A note on singularities in finite time for the L^2 gradient flow of the Helfrich functional

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Abstract. This work investigates the formation of singularities under the steepest descent L^2 -gradient flow of W_{λ_1, λ_2} with zero spontaneous curvature, i.e., the sum of the Willmore energy, λ_1 times the area, and λ_2 times the signed volume of an immersed closed surface without boundary in $\mathbb{R}^3$. We show that in the case that $\lambda_1 > 1$ and $\lambda_2 = 0$, any immersion develops singularities in finite time under this flow. If $\lambda_1 > 0$ and $\lambda_2 > 0$, embedded closed surfaces with energy less than

$$8\pi + \min \left\{ \left(16\pi\lambda_1^3 \right) / \left(3\lambda_2^2 \right), 8\pi \right\}$$

and positive volume evolve singularities in finite time. If in this case the initial surface is a topological sphere and the initial energy is less than 8π , the flow shrinks to a round point in finite time. We furthermore discuss similar results for the case that λ_2 is negative. These results strengthen the ones of McCoy and Wheeler (Commun Anal Geom 24(4):843–886, 2016). For $\lambda_1 > 0$ and $\lambda_2 \geq 0$, they showed that embedded closed spheres with positive volume and energy close to 4π , i.e., close to the Willmore energy of a round sphere, converge to round points in finite time.

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1. Introduction

In [6], Helfrich suggested the functional

$$W_{\lambda_1, \lambda_2}^{H_0}(f) := \int_{\Sigma} |H_f + H_0|^2 d\mu_f + \lambda_1 \mu(f) + \lambda_2 \text{vol}(f)$$

for immersions $f : \Sigma \rightarrow \mathbb{R}^3$ of two-dimensional compact connected surfaces Σ without boundary to study lipid bilayers. Here, $H_f = \frac{1}{2}(\kappa_1 + \kappa_2)$ denotes the mean curvature and μ_f the surface measure on Σ induced by f . The constant $H_0 \in \mathbb{R}$ is called spontaneous curvature and $\text{vol}(f)$ denotes the signed inclosed volume given by

$$\text{vol}(f) = \int_{[0,1] \times \Sigma} \phi_f^*(\text{dvol}),$$

where $\phi_f^*(\text{dvol})$ denotes the pull-back of the standard volume form $\text{dvol} = dx^1 \wedge dx^2 \wedge dx^3$ on $\mathbb{R}^3$ under $\phi_f : [0, 1] \times \Sigma \rightarrow \mathbb{R}^3$, $\phi_f(t, x) := tf(x)$.

Helfrich found that these energies are well suited to explain the characteristic shape of red blood cells: the shape of a biconcave disk. In honor of his work, the functional $W_{\lambda_1, \lambda_2}^{H_0}$ is now called *Helfrich functional*.

In this article, we will restrict our attention to the special case of zero spontaneous curvature H_0 . We only consider

$$W_{\lambda_1, \lambda_2}(f) := W_{\lambda_1, \lambda_2}^0 = \int_{\Sigma} |H_f|^2 d\mu_f + \lambda_1 \mu(f) + \lambda_2 \text{vol}(f), \quad (1.1)$$

in which case the first summand is the *Willmore energy*

$$W(f) := \int_{\Sigma} |H_f|^2 d\mu_f \quad (1.2)$$

We only deal with this case since the Willmore functional is scale invariant—indeed already Blaschke [1] observed that it is invariant under Möbius transformations that leave the surface bounded. Willmore proved that the Willmore energy is always greater or equal to 4π with equality only for a parametrized round sphere.

Helfrich calculated the L^2 -gradient of $W_{\lambda_1, \lambda_2}^{H_0}$. It is known to be equal to

$$\nabla_{L^2} W_{\lambda_1, \lambda_2}^{H_0}(f) = \left(\Delta_f H_f + 2(H_f + H_0) \left(H_f^2 - H_0 H_f - K_f \right) - \lambda_1 H_f - \lambda_2 \right) \nu_f, \quad (1.3)$$

where ν_f denotes the unit normal along f , K_f the Gauß curvature, and Δ_f the Laplace–Beltrami operator. In the case of zero spontaneous curvature, this reads as

$$\nabla_{L^2} W_{\lambda_1, \lambda_2}(f) = \nabla_{L^2} W_{\lambda_1, \lambda_2}^0(f) = \left(\Delta_f H_f + 2H_f \left(H_f^2 - K_f \right) - \lambda_1 H_f + \lambda_2 \right) \nu_f. \quad (1.4)$$

We will consider smooth families of smooth immersions $f_t : \Sigma \rightarrow \mathbb{R}^3$, $t \in [0, T)$ of a compact surface Σ without boundary that are solutions to the steepest L^2 -gradient flow of the Helfrich functional with zero spontaneous curvature $H_0 = 0$, i.e., that solve

$$\partial_t f_t = -\nabla_{L^2} W_{\lambda_1, \lambda_2}(f_t) \quad \forall t \in [0, T). \quad (1.5)$$

Note that such a family of immersions satisfies the equality

$$\frac{d}{dt} W_{\lambda_1, \lambda_2}(f_t) = -\|\nabla_{L^2} W_{\lambda_1, \lambda_2}(f_t)\|_{L^2(d\mu_f)}^2. \quad (1.6)$$

Let us state the following short time existence theorem which is an immediate consequence of Theorem 1.1 in [14] the proof of which was based on the Lions–Lax–Milgram theorem and an analysis of the linearized problem by Polden [7].

THEOREM 1.1. (cf. [14]) *Suppose $f_0 : \Sigma \rightarrow \mathbb{R}^3$ is a compact smoothly immersed surface without boundary. There exists a unique maximal smooth family of immersions $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^3$ solving (1.5) with $f(\cdot, 0) = f_0$.*

For the case of the gradient flow of $\int_{\Sigma} |H_f - H_0|^2 d\mu_f$ constrained to immersions of fixed length and volume, short time existence was proven by Kohsaka and Nagasawa [8]. Note that short time existence of the flow above can for example also be derived using analytic semigroups as carried out for the Willmore flow and the surface diffusion flow in [5, 16]. Liu [12] could bound the lifespan of the flow from below if there is only a small quantum of energy within balls of a given scale thus extending the corresponding result of Kuwert and Schätzle for the Willmore flow [10].

The following two theorems summarize the results of this article. We set for $\lambda_2 \neq 0$

$$\begin{aligned}\varepsilon_1(\lambda_1, \lambda_2) &= \frac{\lambda_1^2}{\lambda_2^2} 16\pi \\ \varepsilon_2(\lambda_1, \lambda_2) &= \frac{\lambda_1^3}{3\lambda_2^2} 16\pi.\end{aligned}\tag{1.7}$$

Then the following theorem on existence of finite time singularities holds.

THEOREM 1.2. *Let us assume that $\lambda_1 > 0$ and let Σ be a smooth compact manifold without boundary.*

- (i) *If $\lambda_2 = 0$, the locally constrained Willmore flow (1.5) starting with any initial immersed surface $f_0 : \Sigma \rightarrow \mathbb{R}^3$ forms a singularity in finite time.*
- (ii) *If $\lambda_2 \neq 0$, the flow starting with an embedding $f_0 : \Sigma \rightarrow \mathbb{R}^3$ with $\text{vol}(f) > 0$ forms a singularity in finite time if either*

$$\mu(f_0) < \varepsilon_1(\lambda_1, \lambda_2) \text{ and } W_{\lambda_1, \lambda_2}(f_0) < 4\pi + \varepsilon_2(\lambda_1, \lambda_2)$$

or

$$\lambda_2 > 0 \text{ and } W_{\lambda_1, \lambda_2}(f_0) < 8\pi + \min\{\varepsilon_2(\lambda_1, \lambda_2), 8\pi\}.$$

This substantially improves the result of McCoy and Wheeler [18] who only considered the case $\lambda_2 \geq 0$, $\text{vol}(f_0) \geq 0$ and had to assume that $W_{\lambda_1, \lambda_2}(f_0) < 4\pi + \varepsilon$ for a universal constant $\varepsilon > 0$, in principle calculable but not further specified. Furthermore our method of proof is based on the different scaling of the three summands that make up the energy and does not rely on sophisticated energy estimates as the proof of the corresponding result by McCoy and Wheeler. Similar effects were for example used in [4] to get control over the elastic flow constrained to fixed length.

For the case of the Helfrich flow of spheres, i.e., in the case that $\Sigma = \mathbb{S}^2$, we can also show that the flow sub-converges to a round point in the sense that there are times

$t_j \rightarrow T$ and a point $x \in \mathbb{R}^n$ such that the rescaled immersions

$$\left(\frac{4\pi}{\mu(f_{t_j})} \right)^{\frac{1}{2}} (f_{t_j} - x)$$

converge to the unit sphere. More precisely, we have

THEOREM 1.3. *If in the situation of Theorem 1.2 furthermore $\Sigma = \mathbb{S}^2$ and*

$$\lim_{t \uparrow T} W_{\lambda_1, \lambda_2}(f_t) < 8\pi,$$

then the flow sub-converges to a round point in finite time $T < \infty$.

This result builds on the construction of a blowup profile in [18]. We show at the end of Sect. 4 that the constant 8π in Theorem 1.3 is sharp.

Let us finish this introduction with an outline of the structure of the remaining article. In Sect. 2, we shortly recapitulate the findings of McCoy and Wheeler on both the existence of critical points of the Helfrich functional and on the formation of singularities in finite time. Section 3 shows how to use the scaling properties of the different summands that make up the energy W_{λ_1, λ_2} to show that the flow (1.5) develops singularities in finite time. In Sect. 4, we recapitulate the construction of a blowup profile at a point singularity from [9, 18]. We again use the different scalings of the components to show that this blowup profile must be a Willmore surfaces without assuming any initial bound on the energy of the initial surface (cf. Theorem 4.2). Combining this with the point removability results of Kuwert and Schätzle [11] and the classification result for Willmore spheres in [3], we can show the sub-convergence to round points as stated in Theorem 1.3.

2. Review of the results of McCoy and Wheeler

In the pioneering paper [17], McCoy and Wheeler completely classified all critical immersions of the functional $W_{\lambda_1, \lambda_2}^{H_0}$ of complete surface without boundary under the assumption that

$$W(f) \leq 4\pi + \varepsilon_0 \tag{2.1}$$

where the constant $\varepsilon_0 > 0$ can in principle be deduced from the constructive nature of their proof. Note that since the Willmore energy of a round sphere is equal to 4π , (2.1) in a certain way says that the distance of f_0 from parametrizing a round sphere is small. They showed

THEOREM 2.1. [17, Theorem 1] *There is an absolute constant $\varepsilon_0 > 0$ such that the following holds: Suppose that $f : \Sigma \rightarrow \mathbb{R}^3$ is a smooth properly immersed complete surface without boundary and*

$$\int_{\Sigma} \|A^0\|^2 d\mu < \varepsilon_0. \tag{2.2}$$

Then the following is true: If $\lambda_1 > 0$

$(\lambda_2 < 0) \quad \nabla_{L^2} W_{\lambda_1, \lambda_2}(f) = 0$ if and only if $f(\Sigma)$ is a sphere of radius $-\frac{2\lambda_1}{\lambda_2}$,
 $(\lambda_2 = 0) \quad \nabla_{L^2} W_{\lambda_1, \lambda_2}(f) = 0$ if and only if $f(\Sigma)$ is a plane,
 $(\lambda_2 > 0) \quad \nabla_{L^2} W_{\lambda_1, \lambda_2}(f) \neq 0$.

It is a straightforward calculation that spheres of radius r_0 remain spheres under the flow where the radii satisfy

$$\partial_t r = -\lambda_1 \frac{1}{r} - \lambda_2.$$

So if λ_1 and λ_2 or both nonnegative and at least one of them is different from zero, the solution to (1.5) sub-converges to a round point in finite time. For the case that $\lambda_1 > 0$, $\lambda_2 \geq 0$, McCoy and Wheeler could extend the above result to the following statement about solutions to the gradient flow (1.5).

THEOREM 2.2. [18] *There is an $\varepsilon > 0$ such that initial immersions $f_0 : \Sigma \rightarrow \mathbb{R}^3$ of a compact manifold without boundary Σ , with positive signed inclosed volume and*

$$W_{\lambda_1, \lambda_2}(f_0) \leq 4\pi + \varepsilon_0$$

sub-converge to a round point under the evolution Eq. (1.5)

One of the main ingredients to the proof of their theorem above is the highly non-trivial fact that under the condition (2.1) we have

$$\|\nabla_{L^2} W_{\lambda_1, \lambda_2}\|_{L^2(\mu_f)}^2 \geq c_0 \|\nabla_{L^2} W\|_{L^2(\mu_f)}^2$$

for some $c_0 > 0$ which they prove using sophisticated energy estimates. We will not make use of this estimate at all. Furthermore, their method did not allow them to treat the case $\lambda_2 < 0$ which we can. Theorem 1.2 reduces the gap between the existence or better nonexistence of critical points in Theorem 2.1 and the existence of finite time singularities in Theorem 2.2.

3. Existence of finite time singularities: an approach based on scaling

Let us now prove the main results of this article on the existence of finite time singularities of the locally constrained Willmore flow. All these results are based on the different scaling behavior of the three terms building W_{λ_1, λ_2} . Apart from this, we will use that, by an inequality of Yau [13], we have

$$4\pi \cdot \#f^{-1}(x) \leq W(f).$$

So especially $W(f) < 8\pi$ implies that f is an embedding. Peter Topping [20, Lemma 1] showed that

$$\text{diam}(f) \leq \frac{2}{\pi} \sqrt{\mu(f) W(f)}. \quad (3.1)$$

This estimate is a sharpened version of [19, Lemma 1.1].

THEOREM 3.1. *Let $\lambda_1 > 0$ and $f_0 : \Sigma \rightarrow \mathbb{R}^3$ be a compact smoothly immersed surface without boundary. Let us further assume that either $\lambda_2 = 0$ or $\lambda_2 > 0$ and f_0 be an embedding with positive enclosed volume nad $W_{\lambda_1, \lambda_2}(f_0) \leq 8\pi$. Then the maximal time of smooth existence T for the Helfrich flow (1.5) with initial data f_0 satisfies*

$$T \leq \frac{(W_{\lambda_1, \lambda_2}(f_0))^2 - (4\pi)^2}{2\pi^2\lambda_1^2}.$$

Proof. Let us start with an observation for a general immersion $f : \Sigma \rightarrow \mathbb{R}^3$. We consider the dilations

$$f_\alpha = \alpha(f - p)$$

around a fixed point p in $f(\Sigma)$. Note that the Willmore energy stays constant while the area behaves like $\alpha^2 \mu(f_1)$ and the volume goes like $\alpha^3 \text{vol}(f_1)$. By the definition of the L^2 gradient, we hence find

$$\int_{\Sigma} \nabla_{L^2} W_{\lambda_1, \lambda_2}(f)(f - p) d\mu_f = \frac{d}{d\alpha} (W_{\lambda_1, \lambda_2}(f_\alpha))|_{\alpha=1} = 2\lambda_1 \mu(f) + 3\lambda_2 \text{vol}(f). \quad (3.2)$$

Using the Cauchy–Schwartz inequality together with (3.1), we have

$$\begin{aligned} \int_{\Sigma} \nabla_{L^2} W_{\lambda_1, \lambda_2}(f)(f - p) d\mu &\leq \| \nabla_{L^2} W_{\lambda_1, \lambda_2}(f) \|_{L^2(\mu)} \cdot \| f - p \|_{L^2(\mu)} \\ &\leq \| \nabla_{L^2} W_{\lambda_1, \lambda_2}(f) \|_{L^2(\mu)} \text{diam}(f) \sqrt{\mu(f)} \\ &\leq \frac{2}{\pi} \sqrt{W(f)} \| \nabla_{L^2} W_{\lambda_1, \lambda_2}(f) \|_{L^2(\mu)} \mu(f). \end{aligned} \quad (3.3)$$

Combining Eq. (3.2) with the estimate (3.3), we get, if $\lambda_2 \text{vol}(f) \geq 0$,

$$\| \nabla_{L^2} W_{\lambda_1, \lambda_2}(f) \|_{L^2(\mu)}^2 \geq \frac{\pi^2 \lambda_1^2}{W(f)}. \quad (3.4)$$

Let us now consider the unique maximal solution f_t , $t \in [0, T)$, of the Helfrich flow (1.5) with initial data f_0 as in the statement of the theorem. Under the assumption that $\lambda_2 \text{vol}(f)$ stays nonnegative along the flow, we get from (1.6) and (3.4) for all $t \in [0, T)$

$$\frac{d}{dt} W_{\lambda_1, \lambda_2}(f_t) = - \| \nabla_{L^2} W_{\lambda_1, \lambda_2}(f_t) \|_{L^2(d\mu)}^2 \leq - \frac{\pi^2 \lambda_1^2}{W_{0,0}(f_t)} \leq - \frac{\pi^2 \lambda_1^2}{W_{\lambda_1, \lambda_2}(f_t)}.$$

Hence the differential inequality

$$\frac{d}{dt} (W_{\lambda_1, \lambda_2})^2 \leq -2\pi^2 \lambda_1^2$$

holds. The maximal time of smooth existence would thus satisfy

$$T \leq \frac{(W_{\lambda_1, \lambda_2}(f_0))^2 - (4\pi)^2}{2\pi^2 \lambda_1^2}$$

as for later times t we would have $W(f_t) \leq W_{\lambda_1, \lambda_2}(f_t) < 4\pi$, which is impossible. This concludes the proof for the case that $\lambda_2 = 0$.

Let us finally show that in the case $\lambda_2 > 0$ every initial embedding with $W_{\lambda_1, \lambda_2}(f_0) \leq 8\pi$ and positive volume stays embedded and hence $\text{vol}(f_t)$ stays non-negative. Otherwise there would be a first time t_0 at which a self-intersection occurs and hence especially $\text{vol}(f_{t_0}) \geq 0$, $\mu(f_{t_0}) > 0$. But then due to the celebrated inequality by Li and Yau [13], we would have $W_{0,0}(f_{t_0}) \geq 8\pi$ which would imply

$$W_{\lambda_1, \lambda_2}(f_{t_0}) > 8\pi \geq W_{\lambda_1, \lambda_2}(f_0).$$

This is impossible since the energy monotonically decreases in time. This concludes the proof. $\square$

Let us contemplate on the proof of Theorem 3.1 a bit further to see what can be saved of it, if we do not assume that $\lambda_2 \text{vol}(f) \geq 0$. Instead, we want to use the different scaling of the surface area and the volume that guarantees that, for $\mu(f)$ small, the surface area dominates the inclosed volume.

The proof of Theorem 3.1 is based on the fact that for any immersion $f : \Sigma \rightarrow \mathbb{R}^3$, $p \in f(\Sigma)$, $f_\alpha = \alpha(f - p)$, we have

$$\frac{d}{d\alpha} W_{\lambda_1, \lambda_2}(f_\alpha)|_{\alpha=1} = 2\lambda_1 \mu(f) + 3\lambda_2 \text{vol}(f)$$

which we have estimated by $2\lambda_1 \mu(f)$ from below under the assumption that $\lambda_2 \text{vol}(f) \geq 0$. If we just assume that $\lambda_1 > 0$, but not $\lambda_2 \geq 0$, we can estimate, using the isoperimetric inequality $\text{vol}(f) \leq \frac{\mu(f)^{\frac{3}{2}}}{6\pi^{\frac{1}{2}}}$,

$$\frac{d}{d\alpha} W_{\lambda_1, \lambda_2}(f_\alpha)|_{\alpha=1} = 2\lambda_1 \mu(f) + 3\lambda_2 \text{vol}(f) \geq \left(2\lambda_1 - |\lambda_2| \frac{\mu(f)^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}}\right) \mu(f).$$

So if $\mu(f) < \varepsilon_1(\lambda_1, \lambda_2) = \frac{\lambda_1^2}{|\lambda_2|^2} 16\pi$, we can still bound this derivative from below by a positive multiple of $\mu(f)$. Combining this with the estimate (3.3), we get

$$\|\nabla_{L^2} W_{\lambda_1, \lambda_2}(f)\|_{L^2(\mu)}^2 \geq \frac{\pi^2 \left(\lambda_1 - |\lambda_2| \frac{\mu(f)^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}}\right)^2}{W(f)}. \quad (3.5)$$

We can only use our control of the energy W_{λ_1, λ_2} to bound the surface area of the immersions along the flow. For this, we again use the isoperimetric inequality to show

$$W_{\lambda_1, \lambda_2}(f) = W(f) + \lambda_1 \mu(f) + \lambda_2 \text{vol}(f) \geq W(f) + \left(\lambda_1 - |\lambda_2| \frac{\mu(f)^{\frac{1}{2}}}{6\pi^{\frac{1}{2}}}\right) \mu(f).$$

Under the assumption that $\mu(f) \leq \varepsilon_1$, we can estimate this further and get

$$\mu(f) \leq \frac{3}{\lambda_1} (W_{\lambda_1, \lambda_2}(f) - W(f)). \quad (3.6)$$

So if we assume that $W_{\lambda_1, \lambda_2}(f) - W(f) < \varepsilon_2(\lambda_1, \lambda_2) = \frac{\lambda_1^3}{3\lambda_2^2} 16\pi$, we can recover the estimate

$$\mu(f) < \frac{\lambda_1^2}{\lambda_2^2} 16\pi.$$

Even for the case of negative $\lambda_2 < 0$, we can show, with the help of the estimates above, the existence of finite time singularities if both $W_{\lambda_1, \lambda_2}(f_0)$ is close to 4π and the area of the initial surface is smaller than $\varepsilon_1(\lambda_1, \lambda_2)$.

THEOREM 3.2. *For $\lambda_1 > 0, \lambda_2 < 0$ let $f_0 : \Sigma \rightarrow \mathbb{R}^3$ be an immersion of a compact surface Σ without boundary satisfying*

$$\mu(f_0) < \varepsilon_1(\lambda_1, \lambda_2), \quad (3.7)$$

$$W_{\lambda_1, \lambda_2}(f_0) < 4\pi + \varepsilon_2(\lambda_1, \lambda_2) \quad (3.8)$$

where as above $\varepsilon_1(\lambda_1, \lambda_2) = \frac{\lambda_1^2}{\lambda_2^2} 16\pi$ and $\varepsilon_2(\lambda_1, \lambda_2) = \frac{16\lambda_1^3}{3\lambda_2^2} \pi$. Then the maximal time of existence T of the Helfrich flow (1.5) with initial data f_0 is finite.

Proof. Note that for all times $t \in [0, T)$ for which $\mu(f_t) \leq \varepsilon_1$ inequality (3.6) and $W(f_t) \geq 4\pi$ imply

$$\begin{aligned} \mu(f_t) &\leq \frac{3}{\lambda_1} (W_{\lambda_1, \lambda_2}(f_t) - W(f_t)) \leq \frac{3}{\lambda_1} (W_{\lambda_1, \lambda_2}(f_0) - 4\pi) \\ &< \frac{3}{\lambda_1} \varepsilon_2(\lambda_1, \lambda_2) = \frac{\lambda_1^2}{\lambda_2^2} 16\pi = \varepsilon_1(\lambda_1, \lambda_2). \end{aligned} \quad (3.9)$$

Let us use this to show that condition (3.7) holds for all $t \in [0, T)$. If not, the intermediate value theorem would give us a time $t_0 \in [0, T]$ such that

$$\mu(f_{t_0}) = \varepsilon_1(\lambda_1, \lambda_2).$$

But then (3.9) would imply $\mu(f_{t_0}) < \varepsilon_1(\lambda_1, \lambda_2)$. So (3.7) holds for all $t \in [0, T)$ and hence also (3.9).

Using (1.6), (3.5), and (3.9), we get

$$\begin{aligned} \frac{d}{dt} W_{\lambda_1, \lambda_2}(f_t) &= -\|\nabla_{L^2} W_{\lambda_1, \lambda_2}(f)\|_{L^2(\mu)}^2 \leq -\frac{\pi^2 \left(\lambda_1 - |\lambda_2| \frac{\mu(f)^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}} \right)^2}{W(f_t)} \\ &\leq -\frac{\pi^2 \left(\lambda_1 - |\lambda_2| \frac{\mu(f)^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}} \right)^2}{W_{\lambda_1, \lambda_2}(f_t)}. \end{aligned}$$

and hence

$$\frac{d}{dt}(W_{\lambda_1, \lambda_2}(f_t))^2 \leq -\pi^2 \left(\lambda_1 - |\lambda_2| \frac{\mu(f)^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}} \right)^2.$$

Since the term on the right-hand side is bounded away from zero by (3.9), a singularity must form in finite time. $\square$

REMARK 3.3. Indeed, the assumption $\mu(f_0) \leq \frac{\lambda_1^2}{\lambda_2^2} 16\pi$ is optimal since, for $\lambda_1 > 0$ and $\lambda_2 < 0$, any sphere of radius $-\frac{2\lambda_1}{\lambda_2}$ is a critical point of W_{λ_1, λ_2} and has area $\frac{\lambda_1^2}{\lambda_2^2} 16\pi$. In contrast to this, we do not expect ε_2 to be optimal.

With essentially the same technique, we can prove that the condition $W_{\lambda_1, \lambda_2}(f_0) \leq 8\pi$ in Theorem 3.1 can be weakened to $W_{\lambda_1, \lambda_2}(f_0) \leq 8\pi + \min\{\varepsilon_2(\lambda_1, \lambda_2), 8\pi\}$. But we additionally have to assume that $\mu(f_0)$ is sufficiently small if the inclosed volume has a negative sign. In this case, we will furthermore face an additional problem: Even if we assume that $\text{vol}(f_0) > 0$ and the initial surface is embedded, the inequality of Li and Yau does not tell us that the surface stays embedded along the flow.

Instead, we will use that every sphere eversion must have a quadruple point (cf. [15]). So if after some time we have the situation that f_{t_0} is an embedding with $\text{vol } f_{t_0} < 0$, then there must have been a quadruple point before t_0 , i.e., due to the inequality of Li and Yau there was a time $t \in (0, t_0)$ such that

$$W(f_t) \geq 16\pi.$$

This allows us to prove

THEOREM 3.4. *For $\lambda_1 > 0, \lambda_2 > 0$ let $f_0 : \Sigma \rightarrow \mathbb{R}^3$ be an embedding of a compact manifold Σ without boundary satisfying $\text{vol}(f_0) > 0$ and*

$$W_{\lambda_1, \lambda_2}(f_0) < 8\pi + \min\{\varepsilon_2(\lambda_1, \lambda_2), 8\pi\}. \quad (3.10)$$

Then the maximal time of existence T of the Helfrich flow (1.5) with initial data f_0 is finite.

Proof. We will give different upper bounds for

$$\frac{d}{dt}(W_{\lambda_1, \lambda_2}(f_t))^2$$

depending on whether the inclosed volume is nonnegative or not. If $\text{vol}(f_t) \geq 0$, we can proceed as in the proof of Theorem 3.1 and get

$$\frac{d}{dt} W_{\lambda_1, \lambda_2}(f_t) = -\|\nabla_{L^2} W_{\lambda_1, \lambda_2}(f_t)\|_{L^2(\mu)}^2 \leq \frac{\pi^2 \lambda_1^2}{W_{0,0}(f_t)} \leq -\frac{\pi^2 \lambda_1^2}{W_{\lambda_1, \lambda_2}(f_t)}.$$

Thus

$$\frac{d}{dt}(W_{\lambda_1, \lambda_2}(f_t))^2 \leq -2\pi^2 \lambda_1^2. \quad (3.11)$$

To deal with the times $t \in [0, T)$ for which $\text{vol}(f_t) < 0$, we first observe that for times t for which $\mu(f_t) \leq \varepsilon_1(\lambda_1, \lambda_2)$ and f_t is not an embedding, we have due to (3.6) and Li and Yau's inequality

$$\mu(f_{t_0}) \leq \frac{3}{\lambda_1} (W_{\lambda_1, \lambda_2}(f_{t_0}) - 8\pi) \leq \frac{3}{\lambda_1} (W_{\lambda_1, \lambda_2}(f_0) - 8\pi) < \frac{\lambda_1^2}{\lambda_2^2} 16\pi = \varepsilon_1(\lambda_1, \lambda_2). \quad (3.12)$$

Furthermore, we note that if for a time $t_0 \in [0, T)$ we have

$$\mu(f_t) \leq \varepsilon_1(\lambda_1, \lambda_2) \quad \forall t \in [0, t_0] \text{ with } \text{vol}(f_t) < 0,$$

then

$$W(f_t) < 16\pi \quad \forall t \in [0, t_0],$$

since for all t with $\text{vol}(f_t) \geq 0$ we have

$$W(f_t) \leq W_{\lambda_1, \lambda_2}(f_t) \leq W_{\lambda_1, \lambda_2}(f_0) < 16\pi$$

and for all t with $\mu(f_t) < \varepsilon_1$ we also get

$$W(f_t) \leq W_{\lambda_1, \lambda_2}(f_t) \leq W_{\lambda_1, \lambda_2}(f_0) < 16\pi.$$

But this implies that f_{t_0} is not embedded as otherwise, due to the result of [15], there would be a time $t \in [0, t_0]$ such that f_t has a quadruple point and hence by the inequality of Li and Yau

$$W(f_t) \geq 16.$$

Combining these two observations, we have shown that for times $t_0 \in [0, T)$ with $\mu(f_{t_0}) \leq \varepsilon_1$ and such that $\mu(f_t) \leq \varepsilon_1$ for all $t \in [0, t_0]$ with $\text{vol}(f_t) < 0$ we have (3.12)

Let us use this to show that for all times $t \in [0, T]$ with $\text{vol}(f_t) < 0$, we obtain

$$\mu(f_t) < \varepsilon_1(\lambda_1, \lambda_2) \quad (3.13)$$

by a continuity argument. We set

$$A := \{t \in (0, T) : \text{vol}(f_t) < 0\},$$

where A of course may be the empty set in which case there is nothing to show.

As $0 \notin A$, we have $\text{vol}(f_t) = 0$ for all $t \in \partial A$ and hence f_t cannot be an embedding. Thus by (3.12)

$$\mu(f_{a_i}) < \varepsilon_1(\lambda_1, \lambda_2).$$

for all $t \in \partial A$. If (3.13) does not hold on A , due to the intermediate value theorem there would be a first time $t_0 \in A$ with $\mu(f_{t_0}) = \frac{\lambda_1^2}{\lambda_2^2} 16\pi$. But then the discussion above implies that we have (3.12), especially

$$\mu(f_{t_0}) < \varepsilon_1,$$

contradicting our choice of t_0 . Thus (3.13) holds for all times $t \in [0, T]$ with $\text{vol}(f_t) < 0$.

From inequality (3.5), we get

$$\frac{d}{dt} W_{\lambda_1, \lambda_2}(f_t) = \|\nabla_{L^2} W_{\lambda_1, \lambda_2}(f_t)\|_{L^2(\mu)}^2 \leq - \frac{\pi^2 \left(\lambda_1 - |\lambda_2| \frac{\mu(f_t)^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}} \right)^2}{W(f_t)}$$

and thus

$$\frac{d}{dt} (W_{\lambda_1, \lambda_2}(f_t))^2 \leq -2\pi^2 \left(\lambda_1 - |\lambda_2| \frac{\mu(f_t)^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}} \right)^2. \quad (3.14)$$

Since $\mu(f_t) \leq \varepsilon_1$ for all $t \in A$, inequality (3.6) holds for all $t \in A$ and shows that the right-hand side in (3.5) is bounded away from zero for all $t \in A$.

Hence, the estimates (3.11) and (3.14) imply that singularities form in finite time. $\square$

Theorems 3.1, 3.2, and 3.4 imply Theorem 1.2.

4. Blowup analysis of finite time singularities

For the convenience of the reader, let us briefly repeat the blowup construction at a singularity in [18] which they perform at the beginning of the proof of Theorem 1.4 on page 25. This result extends the construction of a blowup by Kuwert and Schätzle for the Willmore flow [9, 11].

THEOREM 4.1. *There are constants $\varepsilon_0 > 0$ and $c_0 > 0$ such that the following holds: Let $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^3$ be the maximal solution to (1.5) with $T < \infty$, i.e., singularities occur in finite time. Then there is a sequence of times $t_j \uparrow T$, of radii $r_j \downarrow 0$ and points $x_j \in \mathbb{R}^n$ such that the rescaled flows*

$$f_j : \Sigma \times [0, c_0] \rightarrow \mathbb{R}^3, f_j(p, t) := \frac{1}{r_j} \left(f(p, t_j + r_j^4 t) - x_j \right)$$

satisfy

$$\int_{f_j^{-1}(B_1(0))} \|A_{f_j}\|^2 d\mu_{f_j} \geq \varepsilon_0$$

and converge smoothly locally to a smooth family of proper immersions

$$\tilde{f} : \tilde{\Sigma} \times [0, c_0] \rightarrow \mathbb{R}^3$$

in the following sense: We can represent

$$f_j(\phi_j, t) = \tilde{f} + u_j(\cdot, t)$$

where

- $\phi_j : \tilde{f}^{-1}(B_j(0)) \rightarrow U_j$ is a diffeomorphism,
- $f_j^{-1}(B_R) \subset U_j$ for $j \geq j(R)$,
- $u_j \in C^\infty(\tilde{\Sigma} \times [0, c_0], \mathbb{R}^n)$ is normal along $\tilde{f}$,
- $\|\nabla^k u_j\|_{L^\infty(\tilde{f}^{-1}(B_j(0)))} \rightarrow 0$ as $j \rightarrow \infty$.

We will call such a family of immersion $\tilde{f}$ a blowup limit in the following.

The next theorem shows that possible blowup limits are stationary and parametrize Willmore surfaces. It is an extension of Theorem 4.4 in [18]. Again McCoy and Wheeler have shown the result only under the assumption that the energy of the initial surface is close to the Willmore energy of a sphere 4π .

THEOREM 4.2. *Let $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^3$ be the maximal solution to (1.5) with $T < \infty$, i.e., singularities occur in finite time. Then the blowup limit $\tilde{f} : \tilde{\Sigma}_\infty \rightarrow \mathbb{R}^3$ constructed in Theorem 4.1 does not depend on time and parametrizes a Willmore surface.*

Proof. Using that f satisfies Eq. (1.5) together with

$$\begin{aligned}\Delta_{f_j} H_{f_j} &= r_j^3 \Delta_f H_f, \\ H_{f_j} &= r_j H_f, \\ v_{f_j} &= v_j,\end{aligned}$$

and

$$\partial_t f_j = r_j^3 \partial_t f$$

we get from (1.5) that the f_j satisfy

$$\partial_t f_j = -\nabla_{L^2} W(f_j) + (r_j^2 \lambda_1 H_{f_j} + r_j \lambda_2) v_{f_j}. \quad (4.1)$$

Since f_j converges to $\tilde{f}$ locally smoothly and $r_j \rightarrow 0$, this implies

$$\partial_t \tilde{f} = -\nabla_{L^2} W(\tilde{f}). \quad (4.2)$$

As

$$\begin{aligned}& \int_0^{c_0} \left(\int_\Sigma \|\nabla_{L^2} W f_j(x, t) + (\lambda_1 r_j^2 H_{f_j}(x, t) + \lambda_2 r_j) v_{f_j}\|^2 d\mu_{f_j} \right) dt \\&= \int_{t_j}^{t_j + c_0 r_j^4} \left(\int_\Sigma \|\nabla_{L^2} W_{\lambda_1, \lambda_2} f(x, t)\|^2 d\mu_{f_j} \right) dt \\&= W_{\lambda_1, \lambda_2}(t_j) - W_{\lambda_1, \lambda_2}(t_j + c_0 r_j^4) \\&\rightarrow 0\end{aligned}$$

and $r_j \rightarrow 0$ as $j \rightarrow \infty$, we deduce that $\nabla_{L^2} W \tilde{f} = 0$. □

Combining Theorem 4.2 with the classification of Willmore spheres due to Bryant [3] and the removability of point singularities of Kuwert and Schätzle [11], we get

COROLLARY 4.3. *If $f_0 : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ is an immersion of a sphere that develops a singularity in finite time under the Helfrich flow (1.5) and $\lim_{t \rightarrow T} W(f_t) < 8\pi$, then the blowup limit from Theorem 4.1 is a round sphere.*

Proof. Let us first assume that $\hat{\Sigma}$ is compact. Since then the local convergence of the rescaled solution is in fact global, $\tilde{\Sigma}$ is a topological sphere. So $\tilde{f}$ is a Willmore sphere with energy below 8π and thus is parametrizing a round sphere by the classification result of Bryant [3].

We now lead the case that $\tilde{\Sigma}$ is not compact to a contradiction as in [11]. We can assume without loss of generality that $0 \notin \hat{f}(\hat{\Sigma})$ since $\tilde{f}$ is proper. We consider the images of the f_j under the inversion on the standard sphere $I : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$, $x \mapsto \frac{x}{|x|^2}$, which is well defined for large enough $j \in \mathbb{N}$. The embeddings $\tilde{f}_j = I \circ f_j$ converge locally smoothly to the embedding $I \circ \tilde{f}$ in $\mathbb{R}^3 \setminus \{0\}$ and due to the Möbius invariance of the Willmore energy

$$W(I \circ \tilde{f}) \leq \liminf_{j \rightarrow \infty} W(\tilde{f}^j) = \liminf_{j \rightarrow \infty} W(f^j) < 8\pi.$$

The Möbius invariance of the Willmore energy also implies that $I \circ \tilde{f}$ is a Willmore surface away from 0. Due to the point removability result of Kuwert and Schätzle [11], $I \circ \tilde{f}$ can be extended to a Willmore sphere of Willmore energy less than 8π . Hence, due to a result of Bryant [3], it must parametrize a round sphere. But this would imply that $\tilde{f}$ was a plane—which would contradict

$$\int_{\tilde{\Sigma}} \|A_{\tilde{f}}\|^2 d\mu_{\tilde{f}} > 0.$$

Hence, $\tilde{\Sigma}$ must be compact which concludes the proof. $\square$

Corollary 4.3 implies Theorem 1.3 and the following extensions of the main result in [18].

COROLLARY 4.4. *Let $\lambda_1 > 0$, $\lambda_2 \geq 0$ and $f_0 : \Sigma \rightarrow \mathbb{R}^3$ be a closed smoothly embedded surface without boundary satisfying $W_{\lambda_1, \lambda_2}(f_0) < 8\pi$. Then the Helfrich flow (1.5) with initial data f_0 sub-converges to a round point in finite time.*

REMARK 4.5. The constant 8π in Corollary 4.4 above is sharp which can be shown following the lines of argument in [2]. There we showed that surfaces of revolution exist such that the Gauß map of the profile curve has index equal to three of Willmore energy just slightly larger than 8π . Figure 1 illustrates the construction of such a surface of revolution. As surfaces of revolution remain surfaces of revolution under the flow, the blowup limit can impossibly be a sphere, as otherwise the index of the Gauß map of the initial surface must have been ± 1 . Shrinking the surface if necessary, we see that for

Figure 1. The surface shown above is built out of two round spheres painted in blue and a piece of a catenoid, painted in red. The yellow part is used to connect these pieces. One can adapt the parameters such that the Willmore energy of the resulting surface is arbitrary close to 8π (color figure online)

every $\varepsilon > 0$ we can find an immersed sphere $f : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ with $W_{\lambda_1, \lambda_2}(f) < 8\pi + \varepsilon$ that forms a singularity in finite time in such a way that the blowup limit cannot be a round sphere—instead it consists of a finite number of catenoids and planes.

Acknowledgements

Open access funding provided by Paris Lodron University of Salzburg.

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