightarrow \infty} (A_{t+s}^{v_n}(\omega) - A_s^{v_n}(\omega)) = A_{t+s}(\omega) - A_s(\omega). \quad (8.13)$$

Of course, the same relation holds for A^+ , A^- . Thus A^+ , A^- are positive natural AFs of $\mathbb{X}$ with the defining set Λ and exceptional set N . This implies that M is a martingale AF of $\mathbb{X}$.

Step 4. Let ν^+ (resp. ν^-) be the Revuz measure associated with A^+ (resp. A^-) (see [10, Sect. 8]). To complete the proof it suffices to show that $\nu \in \mathcal{M}_{0,b}(E_{0,T})$ and $\nu = \mu$. Since M is a uniformly integrable martingale,

$$\tilde{u}(x) = E_x A_\zeta, \quad x \in E_{0,T} \setminus N.$$

We have $\tilde{u} = u_+ - u_-$, where

$$u_+(x) := E_x A_\zeta^+, \quad u_-(x) := E_x A_\zeta^-, \quad x \in E_{0,T} \setminus N.$$

It is clear that u_+ , u_- are natural potentials (see [2, Definition IV.4.17] and the comments following the definition). By the construction of functions v , w (see [1, Proposition 4.2]),

$$v(x) \leq u_+(x), \quad w(x) \leq u_-(x), \quad x \in E_{0,T} \setminus N.$$

Therefore v , w are natural potentials, and by [2, Theorem IV.4.22], $v = E.A_\zeta^1$, $w = E.A_\zeta^2$ for some positive natural AFs A^1 , A^2 of $\mathbb{X}$. By the minimality argument,

$$v(x) = u_+(x), \quad w(x) = u_-(x), \quad x \in E_{0,T} \setminus N.$$

From this, (8.5), (8.10) and [10, Theorem 9.3] it follows that for every positive $\eta \in C_b(E_{0,T})$,

$$\frac{1}{t} E_{\eta \cdot m} |A|_t \leq \frac{1}{t} \int_{E_{0,T}} |P_t v - v| \eta \, dm_1 + \frac{1}{t} \int_{E_{0,T}} |P_t w - w| \eta \, dm_1 \leq c \|\eta\|_\infty.$$

By [10, Theorem 8.7], for every positive $\eta \in C_b(E_{0,T})$,

$$\lim_{t \downarrow 0} \frac{1}{t} E_{\eta \cdot m} |A|_t = \int_{E_{0,T}} \eta \, d|\nu|.$$

Therefore $\int_{E_{0,T}} \eta \, d|\nu| < \infty$ for any positive $\eta \in C_b(E_{0,T})$. Consequently, $\nu \in \mathcal{M}_b(E_{0,T})$. By the very definition of ν^+ and ν^- , if $\text{Cap}_\psi(B) = 0$, then $|\nu|(B) = 0$, so $\nu \in \mathcal{M}_{0,b}(E_{0,T})$. By [10, Theorem 9.3], $\tilde{u} = R\nu$ q.e. By the definition of a solution to (6.1) and (5.12),

$$\mathcal{E}(u, \eta) = \int_{E_{0,T}} \eta \, d\mu$$

for $\eta \in \mathcal{W}_0 \cap C_b(E_{0,T})$. On the other hand, for every $\eta \in \mathcal{W}_0 \cap C_b(E_{0,T})$,

$$\begin{aligned} \mathcal{E}(u, \eta) &= \lim_{t \downarrow 0} \frac{1}{t} (u, \eta - \hat{T}_t \eta) = \lim_{t \downarrow 0} \frac{1}{t} (u - T_t u, \eta) \\ &= \lim_{t \downarrow 0} \frac{1}{t} E_{\eta \cdot m_1} \int_0^t dA_r^v = \int_{E_{0,T}} \eta \, d\nu. \end{aligned}$$

Hence $\mu = \nu$. □

9. Decomposition of measures and additive functionals

In this section, we study the structure of the additive functional A^μ in the Revuz correspondence with $\mu \in \mathcal{M}_{0,b}(E_{0,T})$. Specifically, we want to get deeper understanding of the nature of jumps of A^μ and their relation to the decomposition (f, g_1, g_2) of μ given in Theorem 5.6. Before proceeding, remarks concerning two special cases of μ are in order.

If $\mu = f \cdot m_1$, then

$$A_t^\mu = \int_0^t f(X_r) \, dr, \quad t \geq 0.$$

If $\mu = g_1$, then

$$A_t^\mu = -N_t^{[u]}, \quad t \geq 0, \tag{9.1}$$

where $u \in \mathcal{W}_T$ is a unique solution to the Cauchy problem

$$(\partial_t + L_t)u = -g_1, \quad u(T) = 0, \tag{9.2}$$

and $N^{[u]}$ is the continuous AF from Fukushima's decomposition (8.1). Indeed, by Proposition 6.2, $u = R\mu$ m_1 -a.e. In other words, for q.e. $x \in E_{0,T}$ we have

$$\tilde{u}(x) = E_x A_\zeta^\mu. \tag{9.3}$$

Let $M_t^x = E_x(A_\zeta^\mu | \mathcal{F}_t) - \tilde{u}(X_0)$, $t \geq 0$. Because of standard perfection procedure (see, e.g., [8, Lemma A.3.6]), there is a martingale AF M of $\mathbb{X}$ such that $M_t = M_t^x$, $t \geq 0$, P_x -a.s. for q.e. $x \in E_{0,T}$. From (9.3) and the strong Markov property we obtain

$$\tilde{u}(X_t) - \tilde{u}(X_0) = -A_t^\mu + M_t, \quad t \geq 0 \tag{9.4}$$

(see [14, Remark 3.3] for more details). Since $\tilde{u}$ is quasi-continuous and the filtration is quasi-left continuous, from (9.4) it follows that A^μ is continuous. It is clear that M is a martingale AF of $\mathbb{X}$. Let $u_n = nR_n u$. Then

$$u_n(X_t) - u_n(X_0) = N_t^{[u_n]} + M_t^{[u_n]}, \quad t \geq 0, \tag{9.5}$$

and by Itô's formula,

$$u_n(X_t) - u_n(X_0) = -A_t^{\mu_n} + M_t^n, \quad t \geq 0, \quad (9.6)$$

where $\mu_n = n(u_n - u) \cdot m_1$ and $M_t^n = E_x(A_\zeta^{\mu_n} | \mathcal{F}_t) - \tilde{u}_n(X_0)$ P_x -a.s. for q.e. $x \in E_{0,T}$. By an elementary calculation (see, e.g., [8, page 245]), A^{μ_n} is of zero energy. By uniqueness of Fukushima's decomposition, $-A^{\mu_n} = N^{[u_n]}$. We know that $u_n \rightarrow \tilde{u}$ q.e. and $u_n \rightarrow u$ in $\mathcal{V}$, so by Proposition 8.1, $N^{[u_n]} \rightarrow N^{[u]}$. Also, by [11, Theorem 3.3], $A_t^{\mu_n} \rightarrow A_t^\mu$, $t \geq 0$. Thus $-A^\mu = N^{[u]}$.

We see that in both special cases considered above the additive functionals corresponding to μ are continuous. This suggests that the jumps of A^μ stem from the component g_2 of the decomposition of μ . In what follows we will show that this is indeed true and we will make this statement more precise.

Following [12] we adopt the following definition.

DEFINITION. We say that a Borel measurable function u on $E_{0,T}$ is quasi-càdlàg if for q.e. $x \in E_{0,T}$ the process $t \mapsto u(X_t)$ is càdlàg on $[0, T - \tau(0))$ under the measure P_x .

Since A^μ is predictable, by [5, Chapter IV, Theorem 88B], there is a sequence $\{\tau_n\}$ of predictable stopping times exhausting the jumps of A^μ , i.e.

$$\{\Delta A^\mu \neq 0\} = \bigcup_{n=1}^{\infty} [[\tau_n]], \quad [[\tau_n]] \cap [[\tau_m]] = \emptyset, \quad n \neq m,$$

where $[[\tau_n]]$ denotes the graph of τ_n .

In what follows, we denote by $A^{\mu,c}$ the continuous part of A^μ and by $A^{\mu,d}$ the pure jump part of A^μ . For a given càdlàg process Y , we write $\Delta Y_t = Y_t - Y_{t-}$, where $Y_{t-} = \lim_{s \nearrow t} Y_s$.

THEOREM 9.1. Let $\mu \in \mathcal{M}_{0,b}(E_{0,T})$ and (f, g_1, g_2) be a decomposition of μ from Theorem 5.6. Then

- (i) g_2 has a quasi-càdlàg m_1 -version $\tilde{g}_2$.
- (ii) Let $Y = \tilde{g}_2(X)$. For q.e. $x \in E_{0,T}$ we have

$$A_t^{\mu,d} = \sum_{\tau_n \leq t} \Delta Y_{\tau_n} \quad P_x\text{-a.s.} \quad (9.7)$$

Proof. Set $v = \mu - f \cdot m_1$ and

$$v(x) = -E_x \int_0^\zeta dA_t^v, \quad x \in E_{0,T},$$

where A^v is a natural AF of $\mathbb{X}$ in the Revuz correspondence with v . By Proposition 6.2, v is quasi-càdlàg and it is a solution to (6.1) with $f = 0$, $\varphi = 0$ and g replaced by $-v$. By (7.11), $g_2 = v - w$ m_1 -a.e. for some $w \in \mathcal{W}_T$. Write

$$\tilde{g}_2 = v - \tilde{w}. \quad (9.8)$$

By the argument used to prove (9.4), there is a martingale AF M^v of $\mathbb{X}$ such that

$$v(X_t) - v(X_0) = M_t^v + A_t^v, \quad t \geq 0, \quad P_x\text{-a.s.}$$

for q.e. $x \in E_{0,T}$. By Fukushima's decomposition (8.1),

$$\tilde{w}(X_t) - \tilde{w}(X_0) = M_t^{[w]} + N_t^{[w]}, \quad t \geq 0, \quad P_x\text{-a.s.}$$

for q.e. $x \in E_{0,T}$. Hence

$$\tilde{g}_2(X_t) - \tilde{g}_2(X_0) = M_t^v - M_t^{[w]} + A_t^v - N_t^{[w]}, \quad t \geq 0, \quad P_x\text{-a.s.} \quad (9.9)$$

for q.e. $x \in E_{0,T}$. Since the filtration $(\mathcal{F}_t)$ is quasi-left continuous, by [8, Theorems A.3.2, A.3.6], the martingale AFs M^v and $M^{[w]}$ admit no predictable jump. Therefore from (9.9) and continuity of the functionals $N^{[w]}$ and $A^{f \cdot m_1}$ we get the result. $\square$

REMARK 9.2. (i) By (9.8), $\tilde{g}_2$ is quasi-continuous if and only if v is quasi-continuous, which in turn is equivalent to the continuity of A^μ .

(ii) By [10, Theorem 16.8], there is a positive Borel function $h : E_{0,T} \rightarrow \mathbb{R}$ such that

$$\Delta A_t^{\mu,d} = h(X_{t-}), \quad t \geq 0, \quad P_x\text{-a.s.}$$

for q.e. $x \in E_{0,T}$. By this and (9.7),

$$\tilde{g}_2(X_{\tau_n}) - \tilde{g}_2(X)_{\tau_n-} = h(X_{\tau_n-}) = h(X_{\tau_n}) \quad P_x\text{-a.s.},$$

the second equality being a consequence of the quasi-continuity of the filtration $(\mathcal{F}_t)$. We see that if $\tilde{g}_2$ is not quasi-continuous, then the jumps of $\tilde{g}_2(X)$ which are not produced by X are produced by $\tilde{g}_2$. The size of these jumps is described by h .

EXAMPLE 9.3. Assume that $B^{(t)} = B^{(0)}$ for $t \in [0, T]$ and there exists a strictly positive $\beta : E \rightarrow \mathbb{R}$ such that $\beta \in L^1(E; m) \cap C(E)$ (for instance, we may take E to be a bounded open subset D of $\mathbb{R}^d$ and consider the classical Dirichlet form on $L^2(D; dx^0)$ defined as $B^{(t)}(u, v) = \int_D \nabla u(x^0) \nabla v(x^0) dx^0$, $u, v \in H_0^1(D)$). Let $a \in (0, T)$. We define μ on $E_{0,T}$ by

$$\mu(dt dx^0) = (\delta_{\{a\}} + \ell)(dt) \otimes \beta(x^0) m(dx^0),$$

where ℓ is the Lebesgue measure on $[0, T]$. Clearly $\mu \in \mathcal{M}_b(E_{0,T})$. Moreover, if $\text{Cap}_\psi(B) = 0$ for some Borel set $B \subset E_{0,T}$, then $m_1(B) = 0$ and $\text{Cap}_\psi(\{t\} \times B_t) = 0$ for every $t \in (0, T]$, where $B_t = \{x^0 \in E : (t, x^0) \in B\}$. From this, [22, (4.4)] and [23, (6.2.24)] it follows that $m(B_t) = 0$ for $t \in (0, T]$. In particular, $m(B_a) = 0$, and consequently $\mu(B) = 0$. Thus $\mu \in \mathcal{M}_{0,b}(E_{0,T})$. Let

$$\alpha(t) = \mathbf{1}_{[a,T]}(t) + t, \quad g_2(t, x^0) = \alpha(t)\beta(x^0), \quad (t, x^0) \in E_{0,T}.$$

Then

$$\mu = \partial_t g_2.$$

By (2.6), $g_2(X_t) = \alpha(v(t))\beta(X_{v(t)}^0)$, $t \geq 0$, P_x -a.s. for q.e. $x \in E_{0,T}$, from which it follows that g_2 is quasi-càdlàg. Furthermore, under the measure P_x with $x = (s, x^0)$, for every predictable τ we have

$$\Delta g_2(X)_\tau = (\alpha(s + \tau) - \alpha((s + \tau)-))\beta(X_{s+\tau}^0) = \begin{cases} 0, & s + \tau \neq a, \\ \beta(X_a^0), & s + \tau = a. \end{cases}$$

Consequently, g_2 is not quasi-continuous. Finally, we note that

$$\Delta A_t^{\mu,d} = \mathbf{1}_{\{a\}}(s + t)\beta(X_{s+t}^0) = h(X_t), \quad t \in [0, T - s], \quad P_{s,x^0}\text{-a.s.},$$

where $h(t, x^0) = \mathbf{1}_{\{a\}}(t)\beta(x^0)$, $(t, x^0) \in E_{0,T}$.

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