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Anti-periodic solutions for nonlinear evolution inclusions

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Abstract. We consider an anti-periodic evolution inclusion defined on an evolution triple of spaces, driven by an operator of monotone-type and with a multivalued reaction term $F(t, x)$. We prove existence theorem for the "convex" problem (that is, F is convex-valued) and for the "nonconvex" problem (that is, F is nonconvex-valued) and we also show the existence of extremal trajectories (that is, when F is replaced by ext F). Finally, we prove a "strong relaxation" theorem, showing that the extremal trajectories are dense in the set of solutions of the convex problems.

1. Introduction

Let $T = [0, b]$ and let (X, H, X^*) be an evolution triple (Gelfand triple) of spaces. In this paper, we study the following nonlinear anti-periodic evolution inclusion:

$$\begin{cases} -u'(t) \in A(t, u(t)) + F(t, u(t)) & \text{for a.a. } t \in T, \\ u(0) = -u(b). \end{cases} \tag{1.1}$$

In this problem $A: T \times X \longrightarrow 2^{X^*}$ and $F: T \times X \longrightarrow 2^H$ are two set-valued maps. In contrast to earlier works on the subject, we do not assume that $A(t, \cdot)$ is maximal monotone and that $F(t, \cdot)$ is of the subdifferential type. We prove existence results for problems with F being convex-valued ("convex problem") as well as F being nonconvex-valued ("nonconvex problem"). We also produce extremal trajectories, that is, trajectories corresponding to the inclusions in which F is replaced by ext F (the extreme points of F). In the context of control systems, these are the trajectories (states of the system) generated by bang–bang controls. Finally we show that the extremal trajectories are $C(T; H)$-dense in the set of trajectories of the convex problem. Such a result is usually known as "strong relaxation theorem" and again in the framework of control systems it implies that essentially we can have the same outcome by economizing in the controls, using only bang–bang controls.

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The study of anti-periodic evolution equations was initiated by Okochi [18] for subdifferential evolution equations defined on a Hilbert space H (that is, $A(t, x) = A(x) = \partial\varphi(x)$, with $\varphi\colon H \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ being proper, convex, lower semicontinuous) with $F(t, x) = f(t)$ where $f \in L^2(T; H)$. Soon thereafter Haraux [10] and Okochi [19] used a more general forcing term $F(t, x)$. Subsequently Aizicovici–Pavel [1] considered subdifferential evolution equations in a Hilbert space H with $F(t, x) = \partial\psi(x)$, where $\psi\colon H \longrightarrow \overline{\mathbb{R}}$ is proper, convex, lower semicontinuous. Aizicovici–Reich [2] considered anti-periodic subdifferential evolution equations with single-valued time-dependent reaction term, which is not cyclically maximal monotone (that is, it is not of the subdifferential type). We also mention the work of Souplet [23] and more recently those of Chen [6], Liu [17], Liu–Liu [16]. Chen [6] deals with semilinear problems, while Liu [17] and Liu–Liu [16] consider equations with $A(t, x) = A(x)$ maximal monotone. Moreover, in Liu [17], $F(t, x) = G(x)$ with $G\colon X \longrightarrow X^*$ being continuous and weakly continuous. We also mention the recent work on periodic subdifferential evolution equations by Papageorgiou–Rădulescu [21].

We mention that anti-periodic problems arise naturally in the mathematical modeling of a variety of physical processes. We refer to the works of Batchelor et al. [3], Bonilla–Higuera [4], Kulshreshtha et al. [14] for such applications.

2. Mathematical background

Let V, Y be two Banach spaces and assume that V is embedded continuously and densely into Y (denoted $V \hookrightarrow Y$). We have

- Y^* is embedded continuously in V^*;
- if V is reflexive, then $Y^* \hookrightarrow V^*$.

The following notion is central in our considerations.

DEFINITION 2.1. A triple (X, H, X^*) of spaces is said to be an "evolution triple" (or "Gelfand triple"), if the following properties hold:

(a) X is a separable reflexive Banach space and X^* is its topological dual;
(b) H is a separable Hilbert space which is identified with its dual (that is, $H = H^*$) by the Riesz–Fréchet representation theorem;
(c) $X \hookrightarrow H$.

As a consequence of (b) and (c) we also have that $H \hookrightarrow X^*$.

In what follows by $\|\cdot\|$ (respectively $|\cdot|$, $\|\cdot\|_*$) we denote the norm of X (respectively of H, X^*). Also, by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X) and by $(\cdot,\cdot)$ the inner product of H. We will also assume that the embedding $X \hookrightarrow H$ is compact.

We can find constants $\widehat{c}_1, \widehat{c}_2 > 0$ such that

$$|\cdot| \leqslant \widehat{c}_1\|\cdot\| \quad \text{and} \quad \|\cdot\|_* \leqslant \widehat{c}_2|\cdot|.$$

Moreover, we have

$$\langle \cdot, \cdot \rangle\big|_{X \times H} = (\cdot, \cdot).$$

Let $1 < p < +\infty$. The following space is important in the analysis of problem (1.1):

$$W_p(0,b) = \left\{ u \in L^p(T;X) : u' \in L^{p'}(T;X^*) \right\}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$. Here by u' we understand the distributional (weak) derivative of u. From the theory of Lebesgue–Bochner spaces (see Gasiński–Papageorgiou [7, p. 129]), we have

$$L^p(T;X)^* = L^{p'}(T;X^*)$$

and the duality brackets for this pair of spaces are defined by

$$((h,f)) = \int_0^b \langle h(t), f(t) \rangle \, \mathrm{d}t \quad \forall h \in L^{p'}(T;X^*), \ f \in L^p(T;X).$$

If $u \in W_p(0,b)$, then u viewed as an X^*-valued function, is absolutely continuous. So, since X^* is reflexive, $u \colon T \longrightarrow X^*$ is almost everywhere differentiable in the classical sense (see Gasiński–Papageorgiou [7, p. 133]). This derivative coincides with the distributional one. Then we have

$$W_p(0,b) \subseteq AC^{1,p'}(T;X^*) = W^{1,p'}(T;X^*).$$

The space $W_p(0,b)$ becomes a separable reflexive Banach space, when given the norm

$$\|u\|_{W_p} = \left(\|u\|^p_{L^p(T;X)} + \|u'\|^p_{L^{p'}(T;X^*)} \right)^{\frac{1}{p}} \quad \forall u \in W_p(0,b).$$

An equivalent norm is given by

$$|u|_{W_p} = \|u\|_{L^p(T;X)} + \|u'\|_{L^{p'}(T;X^*)}.$$

The following properties of $W_p(0,b)$ will be important in our study of problem (1.1):

- $W_p(0,b) \hookrightarrow C(T;H)$;
- $W_p(0,b) \hookrightarrow L^p(T;H)$ and the embedding is compact;
- if $u, v \in W_p(0,b)$ and $\eta(t) = (u(t), v(t))$ for all $t \in T$, then η is absolutely continuous and

$$\eta'(t) = \langle u'(t), v(t) \rangle + \langle u(t), v'(t) \rangle \quad \text{for a.a. } t \in T \tag{2.1}$$

  ("integration by parts formula").

Suppose that V is a reflexive Banach space, V^* its topological dual and by $\langle \cdot, \cdot \rangle_V$ we denote the duality brackets for the pair (V^*, V). Let $L \colon V \supseteq D(L) \longrightarrow V^*$ be a linear maximal monotone operator and $\mathcal{A} \colon V \longrightarrow 2^{V^*}$. We say that $\mathcal{A}$ is "L-pseudomonotone" if the following conditions hold:

(a) for every $v \in V$, $\mathcal{A}(v) \subseteq V^*$ is nonempty, convex and w-compact;
(b) $\mathcal{A}$ is bounded (that is, maps bounded sets to bounded sets);
(c) if $\{v_n\}_{n\geqslant 1} \subseteq D(L)$, $v_n \xrightarrow{w} v \in D(L)$ in V, $L(v_n) \xrightarrow{w} L(v)$ in V^*, $v_n^* \in \mathcal{A}(v_n)$ for all $n \geqslant 1$, $v_n^* \xrightarrow{w} v^*$ in V^* and $\limsup\limits_{n\to+\infty}\langle v_n^*, v_n - v\rangle_V \leqslant 0$, then $v^* \in \mathcal{A}(v)$ and $\langle v_n^*, v_n\rangle_V \longrightarrow \langle v^*, v\rangle_V$.

The following surjectivity result is due to Papageorgiou et al. [20] and it extends an earlier single-valued result of Lions [15, p. 319].

THEOREM 2.2. *If V is a reflexive Banach space which is strictly convex, $L\colon V \supseteq D(L) \longrightarrow V^*$ is linear, maximal monotone and $\mathcal{A}\colon V \longrightarrow 2^{V^*}$ is L-pseudomonotone and strongly coercive, that is,*

$$\frac{\inf\{\langle v^*, v\rangle_V : v^* \in \mathcal{A}(v)\}}{\|v\|_V} \longrightarrow +\infty \quad \textit{as } \|v\|_V \to +\infty,$$

then $L + \mathcal{A}$ is surjective (that is, $R(L + \mathcal{A}) = V^*$).

Let (Ω, Σ) be a measurable space and Y a separable Banach space. We will use the following notation:

$$P_{f(c)}(Y) = \big\{D \subseteq Y : D \text{ is nonempty, closed (and convex)}\big\},$$
$$P_{(w)k(c)}(Y) = \big\{D \subseteq Y : D \text{ is nonempty, } (w\text{-})\text{compact (and convex)}\big\}.$$

A multifunction (set-valued function) $G\colon \Omega \longrightarrow P_f(Y)$ is said to be "measurable" if for every $y \in Y$, the $\mathbb{R}_+$-valued function

$$\omega \longmapsto d(y, G(\omega)) = \inf\{\|y - u\|_Y : u \in G(\omega)\}$$

is measurable. We say that a multifunction $G\colon \Omega \longrightarrow 2^Y \setminus \{\emptyset\}$ is "graph measurable", if

$$\operatorname{Gr} G = \{(\omega, u) \in \Omega \times Y : u \in G(\omega)\} \in \Sigma \otimes \mathcal{B}(Y).$$

For $P_f(Y)$-valued multifunctions measurability implies graph measurability. For the converse to be true, we need to have $\Sigma = \widehat{\Sigma}$ (with $\widehat{\Sigma}$ being the universal σ-field). Recall that $\Sigma = \widehat{\Sigma}$, if there is a σ-finite measure μ on (Ω, Σ) with respect to which Σ is complete. Now let μ be a σ-finite measure on Σ. For $1 \leqslant p \leqslant +\infty$ we define

$$S_G^p = \big\{u \in L^p(\Omega; Y) : u(\omega) \in G(\omega)\ \mu\text{-a.e.}\big\}.$$

For a graph measurable multifunction $G\colon \Omega \longrightarrow 2^Y \setminus \{\emptyset\}$, the set S_G^p is nonempty if and only if the $\mathbb{R}_+$-valued function $\omega \longmapsto \inf\{\|u\|_Y : u \in G(\omega)\}$ belongs in $L^p(\Omega)_+$. Note that by a corollary to the Yankov-von Neumann–Aumann selection theorem, we can find a sequence $\{g_n\}_{n\geqslant 1}$ of Σ-measurable selectors of G such that

$$G(\omega) \subseteq \overline{\{g_n(\omega)\}}_{n\geqslant 1} \quad \mu\text{-a.e.}$$

The set S_G^p is "decomposable" in the sense that if $(C, g_1, g_2) \in \Sigma \times S_G^p \times S_G^p$, then

$$\chi_C g_1 + \chi_{C^c} g_2 \ \in \ S_G^p$$

with χ_D being the characteristic function for the set $D \in \Sigma$.

On $P_f(Y)$ we can define a generalized metric $h(\cdot, \cdot)$ known as the "Hausdorff metric", by

$$h(C, E) \ = \ \max \Big\{ \sup_{c \in C} d(c, E), \sup_{e \in E} d(e, C) \Big\} \quad \forall C, E \in P_f(Y).$$

Then $(P_f(Y), h)$ is a complete metric space with $P_{fc}(Y)$, $P_k(Y)$ closed subsets. Moreover, $P_k(Y)$ is also separable (therefore $(P_k(Y), h)$ is a Polish space).

Let Z be a Hausdorff topological space. A multifunction $G\colon Z \longrightarrow P_f(Y)$ is said to be "h-continuous", if it is continuous from Z into $(P_f(Y), h)$.

Suppose that Z, W are two Hausdorff topological spaces. A multifunction $G\colon Z \longrightarrow 2^W \setminus \{\emptyset\}$ is said to be

- "upper semicontinuous", if for all closed sets $C \subseteq W$, the set

$$G^-(C) \ = \ \{z \in Z : \ G(z) \cap C \neq \emptyset\}$$

  is closed in Z;

- "lower semicontinuous", if for all closed sets $C \subseteq W$, the set

$$G^+(C) \ = \ \{z \in Z : \ G(z) \subseteq C\}$$

  is closed in Z.

Finally, if E is a Banach space and $\{C_n\}_{n \geqslant 1}$ is a sequence of nonempty subsets of E, then we define

$$w\text{–}\limsup_{n \to +\infty} C_n \ = \ \Big\{ u \in E : \ u = w\text{–}\lim_{k \to +\infty} u_{n_k}, \ u_{n_k} \in C_{n_k}, \ n_1 < n_2 < \ldots \Big\}.$$

For more details on the measurability and continuity properties of multifunctions, we refer to Hu–Papageorgiou [12].

Next we introduce the hypotheses on the map A:

$\underline{H(A)}$: $A\colon T \times X \longrightarrow 2^{X^*}$ is a map such that

(i) for all $x \in X$, $t \longmapsto A(t, x)$ is graph measurable;

(ii) for a.a. $t \in T$, $x \longmapsto A(t, x)$ is pseudomonotone;

(iii) there exist $a_1 \in L^{p'}(T)$ and $c_1 > 0$ such that

$$\|h^*\|_* \ \leqslant a_1(t) + c_1 \|x\|^{p-1} \quad \text{for a.a. } t \in T, \ \text{all } x \in X, \ h^* \in A(t, x),$$

with $2 \leqslant p < +\infty$;

(iv) there exist $a_2 \in L^1(T)_+$ and $c_2 > 0$ such that

$$\langle h^*, x \rangle \ \geqslant \ c_2 \|x\|^p - a_2(t) \quad \text{for a.a. } t \in T, \ \text{all } x \in X, \ h^* \in A(t, x).$$

Finally we mention that using the Troyanski renorming theorem (see, for example, Gasiński–Papageorgiou [7, p. 911]), without any loss of generality we may assume that both X and X^* are locally uniformly convex and so $L^p(T; X)$ and $L^{p'}(T; X^*)$ are strictly convex. Moreover, by $L^1_w(T; H)$ we will denote the Lebesgue–Bochner space $L^1(T; H)$ furnished with the "weak norm" defined by

$$\|u\|_w = \sup\Big\{\Big|\int_t^{t'} u(s)\,\mathrm{d}s\Big| : 0 \leqslant t \leqslant t' \leqslant b\Big\} \quad \forall u \in L^1(T; H).$$

An equivalent definition of the weak norm is the following

$$\|u\|_w = \sup\Big\{\Big|\int_0^{t} u(s)\,\mathrm{d}s\Big| : 0 \leqslant t \leqslant b\Big\} \quad \forall u \in L^1(T; H)$$

(see Gasiński–Papageorgiou [9, p. 234, Definition 2.78 and Remark 2.79]).

3. Convex problem

In this section, we prove the existence of anti-periodic solutions for the case when the multivalued reaction term F is convex-valued. The hypotheses on F are the following:
$\underline{H(F)_1}$: $F : T \times X \longrightarrow P_{fc}(H)$ is a multifunction such that

(i) for all $x \in X$, $t \longmapsto F(t, x)$ is graph measurable;
(ii) for a.a. $t \in T$, Gr $F(t, \cdot)$ is sequentially closed in $X_w \times H_w$ (here by X_w and H_w, we denote the spaces X and H endowed with the weak topologies);
(iii) there exist $a_3 \in L^{p'}(T)_+$ and $c_3 > 0$ such that

$$|F(t, x)| = \sup_{v \in F(t,x)} |v| \leqslant a_3(t) + c_3\|x\|^{p-1} \quad \text{for a.a. } t \in T, \text{ all } x \in X;$$

(iv) there exists $\vartheta \in L^{p'}(T)_+$ such that

$$(v, x) \geqslant -\vartheta(t) \quad \text{for a.a } t \in T, \text{ all } x \in X, \; v \in F(t, x);$$

(v) there exists $M > 0$ such that

$$(v, x) \geqslant a_2(t) \quad \text{for a.a. } t \in T, \text{ all } x \in X, \; |x| = M, \; v \in F(t, x)$$

(see hypothesis $H(A)(iv)$).

REMARK 3.1. If $a_2 \equiv 0$, then hypothesis $H(F)_1(v)$ says that

$$(v, x) \geqslant 0 \quad \text{for a.a. } t \in T, \text{ all } x \in X, \; |x| = M.$$

This condition is known in the literature as "Hartman's condition" and it was first introduced by Hartman [11] in the context of second-order Dirichlet systems in $\mathbb{R}^N$.

Let $\mathcal{A}\colon L^p(T;X) \longrightarrow 2^{L^{p'}(T;X^*)}$ be defined by

$$\mathcal{A}(u) = \left\{h^* \in L^{p'}(T;X^*) : h^*(t) \in \mathcal{A}(t,u(t)) \text{ for a.a. } t \in T\right\}.$$

From the Yankov-von Neumann–Aumann selection theorem (see Hu–Papageorgiou [12, p. 158]) and hypothesis $H(A)(iii)$, we see that

$$\mathcal{A}(u) \in P_{wkc}(L^{p'}(T;X^*)) \quad \forall u \in L^p(T;X).$$

From Hu–Papageorgiou [13, p. 41] (see also Papageorgiou et al. [22, Lemma 5]), we have the following result.

LEMMA 3.2. *If hypotheses $H(A)$ hold, then $\mathcal{A}\colon L^p(T;X) \longrightarrow P_{wkc}(L^{p'}(T;X^*))$ is L-pseudomonotone.*

Let $p_M\colon X \longrightarrow X$ be the M-radial retraction map defined by

$$p_M(x) = \begin{cases} x & \text{if } |x| \leqslant M, \\ \frac{Mx}{|x|} & \text{if } |x| > M. \end{cases}$$

Based on the Hartman condition (see hypothesis $H(F)_1(v)$), we introduce the following modification of the multivalued forcing term

$$\widehat{F}(t,x) = \begin{cases} F(t,x) & \text{if } |x| \leqslant M, \\ (x - p_M(x)) + F(t,p_M(x)) & \text{if } |x| > M, \end{cases} \tag{3.1}$$

for all $(t,x) \in T \times X$. On account of hypothesis $H(F)_1(iii)$ and of (3.1), we have

$$|\widehat{F}(t,x)| = \sup_{\widehat{v}\in\widehat{F}(t,x)} \|\widehat{v}\| \leqslant \widehat{a}(t) + \widehat{c}\|x\| \quad \text{for a.a. } t \in T, \ \text{all } x \in X, \tag{3.2}$$

with $\widehat{a} \in L^{p'}(T)$, $\widehat{c} > 0$. This modification of $F(t,x)$ has the following properties.

PROPOSITION 3.3. *If hypotheses $H(F)_1$ hold, then*

(a) *for all $x \in X$, $t \longmapsto \widehat{F}(t,x)$ is graph measurable;*
(b) *for a.a. $t \in T$, $\mathrm{Gr}\,\widehat{F}(t,\cdot)$ is sequentially closed in $X_w \times H_w$;*
(c) *for a.a. $t \in T$, all $x \in X$ and all $\widehat{v} \in \widehat{F}(t,x)$, we have*

$$(\widehat{v},x) \geqslant \begin{cases} -\vartheta(t) & \text{if } |x| < M, \\ -\frac{\vartheta(t)}{M}|x| & \text{if } |x| \geqslant M; \end{cases}$$

(d) *for a.a. $t \in T$, all $x \in X$ with $|x| = M$ and all $\widehat{v} \in \widehat{F}(t,x)$, we have*

$$(\widehat{v},x) \geqslant a_2(t),$$

with $a_2 \in L^1(T)_+$ as in hypothesis $H(A)(iv)$.

Proof. (a) This follows from hypothesis $H(F)_!(i)$ and (3.1).
(b) Let $\{(x_n, \widehat{v}_n)\}_{n\geqslant 1} \subseteq \operatorname{Gr} \widehat{F}(t, \cdot)$ and assume that

$$x_n \xrightarrow{w} x \quad \text{in } X \quad \text{and} \quad \widehat{v}_n \xrightarrow{w} \widehat{v} \quad \text{in } H,$$

so

$$x_n \longrightarrow x \quad \text{in } H \quad \text{and} \quad \widehat{v}_n \xrightarrow{w} \widehat{v} \quad \text{in } H$$

(recall that $X \hookrightarrow H$ compactly). Then $p_M(x_n) \longrightarrow p_M(x)$ in H and so from (3.1) and hypothesis $H(F)_1(ii)$, it follows that $\operatorname{Gr} \widehat{F}(t, \cdot)$ is sequentially closed in $X_w \times H_w$.
(c) From (3.1) we see that if $x \in X$ satisfies $|x| < M$ and $\widehat{v} \in \widehat{F}(t, x) = F(t, x)$, then

$$(\widehat{v}, x) \geqslant -\vartheta(t)$$

(see hypothesis $H(F)_1(iv)$). If $x \in X$ satisfies $|x| \geqslant M$ and $\widehat{v} \in F(t, x)$, then $\widehat{v} = x - p_M(x) + v$ with $v \in F(t, p_M(x))$. So, we have

$$\begin{aligned}(\widehat{v}, x) = (x - p_M(x) + v, x) &= |x|^2 - (p_M(x), x) + (v, x)\\ &= |x|^2 - M|x| + \Big(v, \frac{Mx}{|x|}\Big)\frac{|x|}{M}\\ &= |x|(|x| - M) - \frac{\vartheta(t)}{M}|x| \geqslant -\frac{\vartheta(t)}{M}|x|.\end{aligned}$$

(d) This is immediate from (3.1) and hypothesis $H(F)_1(v)$. □

Let $\widehat{N}\colon L^p(T; X) \longrightarrow 2^{L^{p'}(T;H)}$ be defined by

$$\widehat{N}(u) = S^{p'}_{\widehat{F}(\cdot,u(\cdot))} = \Big\{\widehat{v} \in L^{p'}(T; H) : \widehat{v}(t) \in \widehat{F}(t, u(t)) \text{ for a.a. } t \in T\Big\}.$$

The Yankov-von Neumann–Aumann selection theorem and (3.2) imply that

$$\widehat{N}(u) \in P_{wkc}(L^{p'}(T; H)) \quad \forall u \in L^p(T; X).$$

Also we consider the linear operator $L\colon L^p(T; X) \supseteq D(L) \longrightarrow L^{p'}(T; X^*)$ defined by

$$L(u) = u' \quad \forall u \in D = D(L) = \{u \in W_p(0, b) : u(0) = -u(b)\}.$$

From Proposition 1 of Liu [17], we have that

$$L \text{ is maximal monotone (hence densely defined).} \tag{3.3}$$

We consider the multivalued map $u \longmapsto \mathcal{A}(u) + \widehat{N}(u)$. On account of (3.3), we can consider the L-pseudomonotonicity of this multivalued map.

PROPOSITION 3.4. *If hypotheses $H(A)$ and $H(F)_1$ hold, then the operator $\mathcal{A} + \widehat{N}\colon L^p(T; X) \longrightarrow P_{wkc}(L^{p'}(T; X^*))$ is L-pseudomonotone.*

Proof. We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq D$ such that

$$u_n \xrightarrow{w} u \quad \text{in } W_b(0,b) \tag{3.4}$$

and $u_n^* \in \mathcal{A}(u_n) + \widehat{N}(u_n)$ for all $n \geqslant 1$ such that

$$u_n^* \xrightarrow{w} u^* \quad \text{in } L^{p'}(T;X^*) \quad \text{and} \quad \limsup_{n\to+\infty}((u_n^*, u_n - u)) \leqslant 0. \tag{3.5}$$

We have

$$u_n^* = h_n^* + \widehat{f}_n, \quad \text{with } h_n^* \in \mathcal{A}(u_n),\ \widehat{f}_n \in \widehat{N}(u_n) \quad \forall n \geqslant 1.$$

Hypothesis $H(A)(iii)$ and (3.4) imply that by passing to a subsequence if necessary, we may assume that

$$h_n^* \xrightarrow{w} h^* \quad \text{in } L^{p'}(T;X^*) \quad \text{and} \quad \widehat{f}_n \xrightarrow{w} \widehat{f} \quad \text{in } L^{p'}(T;H) \quad \text{as } n \to +\infty. \tag{3.6}$$

From (3.4) and since $W_p(0,b) \hookrightarrow C(T;H)$, we have

$$u_n \xrightarrow{w} u \quad \text{in } C(T;H),$$

so

$$u_n(t) \xrightarrow{w} u(t) \quad \text{in } H \quad \forall t \in T. \tag{3.7}$$

We set $\xi_n(t) = \langle h_n^*, u_n(t) - u(t)\rangle$, for all $n \geqslant 1$. Then $\xi_n \in L^1(T)$ for all $n \geqslant 1$ and from Hu–Papageorgiou [13] (proof of Theorem 2.35, p. 41), we have

$$\xi_n(t) \longrightarrow 0 \quad \text{for a.a. } t \in T, \text{ as } n \to +\infty.$$

Hypotheses $H(A)(iii)$ and (iv) imply that

$$\xi_n(t) \geqslant c_2\|u_n(t)\|^p - a_2(t) - \left(a_1(t) + c_1\|u_n(t)\|^{p-1}\right)\|u(t)\| \quad \text{for a.a. } t \in T.$$

So, for a.a. $t \in T$, the sequence $\{u_n(t)\|_{n\geqslant 1} \subseteq X$ is bounded. This fact and (3.7) imply that

$$u_n(t) \xrightarrow{w} u(t) \quad \text{in } X. \tag{3.8}$$

From (3.6) and Proposition 3.9 of Hu–Papageorgiou [12, p. 694], we have

$$\begin{aligned} \widehat{f}(t) &\in \overline{\text{conv}}\, w\text{–}\limsup_{n\to+\infty}\{\widehat{f}_n(t)\}_{n\geqslant 1} \\ &\subseteq \overline{\text{conv}}\, w\text{–}\limsup_{n\to+\infty} \widehat{F}(t, u_n(t)) \\ &\subseteq \widehat{F}(t, u(t)) \quad \text{for a.a. } t \in T \end{aligned} \tag{3.9}$$

(recall that $\widehat{f}_n \in \widehat{N}(u_n)$ for all $n \geqslant 1$ and use Proposition 3.3(b) and (3.8)). From (3.4) and (3.5), we have

$$u_n \xrightarrow{w} u \quad \text{in } W_p(0,b), \tag{3.10}$$

so

$$u_n \longrightarrow u \quad \text{in } L^p(T;H) \tag{3.11}$$

(recall that $W_p(0,b) \hookrightarrow L^p(T;H)$ compactly since $X \hookrightarrow H$ compactly). Then from (3.6) and (3.11), we have

$$\begin{aligned} ((\widehat{f}_n, u_n - u)) &= \int_0^b \langle \widehat{f}_n(t), u_n(t) - u(t)\rangle \, \mathrm{d}t \\ &= \int_0^b (\widehat{f}_n(t), u_n(t) - u(t)) \, \mathrm{d}t \longrightarrow 0 \quad \text{as } n \to +\infty. \end{aligned} \tag{3.12}$$

So, from (3.5) and (3.12), we infer that

$$\limsup_{n\to+\infty} ((h_n^*, u_n - u)) \leqslant 0. \tag{3.13}$$

But from Lemma 3.2 we know that $\mathcal{A}$ is L-pseudomonotone. Therefore from (3.6), (3.10) and (3.13), it follows that

$$h^* \in \mathcal{A}(u) \quad \text{and} \quad ((h_n^*, u_n)) \longrightarrow ((h^*, u)). \tag{3.14}$$

Finally, we have

$$u^* = h^* + \widehat{f}$$

(see (3.5) and (3.6)),

$$h^* \in \mathcal{A}(u), \ \widehat{f} \in \widehat{N}(u)$$

(see (3.14) and (3.9)) and

$$((u_n^*, u_n)) \longrightarrow ((u^*, u))$$

(see (3.14) and (3.12)). We conclude that $u \longmapsto \mathcal{A}(u) + \widehat{N}(u)$ is L-pseudomonotone. □

PROPOSITION 3.5. *If hypotheses $H(A)$ and $H(F)_1$ hold, then $\mathcal{A} + \widehat{N}$ is strongly coercive.*

Proof. Let $h^* \in \mathcal{A}(u)$ and $\widehat{f} \in N(u)$. We have

$$((h^* + \widehat{f}, u)) = ((h^*, u)) + \int_0^b (\widehat{f}(t), u(t)) \, \mathrm{d}t. \tag{3.15}$$

Hypothesis $H(A)(iv)$ implies that

$$((h^*, u)) \geqslant c_2 \|u\|^p_{L^p(T;X)} - \|a_2\|_1. \tag{3.16}$$

Also using Proposition 3.3(b), we have

$$\int_0^b (\widehat{f}(t), u(t)) \, \mathrm{d}t = \int_{\{|u|<M\}} (\widehat{f}(t), u(t)) \, \mathrm{d}t + \int_{\{|u|\geqslant M\}} (\widehat{f}(t), u(t)) \, \mathrm{d}t$$

$$\geqslant -\|\vartheta\|_1 - \frac{1}{M}\|\vartheta\|_{p'}\|u\|_{L^p(T;X)} \tag{3.17}$$

(by Hölder inequality). Returning to (3.15) and using (3.16) and (3.17), we obtain

$$((h^* + \widehat{f}, u)) \geqslant c_2\|u\|^p_{L^p(T;X)} - c_3(\|u\|_{L^p(T;X)} + 1),$$

for some $c_3 > 0$, so $\mathcal{A} + \widehat{N}$ is strongly coercive (recall that $p > 1$). □

We consider the following auxiliary anti-periodic evolution inclusions:

$$\begin{cases} -u'(t) \in A(t, u(t)) + \widehat{F}(t, u(t)) & \text{for a.a. } t \in T, \\ u(0) = -u(b). \end{cases} \tag{3.18}$$

PROPOSITION 3.6. *If hypotheses $H(A)$ and $H(F)_1$ hold, then problem (3.18) admits a solution $u_0 \in W_p(0, b)$.*

Proof. From (3.3) and Propositions 3.3 and 3.4, we see that we can apply Theorem 2.2. So, we have $R(L + \mathcal{A} + \widehat{N}) = L^{p'}(T; X^*)$. Therefore, we can find $u_0 \in D(L)$ such that

$$-u_0' \in \mathcal{A}(u_0) + \widehat{N}(u_0),$$

so $u_0 \in W_p(0, b)$ is a solution of (3.18). □

Since $W_p(0, b) \hookrightarrow C(T; H)$, we have that $u_0 \in C(T; H)$. If we can show that

$$|u_0(t)| \leqslant M \quad \forall t \in T,$$

then on account of (3.1), we will have that $u_0 \in W_p(0, b)$ is a solution of (1.1). We do this in the next proposition exploiting the Hartman condition (see hypothesis $H(F)_1(v)$).

PROPOSITION 3.7. *If hypotheses $H(A)$ and $H(F)_1$ hold and $u_0 \in W_p(0, b)$ is a solution of problem (3.18), then $|u_0(t)| \leqslant M$ for all $t \in T$.*

Proof. First we assume that

$$|u_0(t)| > M \quad \forall t \in T. \tag{3.19}$$

We have

$$-u_0'(t) = h^*(t) + \widehat{f}(t) \quad \text{for a.a. } t \in T,$$

with $h^* \in \mathcal{A}(u_0)$, $\widehat{f} \in \widehat{N}(u_0)$, so

$$-u_0'(t) = h^*(t) + u_0(t) - p_M(u_0(t)) + f(t) \quad \text{for a.a. } t \in T, \tag{3.20}$$

with $f \in N(p_M(u_0)) = S^{p'}_{F(\cdot, p_M(u_0)(\cdot))}$ (see (3.19) and (3.20)). On (3.20) we act with $u_0(t) \in X$ and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_0(t)|^2 + \langle h^*(t), u_0(t)\rangle + |u_0(t)|^2 - (p_M(u_0(t)), u_0(t)) + (f(t), u_0(t)) \tag{3.21}$$

for a.a. $t \in T$. Note that

$$\begin{aligned} & |u_0(t)|^2 - (p_M(u_0(t)), u_0(t)) \\ & \quad = |u_0(t)|^2 - M|u_0(t)| \\ & \quad = |u_0(t)|(|u_0(t)| - M) \ > \ 0 \quad \text{for a.a. } t \in T. \end{aligned} \tag{3.22}$$

Also we have

$$(f(t), u_0(t)) \ = \ \Big(f(t), \frac{Mu_0(t)}{|u_0(t)|}\Big)\frac{|u_0(t)|}{M} \ \geqslant \ a_2(t) \quad \text{for a.a. } t \in T \tag{3.23}$$

(see hypothesis $H(A)(iv)$ and (3.19)). Returning to (3.21) and using (3.22) and (3.23) and hypothesis $H(A)(iv)$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_0(t)|^2 \ < \ 0,$$

so

$$|u_0(b)| \ < \ |u_0(0)|,$$

a contradiction. So, (3.19) cannot occur.

Next assume that there exist $\eta_0, \eta_1 \in T$ with $\eta_0 < \eta_1$, such that

$$|u_0(\eta_0)| \ = \ M \quad \text{and} \quad |u_0(t)| > M \quad \forall t \in (\eta_0, \eta_1). \tag{3.24}$$

Working on the interval (η_0, η_1) as above, we obtain

$$|u_0(\eta_1)| \ < \ |u_0(\eta_0)| \ = \ M$$

(see (3.23)), again a contradiction. Therefore, we conclude that $|u_0(t)| \leqslant M$ for all $t \in T$. □

Let $S_c \subseteq W_p(0, b)$ denote the solution set of problem (1.1) when the multivalued reaction term F is convex-valued. Then Proposition 3.7 and (3.1) lead to the following existence theorem.

THEOREM 3.8. *If hypotheses $H(A)$ and $H(F)_1$ hold, then $S_c \neq \emptyset$.*

4. Nonconvex problem

In this section, we look for solutions of problem (1.1) when F has nonconvex values. Now the hypotheses on F are the following.

$\underline{H(F)_2}$: $F\colon T \times H \longrightarrow P_f(H)$ is a multifunction such that

(i) $(t, x) \longmapsto F(t, x)$ is graph measurable;

(ii) for a.a. $t \in T$, $x \longmapsto F(t, x)$ is lower semicontinuous;

(iii) there exist $a_4 \in L^{p'}(T)_+$ and $c_4 > 0$ such that

$$|F(t,x)| = \sup_{v \in F(t,x)} |v| \leqslant a_4(t) + c_4|x|^{p-1} \quad \text{for a.a. } t \in T, \text{ all } x \in H;$$

(iv) there exists $\vartheta \in L^{p'}(T)_+$ such that

$$(v,x) \geqslant -\vartheta(t) \quad \text{for a.a } t \in T, \text{ all } x \in H, \ v \in F(t,x);$$

(v) there exists $M > 0$ such that

$$(v,x) \geqslant a_2(t) \quad \text{for a.a. } t \in T, \text{ all } x \in H, \ |x| = M, \ v \in F(t,x)$$

(see hypothesis $H(A)(iv)$).

By $S \subseteq W_p(0,b)$ we denote the solution set of problem (1.1) when the multivalued reaction term F is nonconvex-valued.

THEOREM 4.1. *If hypotheses $H(A)$ and $H(F)_2$ hold, then $S \neq \emptyset$.*

Proof. Let $\widehat{F}: T \times H \longrightarrow P_f(H)$ be defined by (3.1) and let $\widehat{N}: L^p(T;H) \longrightarrow 2^{L^{p'}(T;H)}$ be the corresponding multivalued Nemitsky operator defined by

$$\widehat{N}(u) = S^{p'}_{\widehat{F}(\cdot,u(\cdot))} \quad \forall u \in L^p(T;H).$$

As in the proof of Proposition 3.3, we show that

- $(t,x) \longmapsto \widehat{F}(t,x)$ is graph measurable;
- for a.a. $t \in T$, $x \longmapsto \widehat{F}(t,x)$ is lower semicontinuous;
- there exist $a_5 \in L^{p'}(T)_+$ and $c_5 > 0$ such that

$$|\widehat{F}(t,x)| = \sup_{\widehat{v} \in \widehat{F}(t,x)} |\widehat{v}| \leqslant a_5(t) + c_5|x| \quad \text{for a.a. } t \in T, \text{ all } x \in H.$$

These properties and Theorem 7.28 of Hu–Papageorgiou [12, p. 238] imply that $\widehat{N}$ has nonempty, closed and decomposable values and it is lower semicontinuous. So, we can use the Bressan–Colombo [5] selection theorem and produce a continuous function $e: L^p(T;H) \longrightarrow L^{p'}(T;H)$ such that

$$e(u) \in \widehat{N}(u) \quad \forall u \in L^p(T;H). \tag{4.1}$$

We consider the following anti-periodic evolution inclusion:

$$\begin{cases} -u'(t) \in A(t,u(t)) + e(u)(t) & \text{for a.a. } t \in T, \\ u(0) = -u(b). \end{cases} \tag{4.2}$$

Reasoning as in the "convex" case and using (4.1), we show that (4.2) has a solution $\widehat{u} \in W_p(0,b)$ such that $|\widehat{u}(t)| \leqslant M$ for all $t \in T$. From (3.1) we conclude that $\widehat{u} \in S$. $\square$

5. Extremal solutions: strong relaxation

In this section we look for extremal solutions (that is, solutions of the inclusion in which the multivalued forcing term is $\operatorname{ext} F(t,x)$, the set of extremal points of $F(t,x)$). Such solutions are important in control theory in connection with the bang-bang principle.

The anti-periodic evolution inclusion under consideration is the following:

$$\begin{cases} -u'(t) \in A(t,u(t)) + \operatorname{ext} F(t,u(t)) & \text{for a.a. } t \in T, \\ u(0) = -u(b). \end{cases} \tag{5.1}$$

To solve (5.1) we need to strengthen the hypotheses on A and on F. The new conditions on these items are the following.

$\underline{H(A)'}$: $A\colon T \times X \longrightarrow 2^{X^*}$ is a map such that

(i) for all $x \in X$, $t \longmapsto A(t,x)$ is graph measurable;
(ii) for a.a. $t \in T$, $x \longmapsto A(t,x)$ is strictly monotone and maximal monotone;
(iii) there exist $a_6 \in L^{p'}(T)$ and $c_6 > 0$ such that

$$\|h^*\|_* \leqslant a_6(t) + c_6\|x\|^{p-1} \quad \text{for a.a. } t \in T, \text{ all } x \in X, \; h^* \in A(t,x),$$

with $2 \leqslant p < +\infty$;
(iv) there exist $a_7 \in L^1(T)_+$, $c_7 > 0$ such that

$$\langle h^*, x\rangle \geqslant c_7\|x\|^p - a_7(t) \quad \text{for a.a. } t \in T, \text{ all } x \in X, \; h^* \in A(t,x).$$

$\underline{H(F)_3}$: $F\colon T \times H \longrightarrow P_{wkc}(H)$ is a multifunction such that

(i) for all $x \in H$, $t \longmapsto F(t,x)$ is graph measurable;
(ii) for a.a. $t \in T$, $x \longmapsto F(t,x)$ is h-continuous;
(iii) there exist $a_8 \in L^{p'}(T)_+$ and $c_8 > 0$ such that

$$|F(t,x)| = \sup_{v \in F(t,x)} |v| \leqslant a_8(t) + c_8|x|^{p-1} \quad \text{for a.a. } t \in T, \text{ all } x \in H;$$

(iv) there exists $\vartheta \in L^{p'}(T)_+$ such that

$$(v,x) \geqslant -\vartheta(t) \quad \text{for a.a } t \in T, \text{ all } x \in H, \; v \in F(t,x);$$

(v) there exists $M > 0$ such that

$$(v,x) \geqslant a_7(t) \quad \text{for a.a. } t \in T, \text{ all } x \in H, \; |x| = M, \; v \in F(t,x)$$

with $a_7 \in L^1(T)_+$ as in hypothesis $H(A)'(iv)$.

Let $g \in L^{p'}(T;H)$ and consider the following auxiliary anti-periodic problem:

$$\begin{cases} -u'(t) \in A(t,u(t)) + g(t) & \text{for a.a. } t \in T, \\ u(0) = -u(b). \end{cases} \tag{5.2}$$

PROPOSITION 5.1. *If hypotheses $H(A)'$ hold and $g \in L^{p'}(T; H)$, then problem (5.2) admits a unique solution $\xi(g) \in W_p(0, b)$ and the solution map*

$$\xi : L^{p'}(T; H) \longrightarrow C(T; H)$$

is completely continuous (that is, if $g_n \xrightarrow{w} g$ in $L^{p'}(T; H)$, then $\xi(g_n) \longrightarrow \xi(g)$ in $C(T; H)$).

Proof. Existence of a solution follows from Theorem 3.8. For the uniqueness, suppose that $u, v \in W_p(0, b) \subseteq C(T; H)$ are two solutions of (5.2). Then we have

$$u'(t) + h_u^*(t) + g(t) \ = \ 0 \quad \text{and} \quad v'(t) + h_v^*(t) + g(t) \ = \ 0 \quad \text{for a.a. } t \in T,$$

with $h_u^*, h_v^* \in L^{p'}(T; X^*)$ and $h_u^*(t) \in A(t, u(t))$, $h_v^*(t) \in A(t, v(t))$ for a.a. $t \in T$. We have

$$u'(t) - v'(t) + h_u^*(t) - h_v^*(t) \ = \ 0 \quad \text{for a.a. } t \in T.$$

Suppose that $u \neq v$. Taking duality brackets with $u(t) - v(t)$, using the integration by parts formula (see (2.1)), integrating first on $[0, t]$ and then on $[t, b]$, via the strict monotonicity of $A(t, \cdot)$ and the anti-periodic boundary condition, we have

$$|u(t) - v(t)| \ < \ |u(0) - v(0)|$$

and

$$|u(b) - v(b)| \ = \ |u(0) - v(0)| \ < \ |u(t) - v(t)|,$$

a contradiction. So $u \equiv v$. Therefore, the solution of (5.2) is unique.

Next we show that the solution map $\xi : L^{p'}(T; H) \longrightarrow C(T; H)$ is completely continuous. So, suppose that $g_n \xrightarrow{w} g$ in $L^{p'}(T; H)$ and let $u_n = \xi(g_n)$ for all $n \geqslant 1$. We have

$$-u_n'(t) \ = \ h_n^*(t) + g_n(t) \quad \text{for a.a. } t \in T, \quad u_n(0) = -u_n(b), \ n \geqslant 1. \tag{5.3}$$

We take duality brackets with $u_n(t)$ and obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|^2 + \langle h_n^*(t), u(t)\rangle + (g_n(t), u_n(t)) \ = \ 0 \quad \text{for a.a. } t \in T, \ \text{all } n \geqslant 1. \tag{5.4}$$

Integrating (5.4) on $T = [0, b]$ and using the anti-periodic boundary condition, we obtain

$$((h_n^*, u_n)) + \int_0^b (g_n, u_n)\, dt \ = \ 0 \quad \forall n \geqslant 1,$$

so

$$\|u_n\|^p_{L^p(T;X)} \ \leqslant \ c_8(\|u_n\|_{L^p(T;X)} + 1) \quad \forall n \geqslant 1,$$

for some $c_8 > 0$ (see hypothesis $H(A)'(iv)$ and recall that $\{g_n\}_{n\geqslant 1} \subseteq L^{p'}(T; H)$ is bounded), so

$$\text{the sequence } \{u_n\}_{n\geqslant 1} \subseteq L^p(T; H) \text{ is bounded.} \tag{5.5}$$

From (5.5), (5.3) and hypothesis $H(A)'(iii)$, it follows that

$$\text{the sequence } \{u_n'\}_{n\geqslant 1} \subseteq L^{p'}(T; X^*) \text{ is bounded.} \tag{5.6}$$

Then (5.4) and (5.5) imply that

$$\text{the sequence } \{u_n\}_{n\geqslant 1} \subseteq W_p(0, b) \text{ is bounded.}$$

So, by passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_p(0, b). \tag{5.7}$$

We set $u_n^* = h_n^* + g_n$ for $n \geqslant 1$ and have

$$u_n^* \xrightarrow{w} u^* \quad \text{in } L^{p'}(T; X^*).$$

Also, using the integration by parts formula and the anti-periodic boundary condition, we have

$$-((u_n', u_n - u)) = ((u_n', u)) \longrightarrow ((u', u)) = 0,$$

so

$$\lim_{n\to+\infty} ((u_n^*, u_n - u)) = 0. \tag{5.8}$$

Using (5.7) and (5.8) and reasoning as in the proof of Proposition 3.4, we obtain

$$u_n(t) \longrightarrow u(t) = \xi(g)(t) \quad \text{in } H \text{ for all } t \in T \tag{5.9}$$

and

$$u_n \longrightarrow u = \xi(g) \quad \text{in } L^p(T; H) \text{ as } n \to +\infty. \tag{5.10}$$

Hence $u^* = h^* + g$ with $h^* \in \mathcal{A}(u)$. For all $t \in T$, we have

$$\begin{aligned} |u_n(t) - u(t)|^2 \leqslant |u_n(0) - u(0)|^2 + 2\int_0^b |\langle h_n^*, u_n - u\rangle|\,\mathrm{d}t + 2|((h^*, u_n - u))| \\ +2\int_0^b |(g_n - g, u_n - u)|\,\mathrm{d}t. \end{aligned} \tag{5.11}$$

Note that

$$\int_0^b |(g_n - g, u_n - u)|\,\mathrm{d}t \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{5.12}$$

Also, if $\beta_n(t) = \langle h_n^*(t), u_n(t) - u(t)$ for $n \geqslant 1$, then from Hu–Papageorgiou [13, p. 41], we have

$$\beta_n \longrightarrow 0 \quad \text{in } L^1(T),$$

so

$$\int_0^b |\langle h_n^*, u_n - u\rangle|\, \mathrm{d}t \;\longrightarrow\; 0. \tag{5.13}$$

Finally, we have

$$((h_n^*, u_n - u)) \;\longrightarrow\; 0 \quad \text{and} \quad |u_n(0) - u(0)|^2 \;\longrightarrow\; 0 \tag{5.14}$$

(see (5.9)–(5.10)). Returning to (5.11) and using (5.12), (5.13), (5.14), we infer that

$$u_n \;\longrightarrow\; u \quad \text{in } C(T; H),$$

so ξ is completely continuous. □

As before we consider the modification $\widehat{F}$ of F defined by (3.1) and consider the corresponding anti-periodic problem (3.18). From Proposition 3.7 we know that every solution $u \in W_p(0, b)$ of problem (3.18) satisfies

$$|u(t)| \;\leqslant\; M \quad \forall t \in T = [0, b] \tag{5.15}$$

(here $M > 0$ is as in hypothesis $H(F)_3(v)$). Then from (3.2) and (5.15), we see that without any loss of generality, we may assume that

$$|\widehat{F}(t, x)| \;=\; \sup_{\widehat{v}\in\widehat{F}(t,x)} |\widehat{v}| \;\leqslant\; \gamma(t) \quad \text{for a.a. } t \in T, \text{ all } x \in H,$$

with $\gamma \in L^{p'}(T)_+$. Otherwise, we just replace $\widehat{F}$ by

$$\widehat{F}_0(t, x) \;=\; \begin{cases} F(t, x) & \text{if } |x| \leqslant M, \\ F(t, p_M(x)) & \text{if } |x| > M. \end{cases}$$

Then we introduce the set

$$V \;=\; \left\{g \in L^{p'}(T; H) : \; |g(t)| \leqslant \gamma(t) \text{ for a.a. } t \in T\right\}.$$

We have the following result.

PROPOSITION 5.2. *If hypotheses $H(A)'$ and $H(F)_3$ hold, then the set $\xi(V) \subseteq C(T; H)$ is compact.*

Proof. Proposition 5.1 and Proposition 3.1.7 of Gasiński–Papageorgiou [7, p. 268], imply that the solution map ξ is compact. Note that the set $V \subseteq L^{p'}(T; H)$ is w-compact (James theorem). Therefore, the set $\xi(V) \subseteq C(T; H)$ is compact. □

Let $K = \overline{\mathrm{conv}}\, \xi(V) \in P_{kc}(C(T; H))$ (see Gasiński–Papageorgiou [8, p. 852]). We consider the multifunction $\widehat{E} \colon K \longrightarrow P_{wkc}(L^{p'}(T; H))$ defined by

$$\widehat{E}(u) \;=\; S^{p'}_{\widehat{F}(\cdot,u(\cdot))} \quad \forall u \in K.$$

Invoking Theorem 8.31 of Hu–Papageorgiou [12, p. 260], we can find a continuous map $\widehat{e}\colon K \longrightarrow L^1_w(T;H)$ such that

$$\widehat{e}(u) \in \operatorname{ext} \widehat{E}(u) = \operatorname{ext} S^{p'}_{\widehat{F}(\cdot,u(\cdot))} = S^{p'}_{\operatorname{ext}\widehat{F}(\cdot,u(\cdot))} \quad \forall u \in K$$

(see Hu–Papageorgiou [12, p. 191]). We consider the following anti-periodic problem:

$$\begin{cases} -u'(t) \in A(t,u(t)) + \widehat{e}(u)(t) & \text{for a.a. } t \in T, \\ u(0) = -u(b). \end{cases} \tag{5.16}$$

As before (see Theorem 3.8), we obtain a solution $u \in W_p(0,b)$ of (5.16) such that

$$|u(t)| \leqslant M \quad \forall t \in T.$$

On account of (3.1), we see that u is also a solution of problem (5.1). Hence, if by $S_e \subseteq W_p(0,b)$ we denote the solution set of problem (5.1), then we have the following existence result.

THEOREM 5.3. *If hypotheses $H(A)'$ and $H(F)_3$ hold, then $S_e \neq \emptyset$.*

Next we show that every $u \in S_c$ can be approximate in $C(T;H)$ by elements in S_e (that is, $S_c \subseteq \overline{S_e}^{C(T;H)}$). Such a result is known as "strong relaxation theorem" and is important in control theory. The result says that every state of the system can be approximated by states generated by extremal (bang–bang) controls. This way we can economize in the use of control functions. To have such a result, we need to strengthen further the conditions on the multivalued forcing term F. The new hypotheses are the following:

$\underline{H(F)_4}$: $F\colon T \times H \longrightarrow P_{wkc}(H)$ is a multifunction such that

(i) for all $x \in H$, $t \longmapsto F(t,x)$ is graph measurable;

(ii) there exists $k \in L^2(T)$ such that

$$h(F(t,x),F(t,y)) \leqslant k(t)|x-y| \quad \text{for a.a. } t \in T, \text{ all } x,y \in H;$$

(iii) there exist $a_9 \in L^{p'}(T)_+$ and $c_9 > 0$ such that

$$|F(t,x)| = \sup_{v \in F(t,x)} |v| \leqslant a_9(t) + c_9\|x\|^{p-1} \quad \text{for a.a. } t \in T, \text{ all } x \in H;$$

(iv) there exists $\vartheta \in L^{p'}(T)_+$ such that

$$(v,x) \geqslant -\vartheta(t) \quad \text{for a.a } t \in T, \text{ all } x \in H, \ v \in F(t,x);$$

(v) there exists $M > 0$ such that

$$(v,x) \geqslant a_7(t) \quad \text{for a.a. } t \in T, \text{ all } x \in H, \ |x| = M, \ v \in F(t,x)$$

with $a_7 \in L^1(T)_+$ as in hypothesis $H(A)'(iv)$.

THEOREM 5.4. *If hypotheses $H(A)'$ and $H(F)_4$ hold, then $S_c \subseteq \overline{S_e}^{C(T;H)}$.*

Proof. Let $u \in S_c \subseteq W_p(0, b)$. Then by definition, we have

$$\begin{cases} -u'(t) \in A(t, u(t)) + f(t) & \text{for a.a. } t \in T, \\ u(0) = -u(b), \end{cases}$$

with $f \in S^{p'}_{F(\cdot,u(\cdot))}$.

Let $K \in P_{kc}(C(T; H))$ be as in the proof of Theorem 5.3. Let $y \in K$ and $\varepsilon > 0$ and consider the multifunction $G^y_\varepsilon : T \longrightarrow 2^H \setminus \{\emptyset\}$ defined by

$$G^y_\varepsilon(t) = \Big\{v \in F(t, y(t)) : |f(t) - v| < \frac{\varepsilon}{2m_0 b} + d(f(t), F(t, y(t)))\Big\},$$

where $m_0 = \sup\{\|w\|_{C(T;H)} : w \in K\}$.

Hypotheses $H(F)_4(i)$ and (ii) imply that the map $t \longmapsto G^y_\varepsilon(t)$ is graph measurable. So, we can apply the Yankov-von Neumann–Aumann selection theorem (see Hu–Papageorgiou [12, p. 158]) and obtain a measurable selection of the multifunction G^y_ε. Evidently this selection belongs in $L^{p'}(T; H)$ (see hypothesis $H(F)_4(iii)$).

We introduce the multifunction $\Gamma_\varepsilon : K \longrightarrow 2^{L^{p'}(T;H)}$ defined by

$$\Gamma_\varepsilon(y) = S^p_{G^y_\varepsilon(\cdot)} \quad \forall y \in K.$$

We have

- Γ_ε has nonempty and decomposable values;
- Γ_ε is lower semicontinuous (see Hu–Papageorgiou [12, Theorem 7.28, p.238]).

It follows that the map $y \longmapsto \overline{\Gamma_\varepsilon(y)}$ is lower semicontinuous. Then by Bressan–Colombo [5] selection theorem, we can find a continuous map $\gamma_\varepsilon : K \longrightarrow L^{p'}(T; H)$ such that

$$\gamma_\varepsilon(y) \in \overline{\Gamma_\varepsilon(y)} \quad \forall y \in K. \tag{5.17}$$

Then by Theorem 8.31 of Hu–Papageorgiou [12, p. 260], there exists a continuous map $\xi_\varepsilon : K \longrightarrow L^1_w(T; H)$ such that

$$\xi_\varepsilon(y) \in \operatorname{ext} \Gamma_\varepsilon(y) \quad \text{and} \quad \|\xi_\varepsilon(y) - \gamma_\varepsilon(y)\|_w \leqslant \varepsilon \quad \forall y \in K. \tag{5.18}$$

Now, let $\varepsilon_n \searrow 0$ and set $\gamma_n = \gamma_{\varepsilon_n}$, $\xi_n = \xi_{\varepsilon_n}$ and $z = u(0) = -u(b) \in H$. We consider the following Cauchy problem:

$$\begin{cases} -u_n'(t) \in A(t, u_n(t)) + \xi_n(u_n)(t) & \text{for a.a. } t \in T, \\ u_n(0) = z, \ n \geqslant 1. \end{cases} \tag{5.19}$$

We know that (5.19) has a solution $u_n \in W_p(0, b)$ (see Hu–Papageorgiou [13, Theorem 2.2, p. 19]). Exploiting the monotonicity of $A(t, \cdot)$ (see hypothesis $H(A)'(ii)$) and the integration by parts (see (2.1)), we have

$$
\begin{aligned}
&\frac{1}{2}|u_n(t)-u(t)|^2 \\
&\leqslant \int_0^t (f(s)-\xi_n(u_n)(s), u_n(s)-u(s))\,\mathrm{d}s \\
&= \int_0^t (f(s)-\gamma_n(u_n)(s), u_n(s)-u(s))\,\mathrm{d}s \\
&\quad + \int_0^t (\gamma_n(u_n)(s)-\xi_n(u_n)(s), u_n(s)-u(s))\,\mathrm{d}s \\
&\leqslant \varepsilon_n + \int_0^t k(s)|u_n(s)-u(s)|^2\,\mathrm{d}s \\
&\quad + \int_0^t (\gamma_n(u_n)(s)-\xi_n(u_n)(s), u_n(s)-u(s))\,\mathrm{d}s
\end{aligned}
\tag{5.20}
$$

(see (5.17)). From (5.18), we have

$$\|\gamma_n(u_n)-\xi_n(u_n)\|_w \ \leqslant\ \varepsilon_n \quad \forall n \geqslant 1,$$

so

$$\gamma_n(u_n)-\xi_n(u_n) \ \xrightarrow{w}\ 0 \quad \text{in } L^{p'}(T;H) \tag{5.21}$$

(see Hu–Papageorgiou [13, Lemma 2.8, p. 24]). Recall that the sequence $\{u_n\} \subseteq W_p(0,b)$ is bounded and $u_n \in K$ for all $n \geqslant 1$. Since the set $K \subseteq C(T;H)$ is compact, by passing to a suitable subsequence if necessary, we may assume that

$$u_n \ \xrightarrow{w}\ \widehat{u} \ \text{ in } W_p(0,b) \quad \text{and} \quad u_n \ \longrightarrow\ \widehat{u} \ \text{ in } C(T;H), \tag{5.22}$$

so

$$\int_0^t (\gamma_n(u_n)-\xi_n(u_n), u_n-u)\,\mathrm{d}s \ \longrightarrow\ 0 \tag{5.23}$$

(see (5.21)). We return to (5.21), pass to the limit as $n \to +\infty$ and use (5.22) and (5.23). Then

$$\frac{1}{2}|\widehat{u}(t)-u(t)|^2 \ \leqslant\ \int_0^t k(s)|\widehat{u}(s)-u(s)|^2\,\mathrm{d}s,$$

so $\widehat{u}=u$ (by the Gronwall inequality). Therefore, finally we have that $u_n \in S_e$ for all $n \geqslant 1$ and $u_n \longrightarrow u$ in $C(T;H)$. Thus $S_c \subseteq \overline{S_e}^{C(T;H)}$. □

As an example, we consider a distributed parameter control system. Let, $T=[0,b]$ and $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. The system under consideration is the following:

$$
\begin{cases}
\frac{\partial u}{\partial t} - \Delta_p u = f(t,z,u) + (l(z), v)_{\mathbb{R}^m} & \text{on } T \times \Omega, \\
u(0,\cdot) = -u(b,\cdot), \ u|_{T\times\partial\Omega} = 0, \ v(z) \in V(z) \text{ a.e.}
\end{cases}
\tag{5.24}
$$

In this problem, Δ_p is the p-Laplace differential operator defined by

$$\Delta_p u \ =\ \operatorname{div}(|Du|^{p-2}Du).$$

We assume that $2 \leqslant p < +\infty$ (if $p = 2$, then $\Delta_p = \Delta$, the usual Laplace differential operator). Also $v\colon Z \longrightarrow \mathbb{R}^m$ is the control function and V is the control constraint multifunction.

We impose the following conditions on the data of the problem (5.24):

- $f\colon T \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function such that
  (a) there exist $a \in L^{p'}(T \times \Omega)$ and $c \in L^\infty(\Omega)$ such that
  $$|f(t,z,x)| \leqslant a(t,z) + c(z)|x|^{p-1} \quad \text{for a.a. } (t,z) \in T \times \Omega, \text{ all } x \in \mathbb{R};$$
  (b) there exists $k \in L^2(T \times \Omega)$ such that
  $$|f(t,z,x) - f(t,z,x')| \leqslant k(t,z)|x - x'| \quad \text{for a.a. } (t,z) \in \Omega, \text{ all } x, x' \in \mathbb{R};$$
  (c) $f(t,z,x)x \geqslant 0$ for a.a. $(t,z) \in T \times \Omega$, all $x \in \mathbb{R}$;
- $V\colon \Omega \longrightarrow P_{kc}(\mathbb{R}^m)$ is graph measurable and there exists $\vartheta \in L^2(\Omega)$ such that
  $$|V(z)| = \sup_{v \in V(z)} |v| \leqslant \vartheta(z) \quad \text{for a.a. } z \in \Omega;$$
- $l \in L^\infty(\Omega; \mathbb{R}^m)$ and $(l(z), v)_{\mathbb{R}^m} \geqslant 0$ for a.a. $z \in \Omega$, all $v \in V(z)$.

For problem (5.24), the evolution triple is

$$X = W_0^{1,p}(\Omega), \quad H = H^* = L^2(\Omega), \quad X^* = W^{-1,p'}(\Omega)$$

(where $\frac{1}{p} + \frac{1}{p'} = 1$). From the Sobolev embedding theorem, we have that

$$X \hookrightarrow H \text{ compactly}$$

(hence that is true also for $H = H^* \hookrightarrow X^*$).

Let $A\colon X \longrightarrow X^*$ be defined by

$$\langle A(u), v \rangle = \int_A |Du|^{p-2}(Du, Dv)_{\mathbb{R}^N} \, \mathrm{d}z \quad \forall u, v \in X.$$

Then A is strictly monotone, continuous, hence maximal monotone too. Applying Hölder's inequality and Poincaré's inequality, we have

$$\|A(u)\|_* \leqslant \|Du\|_p^{p-1} = \|u\|^{p-1.}$$

Moreover, we have

$$\langle A(u), u \rangle = \|Du\|_p^p = \|u\|^p \quad \forall u \in X.$$

Let $\widehat{f}\colon T \times H \longrightarrow H$ be the Nemitsky operator corresponding to $f(t,z,x)$ and defined by

$$\widehat{f}(t,u)(\cdot) = f(t, \cdot, u(\cdot)) \quad \forall (t,u) \in T \times H.$$

We consider the multifunction $F\colon T \times H \longrightarrow P_{wkc}(H)$ defined by

$$F(t,u) \;=\; \left\{\widehat{f}(t,u) + (l,v)_{L^2(\Omega;\mathbb{R}^m)} \,:\; v \in S^2_{V(\cdot)}\right\}.$$

Then it is easy to see that $F(t,u)$ satisfies hypotheses $H(F)_4$.

We rewrite (5.24) as the following equivalent inclusion:

$$\begin{cases} -u'(t) \;\in A(u(t)) + F(t,u(t)) & \text{for a.a. } t \in T, \\ u(0) \;=\; -u(b). \end{cases}$$

For this inclusion, we can easily apply Theorem 5.4. So, if u is a state of (5.24), then given any $\varepsilon > 0$, we can find a state $\widehat{u}_\varepsilon$ generated by a bang–bang control $\widehat{v}_\varepsilon \in S^2_{\mathrm{ext}\,V}$ such that

$$\|u(t,\cdot) - \widehat{u}(t,\cdot)\|_{L^2(\Omega)} \;\leqslant\; \varepsilon \quad \forall t \in T.$$

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